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Chordal Bipartite Graphs with High Boxicity

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Abstract The boxicity of a graph G is defined as the minimum integer k such that G is an intersection graph of axis-parallel k-dimensional boxes. Chordal bipartite graphs are bipartite graphs that do not contain an induced cycle of length greater than 4. It was conjectured by Otachi, Okamoto and Yamazaki that chordal bipartite graphs have boxicity at most 2. We disprove this conjecture by exhibiting an infinite family of chordal bipartite graphs that have unbounded boxicity.

Keywords Boxicity · Chordal bipartite graphs · Interval graphs · Grid intersection graphs

1 Introduction

A graph G is an *intersection graph* of sets from a family of sets $\mathcal{F}$, if there exists $f : V(G) \to \mathcal{F}$ such that $(u, v) \in E(G) \Leftrightarrow f(u) \cap f(v) \neq \emptyset$. An *interval graph* is an intersection graph in which the set assigned to each vertex is a closed interval on the real line. In other words, interval graphs are intersection graphs of closed intervals on the real line. An *axis-parallel k-dimensional box* in $\mathbb{R}^k$ is the Cartesian product $R_1 \times R_2 \times \cdots \times R_k$, where each R_i is an interval of the form $[a_i, b_i]$ on the real line. *Boxicity* of any graph G (denoted by $\text{box}(G)$) is the minimum integer k such that G

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is an intersection graph of axis-parallel k-dimensional boxes in $\mathbb{R}^k$. Note that interval graphs are exactly those graphs with boxicity at most 1.

The concept of boxicity was introduced by [13]. It finds applications in niche overlap (competition) in ecology and to problems of fleet maintenance in operations research (see [7]). Roberts proved that the boxicity of any graph on n vertices is bounded from above by $\lfloor \frac{n}{2} \rfloor$. He also showed that a complete $\frac{n}{2}$-partite graph with 2 vertices in each part has its boxicity equal to $\frac{n}{2}$. Various other upper bounds on boxicity in terms of graph parameters such as maximum degree and treewidth were proved by Chandran, Francis and Sivadasan. In [3] they showed that, for any graph G on n vertices having maximum degree Δ, $\text{box}(G) \leq (\Delta + 2) \ln n$. They also found an upper bound for boxicity solely in terms of the maximum degree Δ of a graph by showing that $\text{box}(G) \leq 2\Delta^2$ [4]. This means that the boxicity of bounded degree graphs is bounded no matter what the size of the vertex set is. It was shown in [5] by Chandran and Sivadasan that $\text{box}(G) \leq \text{tw}(G) + 2$, where $\text{tw}(G)$ denotes the treewidth of graph G.

Cozzens [6] proved that given a graph, the problem of computing its boxicity is NP-hard. Several attempts have been made to find good upper bounds for the boxicity of special classes of graphs. It was shown by Thomassen [15] that planar graphs have boxicity at most 3. Scheinerman [14] proved that outerplanar graphs have boxicity at most 2. The boxicity of split graphs was investigated by Cozzens and Roberts [7].

1.1 Chordal Bipartite Graphs (CBGs)

A bipartite graph G is a *chordal bipartite graph* (CBG) if G does not have an induced cycle of length greater than 4. In other words, all induced cycles in such a bipartite graph will be of length exactly equal to 4. Chordal bipartite graphs were introduced by Golumbic and Goss [9], as a natural bipartite analogue of chordal graphs. Chordal bipartite graphs are a well studied class of graphs and several characterizations using bisimplicial edges, minimal edge separators, Γ-free matrices etc. (refer [10]) have been found.

1.2 Our Result

In 2007, Otachi et al. [12] proved that P_6-free chordal bipartite graphs have boxicity at most 2. In the same paper, they conjectured that the boxicity of the wider class of chordal bipartite graphs itself is bounded from above by 2. We disprove this conjecture by showing that there exist chordal bipartite graphs with arbitrarily high boxicity.

Let G be a bipartite graph with bipartition $\{A, B\}$. G is a *grid intersection graph* if we can assign horizontal and vertical line segments in the plane to vertices in A and B respectively such that two line segments intersect if and only if their corresponding vertices have an edge between them [11]. From this definition, it is trivial to see that grid intersection graphs have boxicity at most 2. Since it can be easily seen that there are grid intersection graphs that are not chordal bipartite (for example, a cycle on 6 vertices), our result implies that the class of chordal bipartite graphs is incomparable with the class of grid intersection graphs.

2 Definitions and Notations

Let $V(G)$ and $E(G)$ denote the vertex set and edge set respectively of a graph G. For any $S \subseteq V(G)$, let $G - S$ denote the graph induced by the vertex set $V(G) \setminus S$ in G. In this paper, we consider only simple, finite, undirected graphs. In a graph G, for any $u \in V(G)$, $N_G(u)$ denotes its neigbourhood in G, i.e. $N_G(u) = \{v \mid (u, v) \in E(G)\}$. Also, $N_G[u]$ denotes the closed neighbourhood of u in G, i.e. $N_G[u] = N_G(u) \cup \{u\}$. A graph G is a bipartite graph if there is a partition of $V(G)$ into two sets A and B such that both A and B induce independent sets in G. We call $\{A, B\}$ the *bipartition* of the bipartite graph G. Given a tree T and two vertices u and v in T, we denote by uTv the unique path in T between u and v (including u and v). If $G, G_1, G_2, \ldots, G_k$ are $k + 1$ graphs, where $V(G) = V(G_1) = V(G_2) = \cdots = V(G_k)$, then we say that $G = G_1 \cap G_2 \cap \cdots \cap G_k$ if $E(G) = E(G_1) \cap E(G_2) \cap \cdots \cap E(G_k)$.

2.1 Interval Graphs and Boxicity

Since an interval graph is the intersection graph of closed intervals on the real line, for every interval graph I, there exists a function $f : V(I) \to \{X \subseteq \mathbb{R} \mid X \text{ is a closed interval}\}$, such that for $u, v \in V(I)$, $(u, v) \in E(I) \Leftrightarrow f(u) \cap f(v) \neq \emptyset$. The function f is called an *interval representation* of the interval graph I. Note that the interval representation of an interval graph need not be unique. Given a closed interval $X = [y, z]$, we define $l(X) := y$ and $r(X) := z$. For any two intervals $[y_1, z_1], [y_2, z_2]$ on the real line, we say that $[y_1, z_1] < [y_2, z_2]$ if $z_1 < y_2$. Clearly, $[y_1, z_1] \cap [y_2, z_2] = \emptyset$ if and only if $[y_1, z_1] < [y_2, z_2]$ or $[y_2, z_2] < [y_1, z_1]$.

Let I be an interval graph and f an interval representation of I. Let $y, z \in \mathbb{R}$ with $y \leq z$. Then any set of vertices, say $S = \{u_1, u_2, \ldots, u_k\}$ where $S \subseteq V(I)$ and $k > 0$, is said to "overlap in the region $[y, z]$ in f" if each $f(u_i)$ (where $1 \leq i \leq k$) contains the region $[y, z]$, i.e. for each $u_i \in S$, $l(f(u_i)) \leq y \leq z \leq r(f(u_i))$.

A graph G is *chordal* if it does not contain any induced cycle of length greater than 3. The following is a well known fact about interval graphs.

Lemma 1 *All interval graphs are chordal.*

We have seen that interval graphs are intersection graphs of intervals on the real line. The following lemma gives the relationship between intersection graphs of axis-parallel k-dimensional boxes and interval graphs.

Lemma 2 (Roberts[13]) *For any graph G, $\text{box}(G) \leq b$ if and only if there exist b interval graphs $I_1, I_2, \ldots, I_b$, with $V(G) = V(I_1) = V(I_2) = \cdots = V(I_b)$ such that $G = I_1 \cap I_2 \cap \cdots \cap I_b$.*

From the above lemma, we can say that the boxicity of a graph G is the minimum b for which there exist b interval graphs $I_1, \ldots, I_b$ such that $G = I_1 \cap I_2 \cap \cdots \cap I_b$. Note that if $G = I_1 \cap I_2 \cap \cdots \cap I_b$, then each I_i is a supergraph of G and also for every pair of vertices $u, v \in V(G)$ such that $(u, v) \notin E(G)$, $(u, v) \notin E(I_i)$, for some i.

2.2 Strongly Chordal Graphs and Chordal Bipartite Graphs

Two vertices u and v in a graph G are said to be *compatible* if $N_G[u] \subseteq N_G[v]$ or vice versa. A vertex v in a graph G is a *simple vertex* if for any $x, y \in N_G[v]$, x and y are compatible. An ordering $v_1, \ldots, v_n$ of vertices of a graph G is said to be a *simple elimination ordering* if for each i, the vertex v_i is a simple vertex in the graph induced by the vertices $\{v_i, \ldots, v_n\}$ in G. The class of *strongly chordal graphs* were introduced by Farber [8]. The following definition is from page 78 of [2].

Definition 1 A graph is strongly chordal if and only if it admits a simple elimination ordering.

A *split graph* is a graph in which the vertices can be partitioned into a clique and an independent set. For a bipartite graph G with bipartition $\{A, B\}$, we denote by $C_A(G)$ the split graph obtained from G by adding edges between every pair of vertices in A. $C_B(G)$ is defined in a similar way. The following characterization of chordal bipartite graphs appears in [1].

Lemma 3 *Let G be a bipartite graph with bipartition $\{A, B\}$. Then, G is chordal bipartite if and only if $C_A(G)$ is strongly chordal.*

2.3 Bipartite Powers

For any two vertices u, v in a graph G, let $d_G(u, v)$ denote the length of a shortest u-v path in G. Given a bipartite graph G and an odd positive integer k, we define the graph $G^{[k]}$ to be the graph with $V\left(G^{[k]}\right) = V(G)$ and $E\left(G^{[k]}\right) = \{(u, v) \mid u, v \in V(G),\ d_G(u, v)$ is odd and $d_G(u, v) \leq k\}$. The graph $G^{[k]}$ is called the *k-th bipartite power* of the bipartite graph G. Note that the name "bipartite powering" for this special powering operation is motivated by the following observation: if G is a bipartite graph with the bipartition $\{A, B\}$, then $G^{[k]}$ is also a bipartite graph with the bipartition $\{A, B\}$.

3 Bipartite Powers of Trees

Let T be a rooted tree with vertex r being its root. T is clearly a bipartite graph and let $\{A, B\}$ be its bipartition. For any $u, v \in V(T)$, we say $u \preceq v$ in T, if $v \in rTu$. Otherwise, we say $u \not\preceq v$. For $u, v \in V(T)$, we define $P(u, v) := \{x \in V(T) \mid u \preceq x$ and $v \preceq x\}$. The *least common ancestor* (LCA) of any two vertices $u, v \in V(T)$ in T is that vertex $z \in P(u, v)$ such that $\forall y \in P(u, v), z \preceq y$. Note that if z is the LCA of u and v, then $z \in uTv$, $z \in uTr$ and $z \in vTr$. We say that a vertex u is *farthest* from a vertex v in T if $\forall w \in V(T), d_T(v, w) \leq d_T(v, u)$. Note that in this case u will be a leaf of T.

Lemma 4 *Let $x \in V(T)$ be a leaf. Then, $(T - \{x\})^{[k]} = T^{[k]} - \{x\}$.*

Proof For ease of notation, let $G = (T - \{x\})^{[k]}$ and $G' = T^{[k]} - \{x\}$. Let $(u, v) \in E(G')$. Since x is a leaf of T, $x \notin uTv$. Therefore, $(u, v) \in E(G)$. Hence, $(u, v) \in E(G') \Rightarrow (u, v) \in E(G)$. Also, clearly, $(u, v) \in E(G) \Rightarrow (u, v) \in E(G')$. □

Lemma 5 *Let $x \in V(T)$ such that x is farthest from r in T. Assume that $x \in A$. For any odd positive integer k, let $G := C_B(T^{[k]})$. Then, x is a simple vertex in G.*

Proof We shall prove this by proving that, for any two vertices $u_1, u_2 \in N_G[x]$, such that $d_T(r, u_1) \geq d_T(r, u_2)$, $N_G[u_1] \subseteq N_G[u_2]$. Note that we can assume $u_2 \neq x$. Clearly this is so when $d_T(r, u_1) > d_T(r, u_2)$ since x is farthest from r in T. Even when $d_T(r, u_1) = d_T(r, u_2)$ the roles of u_1 and u_2 can be interchanged so that $u_2 \neq x$. Now, since $N_G[x] \cap A = \{x\}$, we have $u_2 \in B$. Let $v \in N_G[u_1]$. If $v \in B$, then $v \in N_G[u_2]$ (since B induces a clique in G).

We will show that even when $v \notin B$, $v \in N_G[u_2]$, thus completing the proof. Note that $d_T(u_2, v)$ is odd since $u_2 \in B$. Thus we only need to prove that $d_T(u_2, v) \leq k$. We prove this as follows.

We know that since $u_1, u_2 \in N_G[x]$ in G, $d_T(u_1, x) \leq k$ and $d_T(u_2, x) \leq k$. Let w be the LCA of u_1 and u_2 in T. Since $d_T(r, u_2) \leq d_T(r, u_1)$, we have

$$d_T(u_2, w) \leq d_T(u_1, w). \tag{1}$$

We now have the following two cases.

Case (i) $v \preceq w$ in T

It is easy to see that $w \in u_1 T x$ or $w \in u_2 T x$. If $w \in u_1 T x$, then $d_T(u_1, x) = d_T(u_1, w) + d_T(w, x) \leq k$ which by (1) implies that $d_T(u_2, w) + d_T(w, x) \leq k$. If $w \in u_2 T x$, $d_T(u_2, w) + d_T(w, x) = d_T(u_2, x) \leq k$. Therefore we always have

$$d_T(u_2, w) + d_T(w, x) \leq k. \tag{2}$$

We know that

$$d_T(r, x) \leq d_T(r, w) + d_T(w, x) \text{ and}$$
$$d_T(r, v) = d_T(r, w) + d_T(w, v) \text{ (as } v \preceq w \text{ in } T).$$

Since $d_T(r, v) \leq d_T(r, x)$ (as x is farthest from r in T), we have $d_T(w, v) \leq d_T(w, x)$. Combining this with (2), we get $d_T(u_2, w) + d_T(w, v) \leq k$. Therefore, $d_T(u_2, v) \leq k$.

Case (ii) $v \npreceq w$ in T

In this case, it is easy to see that $w \in u_1 T v$ and $w \in u_2 T v$. This implies that

$$d_T(u_1, v) = d_T(u_1, w) + d_T(w, v) \tag{3}$$

and

$$\begin{aligned} d_T(u_2, v) &= d_T(u_2, w) + d_T(w, v) \\ &\leq d_T(u_1, w) + d_T(w, v) \text{ (from (1))} \\ &= d_T(u_1, v) \\ &\leq k \text{ (from (3))} \end{aligned}$$

This proves that $N_G[u_1] \subseteq N_G[u_2]$. Hence the lemma. □

Theorem 1 *For any odd positive integer k, $T^{[k]}$ is a CBG.*

Proof Let us prove this by using induction on the number of vertices of T. Let $x \in V(T)$ such that x is farthest from r in T. Assume $x \in A$. Let $G := C_B(T^{[k]})$. Then by Lemma 5, x is a simple vertex in G. Let $G' = G - \{x\}$. Note that $G' = C_B(T^{[k]} - \{x\})$. But from Lemma 4, $T^{[k]} - \{x\} = (T - \{x\})^{[k]}$ which by our induction hypothesis is a CBG. Then, applying Lemma 3, we can say that G' is a strongly chordal graph. Since x is a simple vertex in G and since $G' = G - \{x\}$ is a strongly chordal graph, G is also a strongly chordal graph (recall Definition 1). Therefore by Lemma 3, $T^{[k]}$ is a CBG. □

4 Boxicity of CBGs

Lemma 6 *In an interval graph I, let $S \subseteq V(I)$ be a set of vertices that induces a clique. Then for an interval representation f of I, $\exists y, z \in \mathbb{R}$ with $y \leq z$ such that S overlaps in the region $[y, z]$ in f.*

Proof Proof of the lemma follows directly from the Helly property for intervals on the real line. □

Definition 2 For any bipartite graph G with bipartition $\{A, B\}$, $C_{AB}(G)$ is the graph formed by adding edges between every pair of vertices in A as well as in B. That is, $V(C_{AB}(G)) = V(G)$ and $E(C_{AB}(G)) = E(G) \cup \{(u, v) \mid u, v \in A \text{ and } u \neq v\} \cup \{(u, v) \mid u, v \in B \text{ and } u \neq v\}$.

Lemma 7 *Let G be a bipartite graph with bipartition $\{A, B\}$. Then,* $\text{box}(G) \geq \frac{\text{box}(C_{AB}(G))}{2}$.

Proof Let $\text{box}(G) = b$. Then by Lemma 2 we have a set of interval graphs, say $\mathcal{I} = \{I_1, I_2, \ldots, I_b\}$ with $V(I_1) = \cdots = V(I_b) = V$, such that $G = I_1 \cap I_2 \cap \cdots \cap I_b$. As each I_i is an interval graph, there exists an interval representation f_i for it. For each i, let $s_i = \min_{x \in V} l(f_i(x))$ and $t_i = \max_{x \in V} r(f_i(x))$. Corresponding to each interval graph I_i in $\mathcal{I}$, we construct two interval graphs I_i' and I_i''. We construct the interval representations f_i' and f_i'' for I_i' and I_i''respectively from f_i as follows:

Construction of f_i':

$\forall u \in A,\ f_i'(u) = [s_i, r(f_i(u))]$.
$\forall u \in B,\ f_i'(u) = [l(f_i(u)), t_i]$.

Construction of f_i'':

$\forall u \in A,\ f_i''(u) = [l(f_i(u)), t_i]$.
$\forall u \in B,\ f_i''(u) = [s_i, r(f_i(u))]$.

Let $\mathcal{I}' = \{I_1', \ldots, I_b', I_1'', \ldots, I_b''\}$. Now we claim that $C_{AB}(G) = \bigcap_{I \in \mathcal{I}'} I$.

Let $i \in \mathbb{N}$ such that $1 \leq i \leq b$. Since A overlaps in the region $[s_i, s_i]$ in f_i', the vertices in A induce a clique in I_i'. Similarly, B overlaps in the region $[t_i, t_i]$ in f_i'

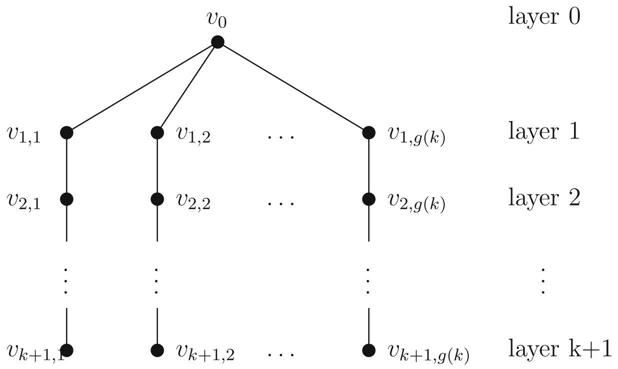

Fig. 1 Tree T_k

and hence the vertices in B also induce a clique in I_i'. Also, for any $u \in A$, $v \in B$, $(u, v) \in E(C_{AB}(G)) \Rightarrow (u, v) \in E(G) \Rightarrow (u, v) \in E(I_i) \Rightarrow (u, v) \in E(I_i')$ (since $\forall u \in V$, $f_i(u) \subseteq f_i'(u)$). Hence each I_i' is a supergraph of $C_{AB}(G)$. It can be shown by proceeding along similar lines that each I_i'' is a supergraph of $C_{AB}(G)$. Therefore, each $I \in \mathcal{I}'$ is a supergraph of $C_{AB}(G)$.

Let $u, v \in V$ such that $(u, v) \notin E(C_{AB}(G))$. Both u and v together cannot be in A or B since both A and B induce cliques in $C_{AB}(G)$. Assume without loss of generality that $u \in A$ and $v \in B$. Now, $(u, v) \notin E(C_{AB}(G)) \Rightarrow (u, v) \notin E(G)$ (this follows from Definition 2). Since $(u, v) \notin E(G)$, $\exists I_i \in \mathcal{I}$ such that $(u, v) \notin E(I_i)$, i.e. $f_i(u) \cap f_i(v) = \emptyset$. If $f_i(u) < f_i(v)$ then $f_i'(u) < f_i'(v)$ and therefore $(u, v) \notin E(I_i')$. Otherwise, if $f_i(v) < f_i(u)$ then $f_i''(v) < f_i''(u)$ and therefore $(u, v) \notin E(I_i'')$. To summarise, for any $(u, v) \notin E(C_{AB}(G))$, $\exists I \in \mathcal{I}'$ such that $(u, v) \notin E(I)$.

Hence we infer that $C_{AB}(G) = \bigcap_{I \in \mathcal{I}'} I$. By Lemma 2, this means that $\text{box}(C_{AB}(G)) \leq 2b = 2 \cdot \text{box}(G)$. □

Let T_k be the tree shown in Fig. 1. Here $k \in \mathbb{N}$ is an odd number and $g(k) = \frac{k+1}{2} \cdot (g(k-2) - 1) + 1$ with $g(1) = 2$. Let $G_k = T_k^{[k]}$. It follows from Theorem 1 that G_k is a CBG. For any $1 \leq i \leq k+1$, let $L_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,g(k)}\}$ denote the set of all vertices in layer i of T_k. Also, let $L_i' = \{v_{i,1}, v_{i,2}, \ldots, v_{i,g(k-2)}\}$. Note that T_k, and consequently G_k, is a bipartite graph with the bipartition $\{A, B\}$ where $A = \{u \in L_i \mid 0 \leq i \leq k+1, i \text{ is an odd number}\}$ and $B = \{u \in L_i \mid 0 \leq i \leq k+1, i \text{ is an even number}\}$. Let $A' = \{u \in L_i' \mid 2 \leq i \leq k \text{ and } i \text{ is an odd number}\}$ and $B' = \{u \in L_i' \mid 2 \leq i \leq k \text{ and } i \text{ is an even number}\}$. Note that since for any i, $L_i' \subseteq L_i$, we have $A' \subseteq A$ and $B' \subseteq B$.

Lemma 8 $\text{box}(C_{AB}(G_k)) > \frac{k+1}{2}$.

Proof For any odd positive integer i, let $X_i := C_{AB}(G_i) - \{v_0\}$.

Since X_k is an induced subgraph of $C_{AB}(G_k)$, $\text{box}(C_{AB}(G_k)) \geq \text{box}(X_k)$. Thus it is enough to prove that $\text{box}(X_k) > \frac{k+1}{2}$. We do this by induction on k. Since X_1 is precisely a cycle of length 4, it is not a chordal graph. Therefore, by Lemma 1, $\text{box}(X_1) > 1$ and thus the statement of the lemma is true for the case $k = 1$.

Let us assume that $\text{box}(X_i) > \frac{i+1}{2}$ for every odd positive integer $i < k$. Now we shall prove by contradiction that $\text{box}(X_k) > \frac{k+1}{2}$. We will show that if $\text{box}(X_k) \leq \frac{k+1}{2}$ then $\text{box}(X_{k-2}) \leq \frac{k-1}{2}$ which is a contradiction to the induction hypothesis.

In order to understand the proof, it is important to make the following observation. *Claim.* X_{k-2} is present as an induced subgraph in X_k.

This is because the graph induced by $A' \cup B'$ in X_k, say Z, is isomorphic to X_{k-2} as we show below. Let $V(X_{k-2}) = \{\overline{v}_{1,1}, \ldots, \overline{v}_{1,g(k-2)}, \overline{v}_{2,1}, \ldots, \overline{v}_{2,g(k-2)}, \ldots, \overline{v}_{k-1,1}, \ldots, \overline{v}_{k-1,g(k-2)}\}$. Then it can be easily verified that this isomorphism is given by the mapping $h : V(Z) \rightarrow V(X_{k-2})$ where, for any $v_{i,j} \in V(Z)$, $h(v_{i,j}) = \overline{v}_{i-1,j}$.

Assume $\text{box}(X_k) \leq r = \frac{k+1}{2}$. Then by Lemma 2, there exists a set of r interval graphs, say $\mathcal{I} = \{I_1, I_2, \ldots, I_r\}$, such that $X_k = I_1 \cap I_2 \cap \cdots \cap I_r$. As each I_i is an interval graph, there exists an interval representation f_i for it. Since the vertices of L_1 induce a clique in X_k, they also induce a clique in each $I_i \in \mathcal{I}$. Let $[s_i, t_i] = \bigcap_{u \in L_1} f_i(u)$. Lemma 6 guarantees that $[s_i, t_i] \neq \emptyset$. We know that for each $v_{k+1,p} \in L_{k+1}$, there exists $v_{1,q} \in L_1$ with $p \neq q$ such that $(v_{k+1,p}, v_{1,q}) \notin E(X_k)$ implying that there exists some $I_j \in \mathcal{I}$ such that $(v_{k+1,p}, v_{1,q}) \notin E(I_j)$. Therefore $f_j(v_{k+1,p}) \cap [s_j, t_j] = \emptyset$.

Now let us partition L_{k+1} into r sets $P_1, P_2, \ldots, P_r$ such that $P_i = \{u \in L_{k+1} \mid f_i(u) \cap [s_i, t_i] = \emptyset$ and for any $j < i$, $f_j(u) \cap [s_j, t_j] \neq \emptyset\}$. Since $|L_{k+1}| = g(k) = r \cdot (g(k-2) - 1) + 1$, there exists some P_α such that $|P_\alpha| \geq g(k-2)$. Without loss of generality, let us also assume that $v_{k+1,1}, v_{k+1,2}, \ldots, v_{k+1,g(k-2)} \in P_\alpha$. So $f_\alpha(v_{k+1,1}) \cap [s_\alpha, t_\alpha] = \emptyset$, $f_\alpha(v_{k+1,2}) \cap [s_\alpha, t_\alpha] = \emptyset$, $\ldots$, $f_\alpha(v_{k+1,g(k-2)}) \cap [s_\alpha, t_\alpha] = \emptyset$. Since $X_k = I_1 \cap I_2 \cap \cdots \cap I_r$, both the sets A and B which induce cliques in X_k also induce cliques in I_α. Let $[y_A, z_A] = \bigcap_{u \in A} f_\alpha(u)$ and $[y_B, z_B] = \bigcap_{u \in B} f_\alpha(u)$. By Lemma 6, $[y_A, z_A] \neq \emptyset$ and $[y_B, z_B] \neq \emptyset$. Since I_α is not a complete graph (recall that $f_\alpha(v_{k+1,1}) \cap [s_\alpha, t_\alpha] = \emptyset$ and therefore $v_{k+1,1}$ is not adjacent to some vertex in L_1 in I_α), $[y_A, z_A] \cap [y_B, z_B] = \emptyset$ implying that either $z_A < y_B$ or $z_B < y_A$. Assume $z_A < y_B$ (the proof is similar in the other case also). Therefore we have

$$y_A \leq z_A < y_B \leq z_B. \tag{4}$$

Since $L_1 \subseteq A$, we have

$$s_\alpha \leq y_A \leq z_A \leq t_\alpha. \tag{5}$$

Since $L'_{k+1} \subseteq L_{k+1} \subseteq B$, for any $u \in L'_{k+1}$,

$$l(f_\alpha(u)) \leq y_B \leq z_B \leq r(f_\alpha(u)). \tag{6}$$

From inequalities 4, 5 and 6, we can see that for any $u \in L'_{k+1}$,

$$s_\alpha < r(f_\alpha(u)). \tag{7}$$

Recall that, by our earlier assumption, $L'_{k+1} \subseteq P_\alpha$ and hence for any $u \in L'_{k+1}$, $f_\alpha(u) \cap [s_\alpha, t_\alpha] = \emptyset$. That is, either

$$t_\alpha < l(f_\alpha(u)) \text{ or } r(f_\alpha(u)) < s_\alpha$$

is true. From inequality 7, we then conclude that for any $u \in L'_{k+1}$,

$$t_\alpha < l(f_\alpha(u)). \tag{8}$$

Let $l_{\min} = \min_{u \in L'_{k+1}} l(f_\alpha(u))$. Combining inequalities 5, 6 and 8, we have

$$s_\alpha \leq y_A \leq z_A \leq t_\alpha < l_{\min} \leq y_B \leq z_B. \tag{9}$$

Since $A' \subseteq A$ and $B' \subseteq B$, A' overlaps in the region $[y_A, z_A]$ in f_α and B' overlaps in the region $[y_B, z_B]$ in f_α. For any $v_{i,j} \in A'$, since $(v_{i,j}, v_{k+1,j}) \in E(X_k) \subseteq E(I_\alpha)$,

$$l(f_\alpha(v_{i,j})) \leq y_A \leq z_A \leq t_\alpha < l_{min} \leq l(f_\alpha(v_{k+1,j})) \leq r(f_\alpha(v_{i,j})). \tag{10}$$

Also, since for any $u \in B'$, $w \in L_1$, $(u, w) \in E(X_k) \subseteq E(I_\alpha)$, we have for any $u \in B'$, $f_\alpha(u) \cap [s_\alpha, t_\alpha] \neq \emptyset$ and therefore,

$$l(f_\alpha(u)) \leq t_\alpha < l_{min} \leq y_B \leq z_B \leq r(f_\alpha(u)). \tag{11}$$

From inequalities 10 and 11, we can say that $A' \cup B'$ overlaps in the region $[t_\alpha, t_\alpha]$ in f_α. Hence in I_α, $A' \cup B'$ induces a clique. Now if Z is the graph induced by $A' \cup B'$ in X_k, $(u, w) \in E(Z) \Leftrightarrow (u, w) \in E(\bigcap_{I \in \mathcal{I} \setminus \{I_\alpha\}} I)$. Hence, Z is the graph induced by $A' \cup B'$ in $\bigcap_{I \in \mathcal{I} \setminus \{I_\alpha\}} I$. Therefore, $\frac{k-1}{2} = r - 1 \geq \text{box}\left(\bigcap_{I \in \mathcal{I} \setminus \{I_\alpha\}} I\right) \geq \text{box}(Z) = \text{box}(X_{k-2})$ (recall that by our earlier claim, Z is isomorphic to X_{k-2}). But this contradicts our induction hypothesis that $\text{box}(X_{k-2}) > \frac{k-1}{2}$. Hence the lemma. □

From Lemmas 7 and 8, we get the following lemma.

Lemma 9 $\text{box}(G_k) > \frac{k+1}{4}$.

Theorem 2 *For any $b \in \mathbb{N}^+$, there exists a chordal bipartite graph G with* $\text{box}(G) > b$.

Proof For any odd positive integer k, let $G_k = T_k^{[k]}$ as defined earlier. G_k is a CBG by Theorem 1. If $k = 4b - 1$, then by Lemma 9, $\text{box}(G_k) > b$. □

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