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ORIGINAL PAPER

Sufficient Condition for the Existence of an Even $[a, b]$-Factor in Graph

Mekkia Kouider · Saliha Ouatiki

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Abstract Let a, b, be two even integers. In this paper, we get a sufficient condition which involves the stability number, the minimum degree of the graph for the existence of an even $[a, b]$-factor.

Keywords Even factor · 2-Edge connected · Minimum degree · Stability number

1 Introduction

We consider finite undirected graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For two vertices u and v of G, let uv and vu denote an *edge* joining u to v. For a subset A of $V(G)$, let $|A|$ be the number of vertices in A. The *order* of G is $|G| = |V(G)| = n$. Given disjoint subsets $A, B \subseteq V(G)$, we write $e_G(A, B)$ for the number of the edges in G with one extremity in A and the other one in B. Thus $e_G(v, V(G) - v) = d_G(v)$ is the *degree* of v and $\delta(G) = \min\{d_G(v) : v \in V(G)\}$ is the *minimum degree* of G. A subgraph of G containing all of $V(G)$ but possibly not all of $E(G)$ is called a *spanning subgraph* of G or a *factor* in G.

Let g, f be mappings from $V(G)$ into the nonnegative integers $\mathbf{N}$ and such that $g(v) \leq f(v)$, for all $v \in V(G)$. Then F is called a $[g, f]$-*factor* of G if F is a factor of G with $g(v) \leq d_F(v) \leq f(v)$ for all v in $V(G)$. For two integers a and b with

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M. Kouider
LRI, Univ. Paris-Sud, UMR 8623, 91405, Orsay Cedex, France
e-mail: km@lri.fr

S. Ouatiki (✉)
LAID3, Faculté de Mathématiques, U.S.T.H.B, BP. 32, El-Alia, Algeria
e-mail: saliha_ouatiki@yahoo.fr

$1 \leq a \leq b$, an $[a, b]$-*factor* of G is defined to be a spanning subgraph F of G such that $a \leq d_F(v) \leq b$ for all $v \in G$.

A factor F satisfying $d_F(v) \equiv 0(mod2)$ for all v in $V(G)$, is called *even*. An edge e in $E(G)$ is a *bridge* if $G - e$ has more components than G. A graph with at least 3 vertices is 2-*edge connected* if it is connected and has no bridge. The minimum number of vertices whose deletion disconnects the graph is said the *connectivity* and it is noted $\kappa(G)$.

For $S \subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S. We write $G - S$ for $G[V(G)\backslash S]$. A vertex set $S \subseteq V(G)$ is called *independent* if $G[S]$ has no edges. Denote by $\alpha(G)$ the *stability number* of a graph G (i.e., the cardinality of a maximum independent set of G).

Consider functions g, f on $V(G)$ with $g(v) \leq f(v)$ for each $v \in V(G)$ and an ordered pair X, Y of disjoint subsets of $V(G)$. A component C of $G - (X \cup Y)$ is called *odd* if $\sum_{v \in V(C)} f(v) + e_G(V(C), Y)$ is an odd number. The number of odd components in $G - (X \cup Y)$ is denoted by $h(X, Y)$.

We recall below the well-known Lovàsz theorem, characterizing graphs having an even $[g, f]$-factor and a fortiori an even $[a, b]$-factor.

Theorem 1 (Lovàsz's parity $[g, f]$-factor Theorem [10]) *Let G be a graph, let g and f be maps from $V(G)$ into the nonnegative integers such that $g(v) \leq f(v), \forall v \in V(G)$, and $g(v) \equiv f(v)(mod2), \forall v \in V(G)$. Then G contains a $[g, f]$-factor F such that $d_F(v) \equiv f(v)(mod2), \forall v \in V(G)$, if and only if, for every ordered pair X, Y of disjoint subsets of $V(G)$,*

$$\sum_{y \in Y} d_G(y) - \sum_{y \in Y} g(y) + \sum_{x \in X} f(x) - h(X, Y) - e_G(X, Y) \geq 0.$$

Let a and $b \geq 2$ be even integers and in the theorem above, let $g(v) = a$, $f(v) = b, \forall v \in V(G)$. Then we immediately obtain.

Corollary 1 *G contains an even $[a, b]$-factor if and only if*

$$\sum_{y \in Y} d_G(y) - a|Y| + b|X| - h(X, Y) - e_G(X, Y) \geq 0,$$

for all ordered pairs X, Y of disjoint subsets of $V(G)$.

2 Known Results

The well-known necessary and sufficient condition for the existence of an $[a, b]$-factor established by Tutte [14] is also a corollary of the (g, f)-factor theorem of Lovàsz in [10]. Many authors have worked on $[a, b]$-factors as it can be seen in [1,9,13,14] but only few results are established for the existence of an $[a, b]$-factor which involves the stability number and the minimum degree. Some ones relate the stability number and the connectivity as that of Nishimura [12] and Neumann-Lara and Rivera-Campo in [11].

Nishimura had established a sufficient condition for the existence of an odd regular factor.

Theorem 2 [12] *Let $r \geq 1$ be an odd integer, and G be a graph of even order of connectivity κ. If $\kappa \geq (r+1)^2/2$, and, $\alpha(G) \leq \frac{4r\kappa}{(r+1)^2}$, then G has an r-factor.*

Studying connected factors was initiated by Kano [3]. A similar result to that of Neumann-Lara and Rivera-Campo for the connected $[2, b]$-factors was shown by Brandt (private communication). For the existence of an f-factor, a condition for the stability number was given in [4]. A sufficient condition on the order and on the minimum degree or on the edge-connectivity for graphs to contain an even $[a, b]$-factor are given by Kouider and Vestergaard in [8]. They prove in [7], that if the graph G of order n is 2-edge connected and each vertex of G has degree at least $\max\{3, \frac{2n}{b+2}\}$ then G has an even $[2, b]$-factor. In [2], they obtain a relationship between the stability number and a connected factor.

Zhou [15] defines a graph G to be (a, b, k)-critical graph if after deleting any k vertices from G, the remaining graph has an even $[a, b]$-factor. He proved that if $\kappa(G) \geq \max\left\{\frac{(a+1)b+2k}{2}; \frac{(a+1)^2\alpha(G)+4bk}{4b}\right\}$ then the graph G is an (a, b, k)-critical. For $k = 0$, we get a condition for the existence of an $[a, b]$-factor in graphs which is close to that established by Kouider and Lonc [5] for the κ-connected graphs. We cite the result of Kouider and Lonc in [5] concerning their condition on the minimum degree and the stability number for the existence of an $[a, b]$-factor in graphs.

Theorem 3 [5] *Let $b \geq a + 1$ and let G be a graph with the minimum degree δ. If $\alpha(G) \leq \begin{cases} 4b(\delta - a + 1)/(a+1)^2, & \text{for } a \text{ odd;} \\ 4b(\delta - a + 1)/a(a+2), & \text{for } a \text{ even.} \end{cases}$ then G has an $[a, b]$-factor.*

Let a, b be two even integers. We give below an example of a graph satisfying the condition of the theorem above, which has an $[a, b]$-factor, but no even $[a, b]$-factor.

Example 1 Let t and p be 2 integers such that $t = 2p+1$. Let $a = 2p$, $b = (2p+2)^3$. We consider $t + 1$ disjoint complete graphs, $G_1, \ldots, G_t$: t copies of K_{2p} and a copy of K_b, furthermore there are 2 external vertices u and v. For each $i \leq t$, let y_i be a fixed vertex of G_i. The vertex v is adjacent to the vertices y_i, $i \leq t$, the vertex u to $V(G_i) - y_i$ for each i. The graph G we obtain has minimum degree $2p$. G has no even $[a, b]$ factor F otherwise we should have $d_F(v) = t$. Nevertheless the condition of the precedent theorem is satisfied.

The existence of an even factor with degrees bounded by the constant a, b is characterized for the complete bipartite graphs by Kouider and Vestergaard [7].

Theorem 4 [7] *For $3 \leq p \leq q$, let $K_{p,q}$ be a complete, bipartite graph and let $b \geq 2$ be an even integer. Then the graph $K_{p,q}$ has an even $[2, b]$-factor if and only if $q \leq \frac{b}{2}p$.*

It follows from this theorem that the complete bipartite graph $G = K_{p,q}$ has an even $[2, b]$-factor if and only if $\alpha(G) \leq \frac{b}{2}\delta(G)$.

A sufficient condition for the existence of an even $[2, b]$-factor for the κ-connected graphs was established by Kouider and Ouatiki in [6].

Theorem 5 [6] *Let $b \geq 6$ be an even integer and let G be a κ-connected graph with the minimum degree δ such that $b \leq \kappa$ and $\alpha(G) < (b-1)(\delta-1)/5$ then G contains an even $[2, b]$-factor.*

3 Main Results

We generalize the result obtained in Theorem 5 to even factors with degrees between a and b where a is an even integer ≥ 2, in the following form.

Theorem 6 *Let a, b be two even integers and let G be a 2-edge connected graph with the minimum degree δ such that $\delta \geq 2a$ and $\alpha(G) \leq \frac{4b(\delta-a)}{(a+1)^2}$, then, G contains an even $[a, b]$-factor.*

Proof We prove this theorem by contradiction. Suppose that G does not contain any even $[a, b]$-factor graph. It follows from the Lovàsz's condition 1 that there exists an ordered pair X, Y of disjoint subsets of $V(G)$ for which

$$\tau(X, Y) = \sum_{y \in Y} d_{G-X}(y) - a|Y| + b|X| - h(X, Y) < 0. \qquad (*)$$

Claim $Y \neq \emptyset$.

Proof If $|Y| = 0$, then $\tau(X, \emptyset) = b|X| < 0$, which is impossible.

Claim $X \neq \emptyset$.

Proof If $|X| = 0$, it follows from $(*)$ that $\tau(\emptyset, Y) = \sum_{y \in Y} d_G(y) - a|Y| - h(\emptyset, Y) < 0$.

We have

$$2a|Y| \leq \delta|Y| \leq \sum_{y \in Y} d_G(y). \qquad (1)$$

Otherwise, as G is 2-edge connected graph, then

$$2h(\emptyset, Y) \leq \sum_{y \in Y} d_G(y). \qquad (2)$$

From the Eqs. (1) and (2), we deduce that

$$2 \sum_{y \in Y} d_G(y) \geq 2a|Y| + 2h(\emptyset, Y).$$

The last inequality implies that

$$\sum_{y \in Y} d_G(y) - a|Y| - h(\emptyset, Y) \geq 0,$$

which is equivalent to $\tau(\emptyset, Y) \geq 0$. So we get a contradiction with $(*)$.

Claim $|Y| > \frac{b}{a}|X|$.

Proof We have $h(X, Y) \leq \sum_{y\in Y} d_{G-X}(y) < a|Y| - b|X| + h(X, Y)$. This implies that $a|Y| - b|X| > 0$ and we get the claim.

We propose the following partition of Y: let x_1 be a vertex in Y such that $d(x_1) = \min_{x\in G[Y]} d(x)$ and let $N_1 = N_G[x_1] \cap Y$ and $Y_1 = Y$. For $i \geq 2$, if $Y - \bigcup_{1\leq j<i} N_j \neq \emptyset$, let $Y_i = Y - \bigcup_{1\leq j<i} N_j$, we take then a vertex x_i in Y_i such that $d(x_i) = \min_{x\in G[Y_i]} d(x)$ and $N_i = N_G[x_i] \cap Y_i$. We continue this process, we will get at the rank $i = r + 1$, $N_i = \emptyset$. It follows from this definition that the set $\{x_1, \ldots, x_r\}$ is an independent set in G.

As $Y \neq \emptyset$ then $r \geq 1$. Let $|N_i| = n_i$, we have $|Y| = \sum_{i=1}^{r} n_i$ and we get

$$\sum_{y\in Y} d_{G-X}(y) \geq \sum_{y\in Y} d_Y(y) + h(X, Y) \geq h(X, Y) + \sum_{i=1}^{r}\sum_{y\in N_i} d_{Y_i}(y)$$
$$\geq h(X, Y) + \sum_{i=1}^{r} n_i(n_i - 1). \quad (3)$$

From the Eqs. $(*)$ and (3), we deduce that

$$h(X, Y) + \sum_{i=1}^{r}[n_i(n_i - 1) - an_i] \leq \sum_{y\in Y} d_{G-X}(y) - a|Y| < h(X, Y) - b|X|. \quad (4)$$

One can easily verify that the function $f(n_i) = n_i^2 - (1 + a)n_i$ has its minimum at $n_1 = \frac{1+a}{2}$ so $f(n_i) \geq f(\frac{1+a}{2})$. Thus,

$$h(X, Y) + \frac{-(1 + a)^2 r}{4} \leq \sum_{y\in Y} d_{G-X}(y) - a|Y| < h(X, Y) - b|X|. \quad (5)$$

On the other hand, we have

$$\alpha(G) \geq \alpha(G[Y]) \geq r. \quad (6)$$

Let us prove the following result.

Claim $|X| < \delta - a$.

Proof Suppose that $|X| \geq \delta - a$. According to the Eqs. (5) and (6), we deduce that

$$b|X| < \frac{(1 + a)^2 r}{4} \leq \frac{(1 + a)^2 \alpha(G)}{4},$$

hence,

$$b(\delta - a) \le b|X| < \frac{(1+a)^2}{4}\alpha(G).$$

So $\alpha(G) > \frac{4b(\delta-a)}{(1+a)^2}$, which is a contradiction.
We can deduce from $(*)$ that

$$(\delta - |X| - a)|Y| \le \sum_{y\in Y} d_{G-X}(y) - a|Y| < -b|X| + h(X,Y). \tag{7}$$

From the Claim 3 and Eq. (7), as $|X| < \delta - a$, then $|Y| < \frac{-b|X|+h(X,Y)}{\delta-|X|-a}$ and $h(X,Y) \le \alpha(G)$, we get $|Y| < \frac{-b|X|+\frac{4b(\delta-a)}{(a+1)^2}}{\delta-|X|-a}$.

We deduce that $|Y| < \frac{4b}{(a+1)^2}\left(1 - \frac{((a+1)^2-4)|X|}{4(\delta-|X|-a)}\right) < \frac{4b}{(a+1)^2}$.

By the Claim 3, we have $|X| < \frac{a}{b}|Y| < \frac{4a}{(a+1)^2} < 1$, which implies that $|X| = 0$ and contradicts the Claim 3. This ends the proof of the theorem.

In the Theorem 6, it is necessary to require that G is 2-edge connected graph as shown in the following example.

Example 2 Let a, b, δ, t be four integers such that $\delta \ge a^2$, $b \ge (a+1)^2$, and $a \le \delta \le t \le 4(\delta - a)$. The integers a and b are even non zero integers.

Let us consider t disjoint copies of a complete graph K_{t+1}, and let x_0 be a vertex with exactly a neighbor on each copy. So, in the resultant graph G, $d(x_0) = t$. The graph G has no even $[a, b]$-factor, since if such factor F exists, F will have at least a components of $F - \{x_0\}$ each of them with exactly one vertex of odd degree.

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