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ORIGINAL PAPER

The Zero-divisor Graphs of Posets and an Application to Semigroups

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Abstract In this paper, we introduce the notion of a compact graph. We show that a simple graph is a compact graph if and only if G is the zero-divisor graph of a poset, and give a new proof of the main result in Halaš and Jukl (Discrete Math 309:4584–4589, 2009) stating that if G is the zero-divisor graph of a poset, then the chromatic number and the clique number of G coincide under a mild assumption. We observe that the zero-divisor graphs of reduced commutative semigroups (rings) are compact, thus provide a large class of graphs G that could be realized as zero-divisor graphs of posets. In addition, using these results, we give some equivalent descriptions for the zero-divisor graphs of posets and reduced commutative semigroups with 0 respectively.

Keywords Compact graph · Partially ordered set · Reduced semigroup · Generalized complete r-partite graph

1 Introduction

Given an algebraic object such as a commutative semigroup or a commutative ring S, one can assign it a graph—the zero-divisor graph $\Gamma(S)$. The zero-divisor graph has been investigated by many authors. The research in this field aims at revealing the relationship between algebraic theory and graph theory and at advancing application of one to the other. Refs. [1–4] may serve as a survey along this line.

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In [11], Halaš and Jukl introduced the zero-divisor graph for a partially ordered set (*poset* for short). Let $(P, \leq)$ be a poset with a least element 0. For any $x, y \in P$, denote $L(x, y) = \{z \in P \mid z \leq x \text{ and } z \leq y\}$. An element $x \in P$ is called a zero-divisor if $L(x, y) = \{0\}$ for some $0 \neq y \in P$. The zero-divisor graph $\Gamma(P)$ of a poset P is the graph G whose vertex set $V(G)$ consists of the nonzero zero-divisors of P, in which a is adjacent to b if and only if $L(a, b) = \{0\}$. It was shown in [11] that for any poset P, the clique number and the chromatic number of $\Gamma(P)$ coincide if $\Gamma(P)$ contains no infinite cliques. We remark that the graph $G = \Gamma(P)$ defined here is slightly different from the one defined in [11], where the vertex set $V(G)$ consists of all the elements of P. More recently, the zero-divisor graph of qosets (quasiordered sets) was studied in [12]. One can refer to [8–10] for the new developments on posets.

In this paper, we introduce the notion of a compact graph. We show that a simple graph is a compact graph if and only if G is the zero-divisor graph of a poset, and give a new proof of the main result in [11] stating that if G is the zero-divisor graph of a poset, then the chromatic number and the clique number of G coincide under a mild assumption. We observe that the zero-divisor graphs of reduced commutative semigroups (rings) are compact, thus provide a large class of graphs G that could be realized as zero-divisor graphs of posets. In addition, using these results, we give some equivalent descriptions for the zero-divisor graphs of posets and reduced commutative semigroups with 0 respectively.

Throughout this paper, P will always denote a poset with a least element 0, S a commutative semigroup with 0, G a simple graph (i.e., an undirected graph without loops and multiple edges). A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Moreover, if for any $a, b \in V(H)$, $\{a, b\} \in E(G)$ implies $\{a, b\} \in E(H)$, then H is called an *induced* subgraph of G. Let x be a vertex of G. The neighborhood of x, denoted by $N(x)$, is the set of vertices adjacent to x. Let n be a (finite or infinite) cardinal number. An *n-partite* graph is a graph whose vertex set can be partitioned into n subsets so that no edge has both ends in the same subset. A *complete n-partite graph* is an n-partite graph in which each vertex is adjacent to every vertex that is not in the same part. An n-partite graph G is called a *proper* n-partite graph if G is not an r-partite graph for any $r < n$. Recall that the *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimal number of colors which can be assigned to the elements of $V(G)$ in such a way that two adjacent vertices have different colors. One easily sees that G is a proper n-partite graph if and only if $\chi(G) = n$. A subgraph C of a graph G is called a *clique* if C is a complete graph. The clique number $\omega(G)$ of G is the least upper bound of the sizes of cliques of G, and clearly $\chi(R) \geq \omega(G)$.

2 Compact Graphs

We begin with the following definition, which plays an important role in this paper.

Definition A simple graph G is called a *compact* graph if G contains no isolated vertices and for each pair x, y of non-adjacent vertices of G, there is a vertex z with $N(x) \cup N(y) \subseteq N(z)$.

First, we give some basic properties of compact graphs.

Proposition 2.1 *Let G be a compact graph. Then for any pair of distinct vertices x, y of G, either $N(x) \cap N(y) \neq \emptyset$, or each vertex in $N(x)$ is adjacent to all the vertices of N(y). In particular, G is connected with diameter at most* 3.

Proof Assume that $N(x) \cap N(y) = \emptyset$. Let $a \in N(x)$ and $b \in N(y)$. If a is not adjacent to b, then there exists a vertex $c \in V(G)$ such that $\{x, y\} \subseteq N(a) \cup N(b) \subseteq N(c)$, and so $c \in N(x) \cap N(y)$, a contradiction. Hence a is adjacent to b and the result follows.

Recall that a vertex x in a graph G is an *end* (vertex) in case there is at most one edge with vertex x. A *cut vertex* is a vertex whose deletion increases the number of connected components.

A graph G is a *star graph* in case there is a vertex x in G such that every other vertex in G is an end adjacent to x. A graph G is a *refinement* of a graph H in case the vertex sets of G and H are the same and every edge in H is an edge of G.

The *core* of G is the union of all the cycles of G.

Proposition 2.2 *Let G be a compact graph.*

(1) *If G is not a star graph, then any vertex in G is either an end or else lies in a cycle of length* ≤ 4. *In particular, the core of G is a union of triangles and rectangles.*
(2) *For each cut vertex c of G, G_c^* is a connected graph, where G_c^* is the subgraph of G induced on the vertex set $V(G) \setminus \{x \in G | x = c$ or x is an end vertex adjacent to $c\}$.*

Proof (1) Assume that $a \in V(G)$ is not an end. Since G is not a star graph, there is a path: $x - a - b - c$ in $V(G)$. In case when $x = c$ or x is adjacent to c, a is a vertex in the triangle $a - b - c - a$ or the rectangle $x - a - b - c - x$, respectively. If x is not adjacent to c, then there is a vertex $d \in V(G)$ such that $\{a, b\} \subseteq N(x) \cup N(c) \subseteq N(d)$ and so a is a vertex in the triangle $a - b - c - a$. This completes the proof of (1).

(2) Assume to the contrary that there exists a cut vertex c such that G_c^* has at least two connected components. Take vertices x, y from distinct components of G_c^*. Then there exists a vertex z such that $N(x) \cup N(y) \subseteq N(z)$. This z must be c and hence G is a refinement of a star graph with the center c. But then we have $c \in N(c)$, a contradiction. This completes the proof.

The rest of this section is devoted to colorings of compact graphs

Proposition 2.3 *Let G be a compact graph and n a positive integer. If G is a proper r-partite graph containing a clique of size $n - 1$, where r is a finite cardinal number with $r \geq n$, then G contains a clique of size n, i.e., $\omega(G) \geq n$.*

Proof Assume that G has parts $A_1, A_2, \ldots, A_{n-1}, A_n, \ldots$, and that $\{a_1, a_2, \ldots, a_{n-1}\}$ is a clique of G with $a_i \in A_i$, $i = 1, \ldots, n-1$. Since G is a proper r-partite graph, there is $a \in A_n$ such that $N(a) \cap A_i \neq \emptyset$ for $i = 1, \ldots, n-1$. If $\{a_1, a_2, \ldots, a_{n-1}\} \subseteq N(a)$, then we are done since $\{a_1, a_2, \ldots, a_{n-1}, a\}$ is a clique of size n. So assume $|N(a) \cap \{a_1, a_2, \ldots, a_{n-1}\}| \leq n - 2$.

Case 1 $|N(a) \cap \{a_1, a_2, \ldots, a_{n-1}\}| = n - 2$. Without loss of generality, we assume $\{a_1, a_2, \ldots, a_{n-2}\} \subseteq N(a)$ and $a_{n-1} \notin N(a)$. Fix $b_{n-1} \in N(a) \cap A_{n-1}$. As a_{n-1} is not adjacent to b_{n-1}, there exists $b \in V(G)$ such that $N(b) \supseteq N(a_{n-1}) \cup N(b_{n-1}) \supseteq \{a_1, \ldots, a_{n-2}, a\}$. It is easy to see that $\{a_1, \ldots, a_{n-2}, a, b\}$ is a clique of size n and we are done.

Case 2 $|N(a) \cap \{a_1, a_2, \ldots, a_{n-1}\}| < n - 2$. Without loss of generality, we assume that a_{n-1} is not adjacent to a. Since $\{a_1, \ldots, a_{n-2}\} \subseteq N(a_{n-1})$, there exists a vertex $b \in V(G)$ such that $N(b) \supseteq \{a_1, a_2, \ldots, a_{n-2}\} \cup N(a)$. Note that $N(b) \supseteq \{a_1, a_2, \ldots, a_{n-2}\}$ and $N(b) \cap A_i \supseteq N(a) \cap A_i \neq \emptyset$ for $1 \leq i \leq n-1$, we obtain a clique of size n by the the same way as in Case 1.

In both cases, we obtain a clique of size n and the result follows.

Corollary 2.4 *For any compact graph G whose clique number $\omega(G)$ is finite, the chromatic number $\chi(G)$ coincides with the clique number $\omega(G)$.*

Proof Write $\omega(G) = n$. If $\chi(G) = r > n$, then G is a proper r-partite graph and so $\omega(G) \geq n + 1$ by Proposition 2.3, a contradiction. Hence $\chi(G) = n$.

For a general simple graph G, the clique number $\omega(G)$ may be infinite even if G contains no infinite cliques. A typical example of this kind can constructed as follows. Let K_n denote a set of n elements for each $n > 0$, satisfying that $K_n \cap K_m = \emptyset$ if $m \neq n$. We use G for the simple graph with vertex set $V(G) = \cup_{n>0} K_n$ and for any distinct elements $x, y \in V(G)$, x is adjacent to y if and only if there is $n > 0$ such that $\{x, y\} \subseteq K_n$. It is clear that G contains no infinite cliques but $\omega(G) = \infty$. Hence, we need the following results to obtain the main theorem of this section.

Lemma 2.5 *For a compact graph G which has no infinite cliques, the ascending chain condition holds for neighborhoods of G.*

Proof Assume to the contrary that there exists a strictly increasing chain of neighborhoods:

$$N(x_1) \subset N(x_2) \subset \cdots \subset N(x_n) \subset \cdots$$

For i, fix $y_i \in N(x_i) \backslash N(x_{i-1})$. Since y_i is not adjacent to x_{i-1}, there exists $a_i \in V(G)$ such that $N(x_{i-1}) \cup N(y_i) \subseteq N(a_i)$. One easily checks that $a_i \in N(x_i) \setminus N(x_{i-1})$ and so $a_i \neq a_j$ if $i \neq j$. For any $j > i$, we have $a_i \in N(x_i) \subseteq N(x_{j-1}) \subseteq N(a_j)$, so a_i is adjacent to a_j. Thus $\{a_i | i \geq 1\}$ is an infinite clique, a contradiction.

Proposition 2.6 *For a compact graph G, the following statements are equivalent*:

(1) *G contains no infinite cliques.*
(2) *The ascending chain condition holds for neighborhoods of G.*
(3) $\omega(G) = n$ *for some positive integer* $n < \infty$.

In addition, if one of the three conditions hold, then $\omega(G)$ is the number of mutually distinct maximal neighborhoods.

Proof (1) $\Rightarrow$ (2). This follows from Lemma 2.5.

(2) $\Rightarrow$ (3). Set $T = \{N(a) | a \in V(G)\}$ and define a partial order on T by set inclusion. By Zorn's lemma, T has maximal elements, thus each $N(x)$ is contained in

a maximal element of T. We claim that if G contains a clique of size k, then T has at least k maximal elements. Indeed, let $\{a_1, a_2, \ldots, a_k\}$ be a clique of size k. For each i, pick up $b_i \in V(G)$ such that $N(a_i) \subseteq N(b_i)$, where $N(b_i)$ is a maximal element in T. We show that $N(b_i) \neq N(b_j)$ if $i \neq j$. If not, then $a_i \in N(a_j) \subseteq N(b_j)$, and so $b_j \in N(a_i) \subseteq N(b_j)$, a contradiction. This proves our claim. Let S be the subset consisting of all maximal elements in T. If $N(b_1) \neq N(b_2) \in S$, then b_1 is adjacent to b_2, for otherwise, there is a vertex $b \in V(G)$ such that $N(b_1) \cup N(b_2) \subseteq N(b)$, which is impossible. Since G contains no infinite cliques, S is finite and it follows that $\omega(G) = n$ by our claim, where $n = |S|$.

(3) $\Rightarrow$ (1). This is obvious. The final statement follows from the proof of (2) $\Rightarrow$ (3).

From the proof of Proposition 2.6, we have the following.

Corollary 2.7 *For a compact graph G, if each neighborhood $N(x)$ is contained in a maximal neighborhood, then the clique number of G coincides with the cardinal number of the set of all maximal neighborhoods in G.*

Combining Corollary 2.4 and Proposition 2.6, we obtain the main result of this section immediately.

Theorem 2.8 *If G is a compact graph which contains no infinite cliques, then $\omega(G) = \chi(G) = n$ for some positive integer $n < \infty$.*

3 The Zero-divisor Graph of a Poset

In this section, we will give a four-way characterization of the zero-divisor graph of a poset.

Theorem 3.1 *A simple graph G is the zero-divisor graph of a poset if and only if G is a compact graph.*

Proof $\Rightarrow$ Assume $G = \Gamma(P)$, where P is a poset with a least element 0. Assume further $V(G) \neq \emptyset$. Clearly, G contains no isolated vertices. If x is not adjacent to y, then $L(x, y) \neq \{0\}$, and so there is an element $0 \neq z \in P$ such that $z \leq x, z \leq y$, which implies $N(x) \cup N(y) \subseteq N(z)$, as desired.

$\Leftarrow$ Assume that G is a compact graph and denote $P = V(G) \cup \{0\}$, where $0 \notin V(G)$. We know that there is an equivalence relation $\sim$ in $V(G)$, i.e., for any $x, y \in V(G)$, $x \sim y$ if and only if $N(x) = N(y)$. Let $\{a_i | i \in I\}$ be a subset of $V(G)$ such that for any $x \in V(G)$, there is a unique $i \in I$ with $x \sim a_i$. Define an order $\leq$ on P as follows: (1) For any $x \in P$, $0 \leq x$ and $x \leq x$, (2) For any $x \neq y \in V(G)$, $x \leq y$ if and only if either $N(y) \subsetneqq N(x)$ or $N(y) = N(x)$ and $x = a_i$ for some $i \in I$. It is routine to check that $(P, \leq)$ is a partially ordered set. Denote $H = \Gamma(P)$ and we proceed to show that the graph H coincides with the graph G.

First, we have to show that $V(G) = V(H)$. Clearly, $V(H) \subseteq V(G)$. Conversely, G is connected by Proposition 2.1. Thus for any $x \in V(G)$, there exists $y \in V(G)$ such that x is adjacent to y. We claim that $L(x, y) = \{0\}$ and hence $x \in V(H)$ and $V(H) = V(G)$. Indeed, if $L(x, y) \neq \{0\}$, then there is an element $z \in V(G)$ such that $N(x) \subseteq N(z)$, $N(y) \subseteq N(z)$. Note that in this case $x \in N(z)$ and so $z \in N(x)$, giving $z \in N(z)$, a contradiction.

Now we show that $E(H) = E(G)$. Assume $x, y \in V(G)$ and x is adjacent to y in G. We have already shown $L(x, y) = \{0\}$, i.e., x is adjacent to y in H. On the other hand, let $x, y \in V(G)$ with $L(x, y) = \{0\}$. If $N(x) = N(y)$, then assume $N(x) = N(a_i)$ for some $i \in I$. Then $a_i \in L(x, y)$, a contradiction. Hence $N(x) \neq N(y)$ and it follows that x is adjacent to y in G, for otherwise, $N(x) \cup N(y) \subseteq N(z)$ for some $z \in V(G)$, which implies $z \in L(x, y)$, a contradiction. This completes the proof.

By Theorem 3.1 and Theorem 2.8, we rediscover the main theorem of [11].

Theorem 3.2 *If G is the zero-divisor graph of a poset which contains no infinite cliques, then $\omega(G) = \chi(G) = n$ for some positive integer $n < \infty$.*

It is interesting to compare Theorem 3.2 with Dilworth's theorem—a famous theorem in extremal combinatorial theory (see [6] or [13]).

Dilworth's theorem *Let P be a finite poset. The minimum number m of the disjoint chains which together contain all elements of P is equal to the maximum number M of elements in an antichain of P.*

Let $T(P)$ denote the graph whose vertex set is P in which a is adjacent to b if and only if a and b are non-comparable. Using this notion, Dilworth's theorem can be restated as follows: *If P is a finite poset, then $\omega(T(P)) = \chi(T(P))$.*

Recall that an equivalence relation $\sim$ was defined for any graph G as follows [1]: $x \sim y$ if and only if $N(x) = N(y)$ for any $x, y \in V(G)$. Let $[x]$ be the equivalence class which contains x, and G_r be the set of equivalence classes. Notice that G_r becomes a graph in the natural way with $[x]$ and $[y]$ adjacent in G_r if and only if x and y are adjacent in G. The graph G_r is called the *reduced graph* of G. Using Theorem 3.2, we can determine the structure of zero-divisor graphs of posets. We begin with the following corollary.

Corollary 3.3 (1) *A graph G is a compact graph if and only if its reduced graph G_r is a compact graph.*

(2) *A graph G is the zero-divisor graph of a poset if and only if its reduced graph G_r is the zero-divisor graph of a poset.*

Proof (1) A straightforward check. (2) This follows from (1) and Theorem 3.1.

For convenience, we give the following definition.

Definition 3.4 Let n be an integer greater than 1. A simple graph G is called an *n-compact graph* in case G is a compact graph with $\omega(G) = n$.

By Theorem 2.8, if G is an n-compact graph, then G is a proper n-partite graph with a clique of size n. To state our main result, we need the following lemmas.

Lemma 3.5 *Let G be an n-compact graph with parts $A_i, i = 1, 2, \ldots, n$, and let $a, b \in V(G)$. If $(N(a) \cup N(b)) \cap A_i \neq \emptyset$ for each $1 \leq i \leq n$, then a is adjacent to b.*

Proof If a is not adjacent to b, then there is a vertex $c \in V(G)$ such that $N(c) \cap A_i \neq \emptyset$ for each $1 \leq i \leq n$, which is impossible.

Let $\Delta_n(G)$ denote the set of all the vertices which lie in some clique of size n of G.

Lemma 3.6 *Let G be an n-compact graph with parts $A_i, i = 1, 2, \ldots, n$. If $a_i \in \Delta_n(G) \cap A_i, i = 1, 2, \ldots, n$, then $\{a_1, a_2, \ldots, a_n\}$ is a clique in G.*

Proof We easily see that if $1 \leq i \neq j \leq n$, then $(N(a_i) \cup N(a_j)) \cap A_k \neq \emptyset$ for $k = 1, 2, \ldots, n$ as either $k \neq i$ or $k \neq j$, and it follows that a_i is adjacent to a_j by Lemma 3.5.

By Lemma 3.6, if G is a n-compact graph, then the induced subgraph on $\Delta_n(G)$ is a complete n-partite graph.

Lemma 3.7 *Let G be an n-compact graph with parts $A_i, i = 1, 2, \ldots, n$. For any $1 \leq k \leq n$ and $h \in V(G) \setminus \Delta_n(G)$, if there exists a vertex $b_0 \in \Delta_n(G) \cap A_k$ such that h is adjacent to b_0, then h is adjacent to all vertices in $\Delta_n(G) \cap A_k$.*

Proof Assume that $\{a_i \in A_i \mid i = 1, 2, \ldots, n\}$ is a clique of G. Then for any $b \in \Delta_n(G) \cap A_k$, we have that $\{a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n\} \subseteq N(b)$ thus $N(h) \cup N(b) \supseteqq \{a_1, \ldots, a_{k-1}, b_0, a_{k+1}, \ldots, a_n\}$ by Lemma 3.6. Hence $(N(h) \cup N(b)) \cap A_i \neq \emptyset$ for $1 \leq i \leq n$, which shows that h is adjacent to b by Lemma 3.5, as desired.

For convenience, we denote $\{1, 2, \ldots, n\}$ by $[1, n]$, and for $a \in V(G)$ we set $W(a) = \{i \in [1, n] | N(a) \cap \Delta_n(G) \cap A_i \neq \emptyset\}$. Lemma 3.7 yields that if $a \in V(G) \setminus \Delta_n(G)$, then $i \in W(a)$ if and only if a is adjacent to any vertex in $\Delta_n(G) \cap A_i$.

Lemma 3.8 *Assume that G is a n-compact graph with parts $A_i, i = 1, 2, \ldots, n$ and let $x, y \in V(G) \setminus \Delta_n(G)$. Then x is adjacent to y if and only if $W(x) \cup W(y) = [1, n]$.*

Proof Let $\{a_i \in A_i | i = 1, 2, \ldots, n\}$ be a clique of G. It is clear that if $W(x) \cup W(y) = [1, n]$, then x is adjacent to y by Lemma 3.5. To prove the converse, assume $W(x) \cup W(y) \subsetneq [1, n]$ and x is adjacent to y. Fix $k \in [1, n] \setminus (W(x) \cup W(y))$. Then as mentioned above x is not adjacent to a_k, hence there exists b such that $N(b) \supseteqq N(x) \cup N(a_k) \supseteqq \{y, a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n\}$. It follows that $b \in \Delta_n(G) \cap A_k$ and y is adjacent to b. However, since $k \notin W(y)$, y is not adjacent to any of the vertices in $\Delta_n(G) \cap A_k$ by Lemma 3.7, a contradiction. This completes the proof.

Definition 3.9 A simple connected graph G is called a *generalized complete n-partite graph* if $V(G)$ is a disjoint union of A and H satisfying the following conditions:

(1) $A = \Delta_n(G)$ and the induced graph on A is a complete $n-$partite graph with parts, say $A_i, i = 1, 2, \ldots, n$.
(2) For any $h \in H$ and $i \in [1, n]$, h is adjacent to some vertex of A_i if and only if h is adjacent to any vertex of A_i.
Set $W(h) = \{1 \leq i \leq n | N(h) \cap A_i \neq \emptyset\}$ for any $h \in H$.
(3) For any $h_1, h_2 \in H$, h_1 is adjacent to h_2 if and only if $W(h_1) \cup W(h_2) = [1, n]$.

Remark 3.10 We will refer the graph in Definition 3.9 as a generalized complete n-partite graph with a partition $V(G) = A \cup H$ (or $V(G) = A_1 \cup A_2 \cup \ldots \cup A_n \cup H$). Note that in Definition 3.9, if $h \in H$, then $|W(h)| \leq n-2$, for otherwise, $h \in \Delta_n(G)$. Hence a generalized complete 2-partite graph is exactly a complete 2-partite graph.

Let $R = \mathbb{Z}_2^n$ be the direct product of n copies of the ring $\mathbb{Z}_2$ of integers modulo 2. Clearly, it is a Boolean ring and it becomes a poset by defining that $e \leq f$ if and only if $ef = e$ for any $e, f \in R$. It is interesting to see that the zero-divisor graphs of R as a ring (or a semigroup) and as a poset coincide. Recall that an element $e \neq 0$ in a poset P is *primitive* if for any $0 \neq f \in P$, $f \leq e$ implies $e = f$. We call a subgraph H of $\Gamma(\mathbb{Z}_2^n)$ *minimal* if H is the induced subgraph on a subset of $V(\Gamma(\mathbb{Z}_2^n))$ which contains all the primitive elements of the poset $\mathbb{Z}_2^n$, and *minimal closed* if H is minimal and $V(H) \cup \{0\}$ is a sub-semigroup of $\mathbb{Z}_2^n$. These concepts are very useful when we describe the structure of the zero-divisor graphs of posets and reduced semigroups.

Now, we are ready to state and prove the main theorem of this section.

Theorem 3.11 *Let G be a graph with $\omega(G) = n < \infty$. Then the following statements are equivalent*:

(1) *G is the zero-divisor graph of a poset.*
(2) *G is an n-compact graph.*
(3) *G is a generalized complete n-partite graph.*
(4) *The reduced graph G_r of G is isomorphic to a minimal subgraph of $\Gamma(\mathbb{Z}_2^n)$.*

Proof (1) $\Leftrightarrow$ (2) It follows from Theorem 3.1.

(2) $\Rightarrow$ (3) It follows from Lemmas 3.6, 3.7 and 3.8.

(3) $\Rightarrow$ (4) Let G be a generalized complete n-partite graph with a partition $A \cup H$, where A has parts $A_1, \ldots, A_n$. Then for any $x, y \in A$, $N(x) = N(y)$ if and only if $x, y \in A_i$ for some i, if and only if $W(x) = W(y)$. For any $x, y \in H$, $N(x) = N(y)$ if and only if $W(x) = W(y)$. For any $x \in A$, $y \in H$, we always have $N(x) \neq N(y)$ and $W(x) \neq W(y)$. In conclusion, for any $x, y \in V(G)$, $W(x) = W(y)$ if and only if $x \sim y$. Let $\{e_i | i \in [1, n]\}$ denote the set of all the primitive elements of $\mathbb{Z}_2^n$. We define a map $\phi : V(G_r) \to \mathbb{Z}_2^n$ by the rule that for any $x \in V(G)$,

$$\phi([x]) = \sum_{i \notin W(x)} e_i.$$

Then $\phi([a_i]) = e_i$ for any $a_i \in A_i$, $i = 1, \ldots, n$ and so $\mathrm{Im}(\phi)$ contains all the primitive elements of $\mathbb{Z}_2^n$. Since the sums on the right side are different for different W's, ϕ is injective. For any $[x], [y] \in V(G_r)$, one easily check that $[x]$ is adjacent to $[y]$ in G_r iff $W(x) \cup W(y) = [1, n]$ iff $\phi([x])\phi([y]) = 0$ iff $\phi([x])$ is adjacent to $\phi([y])$ in $\Gamma(\mathbb{Z}_2^n)$. Thus G_r is isomorphic to a minimal subgraph of $\Gamma(\mathbb{Z}_2^n)$.

(4) $\Rightarrow$ (1) It follows from Corollary 3.3.

By Theorem 3.11, one can easily check whether a simple graph is the zero-divisor graph of a poset.

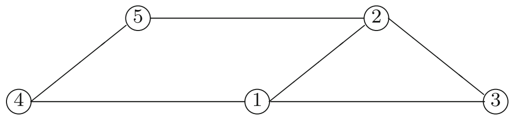

Fig. 1 A graph which is not the graph of any poset

Example 3.12 Let G be the graph of a poset. If G is a bipartite graph, then G is a complete bipartite graph by Remark 3.11. In particular, if G is a tree, i.e., a bipartite graph containing no cycles, then G is a star graph.

Example 3.13 The graph G in Fig. 1 is not the graph of a poset. In fact, if G is the graph of a poset, then $\Delta_n(G) = \{1, 2, 3\}$. Since $W(4) \cup W(5) = \{1, 2\} \neq [1, 3]$, we obtain by Definition 3.9 and Theorem 3.11 that the vertex 4 is not adjacent to 5, a contradiction.

Recall that a simple graph G is called a *split graph* if $V(G)$ can be partitioned into disjoint subsets S and K, such that K is a clique and S is an independent set (i.e., for any distinct vertices $x, y \in S$, x is not adjacent to y). We can determine when a split graph is the graph of a poset immediately using Theorem 3.11.

Corollary 3.14 *Let G be a split graph with a partition $V(G) = K \cup S$. If $|K| < \infty$, then G is the graph of a poset if and only if one of the following conditions hold*:

(1) *For any $a, b \in S$, $N(a) \cup N(b) \subsetneq K$.*
(2) *There exists $a \in S$ such that $N(a) = K$.*

Proof $\Rightarrow$ Note that if $\Delta_n(G) \neq K$, then there exists $a \in S$ such that $N(a) = K$ and in this case $\Delta_n(G) = K \cup \{a\}$. In case $\Delta_n(G) = K$, we have (1) and in case $\Delta_n(G) \neq K$, we have (2) by Definition 3.9 and Theorem 3.11.

$\Leftarrow$ It is immediate from Definition 3.9 and Theorem 3.11.

4 The Zero-divisor Graph of a Reduced Semigroup

In this section, by a semigroup we mean a commutative semigroup with a zero element 0. Recall that a semigroup S is said to be *reduced* if for any $x \in S$ and any positive integer n, $x^n = 0$ implies $x = 0$, and is said to be *idempotent* if for each $x \in S$, $x^2 = x$. An idempotent semigroup is just a so-called semilattice. Recall that a semigroup S is called (von Neumann) *regular* if for every $x \in S$, there exists an element $y \in S$ such that $x = xyx$. Clearly, every idempotent semigroup is von Neumann regular, and each regular semigroup is reduced. For details, we refer to [5,14].

Proposition 4.1 *Let S be a reduced semigroup with 0. Then the zero-divisor graph $\Gamma(S)$ of S is a compact graph.*

Proof The graph $\Gamma(S)$ clearly contains no isolated vertex. For any distinct non-adjacent vertices $x, y \in V(\Gamma(S))$, $xy \neq 0$ by definition and so $xy \in V(\Gamma(S))$. Let $a \in N(x) \cup N(y)$. Then either $ax = 0$ or $ay = 0$, and so $axy = 0$. As $xyxy \neq 0$, we obtain $a \neq xy$, thus $a \in N(xy)$. Hence $N(x) \cup N(y) \subseteq N(xy)$, as desired.

By Proposition 4.1 and Theorem 2.8, the equality $\chi(\Gamma(R)) = \omega(\Gamma(R))$ holds for any reduced ring R, which is one of the main results in [3].

Proposition 4.2 *Let G be any simple connected graph. If the reduced graph G_r of G is the zero-divisor graph of some idempotent semigroup, then so is G.*

Proof Let "$\sim$" be the natural equivalence relation on $V(G)$ and let $\{e_i | i \in I\}$ be a subset of $V(G)$ such that for any $x \in V(G)$, there is a unique $i \in I$ with $x \sim e_i$. Then G_r is (isomorphic to) the subgraph of G induced on the set $\{e_i | i \in I\}$ and so there is a binary operation $\diamond$ on the set $S = \{e_i | i \in I\} \cup \{0\}$ making S into an idempotent semigroup with $\Gamma(S) = G_r$. Now, define a binary operation on the set $T = V(G) \cup \{0\}$ as follows: for any $x, y \in V(G)$ with $x \sim e_i, y \sim e_j$, if $x = y$ then $xy = x$, if $x \neq y$ then $xy = e_i \diamond e_j$; and for any $x \in T$, $0x = x0 = 0$. We first check that the operation is associative. Let $x, y, z \in T$. If one of x, y, z is 0, then $(xy)z = x(yz) = y(xz) = 0$. So we assume $x, y, z \in V(G)$. Then there are $i, j, k \in I$ such that $x \sim e_i$, $y \sim e_j$, $z \sim e_k$. We distinguish three possible cases:

Case 1 Assume that x, y, z are pairwise distinct. If none of the equalities $z = xy$, $y = xz$, $x = yz$ holds, then $(xy)z = x(yz) = y(xz) = e_i \diamond e_j \diamond e_k$; Otherwise, assume without loss of generality that $z = xy$. Then

$$yz = y(e_i \diamond e_j) = \begin{cases} e_i \diamond e_j & \text{if } y = e_i \diamond e_j \\ e_i \diamond (e_i \diamond e_j) = e_i \diamond e_j & \text{if } y \neq e_i \diamond e_j \end{cases} = e_i \diamond e_j.$$

Hence

$$x(yz) = x(e_i \diamond e_j) = \begin{cases} e_i \diamond e_j & \text{if } x = e_i \diamond e_j \\ e_i \diamond (e_i \diamond e_j) = e_i \diamond e_j & \text{if } x \neq e_i \diamond e_j \end{cases}.$$

Thus $x(yz) = e_i \diamond e_j$. Similarly, $(xy)z = y(xz) = e_i \diamond e_j$. This proves $(xy)z = x(yz) = y(xz)$ in this case.

Case 2 Assume that $x = y$ and $x \neq z$. Then

$$x(yz) = x(e_j \diamond e_k) = \begin{cases} e_j \diamond e_k & \text{if } x = e_j \diamond e_k \\ e_i \diamond (e_j \diamond e_k) = e_j \diamond e_k \text{ (as } e_i = e_j) & \text{otherwise} \end{cases}.$$

Thus $x(yz) = e_j \diamond e_k$. Similarly, $(xy)z = y(xz) = e_j \diamond e_k$. This proves $(xy)z = x(yz) = y(xz)$ in this case.

Case 3 Assume that $x = y = z$. Then $(xy)z = x(yz) = y(xz) = x$.

Hence the operation is associative and T is an idempotent semigroup with 0. Finally we check that $\Gamma(T) = G$. Let $x, y \in V(G)$ with $x \neq y$. Then there are $i, j \in I$ such that $x \sim e_i$, $y \sim e_j$. Note that x is adjacent to y in G, if and only if e_i is adjacent to e_j in G_r, if and only if $e_i \diamond e_j = 0$, if and only if $xy = 0$, the result follows.

We conclude the paper by characterizing zero-divisor graphs of reduced semigroups with 0. Recall that a subgraph H of $\mathbb{Z}_2^n$ is called *minimal closed* if H is minimal and $V(H) \cup \{0\}$ is a sub-semigroup of $\mathbb{Z}_2^n$.

Theorem 4.3 *Let G be a simple graph with $\omega(G) = n < \infty$. Then the following statements are equivalent*:

(1) *G is the zero-divisor graph of a reduced semigroup with* 0.
(2) *G is a generalized complete n-partite graph such that for any nonadjacent vertices $a, b \in V(G)$, there is a vertex $c \in V(G)$ with $W(c) = W(a) \cup W(b)$.*
(3) *The reduced graph G_r of G is isomorphic to a minimal closed subgraph of $\Gamma(\mathbb{Z}_2^n)$.*
(4) *G is the zero-divisor graph of a semilattice (or equivalently, idempotent semigroup) with 0.*

Proof (1) ⇒ (2) By Theorem 3.11 and Proposition 4.1, $G = \Gamma(S)$ is a generalized complete n-partite graph. Assume that G has partition $V(G) = A_1 \cup \cdots \cup A_n \cup H$ and let $a, b \in V(G)$ be nonadjacent vertices. If $a, b \in A$, then $a, b \in A_i$ for some i, so $W(a) = W(b) = W(a) \cup W(b)$. If $a \in H$ and $b \in A$, then $W(a) \subsetneqq W(b)$ and so $W(a) \cup W(b) = W(b)$. If $a, b \in H$, we claim $W(a) \cup W(b) = W(ab)$. If not, pick up $k \in W(ab) \setminus (W(a) \cup W(b))$ and take $x \in A_k$. As $W(xa) \supseteqq W(x)$, we have $ax \in A_k$. Likewise, $bx \in A_k$. Hence $axbx \neq 0$. However, since $k \in W(ab)$ and $x \in A_k$, $xab = 0$, a contradiction.

(2) ⇒ (3) As in the proof of Theorem 3.11, we define the map $\phi : V(G_r) \to \mathbb{Z}_2^n$ by the rule that for any $x \in V(G)$,

$$\phi([x]) = \sum_{i \notin W(x)} e_i.$$

It remains to check that $S = \mathrm{Im}(\phi) \cup \{0\}$ is a sub-semigroup of $\mathbb{Z}_2^n$. Let $[a], [b] \in G_r$. If $[a]$ is adjacent to $[b]$, then $\phi([a])\phi([b]) = 0 \in S$. If $[a]$ is not adjacent to $[b]$, then there is $c \in V(G)$ such that $W(c) = W(a) \cup W(b)$. Note that

$$\phi([a])\phi([b]) = \sum_{i \notin W(a)} e_i \sum_{i \notin W(b)} e_i = \sum_{i \notin W(a) \cup W(b)} e_i = \phi([c]),$$

showing S is closed under multiplication and thus is a sub-semigroup of $\mathbb{Z}_2^n$.

(3) ⇒ (4) It follows from Proposition 4.2.

(4) ⇒ (1) Trivially.

Remark 4.4 Using Theorem 3.11 and Theorem 4.3, we easily see that (1) the zero-divisor graph of a reduced semigroup is always the zero-divisor graph of a poset; (2) the zero-divisor graph of a reduced semigroup is the same as the zero-divisor graph of a semilattice (i.e., an idempotent semigroup).

Remark 4.5 Since the zero-divisor graphs of reduced semigroups are the same as zero-divisor graphs of semilattices by Theorem 4.3, one may ask if it is true that a reduced semigroup and its greatest semilattice image have the same zero-divisor graph. Consider the semigroup $(\mathbb{Z}_6, \times) = \{0, 1, 2, 3, 4, 5\}$. Let T be the subset $\{0, 2, 4, 3\}$ of $\mathbb{Z}_6$. Then T is a reduced semigroup as a subsemigroup of $\mathbb{Z}_6$. Clearly $\Gamma(T)$ is isomorphic to $K_{1,2}$, the complete bipartite graph on three vertices. Set $S = \{0, a, b\}$. Then S is an idempodent semigroup (i.e., semilattice) in the operation $a^2 = 2, b^2 = 3, a \cdot b = 0$.

Define $\phi : T \to S$ by $\phi(2) = \phi(4) = a$ and $\phi(3) = b$. It is direct to check that ϕ is a semigroup homomorphism and so S is a greatest semilattice image of T. However $\Gamma(S)$ is an edge. Hence $\Gamma(T) \ncong \Gamma(S)$.

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