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A Description of the Resonance Variety of a Line Combinatorics via Combinatorial Pencils

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Abstract. In this paper we study the resonance variety of a line combinatorics. We introduce the concept of combinatorial pencil, which characterizes the components of this variety and their dimensions. The main theorem in this paper states that there is a correspondence between components of the resonance variety and combinatorial pencils. As a consequence, we conclude that the depth of a component of the resonance variety is determined by its dimension; and that there are no embedded components. This result is useful to study the isomorphisms between fundamental groups of the complements of line arrangements with the same combinatorial type. The definition of combinatorial pencil generalizes the idea of net given by Yuzvinsky and others.

Key words. Line arrangement, Resonance variety, Pencil.

Introduction

The problem of the relationship between the topology of a line arrangement in $\mathbb{CP}^2$ and its combinatorial structure has been one of the most studied in the theory of hyperplane arrangements. After the work of Arnol'd ([1]) and Brieskorn ([4]), Orlik and Solomon showed in [15] that the cohomology algebra of the complement is determined by the intersection lattice. Rybnikov exhibited in [17] the existence of two combinatorially equivalent line arrangements $\mathscr{L}_1$, $\mathscr{L}_2$, whose complements have non-isomorphic fundamental groups. His approach had essentially two parts: On one hand, he stated that an isomorphism between the fundamental groups should preserve the homology classes of meridians (which depend only on the combinatorics). On the other hand, he could distinguish both arrangements using an invariant under such isomorphisms. In the first part is where most details are missing.

In [2] a detailed proof of Rybnikov's result is given. In this paper we generalize the concepts and methods introduced in [2]. The main object of this paper is the study of the resonance variety of a line combinatorics. Historically, its components have been related to combinatorial objects such as neighborly partitions (see [7, 8, 14]), but this relation is not one to one, because there are neighborly partitions that

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do not correspond to components of the resonance variety. Yuzvinsky introduced the concept of net (see [18]), that solved this problem, but presented the opposite, that is, every net corresponds to a component of the resonance variety, but there are components that do not correspond to any net. In this paper, we introduce the concept of admissible class, which characterizes the components of the resonance variety in characteristic zero. The main result of this paper is Theorem 2. It shows that maximal admissible classes are in one-to-one correspondence with combinatorial pencils. This equivalence has not only a theoretical relevance, but also computational: checking the admissibility of a combinatorics involves solving a system of quadratic equations, whereas doing the same for a combinatorial pencil only involves solving linear systems. As a consequence of the main result, we can obtain some results about the structure of the resonance variety of matroids, such as Corollary 1 and Theorem 4. They state that the resonance variety is a finite union of linear subspaces, whose depth coincide with their dimension. These facts were proved for the case of realizable matroids in [14] using geometric tools. Our approach focuses only on the combinatorics, and hence, it can be applied to any line combinatorics, regardless of the existence of realizations.

Section 1 contains the definitions of admissibility for maps, classes and combinatorics. Some examples of admissible combinatorics are shown. It also contains the definition of combinatorial pencil, which captures the combinatorial properties of a line arrangement contained in a pencil of curves. This naturally extends the idea of net allowing weights and arbitrary combinatorics of each fiber. Combinatorial pencils are a particular case of neighborly partitions. The proof of the main theorem is provided in Section 2. It uses the Vinberg classification of real matrices, which is included for completeness. The relationship between admissible classes and components of the resonance variety is also shown in this section. This proof sharpens the one given by Libgober and Yuzvinsky in [14], with some modifications to rule out the indefinite case. This way we obtain the one-to-one correspondence between the irreducible components of the resonance variety and combinatorial pencils[1].

Assume a line combinatorics admits more than one realization. Section 3 describes a method to study the possible isomorphisms between the fundamental groups of their complements. This method involves studying the permutation induced in the set of maximal admissible classes, $\mathrm{Adm}(\mathscr{L}, \mathscr{P})$. Such a permutation must preserve some structure in $\mathrm{Adm}(\mathscr{L}, \mathscr{P})$. In Section 3 we give a formal description of this structure through the function Υ, which allows us to define the concept of triangle of admissible classes. By studying the set of triangles, we can conclude which are the possible permutations of admissible classes. And, if the combinatorics is rich enough, we can also enumerate all the possible isomorphisms induced in the homology. This method follows some of the ideas behind the one given by Falk in [7, Thm. 3.20], but can be used in a more general family of combinatorics. In Section 4 we give an example of how to use it in one of these combinatorics. The condition of strong connectedness is still used, although a weaker condition should work. Section 5 is an appendix that includes, for the sake of completeness, a proof

[1] This fact was independently discovered by Falk and Yuzvnsky in [9]

of the well-known duality between the second level of the lower central series of the fundamental group of a line arrangement, and its Orlik-Solomon algebra.

1. Preliminary Definitions

Definition 1. A **line combinatorics** is a finite set $\mathscr{L} := \{l_1, \ldots, l_n\}$ together with a subset $\mathscr{P} \subseteq \mathcal{P}(\mathscr{L})$ satisfying the following properties:

i) $\#p > 1$, for every $p \in \mathscr{P}$.
ii) for every $l_i, l_j \in \mathscr{L}$, $l_i \neq l_j$, there exists a unique $p \in \mathscr{P}$ such that $l_i, l_j \in p$. This element will be called the intersection of l_i and l_j, and will be denoted by $l_i \cap l_j$.

The elements of $\mathscr{L}$ and $\mathscr{P}$ will be called **lines** and **points** respectively.

This definition captures the incidence properties of the set of lines and the set of points (identifying a point with the set of lines that pass through it) of a line arrangement in a projective plane. Given a line arrangement in the complex projective plane, some of the invariants of its topology depend only on the combinatorics; so we will refer to them as invariants of the combinatorics. In the following we will assume that we have fixed a line combinatorics of n lines. In this context, we will define H as the quotient of the lattice $\mathbb{Z}^n$ by the sublattice generated by the vector $(1, \ldots, 1)$. We fix an order in $\mathscr{L}$ that allows us to establish a bijection between the lines $\{l_1, \ldots, l_n\}$ and the elements of the canonical generating system $\{e_1, \ldots, e_n\}$ of H, (these are the classes in H of the canonical basis of $\mathbb{Z}^n$). For the sake of simplicity, e_l with $l \in \mathscr{L}$ will also denote a canonical generator of H. If there existed a realization of the combinatorics in the complex projective plane, H would be canonically isomorphic to the first homology group of the complement.

Definition 2. Let k be a positive integer greater than 2. A **k-admissible map** is a $\mathbb{Z}$-epimorphism $\alpha : H \to \mathbb{Z}^{k-1}$ such that, for every $p \in \mathscr{P}$ and for every $l_i \in p$, the vectors $\{\alpha(e_i), \sum_{l_j \in p} \alpha(e_j)\}$ are linearly dependent.

Given an admissible map, we define its associated subcombinatorics as the combinatorics whose set of lines is $\mathscr{L}_\alpha := \{l_i \mid \alpha(e_i) \neq 0\}$, and its set of points is $\mathscr{P}_\alpha := \{p \cap \mathscr{L}_\alpha \mid p \in \mathscr{P}, \#(p \cap \mathscr{L}_\alpha) > 1\}$. That reflects the intuitive idea of “deleting" the lines l_i where $\alpha(e_i)$ vanishes.

A line combinatorics is said to be **k-admissible** if it admits a k-admissible map α such that $\alpha(e_l) \neq 0$ for every $l \in \mathscr{L}$.

The group $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^{k-1})$ acts on the set of k-admissible maps by composition. The orbits of this action will be called **k-admissible classes**.

We will say that a k-admissible map is maximal if it can not be obtained by composition of a $(k+1)$-admissible map and a $\mathbb{Z}$-epimorphism $\mathbb{Z}^k \to \mathbb{Z}^{k-1}$. Analogously we will talk about maximal k-admissible combinatorics and classes. The set of maximal k-admissible classes will be denoted by $\mathrm{Adm}_k(\mathscr{L}, \mathscr{P})$; and the set of all maximal admissible classes (that is, $\bigcup_{k>2} \mathrm{Adm}_k(\mathscr{L}, \mathscr{P})$) will be denoted by $\mathrm{Adm}(\mathscr{L}, \mathscr{P})$.

Remark 1. The combinatorics $(\mathscr{L}_\alpha, \mathscr{P}_\alpha)$ associated with an admissible map α is an invariant of its admissible class.

Example 1. Let $l_1, \ldots, l_k$ be k concurrent lines (with $k \geq 3$). One can define a k-admissible map as follows: consider $\{v_1, \ldots, v_{k-1}\}$ a basis of $\mathbb{Z}^{k-1}$. Now let α be the map given by $\alpha(e_j) = v_j$ for $j = 1, \ldots, k-1$, and $\alpha(e_k) = -v_1 - \cdots - v_{k-1}$. This is an admissible map, in fact this k-admissible combinatorics is maximal, we will call it of *point type*. An admissible map whose associated subcombinatorics is of point type is also called an admissible map of *point type*.

Example 2. Let $l_1, \ldots, l_6$ be six lines whose non-double points are $\{l_1, l_2, l_3\}, \{l_1, l_5, l_6\}, \{l_2, l_4, l_6\}, \{l_3, l_4, l_5\}$. Let $\{v_1, v_2\}$ be a basis of $\mathbb{Z}^2$, we can define the following 3-admissible map α given by $\alpha(e_1) = \alpha(e_4) = v_1, \alpha(e_2) = \alpha(e_5) = v_2$, $\alpha(e_3) = \alpha(e_6) = -v_1 - v_2$. It is again easy to check that α is admissible. The admissible classes of this form will be called of *Ceva type*. As in the previous example, we can talk about subcombinatorics of *Ceva type*.

Example 3. Consider a finite field $\mathbb{F}$ of k elements, and consider $\mathbb{F}^2$ the affine plane over $\mathbb{F}$. In this plane there are $k + 1$ directions, and for every direction there are k lines. For each point of the plane passes exactly one line of each direction. So we can construct $(k + 1)$-admissible combinatorics as follows: for each direction D, construct a combinatorics $(\mathscr{L}_D, \mathscr{P}_D)$ using the lines whose direction is D, and consider the combinatorics $(\mathscr{L}, \mathscr{P})$ such that $\mathscr{L}$ is the set of all lines in $\mathbb{F}^2$, and $\mathscr{P}$ is the union of all the $\mathscr{P}_D$ plus the points of $\mathbb{F}^2$. Then choose a basis $(v_1, \ldots, v_k)$ of $\mathbb{Z}^k$, and order the directions of the plane. Now define $v_{k+1} := -v_1 - \cdots - v_k$ and let $\alpha(e_l) = v_i$ if l goes in the i'th direction. This map will be admissible regardless of the election of $\mathscr{P}_D$.

In the case of $\mathbb{Z}/2\mathbb{Z}$, there is only one way of choosing the intersections inside each direction (two lines only have one way to intersect), and the result is the Ceva combinatorics.

In the case of $\mathbb{Z}/3\mathbb{Z}$, and choosing the lines in each direction to be in general position, the result is the combinatorics of the twelve lines joining the nine flexes of a smooth cubic. This one is called the Hesse combinatorics. Explicit equations of a realization can be found, for instance, in [9, Section 3].

There is also a "degenerated" Hesse combinatorics, which is not realizable in the complex plane, where the combinatorics $(\mathscr{L}_D, \mathscr{P}_D)$ of each direction D is chosen to be a triple point.

Example 4. From the Ceva combinatorics, we can construct another classical combinatorics (see [7, Example 4.6] and [9]) by adding lines joining every pair of double points. The result is a combinatorics of nine lines, as shown in Figure 1, and is called *generalized Ceva* combinatorics. In Figure 1 we can see that it is admissible. Note that in this case, there are non-equal proportional vectors.

Its realization in the complex (or real) plane is the union of the three special fibers of the pencil of rational nodal quartics, generated by $x^2(y^2 + z^2)$ and $y^2(x^2 + z^2)$ (up to projective transformation).

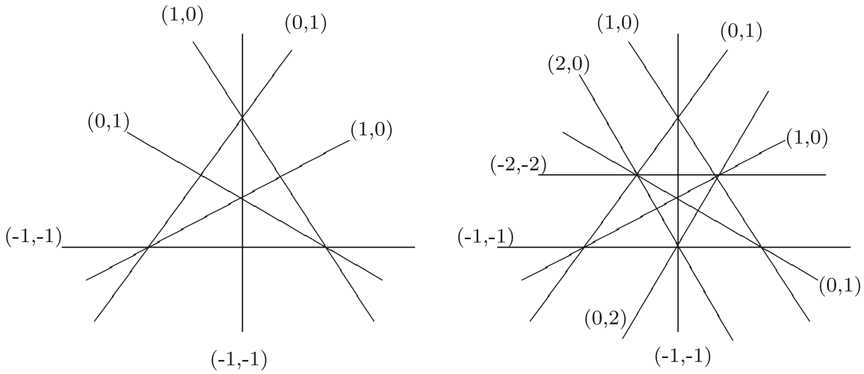


Fig. 1. Ceva and generalized Ceva combinatorics

The concept of admissible map can be presented in a way that is more independent of the election of a basis in H. Consider the $\mathbb{Z}-$submodule $R \leqslant H \wedge H$ generated by the family $\{e_j \wedge \sum_{i \in p} e_i \mid p \in \mathscr{P}, j \in p\}$ (where $\wedge$ denotes the exterior product).

Proposition 1. *An epimorphism* $\alpha : H \to \mathbb{Z}^{k-1}$ *is* k*-admissible if and only if* $\alpha \wedge \alpha(x) = 0 \; \forall x \in R$.

Proof. Let p be a point, $l_j \in p$ and consider a generator $e_j \wedge \sum_{i \in p} e_i$ of R. Its image under $\alpha \wedge \alpha$ is $\alpha(e_i) \wedge \sum_{i \in p} \alpha(e_i)$, which is zero if and only if $\alpha(e_i)$ and $\sum_{i \in p} \alpha(e_i)$ are linearly dependent in $\mathbb{Z}^{k-1}$. □

The definition of R is motivated by the complement of a realization in the complex projective plane: the second factor of the lower central series of its fundamental group is isomorphic to $(H \wedge H)/R$, see [17]. This group also appears in the study of the truncated Alexander invariant of the complement of a line arrangement. In [2] this group was used to study the set of isomorphisms of fundamental groups of both McLane's and Rybnikov's arrangements.

A straightforward consequence of [14, Prop. 7.2] is that a line arrangement whose combinatorics is k-admissible is a union of fibers of a pencil. In particular, Example 1 is trivially the union of some fibers of a pencil of lines; Example 2 is the union of the three singular fibers of a pencil of conics in general position; and Example 4, as explained before, is the union of three non-reduced fibers of a pencil of quartics. The following definition captures the combinatorial properties of a pencil of lines:

Definition 3. A **combinatorial pencil** is a line combinatorics together with a partition $F_1, F_2, \ldots, F_k$ of $\mathscr{L}$ into $k \geq 3$ subsets, and a weight map $w : \mathscr{L} \to \mathbb{Z}^+$ such that at any point $p \in \mathscr{P}$ only one of the following two possibilities occurs:

i) $p \subseteq F_i$ for some $i \in \{1, 2, \ldots, k\}$.
ii) $\forall i \in \{1, 2, \ldots, k\}$, $p \cap F_i \neq \emptyset$ and $\sum_{l \in p \cap F_i} w(l) = \sum_{l \in p \cap F_j} w(l)$ for all F_i, F_j.

The points satisfying property ii) will be called the **base points** of the combinatorial pencil, and the elements F_i of the partition will be called **fibers**.

We will call a combinatorial pencil **maximal** if no other combinatorial pencil can be constructed by refining the partition.

Remark 2. From a combinatorial pencil of k fibers $F_1, \ldots, F_k$, we can construct a k-admissible map as follows: let $\{v_1, \ldots, v_{k-1}\}$ be a basis of $\mathbb{Z}^{k-1}$, and $v_k := -v_1 - \cdots - v_{k-1}$, now consider the map $\alpha : H \to \mathbb{Z}^{k-1}$ given by $\alpha(e_i) := w(l_i) \cdot v_j$, where $l_i \in F_j$. The map α satisfies the condition in Def. 2. Thus, each combinatorial pencil determines an admissible map; the goal of Section 2 is to prove its converse.

Remark 3. It can be shown that, when the combinatorics is realizable, any maximal combinatorial pencil with relatively prime weights is in fact a geometrical pencil (see [14] and [6])

Example 5. From a combinatorial pencil with weight map w and partition Π of k fibers, another one of fewer fibers and with the same base points can be constructed as follows: take a partition of Π, say $\{\Pi_1, \ldots, \Pi_m\}$, and consider the partition $\bar{\Pi} := \{\bigcup \Pi_1, \ldots, \bigcup \Pi_m\}$ of $\mathscr{L}$. Define the weight map given by $\bar{w}(l) = \frac{\operatorname{lcd}(\{\#\Pi_i | i=1,\ldots,m\})}{\#\Pi_j} w(l)$, where $l \in F \in \Pi_j$. Both $\bar{\Pi}$ and $\bar{w}$ determine a combinatorial pencil. This justifies the following

Definition 4. A combinatorial pencil is said to be **maximal** if every two lines in the same fiber can be connected by non-base points.

Remark 4. Every combinatorial pencil is a neighborly partition (see [7, 3.9]), but the converse is not true. A simple example can be constructed by joining two fibers of the Hesse combinatorics as in the previous example, and then deleting one line from one of these two fibers. The result would be a combinatorics of eleven lines with the following multiple points: $\{l_1, l_4, l_8, l_{10}\}$, $\{l_1, l_5, l_9, l_{11}\}$, $\{l_1, l_6, l_7\}$, $\{l_2, l_4, l_9\}$, $\{l_2, l_5, l_7, l_{10}\}$, $\{l_2, l_6, l_8, l_{11}\}$, $\{l_3, l_4, l_7, l_{11}\}$, $\{l_3, l_5, l_8\}$ and $\{l_3, l_6, l_9, l_{10}\}$; the rest of the intersections are in double points. The partition $\{\{l_1, l_2, l_3\}, \{l_4, l_5, l_6\}, \{l_7, \ldots, l_{11}\}\}$ is a neighborly partition (every point either has only lines of the same component or has lines of every component), but one can easily check that it is not a combinatorial pencil.

Falk and Yuzvinsky refined in [9] the concept of neighborly partition to the one of weak multinet, which coincides with the definition of combinatorial pencil. Multinets are exactly the maximal combinatorial pencils with relatively prime weights.

2. Decomposition in Fibers

The goal of this section is to prove that each admissible map determines a combinatorial pencil. In order to do so, we will use some ideas from [14] and the Vinberg classification of matrices (see [11]), which we will include here for completeness. In particular, we will use [11, Thm. 4.3]:

Notation 1. Given a vector $u \in \mathbb{R}^n$, we will write $u \geq 0$ (resp. $>, \leq$ or $<$) to denote that all its entries are nonnegative (resp. positive, nonpositive or negative).

Theorem 1. *Let $M = (a_{i,j})$ be a real $n \times n$ matrix such that:*

- *M is indecomposable.*
- *$a_{i,j} \leq 0$ for $i \neq j$.*
- *$a_{i,j} = 0$ implies $a_{j,i} = 0$.*

Then only one of the following three possibilities hold for both M and its transposed:

(Fin) $\det(M) \neq 0$*; there exists $u > 0$ such that $Mu > 0$; $Mv \geq 0$ implies $v = 0$ or $v > 0$.*
(Aff) $\operatorname{corank}(M) = 1$*; there exists $u > 0$ such that $Mu = 0$; $Mv \geq 0$ implies $Mv = 0$.*
(Ind) there exists $u > 0$ such that $Mu < 0$; $Mv \geq 0$, $v \geq 0$ imply $v = 0$.

In all this section we will assume that $(\mathscr{L}, \mathscr{P})$ is a maximal k-admissible line combinatorics with admissible map α. First consider $\chi_\alpha := \{p \in \mathscr{P} \mid \sum_{l_i \in p} \alpha(e_i) = 0\}$.

If $\#\chi_\alpha = 1$ then $(\mathscr{L}, \mathscr{P})$ is of point type, otherwise there exists a line l_i that does not go through the only point in χ_α. Then for every line l_j, the corresponding vector $\alpha(e_j)$ is proportional to $\alpha(e_i)$, since both are proportional to $\sum_{l_k \in l_i \cap l_j} \alpha(e_k)$, which is not zero. This contradicts the admissibility of α. From Example 1 a maximal k-admissible combinatorics of point type defines a combinatorial pencil with k fibers (one per line).

From now, we will assume that $\#\chi_\alpha \geq 2$. Consider a graph whose vertices are the lines and whose edges join every two lines that intersect outside χ_α. We have a partition of $\mathscr{L}$ given by the connected components of this graph; let us denote such a partition by Π. We can assume that the lines are ordered in a way compatible with Π (that is, if l_i and l_j are in the same component, and $i < k < j$, then l_k is also in the component of l_i and l_j).

Remark 5. This same decomposition is done in [14] with a slightly different approach. Consider Q the $(n \times n)$ matrix whose entries are $Q_{i,i} := \#\{p \in \chi_\alpha \mid l_i \in p\} - 1$ in the diagonal; $Q_{i,j} := 0$ if the intersection of l_i and l_j is in χ_α; and $Q_{i,j} := -1$ otherwise. This is a symmetric matrix that can be decomposed in a direct sum of indecomposable matrices $\oplus_{F \in \Pi} M_F$. This decomposition corresponds to the connected components of the previous graph. It is straightforward to check that $Q_{i,i} \geq 1$ for every i. Another way to define Q is $Q := J^T J - U$, where J is the incidence matrix between χ_α and $\mathscr{L}$, and U is the $\#\mathscr{L} \times \#\mathscr{L}$ matrix whose every entry is 1.

Note that if two lines l_i, l_j are in the same component of Π, the vectors $\alpha(e_i)$ and $\alpha(e_j)$ are linearly dependent, and hence if $F \in \Pi$, there exists a primitive vector $v_F \in \mathbb{Z}^{k-1}$ such that $\forall l \in F$, $\alpha(e_l) = w_l v_F$ for some $w_l \in \mathbb{Z}$.

Lemma 1. *Let $F \in \Pi$, then all the entries of the weight vector $(w_l)_{l \in F}$ have the same sign. In particular $(w_l)_{l \in F}$ can be chosen to be positive.*

Proof. Fix a line l_i, the following equations hold

$$\left\{ \sum_{l_j \in p} \alpha(e_j) = 0 \;\middle|\; p \in \chi_\alpha,\ l_i \in p \right\}. \tag{1}$$

The following properties hold for the system (1)

- $\alpha(e_i)$ appears in $Q_{i,i} + 1$ equations,
- if $l_i \cap l_j \in \chi_\alpha$, $\alpha(l_j)$ appears exactly once,
- if $l_i \cap l_j \notin \chi_\alpha$, $\alpha(l_j)$ does not appear.

Hence if we substract every equation in (1) from the equation

$$\sum_{j=1}^{n} \alpha(e_j) = 0, \tag{2}$$

we obtain that

$$-Q_{i,i}\alpha(e_i) + \sum_{l_i \cap l_j \notin \chi_\alpha} \alpha(e_j) = 0. \tag{3}$$

Since for all such l_j, $\alpha(e_j) = w_j v$ for a certain $v \neq 0$, the equation (3) can be expressed as

$$Q_{i,i} w_i - \sum_{l_i \cap l_j \notin \chi_\alpha} w_j = 0, \tag{4}$$

which means in particular that the weight vector $(w_1, \ldots, w_n)$ is in the kernel of Q. Moreover, for every $F \in \Pi$, the weight vector $(w_l)_{l \in F}$ associated with F is in the kernel of the corresponding indecomposable matrix M_F. Since all these matrices satisfy the hypothesis of Theorem 1, M_F is of one of the three types (Aff), (Fin) or (Ind). We have found a nonzero vector in its kernel, so it cannot be of (Fin) type. Now suppose that, for a certain $G \in \Pi$, the matrix M_G is of (Ind) type. There exists a positive vector $u_G > 0$ such that $M_G u_G < 0$. Now, for every $F \in \Pi \setminus \{G\}$, there exist a vector $u_F < 0$ such that $M_F u_F > 0$ (if M_F is of (Ind) type), or $M_F u_F = 0$ (if M_F is of (Aff) type). By multiplying each u_F by an adequate positive constant, we can assume that $\sum_{l \in \mathscr{L}} u_l = 0$. Consider the vector $u = (u_l)_{l \in \mathscr{L}}$ obtained by concatenation of all the u_F. Since the sum of the entries of u is zero, $Uu = 0$. Then, denoting by $(\bullet, \bullet)$ the standard scalar product, we have:

$$\begin{aligned} 0 &\leq (Ju, Ju) = (Qu, u) + (Uu, u) = (M_G u_G, u_G) + \sum_{F \in \Pi \setminus \{G\}} (M_F u_F, u_F) \\ &\leq (M_G u_G, u_G) < 0 \end{aligned} \tag{5}$$

which is a contradiction. We conclude that all the M_F are of (Aff) type, and since the vectors $(w_l)_{l \in F}$ generate the kernel of M_F, all its entries must have the same sign. □

In particular, since $\alpha(e_1), \ldots, \alpha(e_n)$ generate $\mathbb{Z}^{k-1}$ and $\sum_{i=1}^{n} e_i = 0$, the previous $\{v_F\}_{F \in \Pi}$ is a linearly dependent generating system in $\mathbb{Z}^{k-1}$. So we can conclude that $\#\Pi \geq k$.

Remark 6. By multiplying the v_F's by appropriate constants, we can assume that $\sum_{l\in F} w_l$ is constant for all $F \in \Pi$.

We now will recall the definition of the Orlik-Solomon algebra of a line combinatorics (see [16]) in order to use some of its properties.

Definition 5. Let $(\mathscr{L}, \mathscr{P})$ be a line combinatorics. $\mathscr{L} = \{l_1, \dots, l_n\}$. Consider the $(n-1)$-dimensional vector space E_1 over a field $\mathbb{K}$ generated by $x_1, \dots, x_n$ satisfying the relation $x_1 + \dots + x_n = 0$. Let E be the graded exterior algebra of E_1 (note that $x_1, \dots, x_n$ correspond to the generators $e_1, \dots, e_n$ of H but it is more convenient to use a different notation to distinguish both objects). Now consider the differential $\delta : E_p \to E_{p-1}$ given by

$$\delta(x_{i_1} \wedge \dots \wedge x_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} (x_{i_1} \wedge \dots \wedge \hat{x}_{i_j} \wedge \dots \wedge x_{i_p}).$$

The **Orlik-Solomon algebra** over $\mathbb{K}$ of $(\mathscr{L}, \mathscr{P})$ is defined as the quotient A of E by the ideal generated by $\{\delta(x_{i_1} \wedge \dots \wedge x_{i_n}) \mid n > 3\}$, $\{\delta(x_{i_1} \wedge x_{i_2} \wedge x_{i_3}) \mid l_{i_1} \cap l_{i_2} = l_{i_1} \cap l_{i_3}\}$, and $\{(x_{i_1} \wedge x_{i_2} \wedge x_{i_3}) \mid l_{i_1} \cap l_{i_2} \neq l_{i_1} \cap l_{i_3}\}$.

If $\mathbb{K}$ is not specified, it will be assumed to be $\mathbb{Q}$.

There is a grading in A induced by the grading in E, and $A_1 = E_1$. In the following, we will fix the base $\{x_1, \dots, x_n\}$ to take coordinates. We will say that two vectors $v_1, v_2 \in A_1$ are *orthogonal* if $v_1 \wedge v_2 = 0 \in A$. Given a subspace $C \subset A_1$, we will denote by $C^\perp$ the set of all vectors $v \in A_1$ such that v and w are orthogonal for every $w \in C$; this is a linear subspace that will be called the orthogonal space of C.

Lemma 2. *Let $\alpha : H \to \mathbb{Z}^{k-1}$ be a homomorphism, and let M be the matrix whose columns are $\alpha(e_1), \dots, \alpha(e_n)$. Then α is admissible if and only if the rows of the matrix M (as elements of A_1) are orthogonal in A.*

Proof. We will consider the non-broken-circuit basis of A_2 for the given ordering (see [16]). It can constructed as follows: take the generators $x_i \wedge x_j$ such that l_i is the first line of $l_i \cap l_j$, and the rest of the generators can be expressed in terms of these as follows: let l_i be the first line of the point $l_j \cap l_k$, then if $j < k$, we can use $\delta(x_i \wedge x_j \wedge x_k)$ to see that $x_j \wedge x_k = (x_i \wedge x_k) - (x_i \wedge x_j)$. It is not hard to see that the rest of the relations are a consequence of the previous ones. Now let $a := (a_1, \dots, a_n)$ and $b := (b_1, \dots, b_n)$ be two such rows, $l_j \in p \in \mathscr{P}$, and let l_i be the first line of p. The coefficient of $a \wedge b$ in $x_i \wedge x_j$ is

$$\begin{vmatrix} a_j & \sum_{l_k \in p} a_k \\ b_j & \sum_{l_k \in p} b_k \end{vmatrix}. \tag{6}$$

It is immediate that all such coefficients to be zero is the necessary and sufficient condition for both the admissibility of α and the orthogonality of its rows in A. □

Definition 6. For every element $\omega \in E_1$, we consider the complex

$$\mathbb{K} \xrightarrow{d_\omega} A_1 \xrightarrow{d_\omega} A_2 \longrightarrow 0$$

where d_ω represents the left multiplication by ω. We will denote its cohomology as $H^\bullet(A, \omega)$. The **resonance variety** of A is the set $R_1 := \{\omega \in A_1 \mid H^1(A, \omega) \neq 0\}$.

Remark 7. An element $a \in A$ is in R_1 if and only if there exists another $b \in A \setminus \mathbb{K}a$, such that $a \wedge b = 0$. In that case, the matrix whose rows are a and b defines an admissible map; and vice-versa: the rows of an admissible map are elements of R_1. More precisely, maximal admissible classes correspond exactly to irreducible components of the resonance variety, as will be shown later.

Notation 2. The row vectors of the matrix M in Lemma 2 will be denoted by $s_1, \ldots, s_{k-1}$, and the subspace of E_1 they span will be denoted by C_α; note that this subspace depends only on the admissible class α.

Lemma 3. *For any admissible class α, the space C_α satisfies the following properties:*

i) $C_\alpha \subseteq C_\alpha^\perp$.
ii) if α is maximal, $C_\alpha \subseteq C_\beta \implies \alpha = \beta$ *(as a class).*
iii) if α is maximal, $C_\alpha = C_\alpha^\perp$.
iv) $\dim(C_\alpha) = k - 1$ *if α is maximal k-admissible.*

Proof. Properties i) and ii) are direct consequences of Lemma 2 and the definition of maximal admissible class respectively. To prove iii), suppose that $v \in C_\alpha^\perp$ but $v \notin C_\alpha$. Then v could be appended as a row to the matrix M in Lemma 2; and the columns of the resulting matrix would correspond to an admissible map, which contradicts the maximality of α. The rows of the matrix M give a basis of C_α, which proves iv). □

We can find a basis $\{r_F \mid F \in \Pi\}$ of the kernel of Q formed by the vectors that generate the kernels of the indecomposable submatrices M_F. As in Remark 6, we can choose all these vectors to be positive, and to have the property that the sum of their entries is the same for all of them.

Proposition 2. *The subspace* $C_\alpha = \langle s_1, \ldots, s_{k-1}\rangle_{E_1}$ *is exactly* $\ker(Q) \cap \ker(U)$.

Proof. Equation (3) implies $MQ = 0$. Since Q is symmetric, $QM^t = 0$ and therefore $C_\alpha \subseteq \ker(Q)$. On the other hand, since $e_1 + \cdots + e_n = 0$, $\alpha(e_1) + \cdots + \alpha(e_n) = 0$, and therefore the sum of the coefficients of each s_i equals zero. Note that $\ker(U) = \{(v_1, \ldots, v_n) \in E_1 \mid \sum_{i=1}^n v_i = 0\}$ and hence $\langle s_1, \ldots, s_{k-1}\rangle \subseteq \ker(Q) \cap \ker(U)$.

For the other inclusion, fix a certain $\bar{F} \in \Pi$, and define $\tilde{r}_F := r_F - r_{\bar{F}}$ for all $F \in \Pi \setminus \{\bar{F}\}$. Since the r_F are not in $\ker(U)$, $\ker(Q) \subsetneq \ker(U)$; that, together with the fact that $codim(\ker(U)) = 1$, implies that $\dim(\ker(Q) \cap \ker(U)) = \dim(\ker(Q)) - 1$. Hence, $\{\tilde{r}_F\}_{F \neq \bar{F}}$ forms a basis of $\ker(Q) \cap \ker(U)$. We will prove that they are pairwise orthogonal (as elements of A). Take a point $p \in \mathscr{P}$. If $p \notin \chi_\alpha$, then all the lines in p are in the same component G of Π. The coefficients of $\tilde{r}_F \wedge \tilde{r}_{F'}$ on $x_i \wedge x_j$ are zero for $l_i, l_j \in p$: if $G = \bar{F}$, the coefficient (6) is the determinant of a matrix with two equal rows; if $G = F$ or $G = F'$, one of the rows is zero; and otherwise, both rows are zero.

If $p \in \chi_\alpha$, $\sum_{l \in p} \tilde{r}_{F,l}$ is the dot product of the row of J corresponding to p by $\tilde{r}_F$. Since $\tilde{r}_F$ is in $\ker(Q) \cap \ker(U)$, it is also in $\ker(J^t J)$. But if $J^t J \tilde{r}_F = 0$, then $(J\tilde{r}_F)^t (J\tilde{r}_F) = \tilde{r}_F^t J^t J \tilde{r}_F = \tilde{r}_F^t \bar{0} = 0$. Since wall matrices and vectors have real entries $(J\tilde{r}_F)^t (J\tilde{r}_F) = 0$ implies $J\tilde{r}_F = 0$. Hence, the coefficients of $\tilde{r}_F \wedge \tilde{r}_{F'}$ in the generators of A_2 corresponding to p are again zero.

Therefore, all the $\tilde{r}_F$'s are pairwise orthogonal. In particular, they are orthogonal to all s_i's, and since α is maximal, the space they span must be the same.

Theorem 2. *If α is a k-admissible map, then its corresponding admissible subcombinatorics is a maximal combinatorial pencil of no less than k fibers. Furthermore, if α is maximal, then the number of fibers of the pencil is exactly k.*

Proof. Since there is a basis of $\ker(Q)$ formed by positive vectors, $\ker(Q) \subsetneq \ker(U)$, which means that $\dim(\ker(Q) \cap \ker(U)) = \dim(\ker(Q)) - 1$. This implies that $k = \#\Pi$.

Equation (2) can be expressed as

$$\sum_{F \in \Pi} W_F v_F = 0, \tag{7}$$

where $W_F = \sum_{l \in F} w_l$. We then have a family of k vectors, $\{v_F\}_{F \in \Pi}$ that span $\mathbb{Z}^{k-1}$; that is, all the possible linear combinations satisfied by $\{v_F\}_{F \in \Pi}$ are proportional. By dividing each v_F by a positive integer z_F, we may assume that $\sum_{F \in \Pi} v_F = 0$. The equations $\{\sum_{l \in p} \alpha(l) = 0 \mid p \in \chi_\alpha\}$ can be rewritten as $\{\sum_{F \in \Pi} \sum_{l \in F \cap p} w_l z_F v_F = 0 \mid p \in \chi_\alpha\}$, which means that at each point $p \in \chi_\alpha$, $\sum_{l \in F \cap p} w_l z_F$ must be constant for all $F \in \Pi$. If we denote $\bar{w}_l := z_F w_l$ for $l \in F \in \Pi$, $(\mathscr{L}, \mathscr{P})$, is a combinatorial pencil with the partition Π and the weights $(\bar{w}_l)_{l \in \mathscr{L}}$. □

As a direct consequence of the previous Theorem and Remark 7, we obtain the following result about the resonance variety:

Theorem 3. *Given a combinatorics $(\mathscr{L}, \mathscr{P})$, there is a bijection between the k-dimensional components of its resonance variety and the maximal combinatorial pencils of $k+1$ fibers contained in it.*

Note that $R_1 = \bigcup_{\alpha \in \mathrm{Adm}(\mathscr{L},\mathscr{P})} C_\alpha$. Note also that the number of possible C_α is finite, since each one is determined by the subcombinatorics and the base points. If two different C_α and C_β (in the sense that α and β represent two distinct maximal admissible classes) intersected outside the origin, there would exist a vector $0 \neq v \in E_1$ orthogonal to both C_α an C_β; and hence, it would also be orthogonal to $C_\alpha + C_\beta$. This means in particular that $C_\alpha + C_\beta \subseteq R_1$, but since R_1 is a finite union of subspaces, $C_\alpha + C_\beta$ must be contained in a certain C_γ. By Lemma 3, this implies that $\alpha = \beta = \gamma$, which contradicts the hypothesis that α and β represent two distinct maximal admissible classes. Hence we have immediately the following:

Corollary 1. *The resonance variety R_1 is the union of a finite number of linear subspaces that meet only at the origin. Each one of those subspaces corresponds exactly with a maximal admissible class. Moreover, the orthogonal of a vector in R_1 is exactly the subspace to which it belongs.*

In particular, since the dimension of each component corresponds to the number of fibers of a combinatorial pencil, the dimension of R_1 cannot be greater than the number of lines. Moreover, there are $\#\mathcal{P}(\mathscr{L})$ subcombinatorics, and for each one there are no more than $\#\mathcal{P}(\mathscr{P})$ possible elections of base points. This gives a boundary on the number of components of R_1, answering the question raised in [13, 5.5].

The definition of resonance variety in 6 can be generalized as follows:

Definition 7. For $i \in \mathbb{N}$, the i-th resonance variety is the set $R_i := \{\omega \in A_1 \mid \dim(H^1(A, \omega)) \geq i\}$.

It is straightforward that $R_{i+1} \subseteq R_i$, and that R_i is a union of subspaces of dimension $\geq i + 1$.

Lemma 4. *For every $i \in \mathbb{N}$, the following equality holds:*

$$R_i = \bigcup_{\substack{\alpha \in \mathrm{Adm}_j(\mathscr{L}, \mathscr{P}) \\ j \geq i+2}} C_\alpha \tag{8}$$

Proof. Consider $v \in R_i$, $v \neq 0$. If the dimension of $H^1(A, v)$ is greater or equal to i, $v^\perp$ must have dimension at least $i + 1$. By Corollary 1, $v^\perp$ is exactly C_α for a certain maximal admissible class α. By Lemma 3, α must be at least $(i + 2)$-admissible. Hence, $R_i \subseteq \bigcup_{\substack{\alpha \in \mathrm{Adm}_j(\mathscr{L}, \mathscr{P}) \\ j \geq i+2}} C_\alpha$.

To prove the other inclusion, consider $v \in C_\alpha \setminus \{0\}$ for a certain maximal k-admissible class α with $k \geq i + 2$. Consider the matrix M associated with α; its rows span a $(k - 1)$-dimensional space, whose quotient by $\langle v \rangle$ has dimension $k - 2 \geq i$ and is contained in $H^1(A, v)$. Therefore $v \in R_i$. □

As a consequence of Lemmas 3, 4 and Corollary 1, the following result follows:

Theorem 4. *For every $i \in \mathbb{N}$, $(R_i \setminus R_{i+1}) \cup \{0\} = \bigcup_{\alpha \in \mathrm{Adm}_{i+2}(\mathscr{L}, \mathscr{P})} C_\alpha$, which is a linear variety of pure dimension $i + 1$.*

3. Permutations of the Admissible Classes

Let $\mathrm{Aut}^1(H) := \{\phi \in \mathrm{Aut}(H) \mid \phi \wedge \phi(R) = R\}$. For some combinatorics, this group is as small as it gets, that is, $\{\pm Id\} \times \mathrm{Aut}(\mathscr{L}, \mathscr{P})$ where $\mathrm{Aut}(\mathscr{L}, \mathscr{P})$ is the group of automorphisms of the combinatorics. Such combinatorics are called **homologically rigid**.

In the following, we will see some results that can be helpful to establish the homological rigidity of a given combinatorics.

From Proposition 1 we obtain the following result.

Corollary 2. *Any* $\phi \in \mathrm{Aut}^1(H)$ *induces a permutation* σ_ϕ *of the set of* k*-admissible classes by composition. In fact, there is a group antihomomorphism between* $\mathrm{Aut}^1(H)$ *and the group of permutations of* $\mathrm{Adm}_k(\mathscr{L}, \mathscr{P})$.

These permutations must preserve some structure in the admissible classes. Consider the function $\Upsilon : \mathcal{P}(\mathrm{Adm}(\mathscr{L}, \mathscr{P})) \to \mathbb{Z}$ given by

$$\Upsilon(S) := \mathrm{codim}\left(\bigcap_{\alpha \in S} \ker(\alpha)\right).$$

Note that if two admissible maps belong to the same admissible class, their kernel must be equal, and hence Υ is well defined. Also note that, if α is a k-admissible map, then $\Upsilon(\{\alpha\}) = k - 1$.

The function Υ can be seen as the dual description of the polymatroid determined by the configuration of subspaces $(ker(\alpha)^{\perp})$. It was already used to study isomorphisms of Orlik-Solomon algebras in [7].

For every $\phi \in \mathrm{Aut}^1(H)$, σ_ϕ induces also a permutation $\bar{\sigma}_\phi$ in $\mathcal{P}(\mathrm{Adm}(\mathscr{L}, \mathscr{P}))$. It is straightforward to prove the following.

Lemma 5. *For every* $\phi \in \mathrm{Aut}^1(H)$*, and every* $S \in \mathcal{P}(\mathrm{Adm}(\mathscr{L}, \mathscr{P}))$*,* $\Upsilon(\bar{\sigma}_\phi(S)) = \Upsilon(S)$.

The previous lemma allows us to calculate the set of the possible σ_ϕ (which is a subgroup of the permutations of $\mathrm{Adm}(\mathscr{L}, \mathscr{P})$, in particular it is the image of the morphism mentioned in Corollary 2) as follows: consider the natural action of the group of permutations of $\mathrm{Adm}(\mathscr{L}, \mathscr{P})$ in $\mathcal{P}(\mathrm{Adm}(\mathscr{L}, \mathscr{P}))$. Now, for every couple of positive integers (i, j), consider the subset $P_{i,j} := \{S \in \mathcal{P}(\mathrm{Adm}(\mathscr{L}, \mathscr{P})) \mid \#S = i, \Upsilon(S) = j\}$. Any σ_ϕ must be in the stabilizer of $P_{i,j}$ for each $(i, j) \in \mathbb{Z}^2$. Therefore, calculating the intersection of all such stabilizers gives us a group that contains $\{\sigma_\phi \mid \phi \in \mathrm{Aut}^1(H)\}$ as a subgroup.

But in most cases it is enough to use a particular version of the previous method, by considering only the concept of triangle, which we define below.

Definition 8. Let α_1, α_2 and α_3 be admissible maps. We will say that they form **a triangle** (of admissible classes) if

$$\Upsilon(\{\alpha_1, \alpha_2, \alpha_3\}) = \sum_{i=1}^{3} \Upsilon(\{\alpha_i\}) - 1 \tag{9}$$

If the admissible classes are clear from the context, we can also talk about triangles of admissible subcombinatorics.

Example 6. Let p_1, p_2 and p_3 be three points of multiplicities m_1, m_2 and m_3 greater than two. They are admissible subcombinatorics as in Example 1. These admissible classes form a triangle if and only if $p_i \cap p_j \neq \varnothing$ for $i, j = 1, 2, 3$ but $p_1 \cap p_2 \cap p_3 = \varnothing$. This fact can be easily checked case by case (see [2] for details).

A direct consequence of Lemma 5 is the following.

Lemma 6. *If α_1, α_2 and α_3 are k_1, k_2 and k_3-admissible maps respectively that form a triangle, their images under σ_ϕ are k_1, k_2 and k_3 admissible maps that form a triangle too. Thus σ_ϕ maps triangles to triangles.*

This fact can be used to study the possible permutations induced by ϕ and, in certain cases, that is enough to calculate all the possible automorphisms of H that fix R.

Given $\phi \in \mathrm{Aut}^1(H)$, consider a matrix M that represents ϕ with respect to the generating system $(e_1, \dots, e_n)$. Not that, since the relation $e_1 + \cdots + e_n$ holds, this matrix is not unique. In particular, we can add a multiple of the vector $(1, \dots, 1)$ to each column, and the resulting matrix would represent the same automorphism. By doing so, we can assume that the first row of M has all its entries equal to 0.

Now consider a multiple point $p = \{l_{i_1}, \dots, l_{i_m}\}$, and the corresponding admissible class α_p. If we look at the submatrix M_p obtained by taking the rows $i_1, \dots, i_m$ of M, and see it as a linear map $H \to \mathbb{Z}^m/(1, \dots, 1)$, it is easy to check that this map represents $\sigma_\phi^{-1}(\alpha_p)$. In particular, if a line l_j is not in the subcombinatorics corresponding to $\sigma_\phi^{-1}(\alpha_p)$, the j'th column of M_p must be proportional to $(1, \dots, 1)$, that is, all its entries must be equal.

Using these idea, given a permutation of the admissible classes σ we can see which elements of $\mathrm{Aut}^1(H)$ can induce it. If the combinatorics is rich enough (in the sense of having "enough multiple points"), there will be very few possibilities, as show the following results:

We will now define the concept of strong connetedness, introduced in [7].

Definition 9. A combinatorics $(\mathscr{L}, \mathscr{P})$ is said to be **strongly connected** if for any three lines $l_a, l_b, l_c \in \mathscr{L}$, there exists a chain $l_1, p_1, l_2, p_2, \dots, l_n, p_n, l_{n+1}$ such that:

- $l_1 = l_a$
- $l_{n+1} = l_b$
- $l_i, l_{i+1} \in p_i \; \forall i \in \{1, \dots, n\}$
- $\# p_i \geq 3 \; \forall i \in \{1, \dots, n\}$
- $l_c \notin p_i \; \forall i \in \{1, \dots, n\}$

Theorem 5. *Let $(\mathscr{L}, \mathscr{P})$ be a strongly connected combinatorics, $\sigma \in \mathrm{Aut}(\mathscr{L}, \mathscr{P})$ and $\bar{\sigma}$ the induced permutation of the admissible classes. Then there are only two elements of* $\mathrm{Aut}^1(H)$ *that induce $\bar{\sigma}$, which are given by $\pm\phi(e_i) = \pm e_{\sigma(i)}$.*

Proof. It is immediate that, for any combinatorics, $\pm\phi$ induce $\bar{\sigma}$.

Let's now see the uniqueness. Consider $\phi \in \mathrm{Aut}^1(H)$ inducing the permutation $\bar{\sigma}$, and consider M an $n \times n$ matrix representing ϕ (it does not matter if the first row is zero or not). Take the i'th column of M. In this column, consider two entries $a_{j,i}$ and $a_{k,i}$ such that l_j and l_k intersect in a multiple point p not containing $l_{\sigma(i)}$. Since p maps by $\bar{\sigma}$ to a point not containing l_i, $a_{j,i}$ and $a_{k,i}$ must be equal. By the strong connectedness of $(\mathscr{L}, \mathscr{P})$, we can extend this relation to all the entries of the i'th column except $a_{\sigma(i),i}$. Now we can add to this column a multiple of $(1, \dots, 1)$ in order to obtain all these entries equal to zero.

Repeating the previous process for all the columns, we can assume that each column has only one entry different from zero. Consider a multiple point p, and use the condition given by the fact that the submatrix formed by the rows corresponding to p represents the admissible class $\bar{\sigma}^{-1}(p)$. In particular, the columns corresponding to $\bar{\sigma}^{-1}(p)$ (which are precisely the ones with nonzero entries) must add up to a multiple of $(1, \dots, 1)$. That is, the nonzero entries of the rows corresponding to p are all equal. Again, the strong connectedness allow us to extend this relation to all the rows.

Finally, this matrix must represent an automorphism of $\mathbb{Z}^n/(1, \dots, 1)$ as abelian group, so the nonzero entries must be ± 1. □

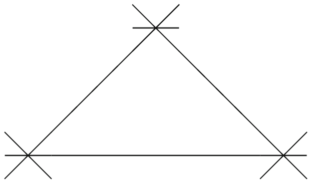
So, if given a strongly connected combinatorics, we can conclude that the only permutations of the admissible classes that preserve Υ are the ones induced by the automorphisms of the combinatorics, then we obtain directly the homological rigidity as a consequence. This happens almost trivially in the following class of combinatorics:

Definition 10. A combinatorics $(\mathscr{L}, \mathscr{P})$ is said to have **enough triangles** if every line passes through at least two multiple points, and every two aligned multiple points are in triangle.

Example 7. Ceva combinatorics is strongly connected. To check this, just notice that when one line is removed, the result is two triple points with a common line; any two lines can be connected through this point.

Moreover, this combinatorics has enough triangles, since any three triple points are in triangle.

Example 8. If we modify the Ceva combinatorics by substituting one triple point with three double points (see figure below), we obtain a combinatorics which does not have enough triangles (every two aligned triple points are in a triangle, but there are lines that go through only one triple point), nor is it strongly connected (if we erase one of the lines that join two triple points, there are two lines that don't go through any triple point, and hence cannot be connected to any other line by multiple points).



Theorem 6. *Let $(\mathscr{L}, \mathscr{P})$ be a strongly connected combinatorics with enough triangles, and let $\phi \in \mathrm{Aut}^1(H)$ be an automorphism such that the induced permutation of the admissible classes preserves the point-type classes.*

Then ϕ is induced by an automorphism of the combinatorics, up to change of sign.

Proof. We can identify each line with the set of multiple points that it crosses. Since each line passes through at least two multiple points, there is no ambiguity in this identification.

Given three multiple points $\{p_1, p_2, p_3\}$ such that the three do not form a triangle, but each two of them are in a triangle (that is, there exist points $\{l_4, l_5, l_6\}$ such that $\{l_1, l_2, l_4\}$, $\{l_1, l_3, l_5\}$ and $\{l_2, l_3, l_6\}$ are triangles), they must be aligned. Hence, every permutation of the multiple points that preserves triangles must preserve lines seen as sets of multiple points. This permutation of the lines preserves points (trivially preserves the multiple points, and double points are precisely the pairs of lines that are not in any multiple point), so it must be an automorphism of the combinatorics.

Using the previous theorem, it is straightforward to check that the element of $\mathrm{Aut}^1(H)$ induced by this permutation must be either ϕ or $-\phi$.

And as a corollary, we have the following criteria for the homological rigidity of certain combinatorics.

Corollary 3. *Every strongly connected combinatorics with enough triangles and such that all permutations of the admissible classes that preserve Υ preserve the point-type classes (for instance, when there are no classes of other type) is homologically rigid.*

4. An Example

The existence of two real line arrangements with the same combinatorial type, but different topology of the embedding was shown in [3]. These arrangements have 11 lines, and are conjugated in $\mathbb{Q}(\sqrt{5})$. They can be constructed after two other arrangements (also conjugated in $\mathbb{Q}(\sqrt{5})$), whose non-generic braid monodromies were shown to be non-equivalent. Here we will show that the combinatorics of these arrangements of ten lines is homologically rigid by studying the possible permutations of the admissible classes induced by $\mathrm{Aut}^1(H)$. This combinatorics is the result of adding one line to the Falk-Sturmfels combinatorics described in [5]. This

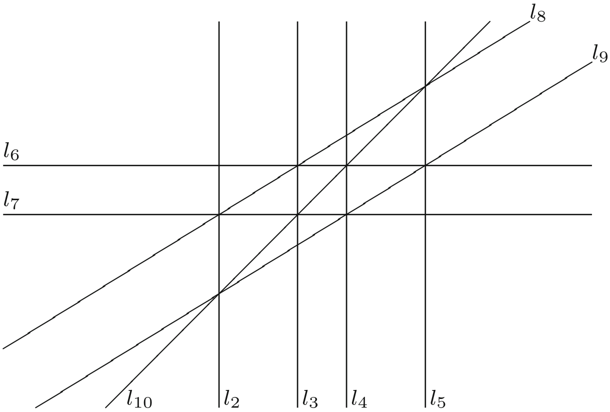


Fig. 2. A realization

example is intended to show how to use Lemmas 5 and 6 to study the homological rigidity of a given combinatorics.

The combinatorics that we will study has ten lines $\{l_1, \ldots, l_{10}\}$, ten triple points: $\{l_1, l_6, l_7\}$, $\{l_1, l_8, l_9\}$, $\{l_2, l_9, l_{10}\}$, $\{l_2, l_7, l_8\}$, $\{l_3, l_6, l_8\}$, $\{l_3, l_7, l_{10}\}$, $\{l_4, l_6, l_{10}\}$, $\{l_4, l_7, l_9\}$, $\{l_5, l_8, l_{10}\}$, and $\{l_5, l_6, l_9\}$, and a quintuple point $\{l_1, l_2, l_3, l_4, l_5\}$. The remaining are double points. A real realization, where l_1 is the line at infinity, can be seen in Figure 2.

We could calculate the admissible classes by solving the quadratic equation system that the coefficients of any admissible map should satisfy, but it is much faster to use Theorem 2, which allows us to calculate the possible combinatorial pencils just by solving systems of linear equations. The result is that the only non-point-type admissible classes are the following ten Ceva type classes:

– $\{l_1, l_4, l_5, l_9, l_6, l_7\}$
– $\{l_2, l_4, l_5, l_6, l_9, l_{10}\}$
– $\{l_1, l_2, l_4, l_7, l_9, l_8\}$
– $\{l_1, l_2, l_3, l_8, l_6, l_7\}$
– $\{l_2, l_3, l_5, l_{10}, l_8, l_7\}$
– $\{l_1, l_3, l_4, l_{10}, l_6, l_7\}$
– $\{l_3, l_4, l_5, l_{10}, l_8, l_6\}$
– $\{l_1, l_2, l_5, l_{10}, l_8, l_9\}$
– $\{l_1, l_3, l_5, l_6, l_9, l_8\}$
– $\{l_2, l_3, l_4, l_7, l_9, l_{10}\}$

There is only one 5-admissible subcombinatorics, which must be preserved by the permutation induced by any $\phi \in \mathrm{Aut}^1(H)$. For each of the 20 3-admissible classes, we can count to how many triangles of maximal 3-admissible maps it belongs. The result is that each point-type combinatorics belongs to 15 such triangles, while each Ceva type belongs to 9. Hence σ_ϕ must induce a permutation of the triple points that preserves triangles. These computations were done in a few seconds in a computer using GAP [10].

Now consider four points p_1, p_2, p_3, p_4 of multiplicity greater than two such that p_i, p_j, p_4 form a triangle for all $i, j \in \{1, 2, 3\}$, $i \neq j$. Then p_1, p_2 and p_3 are aligned if and only if they do not form a triangle. In our combinatorics, for any three aligned points of multiplicity greater than two, there is a fourth one that forms a triangle with any two of them. Since σ_ϕ must preserve triangles, it must also preserve lines considered as sets of multiple points. Up to composition with the automorphism of H induced by σ_ϕ^{-1} (seen as a permutation of the elements of the canonical generating system of H), we may assume that σ_ϕ is the identity.

Now we can consider the basis of H given by $\bar{e}_1, \ldots, \bar{e}_9$ (being $\bar{e}_i$ the class of e_i modulo $(1, \ldots, 1)$) and the matrix $B = (b_{i,j})$ related to ϕ in this basis. Using the fact that σ_ϕ is the identity, we can deduce that for any point p of multiplicity greater than two, $b_{i,k} = b_{j,k}$ for all $l_i, l_j \in p$ and $l_k \notin p$. If $l_{10} \in p$ then these entries are actually 0. These conditions to all the multiple points, forces B to be diagonal. Since B must be an integer matrix with determinant equal to ± 1, all the entries in the diagonal must be ± 1. Now given a multiple point p such that $l_{10} \notin p$, we have that the submatrix of B obtained by selecting the rows and columns corresponding to the lines in p must have columns that add up to a multiple of $(1, \ldots, 1)$, so it means that $b_{i,i} = b_{j,j}$ for all $l_i, l_j \in p$. If we use these conditions in all the multiple points, we obtain that B must be $\pm Id$.

This method also works with the combinatorics of McLane and Rybnikov [17], and the one of eleven lines studied in [3]. This same kind of arguments were used by Falk in [7] to show that strongly connected combinatorics in which all combinatorial pencils are point-type are homologically rigid. In fact this method could be seen as a generalization of his.

5. Apppendix: Duality Between $(H \wedge H)/R$ and the Orlik-Solomon Algebra

The method in the previous section can allow us to calculate $\mathrm{Aut}^1(H)$. Here we will see that this group coincides with the group of automorphisms of the Orlik-Solomon algebra.

Let $(a_{i,j})$ be the matrix that represents an automorphism of H in the basis $\{e_1, \ldots, e_n\}$. Let's denote this automorphism by ϕ, and the automorphism induced in $H \wedge H$ by $\hat{\phi}$. As a sublattice of $H \wedge H$, R is generated by $\{e_i \wedge \sum_{l_j \in p} e_j \mid l_i \in p, p \in \mathscr{P}\}$. Since $\sum_{l_i \in p}(e_i \wedge \sum_{l_j \in p} e_j) = 0$ when $\#p > 2$, we can eliminate the first one and then use these relations to give a basis of the quotient. In particular, for each point $p = \{l_{i_1}, \ldots, l_{i_{\#p}}\}$, we can express the generators of the form $e_{i_1} \wedge e_{i_j}$ as $\sum_{k=j+1}^{\#p} e_{i_j} \wedge e_{i_k} - \sum_{k=1}^{j-1} e_{i_k} \wedge e_{i_j}$. Therefore a basis of the quotient is given by the generators of the form $e_i \wedge e_j$ where $i < j$ and both i and j are not the first line in $l_i \cap l_j$.

The image of $e_{i_1} \wedge e_{i_2}$ under ϕ is

$$\sum_{1 \leq j_1 < j_2 \leq n} \begin{vmatrix} a_{j_1,i_1} & a_{j_1,i_2} \\ a_{j_2,i_1} & a_{j_2,i_2} \end{vmatrix} (e_{j_1} \wedge e_{j_2}),$$

so the image of a generator of R of the form $e_i \wedge \sum_{l_k \in p} e_k$ is

$$\sum_{1 \le j_1 < j_2 \le n} \begin{vmatrix} a_{j_1,i} & \sum_{l_k \in p} a_{j_1,k} \\ a_{j_2,i} & \sum_{l_k \in p} a_{j_2,k} \end{vmatrix} (e_{j_1} \wedge e_{j_2}).$$

If we project this to the quotient, its coefficient in a generator of the form $e_c \wedge e_d$ (if the first line that goes through the intersection point of l_c and l_d is l_b) is

$$\begin{vmatrix} a_{c,i} & \sum_{l_k \in p} a_{c,k} \\ a_{d,i} & \sum_{l_k \in p} a_{d,k} \end{vmatrix} - \begin{vmatrix} a_{b,i} & \sum_{l_k \in p} a_{b,k} \\ a_{d,i} & \sum_{l_k \in p} a_{d,k} \end{vmatrix} + \begin{vmatrix} a_{b,i} & \sum_{l_k \in p} a_{b,k} \\ a_{c,i} & \sum_{l_k \in p} a_{c,k} \end{vmatrix} = \begin{vmatrix} a_{b,i} & \sum_{l_k \in p} a_{b,k} & 1 \\ a_{c,i} & \sum_{l_k \in p} a_{c,k} & 1 \\ a_{d,i} & \sum_{l_k \in p} a_{d,k} & 1 \end{vmatrix}. \quad (10)$$

The extra condition that $\phi \in \mathrm{Aut}^1(H)$ is equivalent to asking (10) to vanish for every point p, every line $l_i \in p$ and every three concurrent lines l_b, l_c, l_d such that l_b is the first line in their intersection point.

Now let's look at the second level of the Orlik-Solomon algebra A^2. The relations we have are of the form $(x_b \wedge x_c) - (x_b \wedge x_d) + (x_c \wedge x_d)$ for every three concurrent lines l_b, l_c, l_d. Now let's suppose that there are three concurrent lines such that the first line that goes through their intersection point is l_a; we can express the relation $(x_b \wedge x_c) - (x_b \wedge x_d) + (x_c \wedge x_d)$ as $((x_a \wedge x_b) - (x_a \wedge x_c) + (x_b \wedge x_c)) - ((x_a \wedge x_b) - (x_a \wedge x_d) + (x_b \wedge x_d)) + ((x_a \wedge x_c) - (x_a \wedge x_d) + (x_c \wedge x_d))$. In particular we only need the relations where the first line in the intersection point appears. These relations allow us to express any $x_b \wedge x_c$ as $(x_a \wedge x_c) - (x_a \wedge x_b)$ (where again $b < c$ and l_a is the first line that goes through $l_b \cap l_c$. Hence a basis of A^2 is given by the $x_i \wedge x_j$ such that l_i is the first line in $l_i \cap l_j$. Let l_b, l_c, l_d be three concurrent lines; the image of $(x_b \wedge x_c) - (x_b \wedge x_d) + (x_c \wedge x_d)$ by $\hat{\phi}$ is

$$\sum_{1 \le i < j \le n} \left(\begin{vmatrix} a_{i,b} & a_{i,c} \\ a_{j,b} & a_{j,c} \end{vmatrix} - \begin{vmatrix} a_{i,b} & a_{i,d} \\ a_{j,b} & a_{j,d} \end{vmatrix} + \begin{vmatrix} a_{i,c} & a_{i,d} \\ a_{j,c} & a_{j,d} \end{vmatrix} \right) (x_i \wedge x_j)$$

$$= \sum_{1 \le i < j \le n} \begin{vmatrix} a_{i,b} & a_{i,c} & a_{i,d} \\ a_{j,b} & a_{j,c} & a_{j,d} \\ 1 & 1 & 1 \end{vmatrix} (x_i \wedge x_j). \quad (11)$$

Now let's take a point p whose first line is l_i and $l_j, l_k \in p$; by the previous relations, $x_j \wedge x_k$ maps by the projection to $x_i \wedge x_k - x_i \wedge x_j$. This means that if we want to calculate the coefficient of the projection of (11) in $x_i \wedge x_j$, we have to add or substract adequately its coefficients in all $x_j \wedge x_k$ such that $l_k \in p$. The result is that the coefficient of (11) in $x_i \wedge x_j$ is

$$\begin{vmatrix} \sum_{k \in p} a_{k,a} & \sum_{k \in p} a_{k,b} & \sum_{k \in p} a_{k,c} \\ a_{j,a} & a_{j,b} & a_{j,c} \\ 1 & 1 & 1 \end{vmatrix}. \quad (12)$$

In order for the matrix $(a_{i,j})$ to induce an automorphism of the Orlik-Solomon algebra, (12) must hold for all three concurrent lines l_b, l_c, l_d, each point p, and each line $l_j \in p$. Comparing (10) and (12), we have the following result.

Proposition 3. *The matrix* $(a_{i,j})$ *induces an automorphism of the Orlik-Solomon algebra if and only if its transpose* $(a_{j,i})$ *induces an automorphism of* $H \wedge H$ *that preserves* R.

Remark 8. The subspace R can be seen as the image of the dual of the product map $A^1 \wedge A^1 \to A^2$, and so, $H \wedge H/R$ is dual to its kernel, which is the degree-two part of the Orlik-Solomon ideal, and coincides with the degree-two part of the holonomy Lie algebra (see [12]).

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