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ORIGINAL PAPER

Linear Coloring of Planar Graphs Without 4-Cycles

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Abstract A proper vertex coloring of a graph G is linear if the graph induced by the vertices of any two color classes is the union of vertex-disjoint paths. The linear chromatic number $\mathrm{lc}(G)$ of G is the smallest number of colors in a linear coloring of G. In this paper, we prove that if G is a planar graph without 4-cycles, then $\mathrm{lc}(G) \leq \lceil \frac{\Delta}{2} \rceil + 8$, where Δ denotes the maximum degree of G.

Keywords Linear coloring · Planar graph · Cycle · Maximum degree

1 Introduction

All graphs considered in this paper are finite simple graphs. For a graph G, we use $V(G)$, $E(G)$, $|G|$, $\delta(G)$ and $\Delta(G)$ (or simply Δ) to denote, respectively, its vertex set, edge set, order, minimum degree, and maximum degree. If G is a plane graph, then we use $F(G)$ to denote the set of faces. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1 u_2 \ldots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices of $b(f)$ in a cyclic order. Repeated occurrences of a vertex are allowed. The degree of a face is the number of edge-steps in its boundary walk. Note that each cut-edge is

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counted twice. For $x \in V(G) \cup F(G)$, let $d_G(x)$ (or simply $d(v)$) denote the degree of v in G, and $N_G(v)$ denote the set of neighbors of v in G. A *k-vertex*, *k^+-vertex*, or *k^--vertex* is a vertex of degree k, at least k, or at most k. Similarly, we can define *k-face*, *k^+-face*, *k^--face*, etc. The *girth* $g(G)$ of a graph G is the length of a shortest cycle in G.

A *proper k-coloring* of a graph G is a mapping c from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that any two adjacent vertices have different colors. A *linear k-coloring* of a graph G is a proper k-coloring of G such that the graph induced by the vertices of any two color classes is the union of vertex-disjoint paths. The *linear chromatic number* $\mathrm{lc}(G)$ of the graph G is the smallest number k such that G has a linear k-coloring.

Yuster [6] introduced the linear coloring of graphs. With the probabilistic method, he proved that $\mathrm{lc}(G) = O(\Delta^{\frac{3}{2}})$ for a general graph G, and constructed graphs G such that $\mathrm{lc}(G) = \Omega(\Delta^{\frac{3}{2}})$. A related concept about linear coloring is the frugal coloring of graphs, considered by Hind et al. [2]. A graph G is *k-frugal* if G can be properly colored so that no color appears k times in any vertex neighborhood. Such a coloring is called a *k-frugal coloring* of G. So a linear coloring is a 3-frugal coloring.

Esperet et al. [1] generalized the linear coloring of graphs to list coloring version. They investigated linear choosability for some classical families of graphs such as trees, grids, bipartite complete graphs, planar graphs, outerplanar graphs, graphs with maximum degree 3 or 4, graphs with given maximum average degree, etc.

Raspaud and Wang [4] investigated the linear coloring of planar graphs. Amang other things, they proved the following result:

Theorem 1 ([4]) *Every planar graph G has* $\mathrm{lc}(G) = \lceil \frac{\Delta(G)}{2} \rceil + 1$ *if there is a pair* $(\Delta, g) \in \{(13, 7), (7, 9), (5, 11), (3, 13)\}$ *such that G satisfies* $\Delta(G) \geq \Delta$ *and* $g(G) \geq g$.

Wang and Li [5] extended the above result by showing the following:

Theorem 2 ([5]) *Every graph G embeddable in surfaces of nonnegative Euler characteristic has* $lc(G) = \lceil \frac{\Delta(G)}{2} \rceil + 1$ *if there is a pair* $(\Delta, g) \in \{(13, 7), (9, 8), (7, 9), (5, 10), (3, 13)\}$ *such that G satisfies* $\Delta(G) \geq \Delta$ *and* $g(G) \geq g$.

Suppose that G is a graph with maximum degree Δ. Li, Wang and Raspaud [3] recently proved the following results:

(i) $\mathrm{lc}(G) \leq \frac{1}{2}(\Delta^2 + \Delta)$ for any graph G;
(ii) $\mathrm{lc}(G) \leq 8$ for any graph G with $\Delta \leq 4$;
(iii) $\mathrm{lc}(G) \leq 14$ for any graph G with $\Delta \leq 5$; and
(iv) $\mathrm{lc}(G) \leq \lfloor 0.9\Delta \rfloor + 5$ if G is planar and $\Delta \geq 52$.

The following problem is put forward in [3]:

Question 1 *Is there a constant C such that every planar graph G has* $\mathrm{lc}(G) \leq \lceil \frac{\Delta}{2} \rceil + C$?

In this paper, we answer positively Question 1 for planar graphs without 4-cycles.

2 Main Results

A vertex v with $2 \le d(v) \le 3$ is called *bad* if it is incident to a 3-face. A 4-vertex v is called *bad* if it is incident to two nonadjacent 3-faces. A vertex v with $2 \le d(v) \le 4$ is called *good* if it is not bad. For a vertex $v \in V(G)$ and an integer $i \ge 1$, let $n_i(v)$ denote the number of i-vertices adjacent to v, and let $n'_i(v)$ denote the number of bad i-vertices adjacent to v for $2 \le i \le 4$. For a face $f \in F(G)$ and an integer $k \ge 3$, let $m_k(f)$ denote the number of k-vertices incident to f. Moreover, for $3 \le k \le 4$, let $m'_k(f)$ denote the number of bad k-vertices incident to f. For a vertex $v \in V(G)$, we use $t(v)$ to denote the number of 3-faces incident to v.

Lemma 3 *Every connected planar graph G without* 4*-cycles contains one of the following configurations:*

(C1) *a 1^--vertex;*
(C2) *a path $x_1x_2x_3x_4$ such that $d(x_2) = d(x_3) = 3$, $d(x_1) \le 6$ and $d(x_4) \le 6$;*
(C3) *a k-vertex v, $2 \le k \le 4$, whose neighbors $v_1, v_2, \ldots, v_k$ satisfy one of the following conditions, assuming $d(v_1) \le d(v_2) \le \cdots \le d(v_k)$:*
 (C3.1) *$k = 2$ and $d(v_1) \le 11$;*
 (C3.2) *$k = 3$, $d(v_1) + d(v_2) \le 14$, and v is bad;*
 (C3.3) *$k = 4$, $d(v_1) + d(v_2) + d(v_3) \le 15$, and v is bad.*

Proof Suppose to the contrary that the lemma is false. Let G be a counterexample. Then we have the following assertions:

(a) $\delta(G) \ge 2$.
(b) Since G does not contain 4-cycles, the following (b1) and (b2) hold:
 (b1) G contains neither 4-faces nor adjacent 3-faces;
 (b2) Every vertex v satisfies $t(v) \le \lfloor \frac{d(v)}{2} \rfloor$.
(c) Since (C3.1) is excluded from G, every face f satisfies $m_2(f) \le \lfloor \frac{d(f)}{2} \rfloor$.

Using Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ and the relations

$$\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E(G)|,$$

we can derive the following identity.

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2d(f) - 6) = -12. \qquad (1)$$

We define the initial weight function w by $w(v) = d(v) - 6$ if $v \in V(G)$ and $w(f) = 2d(f) - 6$ if $f \in F(G)$. We are going to redistribute the weight $w(x)$ to its adjacent or incident elements according to the discharging rules (R1) and (R2). During the process, the total sum of all weights is fixed. However, after the discharging is complete, the new weight function w' satisfies $w'(x) \ge 0$ for all $x \in V(G) \cup F(G)$.

This leads to an obvious contradiction and the proof is complete:

$$0 \leq \sum_{x \in V(G) \cup F(G)} w'(x) = \sum_{x \in V(G) \cup F(G)} w(x) = -12. \tag{2}$$

Our discharging rules (R1) and (R2) are as follows:

(R1) Every 7^+-vertex sends 1 to each adjacent bad 2-vertex, and $(d(v) - 6 - n'_2(v))/(n'_3(v) + n'_4(v))$ to each adjacent bad 3- or 4-vertex.

(R2) Let f be a 5^+-face.

- (R2.1) For each occurrence of a 5^--vertex u in $b(f)$, we transfer from f to u the amount of weight 2 if $d(u) = 2$, 1 if $d(u) = 3$, $\frac{2}{3}$ if $d(u) = 4$, and $\frac{1}{3}$ if $d(u) = 5$.
- (R2.2) For each bad 3- or 4-vertex x in $b(f)$, we further transfer from f to x the amount of weight $\beta(f)/(m'_3(f) + m'_4(f))$, where $\beta(f) = 2d(f) - 6 - 2m_2(f) - m_3(f) - \frac{2}{3}m_4(f) - \frac{1}{3}m_5(f)$.

For $x, y \in V(G) \cup F(G)$, we use $\tau(x \to y)$ to denote the amount of weights discharged from x to y according to the above rules (R1) and (R2.1). For a face f and a bad 3- or 4-vertex x, we use $\rho(f \to x)$ to denote the amount of weights discharged from f to x under (R2.2).

Claim 1 *Suppose that a 12^+-vertex v is adjacent to a bad 3- or 4-vertex u. Then $\tau(v \to u) \geq \frac{1}{2}$; moreover $\tau(v \to u) > \frac{1}{2}$ if either $d(v) \geq 13$, or v is adjacent to a 6^+-vertex x which is not incident to a 3-face $[vxy]$ with $d(y) = 2$.*

Proof Since G contains neither (C3.1) nor adjacent 3-faces, the number of 12^+-vertices adjacent to v is at least $n'_2(v)$. Thus, $n'_3(v) + n'_4(v) \leq d(v) - 2n'_2(v)$. By (R1) and since $d(v) \geq 12$, we have:

$$\begin{aligned}\tau(v \to u) &= \frac{d(v) - 6 - n'_2(v)}{n'_3(v) + n'_4(v)} \geq \frac{d(v) - 6 - n'_2(v)}{d(v) - 2n'_2(v)} \\ &= 1 - \frac{6 - n'_2(v)}{d(v) - 2n'_2(v)} \geq 1 - \frac{6 - n'_2(v)}{12 - 2n'_2(v)} = \frac{1}{2}.\end{aligned}$$

If $d(v) \geq 13$, we derive immediately $\tau(v \to u) > \frac{1}{2}$.

Assume that $d(v) = 12$ and v is adjacent to a 6^+-vertex x which is not incident to a 3-face $[vxy]$ with $d(y) = 2$. Then v gives x nothing by our rules. This implies that $n'_3(v) + n'_4(v) \leq d(v) - 2n'_2(v) - 1 = 11 - 2n'_2(v)$. By (R1) and noting $n'_2(v) \leq 5$ in this case, we have:

$$\tau(v \to u) = \frac{d(v) - 6 - n'_2(v)}{n'_3(v) + n'_4(v)} \geq \frac{6 - n'_2(v)}{11 - 2n'_2(v)} > \frac{1}{2}.$$

This proves Claim 1. □

Since G contains no (C3.1), every vertex v with $7 \leq d(v) \leq 11$ is not adjacent to a 2-vertex.

Claim 2 *Let v be a vertex with $7 \leq d(v) \leq 11$ and u be a bad 3- or 4-vertex adjacent to v. Then (R1) asserts that $\tau(v \to u)$ is at least $\frac{1}{7}$ if $d(v) = 7$, $\frac{1}{4}$ if $d(v) = 8$, $\frac{1}{3}$ if $d(v) = 9$, $\frac{2}{5}$ if $d(v) = 10$, and $\frac{5}{11}$ if $d(v) = 11$.*

Claim 3 *Let v be a vertex with $7 \leq d(v) \leq 11$ and u be a bad 3- or 4-vertex adjacent to v. If v is adjacent to a 6^+-vertex, then (R1) asserts that $\tau(v \to u)$ is at least $\frac{1}{6}$ if $d(v) = 7$, $\frac{2}{7}$ if $d(v) = 8$, $\frac{3}{8}$ if $d(v) = 9$, $\frac{4}{9}$ if $d(v) = 10$, and $\frac{1}{2}$ if $d(v) = 11$.*

We carry out (R1) and (R2) in G. Let w' denote the resultant weight function after discharging. It suffices to prove that $w'(x) \geq 0$ for all $x \in V(G) \cup F(G)$.

Let $f \in F(G)$ such that $f = [x_1 x_2 \dots x_{d(f)}]$. Then $d(f) \neq 4$ by (b1). If $d(f) = 3$, then it holds trivially that $w'(f) = w(f) = 0$. Assume that $d(f) \geq 5$. It suffices to verify that $\beta(f) \geq 0$ by (R2.1) and (R2.2).

First, assume that $d(f) = 5$, then $w(f) = 4$. By (c), $m_2(f) \leq 2$.

- If $m_2(f) = 2$, then we may assume that $d(x_1) = d(x_3) = 2$ and $d(x_i) \geq 12$ for $i = 2, 4, 5$ by (C3.1). Thus $m_3(f)+m_4(f)+m_5(f) = 0$ and so $\beta(f) = 4-2\cdot 2 = 0$ by (R2.1).
- If $m_2(f) = 1$, then $b(f)$ contains at least two 12^+-vertices, thus $\beta(f) \geq 4 - 2 - 2 \cdot 1 = 0$ by (R2.1).
- If $m_2(f) = 0$, then $m_3(f) \leq 3$ by (C2). If $b(f)$ contains a 6^+-vertex, then $\beta(f) \geq 4 - 3 \cdot 1 - \frac{2}{3} = \frac{1}{3}$. Otherwise, assume that $d(x_i) \leq 5$ for all $i = 1, 2, \dots, 5$. If $m_3(f) \leq 2$, then $\beta(f) \geq 4 - 2 \cdot 1 - 3 \cdot \frac{2}{3} = 0$. If $m_3(f) = 3$, then $b(f)$ must contain a configuration (C2), a contradiction.

Next, assume that $d(f) = 6$, then $w(f) = 6$. By (c), $m_2(f) \leq 3$. If $m_2(f) = 3$, then (C3.1) asserts that $b(f)$ contains exactly three 12^+-vertices, thus $\beta(f) = 6-3\cdot 2 = 0$ by (R2.1). If $m_2(f) = 2$, then $b(f)$ contains at least three 12^+-vertices so that $\beta(f) \geq 6 - 2 \cdot 2 - 1 = 1$. If $m_2(f) = 1$, then $b(f)$ contains at least two 12^+-vertices so that $\beta(f) \geq 6 - 2 - 3 \cdot 1 = 1$. If $m_2(f) = 0$, then $\beta(f) \geq 6 - 6 \cdot 1 = 0$.

Finally, assume that $d(f) \geq 7$. Since G contains no (C3.1), the number of 12^+-vertices in $b(f)$ is at least $m_2(f)$. In addition, it is easy to derive that $m_3(f) + m_4(f) + m_5(f) \leq d(f) - 2m_2(f)$. By (R2.1), we have:

$$\begin{aligned}\beta(f) &= w(f) - (2m_2(f) + m_3(f) + \frac{2}{3}m_4(f) + \frac{1}{3}m_5(f)) \\ &\geq 2d(f) - 6 - (2m_2(f) + m_3(f) + m_4(f) + m_5(f)) \\ &\geq 2d(f) - 6 - d(f) = d(f) - 6 \geq 1.\end{aligned}$$

Let $v \in V(G)$. Then $d(v) \geq 2$ by (a). Let $v_1, v_2, \dots, v_{d(v)}$ denote the neighbors of v in a cyclic order.

- If $d(v) = 6$, then it is trivial to conclude that $w'(v) = w(v) = 0$.
- If $d(v) \geq 12$, then Claim 1 implies that $w'(v) \geq 0$.
- If $7 \leq d(v) \leq 11$, then v is not adjacent to any 2-vertex by (C3.1). Since $w(v) = d(v) - 6 \geq 1$, it is easy to derive that $w'(v) \geq 0$ by (R1).
- If $d(v) = 5$, then $w(v) = -1$, and $t(v) \leq 2$ by (b2). Thus, $w'(v) \geq -1+3 \cdot \frac{1}{3} = 0$ by (R2.1).

- If $d(v) = 2$, then $w(v) = -4$, and v is adjacent to two 12^+-vertices by (C3.1). Since G is simple, $t(v) \leq 1$. If $t(v) = 0$, then $w'(v) \geq -4 + 2 \cdot 2 = 0$ by (R2.1). Otherwise, $w'(v) = -4 + 2 + 2 \cdot 1 = 0$.

In what follows, we assume that $3 \leq d(v) \leq 4$. If v is a good 3-vertex, i.e., v is incident to three 5^+-faces, then $w'(f) \geq 3 - 6 + 3 \cdot 1 = 0$ by (R2.1). If v is a good 4-vertex, i.e., v is incident to at least three 5^+-faces, then $w'(f) \geq 4 - 6 + 3 \cdot \frac{2}{3} = 0$ by (R2.1). So assume that v is a bad 3- or a bad 4-vertex.

Let $f = [y_0 y_1 \ldots y_5]$ be a 6-face. For $0 \leq i \leq 5$, let f_i denote the face adjacent to f with $y_i y_{i+1} \in b(f) \cap b(f_i)$, where indices are taken modulo 6. We say that f is of *type 1* if $d(y_i) = 4$ and $d(f_i) = 3$ for all $i = 0, 1, \ldots, 5$, i.e., all vertices in $b(f)$ are bad 4-vertices. The face f is of *type 2* if $d(y_0) = d(y_2) = 3, d(y_1) \geq 5$ and $d(y_i) = 4$ for $i = 3, 4, 5$, and $d(f_i) = 3$ for $i = 2, 3, 4, 5$. Note that y_0, y_2 are bad 3-vertices and y_3, y_4, y_5 are bad 4-vertices.

Claim 4 *Let f be a 6^+-face and x a bad 3- or 4-vertex incident to f. Then $\rho(f \to x) = \frac{1}{3}$ if f is a type 1 6-face, $\rho(f \to x) = \frac{2}{5}$ if f is a type 2 6-face, and $\rho(f \to x) \geq \frac{4}{9}$ otherwise.*

Proof In the following argument, we simply write $m'_k = m'_k(f)$, $m_k = m_k(f)$, etc. By the previous proof, we notice that $\beta(f) \geq d(f) - 6 \geq 1$ when $d(f) \geq 7$. Thus, if $d(f) \geq 11$, then $\rho(f \to x) \geq (d(f) - 6)/d(f) = 1 - (6/d(f)) \geq 1 - \frac{6}{11} = \frac{5}{11}$ by (R2.2). We have to consider some cases.

Case 1 $9 \leq d(f) \leq 10$.

If $b(f)$ contains a vertex that is not a bad 3- and 4-vertex, then $\rho(f \to x) \geq (d(f) - 6)/(d(f) - 1) = 1 - (5/d(f)) \geq 1 - \frac{5}{9} = \frac{4}{9}$ by (R2.2). Otherwise, $m'_3 + m'_4 = d(f)$, implying that $m'_i = m_i$ for $i = 3, 4$, $m_2 = m_i = 0$ for all $i \geq 5$. Since G contains no (C2), $m_3 \leq \lfloor \frac{2}{3} d(f) \rfloor$. Consequently,

$$\begin{aligned}
\rho(f \to x) &= \frac{2d(f) - 6 - m_3 - \frac{2}{3} m_4}{d(f)} \\
&= \frac{2d(f) - 6 - m_3 - \frac{2}{3}(d(f) - m_3)}{d(f)} \\
&= \frac{\frac{4}{3} d(f) - 6 - \frac{1}{3} m_3}{d(f)} \geq \frac{\frac{4}{3} d(f) - 6 - \frac{1}{3} \lfloor \frac{2}{3} d(f) \rfloor}{d(f)} \\
&\geq \frac{10}{9} - \frac{6}{d(f)} \geq \frac{10}{9} - \frac{6}{9} = \frac{4}{9}.
\end{aligned}$$

Case 2 $d(f) = 8$.

Then $w(f) = 10$ and $m_2 \leq 4$ by (C3.1). If $m_2 = 4$, then it is easy to see that $m'_3 = m'_4 = 0$. If $m_2 = 3$, then $b(f)$ contains at least four 12^+-vertices, thus $\rho(f \to x) \geq 10 - 3 \cdot 2 - 1 = 3$. If $m_2 = 2$, then $b(f)$ contains at least three 12^+-vertices, therefore $\rho(f \to x) \geq \frac{1}{3}(10 - 2 \cdot 2 - 3 \cdot 1) = 1$. If $m_2 = 1$, then $b(f)$ contains at least two 12^+-vertices, thus $\rho(f \to x) \geq \frac{1}{5}(10 - 2 - 5 \cdot 1) = \frac{3}{5}$.

Now suppose that $m_2 = 0$. If $b(f)$ contains at least two 6^+-vertices, then $\rho(f \to x) \geq \frac{1}{6}(10 - 6 \cdot 1) = \frac{2}{3}$. If $b(f)$ contains exactly one 6^+-vertex, then $m_3 \leq 5$ by (C2), hence $\rho(f \to x) \geq \frac{1}{7}(10-5\cdot 1-2\cdot\frac{2}{3}) = \frac{11}{21}$. Assume that no 6^+-vertex occurs in $b(f)$. Then $m_3 \leq 4$ by (C2). If $m_3 \leq 3$, then $\rho(f \to x) \geq \frac{1}{8}(10-3\cdot 1-5\cdot\frac{2}{3}) = \frac{11}{24}$. If $m_3 = 4$, then no 3-vertex in $b(f)$ is bad by (C3.2), hence $\rho(f \to x) \geq \frac{1}{4}(10-4\cdot 1-4\cdot\frac{2}{3}) = \frac{5}{6}$.

Case 3 $d(f) = 7$.

Then $w(f) = 8$ and $m_2 \leq 3$ by (C3.1). If $m_2 = 3$, then it is easy to observe that $m'_3 = m'_4 = 0$. If $m_2 = 2$, then $b(f)$ contains at least three 12^+-vertices, hence $\rho(f \to x) \geq \frac{1}{2}(8 - 2\cdot 2 - 2\cdot 1) = 1$. If $m_2 = 1$, then $b(f)$ contains at least two 12^+-vertices, hence $\rho(f \to x) \geq \frac{1}{4}(8 - 2 - 4\cdot 1) = \frac{1}{2}$.

Suppose that $m_2 = 0$. If $b(f)$ has at least two 6^+-vertices, then $\rho(f \to x) \geq \frac{1}{5}(8 - 5\cdot 1) = \frac{3}{5}$.

Assume that $b(f)$ has exactly one 6^+-vertex, then $m_3 \leq 4$. If $m_3 \leq 2$, then $\rho(f \to x) \geq \frac{1}{6}\left(8 - 2\cdot 1 - 4\cdot\frac{2}{3}\right) = \frac{5}{9}$. If $m_3 = 3$, then it is easy to see that at least one 3-vertex in $b(f)$ is not bad since G contains no (C3.2). Thus, $\rho(f \to x) \geq \frac{1}{5}(8 - 3\cdot 1 - 3\cdot\frac{2}{3}) = \frac{3}{5}$. If $m_3 = 4$, then at least two 3-vertices in $b(f)$ are not bad and hence $\rho(f \to x) \geq \frac{1}{4}\left(8 - 4\cdot 1 - 2\cdot\frac{2}{3}\right) = \frac{2}{3}$.

Assume that $b(f)$ has no 6^+-vertex, then $m_3 \leq 3$ by (C2). Since G contains no (C3.2), no 3-vertex in $b(f)$ may be bad. If $m_3 = 0$, then $\rho(f \to x) \geq \frac{1}{7}(8-7\cdot\frac{2}{3}) = \frac{10}{21}$. If $m_3 \geq 1$, then $m_4 \leq 6$, and $\rho(f \to x) \geq (8 - m_3 - \frac{2}{3}m_4)/m'_4 \geq (8 - (d - m_4) - \frac{2}{3}m_4)/m_4 = \frac{1}{3} + \frac{1}{m_4} \geq \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$.

Case 4 $d(f) = 6$.

Then $w(f) = 6$ and $m_2 \leq 3$ by (C3.1). If $m_2 = 3$, then it is easy to observe that $m'_3 = m'_4 = 0$. If $m_2 = 2$, then $b(f)$ has at least three 12^+-vertices, hence $\rho(f \to x) \geq 6 - 2\cdot 2 - 1 = 1$. If $m_2 = 1$, then $b(f)$ has at least two 12^+-vertices, hence $m_3 \leq 3$. When $m_3 \leq 2$, we have $\rho(f \to x) \geq \frac{1}{3}(6 - 2 - 2\cdot 1 - \frac{2}{3}) = \frac{4}{9}$. When $m_3 = 3$, it is easy to see that $m'_3 \leq 2$ since G contains no (C3.2), thus $\rho(f \to x) \geq \frac{1}{2}(6 - 2 - 3\cdot 1) = \frac{1}{2}$.

Suppose that $m_2 = 0$. If $b(f)$ has at least two 6^+-vertices, then $\rho(f \to x) \geq \frac{1}{4}(6 - 4\cdot 1) = \frac{1}{2}$.

Assume that $b(f)$ has exactly one 6^+-vertex, then $m_3 \leq 4$ by (C2) and $m'_3 \leq 2$ by (C3.2). If $m_3 \leq 1$, then $\rho(f \to x) \geq \frac{1}{5}(6 - 1 - 4\cdot\frac{2}{3}) = \frac{7}{15}$. If $3 \leq m_3 \leq 4$, then it is easy to see that $m'_3 + m'_4 \leq 3$, hence $\rho(f \to x) \geq \frac{1}{3}(6 - 4\cdot 1 - \frac{2}{3}) = \frac{4}{9}$. Let $m_3 = 2$. If $m'_3 + m'_4 \leq 4$, then $\rho(f \to x) \geq \frac{1}{4}(6 - 2\cdot 1 - 3\cdot\frac{2}{3}) = \frac{1}{2}$. If $m'_3 + m'_4 = 5$, then f must be a type 2 6-face, so that $\rho(f \to x) = \frac{1}{5}(6 - 2\cdot 1 - 3\cdot\frac{2}{3}) = \frac{2}{5}$.

Assume that $b(f)$ has no 6^+-vertex, then $m_3 \leq 3$ by (C2) and $m'_3 = 0$ by (C3.2). If $m_3 = 3$, then $m'_4 = 0$. If $1 \leq m_3 \leq 2$, then $m'_4 \leq 3$, and $\rho(f \to x) \geq \frac{1}{3}(6 - 2\cdot 1 - 3\cdot\frac{2}{3}) = \frac{2}{3}$. Let $m_3 = 0$. If $m'_4 \leq 4$, then $\rho(f \to x) \geq \frac{1}{4}(6 - 6\cdot\frac{2}{3}) = \frac{1}{2}$. If $m'_4 = 6$, then f is a type 1 6-face and $\rho(f \to x) = \frac{1}{6}(6 - 6\cdot\frac{2}{3}) = \frac{1}{3}$. Suppose that $m'_4 = 5$. If $m_4 = 5$, then $\rho(f \to x) \geq \frac{1}{5}(6 - 5\cdot\frac{2}{3} - \frac{1}{3}) = \frac{7}{15}$. If $m_4 = 6$, it follows easily that $m'_4 = m_4 = 6$, contradicting the assumption. This proves Claim 4. □

Now suppose that v is a bad 3-vertex with neighbors v_1, v_2, v_3 such that $[vv_1v_2]$ is a 3-face with $d(v_1) \leq d(v_2)$. Let f_1 and f_2 be two other incident faces of v different from $[vv_1v_2]$ such that $vv_1 \in b(f_1)$ and $vv_2 \in b(f_2)$. By (R2.1), each of f_1 and f_2 gives 1 to v. In order to prove that $w'(v) = -3 + 2 \cdot 1 + \sigma \geq 0$, it suffices to confirm that $\sigma = \rho(f_1 \to v) + \rho(f_2 \to v) + \tau(v_1 \to v) + \tau(v_2 \to v) + \tau(v_3 \to v) \geq 1$.

Since G contains no (C3.1), we see that $d(v_i) \geq 3$ for $i = 1, 2, 3$. Since G contains no (C3.2), $d(v_k) + d(v_j) \geq 15$ for any two distinct $k, j \in \{1, 2, 3\}$. If $d(v_1) = 3$, then $d(v_i) \geq 12$, and $\tau(v_i \to v) \geq \frac{1}{2}$ for $i = 2, 3$ by Claim 1, therefore $\sigma \geq \frac{1}{2} + \frac{1}{2} = 1$. If $d(v_2) = 3$, we have a similar discussion. If $d(v_3) \leq 4$, then $d(v_i) \geq 11$, and $\tau(v_i \to v) \geq \frac{1}{2}$ by Claims 1 and 3 for $i = 1, 2$, therefore $\sigma \geq 1$. Thus, we assume that $d(v_1), d(v_2) \geq 4$ and $d(v_3) \geq 5$.

Claim 5 *Let $i \in \{1, 2\}$. If $d(f_i) = 5$, then $\rho(f_i \to v) \geq \frac{1}{6}$ if $d(v_i) = 4$, and $\rho(f_i \to v) \geq \frac{1}{3}$ otherwise.*

Proof By symmetry, we only give the proof for the case $i = 1$. Assume that $f_1 = [v_1vv_3yx]$. We consider three cases as follows.

Case 1 $d(v_1) = 4$.

Then $d(v_3) \geq 11$, and $d(x) \geq 3$ by (C3.1). If $d(y) = 2$, then $d(x) \geq 12$, and $\rho(f_1 \to v) \geq \frac{1}{2}(4 - 2 - 1 - \frac{2}{3}) = \frac{1}{6}$. So assume that $d(y) \geq 3$. If $d(x) \geq 4$ or $d(y) \geq 4$, then $\rho(f_1 \to v) \geq \frac{1}{4}(4 - 2 \cdot 1 - 2 \cdot \frac{2}{3}) = \frac{1}{6}$. If $d(x) = d(y) = 3$, then x is not bad by (C3.2), and further v_1 is not bad, thus $\rho(f_1 \to v) \geq \frac{1}{2}(4 - 3 \cdot 1 - \frac{2}{3}) = \frac{1}{6}$.

Case 2 $d(v_1) = 5$.

Then $d(v_3) \geq 10$, and $d(x) \geq 3$ by (C3.1). If $d(y) = 2$, then $d(x) \geq 12$, and $\rho(f_1 \to v) \geq 4 - 2 - 1 - \frac{1}{3} = \frac{2}{3}$. So assume that $d(y) \geq 3$. If $d(x) \geq 4$ or $d(y) \geq 4$, then $\rho(f_1 \to v) \geq \frac{1}{3}(4 - 2 \cdot 1 - \frac{2}{3} - \frac{1}{3}) = \frac{1}{3}$. If $d(x) = d(y) = 3$, then x is not bad by (C3.2), thus $\rho(f_1 \to v) \geq \frac{1}{2}\left(4 - 3 \cdot 1 - \frac{1}{3}\right) = \frac{1}{3}$.

Case 2 $d(v_1) \geq 6$.

If $d(x) = 2$ or $d(y) = 2$, then $\rho(f_1 \to v) \geq 4 - 2 - 1 - \frac{1}{3} = \frac{2}{3}$. So assume that $d(x), d(y) \geq 3$. If $d(x) \geq 4$ or $d(y) \geq 4$, then $\rho(f_1 \to v) \geq \frac{1}{3}(4 - 2 \cdot 1 - \frac{2}{3} - \frac{1}{3}) = \frac{1}{3}$. Assume that $d(x) = d(y) = 3$. If at least one of x and y is not bad, then $\rho(f_1 \to v) \geq \frac{1}{2}(4 - 3 \cdot 1 - \frac{1}{3}) = \frac{1}{3}$. Otherwise, it follows from (C3.2) that $d(v_3) \geq 12$ and thus $\rho(f_1 \to v) \geq \frac{1}{3}(4 - 3 \cdot 1) = \frac{1}{3}$. This completes the proof of Claim 5. □

It remains to show that $\sigma \geq 1$ by applying the previous Claims 1–5. Without loss of generality, we may assume that $d(v_1) \leq d(v_2)$.

If $4 \leq d(v_1) \leq 6$, then $d(v_i) \geq 9$ and $\tau(v_i \to v) \geq \frac{1}{3}$ for $i = 2, 3$ by Claims 1 and 2. By Claims 4 and 5, each of f_1, f_2 gives at least $\frac{1}{6}$ to v. Consequently, $\sigma \geq 2 \cdot \frac{1}{3} + 2 \cdot \frac{1}{6} = 1$.

If $d(v_1) \geq 7$, then $\tau(v_i \to v) \geq \frac{1}{6}$ for $i = 1, 2$ by Claims 1 and 3, and $\rho(f_i \to v) \geq \frac{1}{3}$ for $i = 1, 2$ by Claims 4 and 5. Thus, $\sigma \geq 2 \cdot \frac{1}{6} + 2 \cdot \frac{1}{3} = 1$.

Finally, suppose that v is a bad 4-vertex such that $[vv_1v_2]$ and $[vv_3v_4]$ are 3-faces. Let f_1 and f_2 be the two other incident faces of v different from $[vv_1v_2]$ and $[vv_3v_4]$ such that $vv_1, vv_4 \in b(f_1)$ and $vv_2, vv_3 \in b(f_2)$. Then $d(f_i) \geq 5$ and $\tau(f_i \to v) = \frac{2}{3}$ by (R2.1). In order to show that $w'(v) = -2 + 2 \cdot \frac{2}{3} + \sigma^* \geq 0$, it suffices to inspect that

$\sigma^* = \rho(f_1 \to v) + \rho(f_2 \to v) + \tau(v_1 \to v) + \tau(v_2 \to v) + \tau(v_3 \to v) + \tau(v \to v_4) \geq \frac{2}{3}$.

Since G contains no (C3.1), we see that $d(v_i) \geq 3$ for $i = 1, 2, 3, 4$. Since G contains no (C3.3), $d(v_i)+d(v_j)+d(v_k) \geq 16$ for any three mutually distinct indices $i, k, j \in \{1, 2, 3, 4\}$.

Claim 6 *If $d(f_1) = 5$, then the following statements hold:*

(1) $\rho(f_1 \to v) \geq \frac{2}{15}$ *if $d(v_1) = d(v_4) = 4$;*
(2) $\rho(f_1 \to v) \geq \frac{1}{6}$ *if either $d(v_1) = 3$ and $d(v_4) \geq 4$, or $d(v_4) = 3$ and $d(v_1) \geq 4$;*
(3) $\rho(f_1 \to v) \geq \frac{1}{4}$ *if either $d(v_1) = 4$ and $d(v_4) \geq 5$, or $d(v_4) = 4$ and $d(v_1) \geq 5$;*
(4) $\rho(f_1 \to v) \geq \frac{4}{9}$ *if either $d(v_1) = d(v_4) = 3$, or $d(v_1) \geq 5$ and $d(v_4) \geq 5$.*

Proof Assume that $f_1 = [v_1 v v_4 y x]$. Without loss of generality, assume that $d(v_1) \leq d(v_4)$. We need to consider some cases as follows.

Case 1 $d(v_1) = 3$.

Since G contains no (C3.2), $d(x) \geq 15 - d(v) = 15 - 4 = 11$. If $d(v_4) = 3$, then similarly $d(v_4) \geq 11$ by (C3.2), and $\rho(f_1 \to v) \geq \frac{1}{3}(4-2\cdot 1-\frac{2}{3}) = \frac{4}{9}$. If $d(v_4) \geq 4$, then $\rho(f_1 \to v) \geq \frac{1}{4}(4 - 2 \cdot 1 - 2 \cdot \frac{2}{3}) = \frac{1}{6}$.

Case 2 $d(v_1) = 4$.

Since G contains no (C3.1), $d(x) \geq 3$. The proof is split into some subcases.

(2.1) $d(v_4) = 4$. Then $d(y) \geq 3$ by (C3.1). By symmetry, we may suppose that $d(x) \leq d(y)$. If $d(x) \geq 5$, then $\rho(f_1 \to v) \geq \frac{1}{3}(4 - 3 \cdot \frac{2}{3} - 2 \cdot \frac{1}{3}) = \frac{4}{9}$. If $d(x) = 4$, then $\rho(f_1 \to v) \geq \frac{1}{5}(4 - 5 \cdot \frac{2}{3}) = \frac{2}{15}$. Assume that $d(x) = 3$, then $d(y) \geq 4$ by (C2). If $d(y) \geq 5$, then $\rho(f_1 \to v) \geq \frac{1}{4}(4 - 1 - 3 \cdot \frac{2}{3} - \frac{1}{3}) = \frac{1}{6}$. If $d(y) = 4$, then x is not a bad 3-vertex and further y, v_1 are not bad 4-vertices, thus $\rho(f_1 \to v) \geq \frac{1}{2}(4 - 1 - 4 \cdot \frac{2}{3}) = \frac{1}{6}$.

(2.2) $d(v_4) = 5$. Then $d(y) \geq 3$ by (C3.1). If $d(x), d(y) \geq 4$, then $\rho(f_1 \to v) \geq \frac{1}{4}(4 - 4 \cdot \frac{2}{3} - \frac{1}{3}) = \frac{1}{4}$.

Assume that $d(x) = 3$. Then $d(y) \geq 4$ by (C2). If $d(y) \geq 5$, then $\rho(f_1 \to v) \geq \frac{1}{3}(4 - 1 - 2 \cdot \frac{2}{3} - 2 \cdot \frac{1}{3}) = \frac{1}{3}$. If $d(y) = 4$, then x is not a bad 3-vertex, further v_1, y are not bad 4-vertices. Thus, $\rho(f_1 \to v) \geq 4 - 1 - 3 \cdot \frac{2}{3} - \frac{1}{3} = \frac{2}{3}$.

Assume that $d(y) = 3$. Then $d(x) \geq 4$ by (C2). If $d(x) \geq 5$, then $\rho(f_1 \to v) \geq \frac{1}{3}(4 - 1 - 2 \cdot \frac{2}{3} - 2 \cdot \frac{1}{3}) = \frac{1}{3}$. If $d(x) = 4$, then y and x are not bad, thus $\rho(f_1 \to v) \geq \frac{1}{2}(4 - 1 - 3 \cdot \frac{2}{3} - \frac{1}{3}) = \frac{1}{3}$.

(2.3) $d(v_4) \geq 6$. If $d(y) = 2$, then $d(x) \geq 12$, and hence $\rho(f_1 \to v) \geq \frac{1}{2}(4 - 2 - 2 \cdot \frac{2}{3}) = \frac{1}{3}$. If $d(y) \geq 4$, then $\rho(f_1 \to v) \geq \frac{1}{4}(4 - 1 - 3 \cdot \frac{2}{3}) = \frac{1}{4}$. Assume that $d(y) = 3$. If $d(x) \geq 4$, then $\rho(f_1 \to v) \geq \frac{1}{4}(4 - 1 - 3 \cdot \frac{2}{3}) = \frac{1}{4}$. If $d(x) = 3$, then x, v_1 are not bad, thus $\rho(f_1 \to v) \geq \frac{1}{2}(4 - 2 \cdot 1 - 2 \cdot \frac{2}{3}) = \frac{1}{3}$.

Case 3 $d(v_1) = 5$.

Since G contains no (C3.1), $d(x) \geq 3$. There are two subcases:

(3.1) $d(v_4) = 5$. Then $d(y) \geq 3$ by (C3.1). By symmetry, we may suppose that $d(x) \leq d(y)$. Since G contains no (C2), we derive that $d(y) \geq 4$. If $d(x) \geq 4$, then $\rho(f_1 \to v) \geq \frac{1}{3}(4 - 3 \cdot \frac{2}{3} - 2 \cdot \frac{1}{3}) = \frac{4}{9}$. If $d(x) = 3$, then x is not bad, thus $\rho(f_1 \to v) \geq \frac{1}{2}(4 - 1 - 2 \cdot \frac{2}{3} - 2 \cdot \frac{1}{3}) = \frac{1}{2}$.

(3.2) $d(v_4) \geq 6$. If $d(y) = 2$, then $d(x) \geq 12$ and $\rho(f_1 \to v) \geq 4 - 2 - \frac{2}{3} - \frac{1}{3} = 1$. If $d(y) \geq 4$, then $\rho(f_1 \to v) \geq \frac{1}{3}(4 - 1 - 2 \cdot \frac{2}{3} - \frac{1}{3}) = \frac{4}{9}$. Assume that $d(y) = 3$. If $d(x) \geq 4$, we have similarly $\rho(f_1 \to v) \geq \frac{4}{9}$. If $d(x) = 3$, then x is not bad, thus $\rho(f_1 \to v) \geq \frac{1}{2}(4 - 2 \cdot 1 - \frac{2}{3} - \frac{1}{3}) = \frac{1}{2}$.

Case 4 $d(v_1) \geq 6$.

If $d(x) = 2$, then $d(y) \geq 12$, thus $\rho(f_1 \to v) \geq 4 - 2 - \frac{2}{3} = \frac{4}{3}$. If $d(y) = 2$, we have a similar proof. If $d(x), d(y) \geq 3$, then $\rho(f_1 \to v) \geq \frac{1}{3}(4 - 2 \cdot 1 - \frac{2}{3}) = \frac{4}{9}$. This proves Claim 6. □

With the same reason, we can prove the following:

Claim 7 *If $d(f_2) = 5$, then the following statements hold:*

(1) $\rho(f_2 \to v) \geq \frac{2}{15}$ *if $d(v_2) = d(v_3) = 4$;*
(2) $\rho(f_2 \to v) \geq \frac{1}{6}$ *if either $d(v_2) = 3$ and $d(v_3) \geq 4$, or $d(v_3) = 3$ and $d(v_2) \geq 4$;*
(3) $\rho(f_2 \to v) \geq \frac{1}{4}$ *if either $d(v_2) = 4$ and $d(v_3) \geq 5$, or $d(v_3) = 4$ and $d(v_2) \geq 5$;*
(4) $\rho(f_2 \to v) \geq \frac{4}{9}$ *if either $d(v_2) = d(v_3) = 3$, or $d(v_2) \geq 5$ and $d(v_3) \geq 5$.*

Using Claims 1–4, 6 and 7, we will show that $\sigma^* \geq \frac{2}{3}$. In fact, if $d(v_i) \geq 5$ for all $i = 1, 2, 3, 4$, then $\rho(f_j \to v) \geq \frac{1}{3}$ for $j = 1, 2$ by Claims 4, 6 and 7, thus $\sigma^* \geq \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. Thus, we may assume that $d(v_1) = \min_{1 \leq i \leq 4}\{d(v_i)\} \leq 4$. The proof is split into the following cases:

Case 1 $d(v_2) \leq d(v_i)$ for $i = 3, 4$.

(1.1) $d(v_1) = 3$.

If $d(v_2) \leq 6$, then $d(v_i) \geq 16 - d(v_1) - d(v_2) \geq 16 - 3 - 6 = 7$ for $i = 3, 4$. By Claims 1 and 3, $\tau(v_i \to v) \geq \frac{1}{6}$. By Claims 4,6,7, $\rho(f_i \to v) \geq \frac{1}{6}$ for $i = 1, 2$. Thus, $\sigma^* \geq 4 \cdot \frac{1}{6} = \frac{2}{3}$.

If $d(v_2) \geq 7$, then $d(v_3), d(v_4) \geq 7$, and $\tau(v_2 \to v) \geq \frac{1}{7}$, $\tau(v_i \to v) \geq \frac{1}{6}$ for $i = 3, 4$. Noting that $\rho(f_i \to v) \geq \frac{1}{6}$ for $i = 1, 2$, we have $\sigma^* \geq \frac{1}{7} + 4 \cdot \frac{1}{6} = \frac{17}{21}$.

(1.2) $d(v_1) = 4$.

If $4 \leq d(v_2) \leq 5$, then $d(v_i) \geq 7$ and $\tau(v_i \to v) \geq \frac{1}{6}$ for $i = 3, 4$. Since $\rho(f_i \to v) \geq \frac{1}{4}$ for $i = 1, 2$, we have $\sigma^* \geq 2 \cdot \frac{1}{6} + 2 \cdot \frac{1}{4} = \frac{5}{6}$.

If $d(v_2) \geq 6$, then $d(v_i) \geq 6$ for $i = 3, 4$. Since $\rho(f_1 \to v) \geq \frac{1}{4}$, $\rho(f_2 \to v) \geq \frac{4}{9}$, we have $\sigma^* \geq \frac{1}{4} + \frac{4}{9} = \frac{25}{36}$.

Case 2 $d(v_3) \leq d(v_i)$ for $i = 2, 4$.

(2.1) $d(v_1) = 3$.

If $d(v_3) \leq 5$, then $d(v_i) \geq 8$ and $\tau(v_i \to v) \geq \frac{1}{4}$ for $i = 2, 4$. Since $\rho(f_i \to v) \geq \frac{1}{6}$ for $i = 1, 2$, we have $\sigma^* \geq 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{6} = \frac{5}{6}$.

If $d(v_3) = 6$, then $d(v_i) \geq 7$ and $\tau(v_i \to v) \geq \frac{1}{7}$ for $i = 2, 4$. Since $\rho(f_1 \to v) \geq \frac{1}{6}$ and $\rho(f_2 \to v) \geq \frac{4}{9}$, we have $\sigma^* \geq 2 \cdot \frac{1}{7} + \frac{1}{6} + \frac{4}{9} = \frac{113}{126}$.

If $d(v_3) \geq 7$, then $\tau(v_4 \to v) \geq \frac{1}{7}$ and $\tau(v_i \to v) \geq \frac{1}{6}$ for $i = 2, 3$. Since $\rho(f_i \to v) \geq \frac{1}{6}$ for $i = 1, 2$, we have $\sigma^* \geq \frac{1}{7} + 4 \cdot \frac{1}{6} = \frac{17}{21}$.

(2.2) $d(v_1) = 4$.

If $4 \leq d(v_3) \leq 5$, then $d(v_i) \geq 7$ and $\tau(v_i \to v) \geq \frac{1}{7}$ for $i = 2, 4$. Since $\rho(f_i \to v) \geq \frac{1}{4}$ for $i = 1, 2$, we have $\sigma^* \geq 2 \cdot \frac{1}{7} + 2 \cdot \frac{1}{4} = \frac{11}{14}$.

If $d(v_3) \geq 6$, then $d(v_i) \geq 6$ for $i = 2, 4$. Since $\rho(f_1 \to v) \geq \frac{1}{4}$, $\rho(f_2 \to v) \geq \frac{4}{9}$, we have $\sigma^* \geq \frac{1}{4} + \frac{4}{9} = \frac{25}{36}$.

Case 3 $d(v_4) \leq d(v_i)$ for $i = 2, 3$.

If $d(v_1) = 4$ and $4 \leq d(v_4) \leq 5$, then $d(v_i) \geq 7$ and $\tau(v_i \to v) \geq \frac{1}{7}$ for $i = 2, 3$. Since $\rho(f_1 \to v) \geq \frac{2}{15}$ and $\rho(f_2 \to v) \geq \frac{4}{9}$, we have $\sigma^* \geq 2 \cdot \frac{1}{7} + \frac{2}{15} + \frac{4}{9} > \frac{2}{3}$. The rest proof is similar to that of Case 2. □

Given a partial linear coloring c of a graph G using the color set C and a vertex $v \in V(G)$, we use $C_2(v)$ to denote the set of colors which appear exactly twice in $N_G(v)$. For $V_0 \subseteq V(G)$, we use $C_{\geq 2}(V_0)$ to denote the set of colors which appear at least twice in V_0.

Theorem 4 *If G is a planar graph without 4-cycles, then* $\mathrm{lc}(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + 8$.

Proof We prove the theorem by induction on the vertex number $|G|$. If $|G| \leq 8$, the theorem holds obviously. Let G be a connected planar graph with $|G| \geq 9$ and without 4-cycles. By Lemma 3, G contains one of the configurations (C1)-(C3). We have to handle each of these cases separately. Let $C = \{1, 2, \ldots, \lceil \frac{\Delta}{2} \rceil + 8\}$ denote the set of colors used in the following argument, where $\Delta = \Delta(G)$.

Case 1 G contains a 1-vertex v adjacent to a vertex u.

Let $H = G - v$. Then H is a planar graph with $\Delta(H) \leq \Delta$ and $|H| < |G|$ and without 4-cycles. By the induction assumption, H has a linear coloring c using the color set C. We color v with a color different from $c(u)$ and those colors in $C_2(u)$ to extend c to the whole graph G. Since $|C_2(u)| \leq \lfloor \frac{d(u)-1}{2} \rfloor \leq \lfloor \frac{\Delta-1}{2} \rfloor = \lceil \frac{\Delta}{2} \rceil - 1$, such color always exists.

Case 2 G contains a path $x_1x_2x_3x_4$ such that $d(x_2) = d(x_3) = 3$, $d(x_1) \leq 6$ and $d(x_4) \leq 6$.

For $i = 2, 3$, let x_i' denote the neighbor of x_i that is not on the path $x_1x_2x_3x_4$. Let $H = G - \{x_2, x_3\}$. By the induction hypothesis, H admits a linear coloring c using the color set C. We color x_2 with a color $a \in C \backslash (C_{\geq 2}(N_H(x_1) \cup N_H(x_2')) \cup \{c(x_1), c(x_4), c(x_2'), c(x_3')\})$ and x_3 with a color $b \in C \backslash (C_{\geq 2}(N_H(x_4) \cup N_H(x_3')) \cup \{a, c(x_1), c(x_4), c(x_2'), c(x_3')\})$. Since

$$
\begin{aligned}
&|C \backslash (C_{\geq 2}(N_H(x_1) \cup N_H(x_2')) \cup \{c(x_1), c(x_4), c(x_2'), c(x_3')\})| \\
&\quad \geq |C| - |C_{\geq 2}(N_H(x_1) \cup N_H(x_2')| - 4 \\
&\quad \geq \left\lceil \frac{\Delta}{2} \right\rceil + 8 - \left\lfloor \frac{(d(x_1) - 1) + (d(x_2') - 1)}{2} \right\rfloor - 4 \\
&\quad \geq \left\lceil \frac{\Delta}{2} \right\rceil + 4 - \left\lfloor \frac{\Delta + 6 - 2}{2} \right\rfloor \geq 2,
\end{aligned}
$$

and similarly $|C \backslash (C_{\geq 2}(N_H(x_4) \cup N_H(x_3')) \cup \{a, c(x_1), c(x_4), c(x_2'), c(x_3')\})| \geq 1$, the colors a and b exist.

Case 3 G contains a k-vertex v, $2 \leq k \leq 4$, whose neighbors $v_1, v_2, \ldots, v_k$ satisfy one of the following conditions, assuming $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_k)$:

(C3.1) $k = 2$ and $d(v_1) \leq 11$;
(C3.2) $k = 3$, $d(v_1) + d(v_2) \leq 14$, and v is bad;

(C3.3) $k = 4$, $d(v_1) + d(v_2) + d(v_3) \leq 15$, and v is bad.

Let $H = G - v$. By the induction assumption, H has a linear coloring c with the color set C. If (C3.1) holds, we color v with a color $a \in C\backslash(C_{\geq 2}(N_H(v_1) \cup N_H(v_2)) \cup \{c(v_1), c(v_2)\})$. Since $|C_{\geq 2}(N_H(v_1) \cup N_H(v_2)) \cup \{c(v_1), c(v_2)\}| \leq 2 + \lfloor \frac{(d(v_1)-1)+(d(v_2)-1)}{2} \rfloor \leq 2 + \lfloor \frac{11+\Delta-2}{2} \rfloor \leq \lceil \frac{\Delta}{2} \rceil + 6$, a exists.

If (C3.2) holds, assuming that v is incident to the 3-face $[vv_1v_2]$, we color v with a color $a \in C\backslash(C_{\geq 2}((N_H(v_1)\backslash\{v_2\}) \cup (N_H(v_2)\backslash\{v_1\}) \cup N_H(v_3)) \cup \{c(v_1), c(v_2), c(v_3)\})$. It is easy to see that $|C_{\geq 2}((N_H(v_1)\backslash\{v_2\}) \cup (N_H(v_2)\backslash\{v_1\}) \cup N_H(v_3)) \cup \{c(v_1), c(v_2), c(v_3)\}| \leq 3 + \lfloor \frac{(d(v_1)-2)+(d(v_2)-2)+(d(v_3)-1)}{2} \rfloor \leq 3 + \lfloor \frac{14+\Delta-5}{2} \rfloor = \lceil \frac{\Delta}{2} \rceil + 7$, hence a exists. Since $c(v_1) \neq c(v_2)$ and so $|\{c(v_1), c(v_2), c(v_3)\}| \geq 2$, the coloring is available.

If (C3.3) holds, assuming that v is incident to two 3-faces $[vv_1v_2]$ and $[vv_3v_4]$, we color v with a color $a \in C\backslash(C_{\geq 2}((N_H(v_1)\backslash\{v_2\}) \cup (N_H(v_2)\backslash\{v_1\}) \cup (N_H(v_3)\backslash\{v_4\}) \cup (N_H(v_4)\backslash\{v_3\})) \cup \{c(v_1), c(v_2), c(v_3), c(v_4)\})$. It is easy to see that $|C_{\geq 2}((N_H(v_1)\backslash\{v_2\}) \cup (N_H(v_2)\backslash\{v_1\}) \cup (N_H(v_3)\backslash\{v_4\}) \cup (N_H(v_4)\backslash\{v_3\})) \cup \{c(v_1), c(v_2), c(v_3), c(v_4)\}| \leq 4 + \left\lfloor \frac{(d(v_1)-2)+(d(v_2)-2)+(d(v_3)-2)+(d(v_4)-2)}{2} \right\rfloor \leq 4 + \lfloor \frac{15+\Delta-8}{2} \rfloor = \lceil \frac{\Delta}{2} \rceil + 7$, hence a exists. Since $c(v_1) \neq c(v_2)$ and $c(v_3) \neq c(v_4)$, no three elements in $\{c(v_1), c(v_2), c(v_3), c(v_4)\}$ are same, the coloring is available. This completes the proof of the theorem. □

We fell that the upper bound $\lceil \frac{\Delta(G)}{2} \rceil + 8$ in Theorem 4 is not best possible. It is straightforward to see that for any graph G, $\mathrm{lc}(G) \geq \lceil \frac{\Delta(G)}{2} \rceil + 1$. Thus, it is interesting to determine the smallest constant c^* such that every planar graph G without 4-cycles has $\mathrm{lc}(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + c^*$. Our Theorem 4 and the fact that every odd cycle C_{2n+1} has $\mathrm{lc}(C_{2n+1}) = 3 = \lceil \frac{\Delta(C_{2n+1})}{2} \rceil + 2$ imply that $2 \leq c^* \leq 8$.

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