Graphs and Combinatorics (2013) 29:275–280
DOI 10.1007/s00373-011-1108-0

ORIGINAL PAPER

Spanning Eulerian Subgraphs of 2-Edge-Connected Graphs

Xiangwen Li · Chunxiang Wang · Qiong Fan · Zhaohong Niu · Liming Xiong

Received: 31 July 2010 / Revised: 16 October 2011 / Published online: 8 November 2011

Abstract For integers l and k with $l > 0$ and $k > 0$, let $\mathscr{C}(l, k)$ denote the family of 2-edge-connected graphs G such that for each bond cut $|S| \leq 3$, each component of $G - S$ has at least $(|V(G)| - k)/l$ vertices. In this paper we prove that if $G \in \mathscr{C}(7, 0)$, then G is not supereulerian if and only if G can be contracted to one of the nine specified graphs. Our result extends some earlier results (Catlin and Li in J Adv Math 160:65–69, 1999; Broersma and Xiong in Discrete Appl Math 120:35–43, 2002; Li et al. in Discrete Appl Math 145:422–428, 2005; Li et al. in Discrete Math 309:2937–2942, 2009; Lai and Liang in Discrete Appl Math 159:467–477, 2011).

Keywords Supereulerian · Collapsible · Eulerian graphs

1 Introduction

Graphs in this paper are finite, undirected, and loopless. Graphs may have multiple edges. A graph G is *nontrivial* if it contains at least one edge. We follow Bondy and Murty [1] for undefined notation and terminology.

Xiangwen Li was supported by the Natural Science Foundation of China (11171129). Liming Xiong was supported by the Natural Science Fundation of China (11071016).

X. Li (✉) · C. Wang · Q. Fan
Department of Mathematics, Huazhong Normal University, Wuhan 430079, China
e-mail: xwli68@yahoo.cn

Z. Niu
School of Mathematical Sciences, Shanxi University, Taiyuan 030006, China

L. Xiong
Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China

L. Xiong
Department of Mathematics, Jiangxi Normal University, Nanchang 330027, China

Let H be a subgraph of a graph G. For $u \in V(H)$, let $d_H(u)$ denote the degree of u in H. For a subgraph H of G, let $\partial(H)$ denote the number of edges with one endpoint in H and the other endpoint in $G - V(H)$. An edge cut $X \subset E(G)$ is a *bond* if X is a minimal edge cut. A path $P : v_1 v_2 \dots v_t$ of a graph G is a *branch* of G if $d(v_1) \geq 3$, $d(v_t) \geq 3$ and $d(v_i) = 2$ for each $i \in \{2, 3, \dots, t-1\}$. P is *trivial* if $t = 2$ and a *nontrivial branch* otherwise.

For a graph G, let $O(G)$ denote the set of all odd degree vertices of G. For $F \subset E(G)$, the *contraction* G/F is obtained from G by contracting each edge of F and deleting the resulting loops. For a subgraph H of G, we write G/H for $G/E(H)$. A graph G is *eulerian* if it is a connected graph with $O(G) = \emptyset$. A graph is *supereulerian* if it has a spanning eulerian subgraph. In particular, K_1 is both eulerian and supereulerian. For integers l and k with $l > 0$ and $k > 0$, let $\mathscr{C}(l, k)$ denote the family of 2-edge-connected graphs G such that for every bond S with two or three edges, each component of $G - S$ has at least $(|V(G)| - k)/l$ vertices.

It is well known that the following graphs are not supereulerian, some of which are mentioned in [5,2,7–9].

For simplicity, we define

$$\mathscr{F}' = \{G_1, G_2, G_5, G_6, G_8, G_9\}$$

and

$$\mathscr{F} = \{G_1, G_2, G_3, \dots, G_9\}.$$

Jaeger [6] showed that every 4-edge-connected graph is supereulerian. It is well known that if G is supereulerian, then G is 2-edge-connected. Thus, for supereulerianity, one has to investigate k-edge-connected graphs, where $k = 2, 3$. For this purpose, Catlin and Li [5] investigated the family graphs $\mathscr{C}(5, 0)$ and proved that $G \in \mathscr{C}(5, 0)$ is not supereulerian if and only if G can be contracted to $K_{2,3}$. This result was improved by Broersma and Xiong [2], who proved that $G \in \mathscr{C}(5, 2)$ with $|V(G)| \geq 13$ is not supereulerian if and only if G can be contracted to $K_{2,3}$ or $K_{2,5}$. Li et al.[8] generalized the results of Catlin and Li, Broersma and Xiong by showing that $G \in \mathscr{C}(6, 0)$ with $|V(G)|$ sufficient large is not supereulerian if and only if G can be contracted to $K_{2,3}$ or $K_{2,5}$ or G_2 in Fig.1. Li et al. [9] proved that $G \in \mathscr{C}(6, 5)$ is not supereulerian if and only if G can be contracted to one of G_i, $1 \leq i \leq 6$, in Fig.1, where $|V(G)| > 35$. Recently, Lai and Liang [7] improved this result by showing that there exist an integer $N(k) \leq 7k$, where $k > 0$ is an integer, such that, for any graph $G \in \mathscr{C}(6, k)$ with $|V(G)| > N(k)$, G is not supereulerian if and only if G can be contracted to a member in $\mathscr{F}'$. Motivated by these results, we prove the following result in this paper.

Theorem 1.1 *If $G \in \mathscr{C}(7, 0)$, then G is not supereulerian if and only if G can be contracted to a member in $\mathscr{F}$.*

2 Catlin's Reduction and Previous Results

A graph G is *collapsible* if for every set $X \subset V(G)$ with $|X|$ even, G has a spanning connected subgraph G_X with $O(G_X) = X$. Thus, K_1 is both supereulerian and collapsible, and every collapsible graph is also supereulerian.

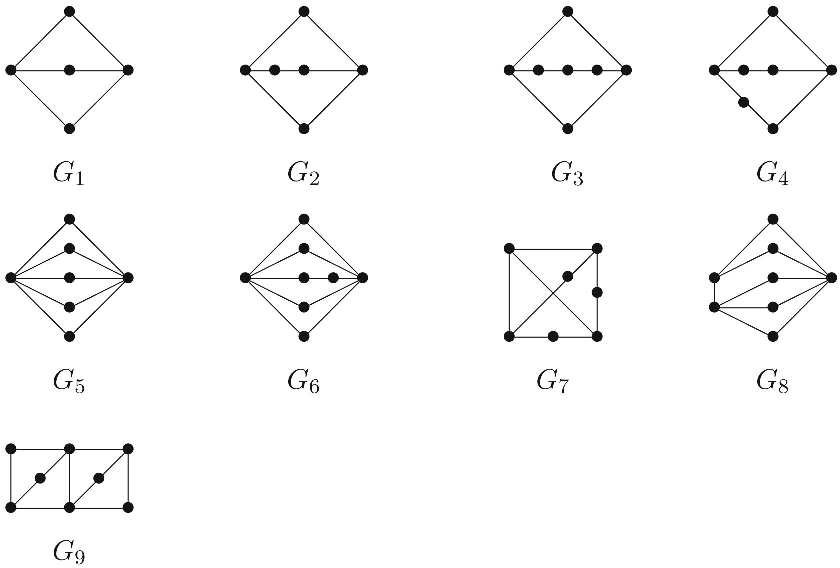

Fig. 1 9 non-supereulerian graphs

Catlin [3] showed that every graph G has a unique collection of pairwise disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. The contraction of G obtained from G by contracting each H_i, where $1 \le i \le c$, into a single vertex v_i, is called the *reduction* of G. The vertex v_i is said to be the *image* of H_i and H_i is called the *preimage* of v_i. A graph is *reduced* if it is the reduction of itself. Thus, a reduced graph does not have a nontrivial collapsible subgraph.

Theorem 2.1 (Catlin [3]) *If G be a connected graph, then each of the following holds.*

- *(i) If H is a collapsible subgraph of a graph G, then G is supereulerian if and only if G/H is supereulerian.*
- *(ii) Let G' be the reduction of G. G is supereulerian if and only if G' is supereulerian, and G is collapsible if and only if $G' = K_1$.*

Theorem 2.2 (Catlin et al. [4]) *Let G be a connected reduced graph. If $2|V(G)| - |E(G)| \le 4$, then either G is K_1, or K_2 or a $K_{2,t}$ for some inter $t \ge 1$.*

For each integer $i \ge 1$, denote $d_i = |D_i(G)|$, where $D_i(G) = \{v \in V(G) | d_G(v) = i\}$.

Theorem 2.3 (Catlin [3]) *If G is a graph, then each of the following holds.*

- *(i) G is reduced if and only if G has no nontrivial collapsible subgraph.*
- *(ii) If G is reduced and $\kappa'(G) \ge 2$, then $d_2 + d_3 \ge 4$, and when $d_2 + d_3 = 4$, G must be eulerian.*
- *(iii) If G is reduced, then G is simple, K_3-free, and G cannot have a nontrivial subgraph with 2 edge-disjoint spanning trees, and for any subgraph H of G, either $H \in \{K_1, K_2\}$ or $|E(H)| \le 2|V(H)| - 4$.*

The following theorem is due to Lai and Liang [7], which will play a key role in the proof of our main theorem.

Theorem 2.4 *If G is a 2-edge-connected reduced graph which satisfies*

(i) $d_2 + d_3 \leq 6$, *and*
(ii) $d_3 + d_5 \leq 2$,
then G is supereulerian if and only if G is not a member in $\mathscr{F}'$.

3 Proof of Theorem 1.1

By Theorem 2.1, it is sufficient for us to show that if G cannot be contracted to a member in $\mathscr{F}$, then G is supereulerian. We will proceed our proof by contradiction, and assume that $G \in \mathscr{C}(7, 0)$ cannot be contracted to any G_i in $\mathscr{F}$, where $1 \leq i \leq 9$, such that

$$G \text{ is a counterexample to Theorem 1.1.} \tag{1}$$

Let G' be the reduction of G. If $G' = K_1$, then G is supereulerian, contrary to (1). Assume that $G' \neq K_1$, Since G is 2-edge-connected, then G' is 2-edge-connected and nontrivial. Define $V_4' = \{v \in G' : d_{G'}(v) \geq 4\}$ and $d_i(G') = |D_i(G')|$, where $D_i(G') = \{v \in V(G') : d_{G'}(v) = i\}$. For simplicity, we use D_i and d_i for $D_i(G')$ and $d_i(G')$, respectively, in this section. We establish the following lemmas.

Lemma 3.1 $d_2 + d_3 \leq 7$. *Moreover, if* $d_2 + d_3 = 7$, *then* $|V_4'| = 0$.

Proof Suppose, to the contrary, that $d_2+d_3 \geq 8$. Let $v_1, v_2, \ldots, v_8$ denote the vertices of $V(G')$ such that $d_{G'}(v_i) \leq 3$, where $1 \leq i \leq 8$, and let H_i denote the preimage of v_i, where $1 \leq i \leq 8$. For $1 \leq i \leq 8$, $\partial(H_i) \leq 3$. Since $G \in \mathscr{C}(7, 0)$, $|V(H_i)| \geq \frac{n}{7}$ and hence $n = |V(G)| \geq \sum_{i=1}^{8} |V(H_i)| \geq \frac{8n}{7}$. This contradiction establishes that $d_2 + d_3 \leq 7$. Moreover by $G \in \mathscr{C}(7, 0)$, $n = |V(G)| \geq \sum_{i=1}^{7} |V(H_i)| + |V_4'| \geq n + |V_4'|$ which implies that $|V_4'| = 0$. □

Lemma 3.2 *Each of the following holds.*

(i) $2d_2 + d_3 \geq 10 + \sum_{i\geq 4}(i - 4)d_i$.
(ii) $d_2 \geq 4$ *and* $d_3 \leq 3$.
(iii) $5 \leq d_2 + d_3 \leq 7$.
(iv) $d_i = 0$ *for* $i \geq 7$.

Proof (i). Assume first that $2|V(G')|-|E(G')| \leq 4$. Since G is 2-edge-connected, G' has no cut edge. By Theorem 2.2 and Lemma 3.1, G' is K_1 or $G' = K_{2,t}$ for some integer $t \geq 1$. In the former case, G is collapsible and so G is supereulerian, contrary to (1). In the later case, when t is even, G' is supereulerian. By Theorem 2.1, G is supereulerian, contrary to (1). When t is odd, since $G \in \mathscr{C}(7, 0)$, $s = 3, 5$, that is, G' is G_1 or G_5, contrary to (1).

Thus, we assume that $2|V(G')| - |E(G')| \geq 5$. Since G' is 2-edge-connected, $|V(G')| = \sum_{i\geq 2} d_i$ and $2|E(G')| = \sum_{i\geq 2} i d_i$. It follows that $2\sum_{i\geq 2} d_i - (\sum_{i\geq 2} i d_i)/2 \geq 5$, that is, $\sum_{i\geq 2}(4d_i - i d_i) \geq 10$, which implies

$$2d_2 + d_3 \geq 10 + \sum_{i\geq 4}(i-4)d_i.$$

(ii). By (i), $2d_2+d_3 \geq 10$. By Lemma 3.1, $d_2+d_3 \leq 7$. Thus, $d_2 \geq 10-(d_2+d_3) \geq 3$. We show that $d_2 \neq 3$. Suppose otherwise that $d_2 = 3$ and $D_2 = \{v_1, v_2, v_3\}$. By (i), $d_3 \geq 4$. By Lemma 3.1, $d_3 \leq 4$. Thus, $d_3 = 4$. In this case, by Lemma 3.1, $|V_4'| = 0$. Denote by G'' the graph obtained from G' by replacing each branch with an edge. Thus, $G'' \cong K_4$ and G'' is supereulerian. By Theorem 2.1, we may assume that G' has no spanning eulerian subgraph containing all three vertices v_1, v_2, v_3. It follows that G'' contains either a vertex incident with three edges, each of which has exactly one of v_1, v_2, v_3, or a triangle, each edge of which has exactly one vertex of v_1, v_2, v_3. In the latter case, G' contains a triangle, contrary to Theorem 2.3. In the former case, G can be contracted to G_7, contrary to (1). Thus, $d_2 \geq 4$ as desired. By Lemma 3.1, $d_3 \leq 3$.

(iii). By Lemma 3.1, $d_2 + d_3 \leq 7$. By (ii), $d_2 \geq 4$. Thus, $4 \leq d_2 + d_3 \leq 7$. If $d_2 + d_3 = 4$, it follows from Theorem 2.3 that G' is spereulerian, contrary to (1). We conclude that $5 \leq d_2 + d_3 \leq 7$.

(iv). By (i), (iii) and (1), $14 \geq 2(d_2+d_3) \geq 2d_2+d_3 \geq 10+\sum_{i\geq 4}(i-4)d_i$, which implies that $d_i = 0$ for $i \geq 9$. If $d_7 \neq 0$, then $2d_2 + d_3 \geq 13$. Since $d_7 \neq 0$, by Lemma 3.1, $d_2 + d_3 \leq 6$. Thus, $13 \leq d_2 + (d_2 + d_3) \leq d_2 + 6$, which implies $d_2 \geq 7$. In this case, $n = \sum_{i=1} |V(H_i)| \geq 7\frac{n}{7} + 1 > n$, a contradiction. If $d_8 \neq 0$, by (iii), then $d_2 = 7$. Similarly, we get a contradiction. □

Lemma 3.3 $d_3 + d_5 = 2$.

Proof We first prove $d_3 \leq 2$. By Lemma 3.2 (ii), $d_3 \leq 3$. It is sufficient to show that $d_3 \neq 3$. Suppose otherwise that $d_3 = 3$. Since the number of vertices of odd degree is even, it follows by Lemma 3.2 (iv) that $d_5 \geq 1$ and d_5 is odd. By Lemma 3.2 (i), $2d_2 + d_3 \geq 10 + (5-4)d_5 \geq 11$, which implies that $d_2 \geq 4$. By Lemma 3.2 (iii), $d_2 = 4$. Thus, $d_2 + d_3 = 7$ and $|V_4'| \geq 1$, contrary to Lemma 3.1.

Next, we prove $d_5 \leq 2$. Suppose otherwise that $d_5 \geq 3$. By Lemma 3.2 (i), $2d_2 + d_3 \geq 10 + (5-4)d_5 \geq 13$. Since $D_5 \subset V_4'$ and $|V_4'| \geq 3$, by Lemma 3.1, $d_2 + d_3 \leq 6$. Applying the inequality $d_2 + d_3 \leq 6$ to the inequality $2d_2 + d_3 \geq 13$, we get $d_2 \geq 7$. Thus, $d_2 + d_3 \geq 7$, contrary to Lemma 3.1.

Finally, we prove that $d_3 + d_5 = 2$. It is sufficient to show that there does not exist the case when $d_3 = 2$ and $d_5 = 2$. Suppose otherwise that such case exists. Since $d_2 + d_3 \leq 7$, by Lemma 3.2, $4 \leq d_2 \leq 5$. If $d_2 = 5$, then $d_2 + d_3 = 7$, contrary to Lemma 3.1. Thus, $d_2 = 4$. In this case, by Lemma 3.2 (i), $10 = 2d_2 + d_3 \geq 10 + (5-4)d_5 = 12$, a contradiction. □

Proof of Theorem 1.1 Note that G is 2-edge-connected, so is G'. By Lemma 3.1, $d_2 + d_3 \leq 7$. Suppose first that $d_2 + d_3 \leq 6$. By Lemma 3.3, $d_3 + d_5 = 2$. By Theorem 2.4 and by the assumption that G cannot be contracted to a member in $\mathscr{F}$, G'

is supereulerian. It follows by Theorem 2.1 that G is supereulerian, contrary to (1). Thus, we may assume that $d_2 + d_3 = 7$.

Thus, $|V(G')| = 7$. By Lemma 3.1, $V_4' = \emptyset$. It follows that $d_5 = 0$. By Lemma 3.3, $d_3 = 2$. Let u and v denote two vertices of degree 3 in G'. If $uv \in E(G')$, then let $N_{G'}(u) = \{v, u_1, u_2\}$ and $N_{G'}(v) = \{u, v_1, v_2\}$. Since G' is reduced, by Theorem 2.3, G' is K_3-free, and hence $\{u_1, u_2\} \cap \{v_1, v_2\} = \emptyset$. Since G' is 2-edge-connected, there are two edge-disjoint branches from $\{u_1, u_2\}$ to $\{v_1, v_2\}$, which implies that G' is hamiltonian and hence supereulerian. By Theorem 2.1, G is supereulerian, contrary to (1). Thus, $uv \notin E(G')$. In this case, let $N_{G'}(u) = \{u_1, u_2, u_3\}$ and $N_{G'}(v) = \{v_1, v_2, v_3\}$. Note that $d_2 = 5$ and $\{u_1, u_2, u_3, v_1, v_2, v_3\} \subseteq D_2$. It follows that $\{u_1, u_2, u_3\} \cap \{v_1, v_2, v_3\} \neq \emptyset$. Since $d_2 = 5$, $|\{u_1, u_2, u_3\} \cap \{v_1, v_2, v_3\}| \leq 2$. If $|\{u_1, u_2, u_3\} \cap \{v_1, v_2, v_3\}| = 1$, then G' is isomorphic to G_4 in Fig.1, a contradiction; if $|\{u_1, u_2, u_3\} \cap \{v_1, v_2, v_3\}| = 2$, then G' is isomorphic to G_3 in Fig.1, a contradiction. □

Acknowledgments The authors would like to thank the referees for the valuable comments and suggestions which leads to improve the representation. In particular, one of the referees suggested to use the result [7, Theorem 3.1] to shorten the proof of our main theorem.

References

1. Bondy, J.A., Murty, U.S.R.: Graph Theory with Applications. American Elsevier, New York (1976)
2. Broersma, H.J., Xiong, L.: A note on minimun degree conditions for supereulerian graphs. Discrete Appl. Math. **120**, 35–43 (2002)
3. Catlin, P.A.: A reduction method to find spanning eulerian subgraphs. J. Graph Theory **12**, 29–45 (1998)
4. Catlin, P.A., Han, Z.Y., Lai, H.-J.: Graphs without spanning closed trails. Discrete Math. **160**, 81–91 (1996)
5. Catlin, P.A., Li, X.: Supereulerian graphs of minimum degree at least 4. J. Adv. Math. **160**, 65–69 (1999)
6. Jaeger, F.: A note on sub-Eulerian graphs. J. Graph Theory **3**, 91–93 (1979)
7. Lai, H.-J., Liang, Y.: Supereulerian graphs in the graph family $C_2(6, k)$. Discrete Appl. Math. **159**, 467–477 (2011)
8. Li, D., Lai, H.-J., Zhan, M.: Eulerian subgraphs and Hamilton-connected line graphs. Discrete Appl. Math. **145**, 422–428 (2005)
9. Li, X., Li, D., Lai, H.-J.: The supereulerian graphs in the graph family $C(l, k)$. Discrete Math. **309**, 2937–2942 (2009)