Graphs and Combinatorics (2009) 25:583–600
Digital Object Identifier (DOI) 10.1007/s00373-009-0859-3

Graphs and Combinatorics

On Substructure Densities of Hypergraphs

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Abstract. A real number $\alpha \in [0, 1)$ is a jump for an integer $r \geq 2$ if there exists $c > 0$ such that for any $\epsilon > 0$ and any integer $m \geq r$, there exists an integer n_0 such that any r-uniform graph with $n > n_0$ vertices and density $\geq \alpha + \epsilon$ contains a subgraph with m vertices and density $\geq \alpha + c$. It follows from a fundamental theorem of Erdös and Stone that every $\alpha \in [0, 1)$ is a jump for $r = 2$. Erdös also showed that every number in $[0, r!/r^r)$ is a jump for $r \geq 3$ and asked whether every number in $[0, 1)$ is a jump for $r \geq 3$ as well. Frankl and Rödl gave a negative answer by showing a sequence of non-jumps for every $r \geq 3$. Recently, more non-jumps were found for some $r \geq 3$. But there are still a lot of unknowns on determining which numbers are jumps for $r \geq 3$. The set of all previous known non-jumps for $r = 3$ has only an accumulation point at 1. In this paper, we give a sequence of non-jumps having an accumulation point other than 1 for every $r \geq 3$. It generalizes the main result in the paper 'A note on the jumping constant conjecture of Erdös' by Frankl, Peng, Rödl and Talbot published in the Journal of Combinatorial Theory Ser. B. 97 (2007), 204–216.

Key words. Extremal problems in hypergraphs, Turán density, Erdös jumping constant conjecture.

1. Introduction

For a finite set V and a positive integer r we denote by $\binom{V}{r}$ the family of all r-subsets of V. An r-*uniform graph* G is a set $V(G)$ of vertices together with a set $E(G) \subseteq \binom{V(G)}{r}$ of edges. The *density* of G is defined by $d(G) = |E(G)| / \left|\binom{V(G)}{r}\right|$. An r-uniform graph H is called a *subgraph* of an r-uniform graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $H \subseteq G$ if H is a subgraph of G. A subgraph H of G is called *induced* if $E(H) = E(G) \cap \binom{V(H)}{r}$. An argument in [6] by Katona, Nemetz, and Simonovits shows the following fact:

Fact 1.1. [6] *Let G be an r-uniform graph and $m \geq r$ be an integer. Then the average density of all induced subgraphs of G with m vertices is $d(G)$.*

Therefore, G always contains a subgraph with any given order ($\geq r$) and density $\geq d(G)$. A question is whether there exists a subgraph of any given order with density $\geq d(G) + c$, where $c > 0$ is a constant? To be precise, the concept of 'jump' is given below.

Definition 1.1. *A real number $\alpha \in [0, 1)$ is a* jump *for an integer $r \geq 2$ if there exists a constant $c > 0$ such that for any $\epsilon > 0$ and any integer $m \geq r$, there exists an integer $n_0(\epsilon, m)$ such that any r-uniform graph with $n > n_0(\epsilon, m)$ vertices and density $\geq \alpha + \epsilon$ contains a subgraph with m vertices and density $\geq \alpha + c$.*

The study of jump is closely related to the study of Turán density. Finding good estimation for Turán densities in hypergraphs ($r \geq 3$) is believed to be one of the most challenging problems in extremal set theory. For a family $\mathcal{F}$ of r-uniform graphs, the *Turán density* [17] of $\mathcal{F}$, denoted by $\gamma(\mathcal{F})$ is the limit of the maximum density of an r-uniform graph of order n not containing any member of $\mathcal{F}$ as $n \to \infty$, i.e.,

$$\gamma(\mathcal{F}) = \lim_{n\to\infty} \frac{\max\{|E| : G = (V, E) \text{ is an } \mathcal{F} - \text{free } r\text{−uniform graph of order } n\}}{\binom{n}{r}}.$$

Such a limit exists since the sequence in the right hand side decreases as n increases by Fact 1.1. The set of all possible Turán densities for $r \geq 2$ is denoted by Γ_r, i.e.,

$$\Gamma_r = \{\gamma(\mathcal{F}) : \mathcal{F} \text{ is a family of } r - \text{uniform graphs}\}.$$

It was shown in [4] that α is a jump for r if and only if there exists $c > 0$ such that $\Gamma_r \cap (\alpha, \alpha + c) = \emptyset$. Consequently, Γ_r is a well-ordered set if and only if every $\alpha \in [0, 1)$ is a jump for r. Erdös and Stone[2] showed that every $\alpha \in [0, 1)$ is a jump for $r = 2$. For $r \geq 3$, Erdös [1] proved that every $\alpha \in [0, r!/r^r)$ is a jump. Furthermore, Erdös proposed the *jumping constant conjecture*: Every $\alpha \in [0, 1)$ is a jump for every integer $r \geq 2$. In [4], Frankl and Rödl disproved the Conjecture by showing that

Theorem 1.2. [4] $1 - \frac{1}{l^{r-1}}$ *is not a jump for r if $r \geq 3$ and $l > 2r$.*

However, there are still a lot of unknowns on determining whether a number is a jump for $r \geq 3$. A well-known open question of Erdös is whether $r!/r^r$ is a jump for $r \geq 3$ and what is the smallest non-jump? Another question raised in [5] is whether there is an interval of non-jumps for some $r \geq 3$? Both questions seem to be very challenging. Regarding the first question, the following was shown in [5].

Theorem 1.3. [5] $\frac{5r!}{2r^r}$ *is a non-jump for $r \geq 3$.*

At this moment, this is the smallest known non-jump. Some efforts were made in finding more non-jumps for some $r \geq 3$: One more infinite sequence of non-jumps (converging to 1) for $r = 3$ was given in [5]. Several infinite sequences of non-jumps (converging to 1) for $r = 4$ were found in [8], [9], [10], [11] and [12]. Then every non-jump in the above papers was extended to many sequences of non-jumps (still converging to 1) in [14]. The approach in the above papers is still based on the approach developed by Frankl and Rödl in [4]. In [13], a way to generate non-jumps for every $p \geq r$ based on a non-jump for r was given. The following result was shown there.

Theorem 1.4. [13] *Let $p \geq r \geq 3$ be positive integers. If $\alpha \cdot \frac{r!}{r^r}$ is a non-jump for r, then $\alpha \cdot \frac{p!}{p^p}$ is a non-jump for p.*

From the definition of a 'jump', if a number α is a jump, then there exists a constant c such that every number in $[\alpha, \alpha + c)$ is a jump. Consequently, if there is a set A of non-jumps such that the closure of A is an interval $[a, b]$, then every number in the interval $[a, b)$ is a non-jump. We do not know whether or not such a dense set of non-jumps exists and it seems to be very challenging to answer this question.

We feel that there might exist a dense set of non-jumps for sufficiently large r. If this is true, more non-jumps should be found in addition to the current known non-jumps. The set of the known non-jumps for some $r \geq 3$ before Theorem 1.4 has only an accumulation point at 1. Combining Theorem 1.4 and results in [5], [8], [9], [10], [11], [12], and [14], we obtain several sequences of non-jumps for $r \geq 4$ with accumulation points different from 1 in [13]. But the set of all previous known non-jumps for $r = 3$ has only an accumulation point at 1. In this paper we first give a sequence of non-jumps for $r = 3$ with accumulation point other than 1. Since it is difficult to determine jumps or non-jumps for $r \geq 3$ in general and the set of jumps or non-jumps remains a lot of mystery to us, such a new sequence of non-jumps with accumulation point other than 1 might be interesting. Our main result is

Theorem 1.5. *Let l be any positive integer. Then $\frac{7}{12} - \frac{1}{4 \cdot 9^l}$ is not a jump for $r = 3$.*

The proof of Theorem 1.5 will be given in Section 3. Combining Theorem 1.5 and Theorem 1.4, we obtain the following

Corollary 1.6. *Let l be any positive integer. Then $\left(\frac{21}{8} - \frac{1}{8 \cdot 9^{l-1}}\right) r!/r^r$ is not a jump for $r \geq 3$.*

Taking $l = 1$ in Corollary 1.6, we obtain Theorem 1.3.

In the following section, we will introduce some preliminary results and sketch the general idea of our proof. The general method is still based on the approach developed by Frankl and Rödl in [4].

2. Preliminaries and Sketch of the General Approach

We first give the definition of the Lagrangian of an r-uniform graph, a helpful tool in our approach. More studies of Lagrangians can be found in [3], [4], [7] and [16].

Definition 2.1. *For an r-uniform graph G with vertex set $\{1, 2, \ldots, m\}$, edge set $E(G)$ and a vector $\vec{x} = (x_1, \ldots, x_m) \in R^m$, define*

$$\lambda(G, \vec{x}) = \sum_{\{i_1, \ldots, i_r\} \in E(G)} x_{i_1} x_{i_2} \ldots x_{i_r}.$$

x_i is called the *weight* of the vertex i.

Definition 2.2. *Let* $S = \{\vec{x} = (x_1, x_2, \ldots, x_m) : \sum_{i=1}^{m} x_i = 1, x_i \geq 0$ for $i = 1, 2, \ldots, m\}$. *The Lagrangian of* G*, denoted by* $\lambda(G)$*, is defined as*

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in S\}.$$

A vector $\vec{y} \in S$ is called an *optimal vector* for $\lambda(G)$ if $\lambda(G, \vec{y}) = \lambda(G)$.

The following fact is easily implied by the definition of the Lagrangian.

Fact 2.1. *If* $G_1 \subseteq G_2$*, then*

$$\lambda(G_1) \leq \lambda(G_2).$$

For an r-uniform graph G and $i \in V(G)$ we define G_i to be the $(r-1)$-uniform graph on $V - \{i\}$ with edge set $E(G_i)$ given by $e \in E(G_i)$ if and only if $e \cup \{i\} \in E(G)$.

We call two vertices i, j of an r-uniform graph G equivalent if for all $f \in \binom{V(G)-\{i,j\}}{r-1}$, $f \in E(G_i)$ if and only if $f \in E(G_j)$.

The following lemma (given in [4]) will be useful when calculating Lagrangians of certain graphs.

Lemma 2.2. (c.f. [4]) *Suppose* G *is an* r*-uniform graph on vertices* $\{1, 2, \ldots, m\}$.

1. *If vertices* $i_1, i_2, \ldots, i_t$ *are pairwise equivalent, then there exists an optimal vector* $\vec{y} = (y_1, y_2, \ldots, y_m)$ *of* $\lambda(G)$ *such that* $y_{i_1} = y_{i_2} = \cdots = y_{i_t}$.
2. *Let* $\vec{y} = (y_1, y_2, \ldots, y_m)$ *be an optimal vector of* $\lambda(G)$ *and* $y_i > 0$. *Let* $\hat{y}_i$ *be the restriction of* $\vec{y}$ *on* $\{1, 2, \ldots, m\} \setminus \{i\}$. *Then* $\lambda(G_i, \hat{y}_i) = r\lambda(G)$. *To simplify the notation, we write* $\lambda(G_i, \hat{y}_i)$ *as* $\lambda(G_i, \vec{y})$ *sometimes.*

We also note that for an r-uniform graph G with m vertices, if we take $\vec{u} = (u_1, \ldots, u_m)$, where each $u_i = 1/m$, then

$$\lambda(G) \geq \lambda(G, \vec{u}) = \frac{|E(G)|}{m^r} \geq \frac{d(G)}{r!} - \epsilon$$

for $m \geq m'(\epsilon)$, where $m'(\epsilon)$ is a sufficiently large integer.

On the other hand, we introduce the blow-up of an r-uniform graph G which will allow us to construct r-uniform graphs with large number of vertices and density close to $r!\lambda(G)$.

Definition 2.3. *Let* G *be an* r*-uniform graph with* $V(G) = \{1, 2, \ldots, m\}$ *and* $(n_1, \ldots, n_m)$ *be a positive integer vector. Define the* $(n_1, \ldots, n_m)$ *blow-up of* G*,* $(n_1, \ldots, n_m) \otimes G$ *as an* m*-partite* r*-uniform graph with vertex set* $V_1 \cup \ldots \cup V_m$, $|V_i| = n_i$, $1 \leq i \leq m$*, and edge set* $E((n_1, \ldots, n_m) \otimes G) = \{\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$, where $\{i_1, i_2, \ldots, i_r\} \in E(G)$ and $v_{i_k} \in V_{i_k}$ for $1 \leq k \leq r\}$.

Remark 2.3. [4] Let G be an r-uniform graph with m vertices and $\vec{y} = (y_1, \ldots, y_m)$ be an optimal vector for $\lambda(G)$. Then for any $\epsilon > 0$, there exists an integer $n_1(\epsilon)$, such that for any integer $n \geq n_1(\epsilon)$,

$$d((\lceil ny_1 \rceil, \lceil ny_2 \rceil, \ldots, \lceil ny_m \rceil) \otimes G) \geq r!\lambda(G) - \epsilon. \qquad (1)$$

Let us also state a fact which follows directly from the definition of the Lagrangian.

Fact 2.4. [4] *For every r-uniform graph G and every positive integer n, $\lambda((n, n, \ldots, n) \otimes G) = \lambda(G)$ holds.*

Lemma 2.5 proved in [4] gives a necessary and sufficient condition for a number α to be a jump.

Lemma 2.5. [4] *The following two properties are equivalent.*

1. *α is a jump for r.*
2. *There exists some finite family $\mathcal{F}$ of r-uniform graphs satisfying $\gamma(\mathcal{F}) \leq \alpha$ and $\lambda(F) > \dfrac{\alpha}{r!}$ for all $F \in \mathcal{F}$.*

We also need the following lemma from [4] in the proof of our main result.

Lemma 2.6. [4] *For any $\sigma \geq 0$ and any integer $k \geq r$, there exists $t_0(k, \sigma)$ such that for every $t \geq t_0(k, \sigma)$, there exists an r-uniform graph $A = A(k, \sigma, t)$ satisfying:*

1. $|V(A)| = t$,
2. $|E(A)| \geq \sigma t^{r-1}$,
3. *For all $V_0 \subset V(A)$, $r \leq |V_0| \leq k$, we have $|E(A) \cap \binom{V_0}{r}| \leq |V_0| - r + 1$.*

The general approach in proving Theorem 1.5 is sketched as follows: Let α be the number to be proved to be a non-jump for $r = 3$. Assuming that α is a jump for $r = 3$, we will derive a contradiction by the following steps.

Step1. Construct a 3-uniform graph with the Lagrangian slightly smaller than $\frac{\alpha}{6}$, then use Lemma 2.6 to add a 3-uniform graph with a large enough number of edges but sparse enough (guaranteed by properties 2 and 3 in Lemma 2.6) and obtain a 3-uniform graph with the Lagrangian at least $\frac{\alpha}{6} + \epsilon$ for some $\epsilon > 0$. Then we 'blow up' it to a 3-uniform graph, say H with a large enough number of vertices and density $\geq \alpha + \epsilon$ (see Remark 2.3). If α is a jump, then by Lemma 2.5, $\gamma(\mathcal{F}) \leq \alpha$ for some finite family $\mathcal{F}$ of 3-uniform graphs with Lagrangians $> \frac{\alpha}{6}$. So H must contain some member of $\mathcal{F}$ as a subgraph.

Step 2. We show that any subgraph of H with the number of vertices not greater than $\max\{|V(F)|, F \in \mathcal{F}\}$ has the Lagrangian $\leq \frac{\alpha}{6}$ and derive a contradiction.

We would like to point out that it is certainly nontrivial to construct an r-uniform graph satisfying the properties in both Steps 1 and 2. Generally, whenever we find such a construction, we can obtain a corresponding non-jump. This method was first developed by Frankl and Rödl in [4], then it was used in [5], [8], [9], [10], [11] and [12] to find more non-jumps by giving this type of construction. The critical and technical part in the proof of the main theorem in this paper is to show that the construction satisfies the property in Step 2 (Lemma 3.1 in Section 3.1).

3. Proof of Theorem 1.5

We first define a 3-uniform graph G on 3 disjoint sets V_1, V_2, V_3, each of cardinality

$$s = 3^{l-1}t,$$

where t is an integer. For each V_{i_1}, $1 \le i_1 \le 3$, we partition V_{i_1} into 3 equal parts, denoted by $V_{i_1 i_2}$, $1 \le i_2 \le 3$. Then partition each $V_{i_1 i_2}$ into 3 equal parts, denoted by $V_{i_1 i_2 i_3}$, $1 \le i_3 \le 3$. Continue the process until we have 3^l disjoint equal parts $V_{i_1 i_2 \dots i_l}$, $1 \le i_j \le 3$, $1 \le j \le l$.

For 2 sets U and V, let $\binom{U}{2} \times V$ be the set of 3 vertices choosing 2 from U and 1 from V.

The edge set of G is

$$(V_1 \times V_2 \times V_3) \bigcup \left(\binom{V_1}{2} \times V_2\right) \bigcup \left(\binom{V_2}{2} \times V_3\right) \bigcup \left(\binom{V_3}{2} \times V_1\right) \bigcup_{i_1=1}^{3} (V_{i_1 1} \times V_{i_1 2} \times V_{i_1 3})$$
$$\bigcup_{1 \le i_1, i_2 \le 3} (V_{i_1 i_2 1} \times V_{i_1 i_2 2} \times V_{i_1 i_2 3}) \bigcup \cdots \bigcup_{1 \le i_1, \dots, i_{l-1} \le 3} (V_{i_1 \dots i_{l-1} 1} \times V_{i_1 \dots i_{l-1} 2} \times V_{i_1 \dots i_{l-1} 3}).$$

Note that the density of G is close to

$$\alpha = \frac{7}{12} - \frac{1}{4 \cdot 9^l}$$

if s is large enough. In fact,

$$|E(G)| = s^3 + 3\binom{s}{2}s + s^3\left(\sum_{i=1}^{l-1} \frac{1}{9^i}\right)$$
$$= \left(\frac{21}{8} - \frac{1}{8 \cdot 9^{l-1}}\right)s^3 - c_0 s^2 + o(s^2), \tag{2}$$

where c_0 is positive (we omit giving the precise calculation here). Let $\vec{u} = (u_1, \dots, u_{3s})$, where $u_i = 1/(3s)$ for each i, $1 \le i \le 3s$, then

$$\lambda(G) \ge \lambda(G, \vec{u}) = \frac{|E(G)|}{(3s)^3} = \frac{7}{72} - \frac{1}{24 \cdot 9^l} - \frac{c_0}{27s} + o\left(\frac{1}{s}\right) = \frac{\alpha}{6} - \frac{c_0}{27s} + o\left(\frac{1}{s}\right)$$

which is close to $\frac{\alpha}{6}$ when s is large enough.

We will use Lemma 2.6 to add 3^l 3-uniform graphs to G so that the Lagrangian of the resulting 3-uniform graph is $> \frac{\alpha}{6} + \epsilon(s)$ for some $\epsilon(s) > 0$. The precise argument is given below.

Suppose that α is a jump. In view of Lemma 2.5, there exists a finite collection $\mathcal{F}$ of 3-uniform graphs satisfying the following:

i) $\lambda(F) > \dfrac{\alpha}{6}$ for all $F \in \mathcal{F}$, and
ii) $\gamma(\mathcal{F}) \le \alpha$.

Set $k_0 = \max_{F\in\mathcal{F}}|V(F)|$ and $\sigma_0 = 3^l c_0$. Let $r = 3$ in Lemma 2.6 and $t_0(k_0, \sigma_0)$ be given as in Lemma 2.6. Take an integer $t > \max(t_0, t_1)$, where t_1 is determined in (3). For each $(i_1, i_2, \dots, i_l)$, $1 \le i_1, i_2, \dots, i_l \le 3$, take a 3-uniform graph $A(k_0, \sigma_0)$ satisfying the conditions in Lemma 2.6 with $V(A(k_0, \sigma_0)) = V_{i_1 i_2 \dots i_l}$. The 3-uniform graph H is obtained by adding all $A(k_0, \sigma_0)$ to the 3-uniform hypegraph G. Then

$$\lambda(H) \ge \frac{|E(H)|}{(3s)^3}.$$

In view of the construction of H and equation (2), we have

$$\begin{aligned}
\frac{|E(H)|}{(3s)^3} &\ge \frac{|E(G)| + 3^l\sigma_0 t^2}{(3s)^3} \\
&\overset{(2)}{=} \frac{\left(\frac{21}{8} - \frac{1}{8\cdot 9^{l-1}}\right)s^3 - c_0(3^{l-1}t)^2 + 3^l \cdot 3^l c_0 t^2 + o(1/s)}{(3s)^3} \\
&\ge \frac{1}{6}\left(\frac{7}{12} - \frac{1}{4\cdot 9^l}\right) + \frac{c_0}{3^{3l}t} \\
&= \frac{\alpha}{6} + \frac{c_0}{3^{3l}t}
\end{aligned} \tag{3}$$

for $t \ge t_1$, where t_1 is a sufficiently large integer. Consequently,

$$\lambda(H) \ge \frac{\alpha}{6} + \frac{c_0}{3^{3l}t} \tag{4}$$

for $t \ge t_1$.

Now suppose $\vec{y} = (y_1, y_2, \dots, y_{3^l t})$ is an optimal vector of $\lambda(H)$. Let $\epsilon = \frac{3c_0}{3^{3l}t}$ and $n > n_1(\epsilon)$ as in Remark 2.3. Then 3-uniform graph $S_n = (\lfloor ny_1\rfloor, \dots, \lfloor ny_{3^l t}\rfloor) \otimes H$ has density at least $\alpha + \epsilon$. Since $\gamma(\mathcal{F}) \le \alpha$, some member F of $\mathcal{F}$ is a subgraph of S_n for n sufficiently large. For such $F \in \mathcal{F}$, there exists a subgraph M of H with $|V(M)| \le |V(F)| \le k_0$ so that $F \subset (n, n, \dots, n) \otimes M$. By Fact 2.1 and Fact 2.4, we have

$$\lambda(F) \overset{\text{Fact 2.1}}{\le} \lambda((n, n, \dots, n) \otimes M) \overset{\text{Fact 2.4}}{=} \lambda(M). \tag{5}$$

Theorem 1.5 will follow from the following lemma to be proved in Section 3.1.

Lemma 3.1. *Let M be any subgraph of H with $|V(M)| \le k_0$. Then*

$$\lambda(M) \le \frac{1}{6}\alpha = \frac{7}{72} - \frac{1}{24\cdot 9^l} \tag{6}$$

holds.

Assuming that Lemma 3.1 is true and applying Lemma 3.1 to (5), we have

$$\lambda(F) \le \frac{1}{6}\alpha$$

which contradicts our choice of F, i.e., contradicts that $\lambda(F) > \frac{1}{6}\alpha$ for all $F \in \mathcal{F}$.

To complete the proof of Theorem 1.5, what remains is to show Lemma 3.1.

3.1. Proof of Lemma 3.1

By Fact 2.1, we may assume that M is an induced subgraph of H. For each $(i_1, i_2, ..., i_l)$, where $1 \le i_j \le 3$ and $1 \le j \le l$, let

$$U_{i_1 i_2 \dots i_l} = V(M) \cap V_{i_1 i_2 \dots i_l} = \{v^1_{i_1 i_2 \dots i_l}, v^2_{i_1 i_2 \dots i_l}, \dots, v^p_{i_1 i_2 \dots i_l}\}.$$

where p might be different for different $(i_1, i_2, ..., i_l)$. To make the notation shorter, we simply write a uniform p without affecting the proof. We will apply the following Claim proved in [4].

Claim 3.2. (c.f. [4]) *If N is the 3-uniform graph formed from M by removing the edges contained in each $U_{i_1 i_2 \dots i_l}$ and inserting the edges $\{\{v^1_{i_1 i_2 \dots i_l}, v^2_{i_1 i_2 \dots i_l}, v^q_{i_1 i_2 \dots i_l}\} : 3 \le q \le p\}$ then $\lambda(M) \le \lambda(N)$.*

By Claim 3.2 the proof of Lemma 3.1 will be completed if we show that $\lambda(N) \le \frac{\alpha}{6}$. Since $v^1_{i_1 i_2 \dots i_l}, v^2_{i_1 i_2 \dots i_l}$ are pairwise equivalent and $v^3_{i_1 i_2 \dots i_l}, \dots v^p_{i_1 i_2 \dots i_l}$ are pairwise equivalent we can use Lemma 2.2 (part 1) to obtain an optimal vector $\vec{z}$ of $\lambda(N)$ such that

$$\begin{aligned} w(v^1_{i_1 i_2 \dots i_l}) = w(v^2_{i_1 i_2 \dots i_l}) &\overset{\text{def}}{=} \rho_{i_1 i_2 \dots i_l}, \quad w(v^3_{i_1 i_2 \dots i_l}) \\ &= w(v^4_{i_1 i_2 \dots i_l}) = \cdots = w(v^p_{i_1 i_2 \dots i_l}) \overset{\text{def}}{=} \zeta_{i_1 i_2 \dots i_l}, \end{aligned} \tag{7}$$

where $w(v)$ denotes the component of $\vec{z}$ corresponding to vertex v.

Let $a_{i_1 i_2 \dots i_l}$ be the sum of the components of $\vec{z}$ corresponding to all vertices in $U_{i_1 i_2 \dots i_l}$. Note that

$$\sum_{i=1}^{3} a_i = 1,$$

and

$$\sum_{i_{j+1}=1}^{3} a_{i_1 i_2 \dots i_j i_{j+1}} = a_{i_1 i_2 \dots i_j} \tag{8}$$

holds for each $i_1 i_2 \dots i_j$, $1 \le j \le l-1$.

We will apply the following 2 claims.

Claim 3.3. *We may assume that $a_i > 0$ for each $i = 1, 2, 3$.*

Claim 3.4. *Let $1 \le q \le l$ be an integer. If $a_{i_1 i_2 \dots i_q} > 0$, then*

$$\begin{aligned} 3\lambda(N) \le\ & a_{i_1+1} a_{i_1+2} + a_{i_1} a_{i_1+1} + \left(\frac{1}{2} a^2_{i_1-1} - \sum_{1 \le i_2, \dots, i_l \le 3} \rho^2_{i_1-1 i_2 \dots i_l} \right) \\ & + a_{i_1 i_2+1} a_{i_1 i_2+2} + a_{i_1 i_2 i_3+1} a_{i_1 i_2 i_3+2} + \cdots + a_{i_1 i_2 i_3 \dots i_{q-1} i_q+1} a_{i_1 i_2 i_3 \dots i_{q-1} i_q+2} \\ & + \frac{1}{8} \left(1 - \frac{1}{9^{l-q}} \right) a^2_{i_1 i_2 \dots i_q} + \sum_{1 \le i_{q+1}, i_{q+2} \dots, i_l \le 3} \rho^2_{i_1 i_2 \dots i_l}, \end{aligned}$$

where all subscripts are mod 3. For example, $a_{1-1} = a_3$ *and* $a_{3+1} = a_1$. *Note that when* $q = 1$, *terms* $a_{i_1 i_2+1} a_{i_1 i_2+2} + a_{i_1 i_2 i_3+1} a_{i_1 i_2 i_3+2} + \cdots + a_{i_1 i_2 i_3 \ldots i_{q-1} i_q+1} a_{i_1 i_2 i_3 \ldots i_{q-1} i_q+2}$ *are vacant.*

Proofs of Claims 3.3 and 3.4 will be given later. Now let us assume that Claims 3.3 and 3.4 hold and we will apply them to prove Lemma 3.1.

Proof of Lemma 3.1. By Claim 3.3, we can assume that $a_i > 0$ for each $i = 1, 2, 3$. Applying Claim 3.4 to a_1, a_2 and a_3 respectively, we have

$$3\lambda(N) \le a_2 a_3 + a_1 a_2 + \left(\frac{1}{2} a_3^2 - \sum_{1 \le i_2, \ldots, i_l \le 3} \rho^2_{3 i_2 \ldots i_l}\right) + \frac{1}{8}\left(1 - \frac{1}{9^{l-1}}\right) a_1^2 + \sum_{1 \le i_2, \ldots, i_l \le 3} \rho^2_{1 i_2 \ldots i_l}; \tag{9}$$

$$3\lambda(N) \le a_1 a_3 + a_2 a_3 + \left(\frac{1}{2} a_1^2 - \sum_{1 \le i_2, \ldots, i_l \le 3} \rho^2_{1 i_2 \ldots i_l}\right) + \frac{1}{8}\left(1 - \frac{1}{9^{l-1}}\right) a_2^2 + \sum_{1 \le i_2, \ldots, i_l \le 3} \rho^2_{2 i_2 \ldots i_l}; \tag{10}$$

and

$$3\lambda(N) \le a_1 a_2 + a_1 a_3 + \left(\frac{1}{2} a_2^2 - \sum_{1 \le i_2, \ldots, i_l \le 3} \rho^2_{2 i_2 \ldots i_l}\right) + \frac{1}{8}\left(1 - \frac{1}{9^{l-1}}\right) a_3^2 + \sum_{1 \le i_2, \ldots, i_l \le 3} \rho^2_{3 i_2 \ldots i_l}. \tag{11}$$

Adding equations (9), (10) and (11), we have

$$\begin{aligned}
9\lambda(N) &\le 2(a_1 a_2 + a_2 a_3 + a_1 a_3) + \left(\frac{1}{2} + \frac{1}{8} - \frac{1}{8 \cdot 9^{l-1}}\right)(a_1^2 + a_2^2 + a_3^2) \\
&= \left(\frac{5}{8} - \frac{1}{8 \cdot 9^{l-1}}\right)(a_1^2 + a_2^2 + a_3^2 + 2a_1 a_2 + 2a_2 a_3 + 2a_1 a_3) \\
&\quad + \left(2 - \frac{5}{4} + \frac{1}{4 \cdot 9^{l-1}}\right)(a_1 a_2 + a_2 a_3 + a_1 a_3) \\
&= \frac{5}{8} - \frac{1}{8 \cdot 9^{l-1}} + \left(2 - \frac{5}{4} + \frac{1}{4 \cdot 9^{l-1}}\right)(a_1 a_2 + a_2 a_3 + a_1 a_3).
\end{aligned}$$

Notice that $a_1 a_2 + a_2 a_3 + a_1 a_3$ has the maximum $\frac{1}{3}$ when $a_1 = a_2 = a_3 = \frac{1}{3}$, therefore,

$$9\lambda(N) \le \frac{5}{8} - \frac{1}{8 \cdot 9^{l-1}} + \frac{1}{3}\left(2 - \frac{5}{4} + \frac{1}{4 \cdot 9^{l-1}}\right) = \frac{7}{8} - \frac{1}{24 \cdot 9^{l-1}}.$$

Consequently,

$$\lambda(N) \leq \frac{7}{72} - \frac{1}{24 \cdot 9^l}.$$

This completes the proof of Lemma 3.1. □

What left is to show Claims 3.3 and 3.4.

3.1.1. Proof of Claim 3.3

Proof of Claim 3.3. If one of the a_i is 0, without loss of generality, assuming that $a_1 = 0$, then in view of the possible edges of N, we have

$$\lambda(N) \leq \frac{1}{2}a_2^2 a_3 + \lambda(N[V_2], \vec{z}|N[V_2]) + \lambda(N[V_3], \vec{z}|N[V_3]),$$

where $\vec{z}|N[V_2]$ ($\vec{z}|N[V_3]$) is the restriction of $\vec{z}$ on $N[V_2]$ ($N[V_3]$). To shorten the notation, we write $\lambda(F, \vec{z}|F)$ as $\lambda(F, \vec{z})$ in general for any subgraph F. Using this simplified notation, we rewrite the above inequality as

$$\lambda(N) \leq \frac{1}{2}a_2^2 a_3 + \lambda(N[V_2], \vec{z}) + \lambda(N[V_3], \vec{z}). \tag{12}$$

Claim 3.5.

$$\lambda(N[V_i], \vec{z}) \leq \frac{1}{24}\left(1 - \frac{1}{9^l}\right) a_i^3$$

holds for $i = 2, 3$.

The proof of Claim 3.5 will be given later. Assume that Claim 3.5 holds. Then in view of (12), we have

$$\begin{aligned}
\lambda(N) &\leq \frac{1}{2}a_2^2 a_3 + \frac{1}{24}\left(1 - \frac{1}{9^l}\right)(a_2^3 + a_3^3) \\
&= \frac{1}{24}\left(1 - \frac{1}{9^l}\right)(a_2^3 + a_3^3 + 3a_2^2 a_3 + 3a_2 a_3^2) + \frac{1}{2}a_2^2 a_3 \\
&\quad -\frac{1}{8}\left(1 - \frac{1}{9^l}\right) a_2 a_3 (a_2 + a_3) = \frac{1}{24}\left(1 - \frac{1}{9^l}\right) + \frac{1}{2}a_2 a_3 \left[a_2 - \frac{1}{4}\left(1 - \frac{1}{9^l}\right)\right] \\
&\leq \frac{1}{24}\left(1 - \frac{1}{9^l}\right) + \frac{1}{2}a_2 a_3 \left[a_2 - \frac{1}{4} \cdot \frac{8}{9}\right] \\
&= \frac{1}{24} - \frac{1}{24 \cdot 9^l} + \frac{1}{2}a_2 a_3 \left[a_2 - \frac{2}{9}\right] \\
&= \frac{1}{24} - \frac{1}{24 \cdot 9^l} + \frac{2}{3} \cdot \frac{1}{2}a_2 \cdot \frac{3}{2}a_3 \left(a_2 - \frac{2}{9}\right)
\end{aligned}$$

$$\le \frac{1}{24} - \frac{1}{24\cdot 9^l} + \frac{2}{3}\left(\frac{\frac{1}{2}a_2 + \frac{3}{2}a_3 + a_2 - \frac{2}{9}}{3}\right)^3$$
$$= \frac{1}{24} + \frac{2}{3}\left(\frac{23}{54}\right)^3 - \frac{1}{24\cdot 9^l}$$
$$< \frac{7}{72} - \frac{1}{24\cdot 9^l}.$$

Consequently, Lemma 3.1 holds. Therefore, we may assume that all $a_i > 0$ and this completes the proof of Claim 3.3. □

Now we give a proof of Claim 3.5.

Proof of Claim 3.5. We only show it for $i = 2$. The proof for $i = 3$ is exactly the same. In view of the edge set of $N[V_2]$, we have

$$\lambda(N[V_2], \vec{z}) = a_{21}a_{22}a_{23} + \sum_{1\le i_2\le 3} a_{2i_21}a_{2i_22}a_{2i_23} + \sum_{1\le i_2,i_3\le 3} a_{2i_2i_31}a_{2i_2i_32}a_{2i_2i_33}$$
$$+\cdots+ \sum_{1\le i_2,\dots i_{l-1}\le 3} a_{2i_2\dots i_{l-1}1}a_{2i_2\dots i_{l-1}2}a_{2i_2\dots i_{l-1}3} + \sum_{1\le i_2,\dots,i_l\le 3} \rho^2_{2i_2\dots i_l}(a_{2i_2\dots i_l} - 2\rho_{2i_2\dots i_l}).$$

Since

$$\rho^2_{2i_2\dots i_l}(a_{2i_2\dots i_l} - 2\rho_{2i_2\dots i_l}) \le \left(\frac{a_{2i_2\dots i_l}}{3}\right)^3,$$

we have

$$\lambda(N[V_2], \vec{z}) \le a_{21}a_{22}a_{23} + \sum_{1\le i_2\le 3} a_{2i_21}a_{2i_22}a_{2i_23} + \sum_{1\le i_2,i_3\le 3} a_{2i_2i_31}a_{2i_2i_32}a_{2i_2i_33}$$
$$+\cdots+ \sum_{1\le i_2,\dots i_{l-1}\le 3} a_{2i_2\dots i_{l-1}1}a_{2i_2\dots i_{l-1}2}a_{2i_2\dots i_{l-1}3} + \sum_{1\le i_2,\dots,i_l\le 3} \left(\frac{a_{2i_2\dots i_l}}{3}\right)^3. \quad (13)$$

For $1 \le q \le l-1$, let

$$f_q = \sum_{i_1=2,1\le i_2,\dots,i_q\le 3} a_{i_1i_2\dots i_q1}a_{i_1i_2\dots i_q2}a_{i_1i_2\dots i_q3}$$
$$+ \sum_{i_1=2,1\le i_2,\dots,i_{q+1}\le 3} a_{i_1i_2\dots i_{q+1}1}a_{i_1i_2\dots i_{q+1}2}a_{i_1i_2\dots i_{q+1}3}$$
$$+\cdots+ \sum_{i_1=2,1\le i_2,\dots,i_{l-1}\le 3} a_{i_1i_2\dots i_{l-1}1}a_{i_1i_2\dots i_{l-1}2}a_{i_1i_2\dots i_{l-1}3}$$
$$+ \sum_{i_1=2,1\le i_2,\dots,i_l\le 3} \left(\frac{a_{i_1i_2\dots i_l}}{3}\right)^3.$$

Let

$$f_l = \sum_{i_1=2,1\le i_2,\dots,i_l\le 3} \left(\frac{a_{i_1i_2\dots i_l}}{3}\right)^3.$$

The proof of Claim 3.5 will be completed by proving the following:

Claim 3.6. *For $1 \leq q \leq l$, the following holds:*

$$f_q \leq \frac{1}{24}\left(1 - \frac{1}{9^{l-q+1}}\right) \sum_{i_1=2, 1\leq i_2,\ldots,i_q\leq 3} (a_{i_1 i_2 \ldots i_q})^3. \tag{14}$$

In view of (13), $\lambda(N[V_2], \vec{z}) \leq f_1$. Assuming that Claim 3.6 holds and applying Claim 3.6 by taking $q = 1$, we get

$$\lambda(N[V_2], \vec{z}) \leq \frac{1}{24}\left(1 - \frac{1}{9^l}\right) a_2^3$$

and Claim 3.5 holds. □

Now we need to prove Claim 3.6.

Proof of Claim 3.6. Use induction on q. If $q = l$, then

$$f_l = \frac{1}{27} \sum_{i_1=2, 1\leq i_2,\ldots,i_l\leq 3} a^3_{2 i_2 \ldots i_l}$$

and (14) holds for $q = l$.

Now assume that

$$f_{q+1} \leq \frac{1}{24}\left(1 - \frac{1}{9^{l-q}}\right) \sum_{i_1=2, 1\leq i_2,\ldots,i_{q+1}\leq 3} (a_{i_1 i_2 \ldots i_{q+1}})^3 \tag{15}$$

holds for $1 \leq q \leq l - 1$. Need to show that

$$f_q \leq \frac{1}{24}\left(1 - \frac{1}{9^{l-q+1}}\right) \sum_{i_1=2, 1\leq i_2,\ldots,i_q\leq 3} (a_{i_1 i_2 \ldots i_q})^3$$

holds. Notice that

$$f_q = \sum_{i_1=2, 1\leq i_2,\ldots,i_q\leq 3} a_{i_1 i_2 \ldots i_q 1} a_{i_1 i_2 \ldots i_q 2} a_{i_1 i_2 \ldots i_q 3} + f_{q+1}.$$

Applying induction assumption (i.e. (15)) to f_{q+1}, we have

$$\begin{aligned} f_q \leq & \sum_{i_1=2, 1\leq i_2,\ldots,i_q\leq 3} a_{i_1 i_2 \ldots i_q 1} a_{i_1 i_2 \ldots i_q 2} a_{i_1 i_2 \ldots i_q 3} \\ & + \frac{1}{24}\left(1 - \frac{1}{9^{l-q}}\right) \sum_{i_1=2, 1\leq i_2,\ldots,i_q\leq 3} (a^3_{i_1 i_2 \ldots i_q 1} + a^3_{i_1 i_2 \ldots i_q 2} + a^3_{i_1 i_2 \ldots i_q 3}). \end{aligned} \tag{16}$$

Then we will apply the following Claim proved in [15]. Its proof will be also given later.

Claim 3.7. *Let* μ *be a constant in* $[0, \frac{1}{24}]$ *and* c *be a positive constant. Then*

$$\begin{aligned} g(c_1, c_2, c_3) &= c_1c_2c_3 + \mu(c_1^3 + c_2^3 + c_3^3) \\ &\le g\left(\frac{c}{3}, \frac{c}{3}, \frac{c}{3}\right) = \frac{1}{27}(1 + 3\mu)c^3 \end{aligned}$$

holds under the constraints $c_1 + c_2 + c_3 = c$ *and each* $c_i \ge 0$.

Applying Claim 3.7 into (16) and noting that $a_{i_1i_2\ldots i_q1} + a_{i_1i_2\ldots i_q2} + a_{i_1i_2\ldots i_q3} = a_{i_1i_2\ldots i_q}$, we obtain that

$$\begin{aligned} f_q &\le \frac{1}{27}\left[1 + 3 \times \frac{1}{24}\left(1 - \frac{1}{9^{l-q}}\right)\right] \sum_{i_1=2, 1\le i_2,\ldots,i_q\le 3} a^3_{i_1i_2\ldots i_q} \\ &= \frac{1}{27}\left(\frac{9}{8} - \frac{1}{8 \cdot 9^{l-q}}\right) \sum_{i_1=2, 1\le i_2,\ldots,i_q\le 3} a^3_{i_1i_2\ldots i_q} \\ &= \frac{1}{24}\left(1 - \frac{1}{9^{l-q+1}}\right) \sum_{i_1=2, 1\le i_2,\ldots,i_q\le 3} a^3_{i_1i_2\ldots i_q}. \end{aligned}$$

This completes the proof of Claim 3.6. □

Next we give a proof of Claim 3.7.

Proof of Claim 3.7. Since every term in g has degree 3, we will assume that $c = 1$. Suppose that g reaches the maximum at (b_1, b_2, b_3). If one of b_i's equals 0, say $b_3 = 0$ (without loss of generality), then $g(b_1, b_2, 0) = \mu(b_1^3 + b_2^3) \le \mu \le \frac{1}{27}(1 + 3\mu)$ since $\mu \le \frac{1}{24}$. So assume that $b_1, b_2, b_3 > 0$. We show that for any pair i, j, where $1 \le i < j \le 3$, $b_i = b_j$. Otherwise, without loss of generality, assume that $0 < b_1 < b_2$. Then

$$\begin{aligned} &g(b_1 + \epsilon, b_2 - \epsilon, b_3) - g(b_1, b_2, b_3) \\ &\quad = [(b_1 + \epsilon)(b_2 - \epsilon)b_3 - b_1b_2b_3)] + \mu[(b_1 + \epsilon)^3 + (b_2 - \epsilon)^3 - b_1^3 - b_2^3] \\ &\quad = [(b_2 - b_1)b_3\epsilon - \epsilon^2 b_3] + \mu[3\epsilon(b_1^2 - b_2^2) + 3\epsilon^2(b_1 + b_2)] \\ &\quad = \epsilon(b_2 - b_1)[b_3 - 3\mu(b_1 + b_2)] + \epsilon^2[3\mu(b_1 + b_2) - b_3]. \end{aligned} \tag{17}$$

If $b_3 - 3\mu(b_1 + b_2) > 0$, then the right hand side of equation (17) > 0 for small enough $\epsilon > 0$. If $b_3 - 3\mu(b_1 + b_2) < 0$, then the right hand side of equation (17) > 0 for $\epsilon < 0$ with $|\epsilon|$ small enough. In either case it contradicts the assumption that g reaches the maximum at (b_1, b_2, b_3). Now assume that $b_3 - 3\mu(b_1 + b_2) = 0$, it follows that $b_3 = \frac{3\mu}{3\mu+1}$ and $b_1 + b_2 = \frac{1}{3\mu+1}$. Then

$$
\begin{aligned}
g(b_1, b_2, b_3) &= \frac{3\mu}{3\mu+1} b_1 b_2 + \mu(b_1^3 + b_2^3) + \mu \left(\frac{3\mu}{3\mu+1}\right)^3 \\
&= 3\mu b_1 b_2 (b_1 + b_2) + \mu(b_1^3 + b_3^3) + \mu \left(\frac{3\mu}{3\mu+1}\right)^3 \\
&= \mu[b_1^3 + 3b_1^2 b_2 + 3b_1 b_2^2 + b_2^3] + \mu \left(\frac{3\mu}{3\mu+1}\right)^3 \\
&= \mu(b_1 + b_2)^3 + \mu \left(\frac{3\mu}{3\mu+1}\right)^3 \\
&= \mu \left[\left(\frac{1}{3\mu+1}\right)^3 + \left(\frac{3\mu}{3\mu+1}\right)^3\right].
\end{aligned} \tag{18}
$$

Observe that

$$
\begin{aligned}
&\mu \left[\left(\frac{1}{3\mu+1}\right)^3 + \left(\frac{3\mu}{3\mu+1}\right)^3\right] - \frac{1}{27}(1+3\mu) \\
&= \mu \left[\left(\frac{1}{3\mu+1}\right)^3 + \left(\frac{3\mu}{3\mu+1}\right)^3 - \frac{1}{9}\right] - \frac{1}{27} \\
&\le \begin{cases} \frac{1}{24}\left[\left(\frac{9}{10}\right)^3 + \left(\frac{1}{9}\right)^3 - \frac{1}{9}\right] - \frac{1}{27}, & if \ \frac{1}{27} \le \mu \le \frac{1}{24} \\ \frac{1}{27}\left[\left(\frac{3\mu+1}{3\mu+1}\right)^3 - \frac{1}{9}\right] - \frac{1}{27}, & if \ 0 \le \mu < \frac{1}{27}. \end{cases} \\
&< 0.
\end{aligned} \tag{19}
$$

Therefore, $g(b_1, b_2, b_3) \le \frac{1}{27}(1+3\mu)$. □

3.1.2. Proof of Claim 3.4

Proof of Claim 3.4. Use induction on q. If $q = l$ and $a_{i_1 i_2 \ldots i_l} > 0$, then take $u_{i_1 i_2 \ldots i_l} = v^3_{i_1 i_2 \ldots i_l}$ if $\zeta_{i_1 i_2 \ldots i_l} > 0$, take $u_{i_1 i_2 \ldots i_l} = v^1_{i_1 i_2 \ldots i_l}$ otherwise (see (7)). Then by Lemma 2.2 part 2, we have

$$3\lambda(N) = \lambda(u_{i_1 i_2 \ldots i_l}, \vec{z}).$$

Considering all possible edges incident to $u_{i_1 i_2 \ldots i_l}$, we have

$$
\begin{aligned}
\lambda(u_{i_1 i_2 \ldots i_l}, \vec{z}) \le\ & a_{i_1+1} a_{i_1+2} + a_{i_1} a_{i_1+1} + \left(\frac{1}{2} a_{i_1-1}^2 - \sum_{1 \le i_2, \ldots, i_l \le 3} \rho_{i_1-1 i_2 \ldots i_l}^2\right) \\
& + a_{i_1 i_2+1} a_{i_1 i_2+2} + a_{i_1 i_2 i_3+1} a_{i_1 i_2 i_3+2} + \cdots \\
& + a_{i_1 i_2 i_3 \ldots i_{l-1} i_l+1} a_{i_1 i_2 i_3 \ldots i_{l-1} i_l+2} + \rho_{i_1 i_2 \ldots i_l}^2.
\end{aligned}
$$

In the above inequality, $\sum_{1\le i_2,\ldots,i_l\le 3}\rho^2_{i_1-1i_2\ldots i_l}$ is subtracted from $\frac{1}{2}a^2_{i_1-1}$ since $w(u_{i_1i_2\ldots i_l})\cdot w(u_{i_1i_2\ldots i_l}) = \rho^2_{i_1-1i_2\ldots i_l}$ does not contribute to $\lambda(u_{i_1i_2\ldots i_l},\vec{z})$ but it is calculated in $\frac{1}{2}a^2_{i_1-1}$. By the above inequality, Claim 3.4 holds for $q=l$.

Now assume that the conclusion is true for $q+1$, i.e., if $a_{i_1i_2\ldots i_{q+1}}>0$, then

$$3\lambda(N)\le a_{i_1+1}a_{i_1+2}+a_{i_1}a_{i_1+1}+\left(\frac{1}{2}a^2_{i_1-1}-\sum_{1\le i_2,\ldots,i_l\le 3}\rho^2_{i_1-1i_2\ldots i_l}\right)$$
$$+a_{i_1i_2+1}a_{i_1i_2+2}+a_{i_1i_2i_3+1}a_{i_1i_2i_3+2}+\cdots+a_{i_1i_2i_3\ldots i_qi_{q+1}+1}a_{i_1i_2i_3\ldots i_qi_{q+1}+2}$$
$$+\frac{1}{8}\left(1-\frac{1}{9^{l-q-1}}\right)a^2_{i_1i_2\ldots i_{q+1}}+\sum_{1\le i_{q+2},i_{q+3}\ldots,i_l\le 3}\rho^2_{i_1i_2\ldots i_l} \tag{20}$$

holds. We need to show that if $a_{i_1i_2\ldots i_q}>0$, then

$$3\lambda(N)\le a_{i_1+1}a_{i_1+2}+a_{i_1}a_{i_1+1}+\left(\frac{1}{2}a^2_{i_1-1}-\sum_{1\le i_2,\ldots,i_l\le 3}\rho^2_{i_1-1i_2\ldots i_l}\right)$$
$$+a_{i_1i_2+1}a_{i_1i_2+2}+a_{i_1i_2i_3+1}a_{i_1i_2i_3+2}+\cdots+a_{i_1i_2i_3\ldots i_{q-1}i_q+1}a_{i_1i_2i_3\ldots i_{q-1}i_q+2}$$
$$+\frac{1}{8}\left(1-\frac{1}{9^{l-q}}\right)a^2_{i_1i_2\ldots i_q}+\sum_{1\le i_{q+1},i_{q+2}\ldots,i_l\le 3}\rho^2_{i_1i_2\ldots i_l} \tag{21}$$

holds.

Case 1. If $a_{i_1i_2\ldots i_q1}>0$, $a_{i_1i_2\ldots i_q2}>0$, and $a_{i_1i_2\ldots i_q3}>0$, then applying induction assumption ((20)) to each $a_{i_1i_2\ldots i_qi_{q+1}}$, $1\le i_{q+1}\le 3$, we have

$$3\lambda(N)\le a_{i_1+1}a_{i_1+2}+a_{i_1}a_{i_1+1}+\left(\frac{1}{2}a^2_{i_1-1}-\sum_{1\le i_2,\ldots,i_l\le 3}\rho^2_{i_1-1i_2\ldots i_l}\right)$$
$$+a_{i_1i_2+1}a_{i_1i_2+2}+a_{i_1i_2i_3+1}a_{i_1i_2i_3+2}+\cdots+a_{i_1\ldots i_{q-1}i_q+1}a_{i_1\ldots i_{q-1}i_q+2}$$
$$+a_{i_1i_2i_3\ldots i_qi_{q+1}+1}a_{i_1i_2i_3\ldots i_qi_{q+1}+2}$$
$$+\frac{1}{8}\left(1-\frac{1}{9^{l-q-1}}\right)a^2_{i_1\ldots i_qi_{q+1}}+\sum_{1\le i_{q+2},i_{q+3}\ldots,i_l\le 3}\rho^2_{i_1i_2\ldots i_l}$$

holds for each $1\le i_{q+1}\le 3$. Taking $\sum_{1\le i_{q+1}\le 3}$ in both sides of the above inequalities and dividing by 3, we get

$$3\lambda(N)\le a_{i_1+1}a_{i_1+2}+a_{i_1}a_{i_1+1}+\left(\frac{1}{2}a^2_{i_1-1}-\sum_{1\le i_2,\ldots,i_l\le 3}\rho^2_{i_1-1i_2\ldots i_l}\right)$$
$$+a_{i_1i_2+1}a_{i_1i_2+2}+a_{i_1i_2i_3+1}a_{i_1i_2i_3+2}+\cdots+a_{i_1\ldots i_{q-1}i_q+1}a_{i_1\ldots i_{q-1}i_q+2}$$
$$+\frac{a_{i_1i_2i_3\ldots i_q1}a_{i_1i_2i_3\ldots i_q2}+a_{i_1i_2i_3\ldots i_q2}a_{i_1i_2i_3\ldots i_q3}+a_{i_1i_2i_3\ldots i_q1}a_{i_1i_2i_3\ldots i_q3}}{3}$$
$$+\frac{1}{24}\left(1-\frac{1}{9^{l-q-1}}\right)[a^2_{i_1i_2\ldots i_q1}+a^2_{i_1i_2\ldots i_q2}+a^2_{i_1i_2\ldots i_q3}]+\sum_{1\le i_{q+1},i_{q+2}\ldots,i_l\le 3}\rho^2_{i_1i_2\ldots i_l}.$$

Note that

$$
\begin{aligned}
&\frac{a_{i_1i_2i_3\dots i_q1}a_{i_1i_2i_3\dots i_q2}+a_{i_1i_2i_3\dots i_q2}a_{i_1i_2i_3\dots i_q3}+a_{i_1i_2i_3\dots i_q1}a_{i_1i_2i_3\dots i_q3}}{3}\\
&\quad+\frac{1}{24}\left(1-\frac{1}{9^{l-q-1}}\right)[a^2_{i_1i_2\dots i_q1}+a^2_{i_1i_2\dots i_q2}+a^2_{i_1i_2\dots i_q3}]\\
&=\frac{1}{24}\left(1-\frac{1}{9^{l-q-1}}\right)[a^2_{i_1i_2\dots i_q1}+a^2_{i_1i_2\dots i_q2}+a^2_{i_1i_2\dots i_q3}\\
&\quad+2a_{i_1i_2\dots i_q1}a_{i_1i_2\dots i_q2}+2a_{i_1i_2\dots i_q2}a_{i_1i_2\dots i_q3}+2a_{i_1i_2\dots i_q1}a_{i_1i_2\dots i_q3}]\\
&\quad+\left[\frac{1}{3}-\frac{1}{12}\left(1-\frac{1}{9^{l-q-1}}\right)\right]\\
&\quad\times(a_{i_1i_2\dots i_q1}a_{i_1i_2\dots i_q2}+a_{i_1i_2\dots i_q2}a_{i_1i_2\dots i_q3}+a_{i_1i_2\dots i_q1}a_{i_1i_2\dots i_q3})\\
&=\frac{1}{24}\left(1-\frac{1}{9^{l-q-1}}\right)a^2_{i_1i_2\dots i_q}\\
&\quad+\left[\frac{1}{3}-\frac{1}{12}\left(1-\frac{1}{9^{l-q-1}}\right)\right]\\
&\qquad\left(a_{i_1i_2\dots i_q1}a_{i_1i_2\dots i_q2}+a_{i_1i_2\dots i_q2}a_{i_1i_2\dots i_q3}+a_{i_1i_2\dots i_q1}a_{i_1i_2\dots i_q3}\right)\\
&\le\frac{1}{24}\left(1-\frac{1}{9^{l-q-1}}\right)a^2_{i_1i_2\dots i_q}+\left[\frac{1}{3}-\frac{1}{12}\left(1-\frac{1}{9^{l-q-1}}\right)\right]\frac{1}{3}a^2_{i_1i_2\dots i_q}\\
&=\frac{1}{8}\left(1-\frac{1}{9^{l-q}}\right)a^2_{i_1i_2\dots i_q}
\end{aligned}
$$

since $a_{i_1i_2\dots i_q1}a_{i_1i_2\dots i_q2}+a_{i_1i_2\dots i_q2}a_{i_1i_2\dots i_q3}+a_{i_1i_2\dots i_q1}a_{i_1i_2\dots i_q3}$ has the maximum value $\frac{1}{3}a^2_{i_1i_2\dots i_q}$ when $a_{i_1i_2\dots i_q1}=a_{i_1i_2\dots i_q2}=a_{i_1i_2\dots i_q3}=\frac{a_{i_1i_2\dots i_q}}{3}$. Combining the above two inequalities, we obtain that

$$
\begin{aligned}
3\lambda(N)\le\ & a_{i_1+1}a_{i_1+2}+a_{i_1}a_{i_1+1}+\left(\frac{1}{2}a^2_{i_1-1}-\sum_{1\le i_2,\dots,i_l\le 3}\rho^2_{i_1-1i_2\dots i_l}\right)\\
&+a_{i_1i_2+1}a_{i_1i_2+2}+a_{i_1i_2i_3+1}a_{i_1i_2i_3+2}+\cdots+a_{i_1\dots i_{q-1}i_q+1}a_{i_1\dots i_{q-1}i_q+2}\\
&+\frac{1}{8}\left(1-\frac{1}{9^{l-q}}\right)a^2_{i_1i_2\dots i_q}+\sum_{1\le i_{q+1},i_{q+2}\dots,i_l\le 3}\rho^2_{i_1i_2\dots i_l}.
\end{aligned}
$$

Case 2. Two of $a_{i_1i_2\dots i_q1}$, $a_{i_1i_2\dots i_q2}$, and $a_{i_1i_2\dots i_q3}$ are positive and one of them is 0. Without loss of generality, assume that $a_{i_1i_2\dots i_q1}>0$, $a_{i_1i_2\dots i_q2}>0$, and $a_{i_1i_2\dots i_q3}=0$. Applying induction assumption to $a_{i_1i_2\dots i_qi_{q+1}}$, where $i_{q+1}=1,2$ (see (20)), we have

$$
\begin{aligned}
3\lambda(N)\le\ & a_{i_1+1}a_{i_1+2}+a_{i_1}a_{i_1+1}+\left(\frac{1}{2}a^2_{i_1-1}-\sum_{1\le i_2,\dots,i_l\le 3}\rho^2_{i_1-1i_2\dots i_l}\right)\\
&+a_{i_1i_2+1}a_{i_1i_2+2}+a_{i_1i_2i_3+1}a_{i_1i_2i_3+2}+\cdots+a_{i_1\dots i_{q-1}i_q+1}a_{i_1\dots i_{q-1}i_q+2}\\
&+\frac{1}{8}\left(1-\frac{1}{9^{l-q-1}}\right)a^2_{i_1i_2\dots i_qi_{q+1}}+\sum_{1\le i_{q+2},i_{q+3}\dots,i_l\le 3}\rho^2_{i_1i_2\dots i_l}
\end{aligned}
$$

holds for each $i_{q+1} = 1, 2$. (observing that $a_{i_1 i_2 \ldots i_q i_{q+1}+1} a_{i_1 i_2 \ldots i_q i_{q+1}+2} = 0$ for each $i_{q+1} = 1, 2$).

Taking $\sum_{i_{q+1}=1}^{2}$ in both sides of the above inequality and dividing by 2, we have

$$3\lambda(N) \leq a_{i_1+1}a_{i_1+2} + a_{i_1}a_{i_1+1} + \left(\frac{1}{2}a_{i_1-1}^2 - \sum_{1\leq i_2,\ldots,i_l\leq 3} \rho_{i_1-1 i_2\ldots i_l}^2\right)$$
$$+ a_{i_1 i_2+1}a_{i_1 i_2+2} + a_{i_1 i_2 i_3+1}a_{i_1 i_2 i_3+2} + \cdots + a_{i_1\ldots i_{q-1} i_q+1}a_{i_1\ldots i_{q-1} i_q+2}$$
$$+ \frac{1}{8}\left(1 - \frac{1}{9^{l-q-1}}\right)\frac{a_{i_1 i_2\ldots i_q 1}^2 + a_{i_1 i_2\ldots i_q 2}^2}{2} + \sum_{1\leq i_{q+1},\ldots,i_l\leq 3} \rho_{i_1 i_2\ldots i_l}^2.$$

Since

$$\frac{1}{8}\left(1 - \frac{1}{9^{l-q-1}}\right)\frac{a_{i_1 i_2\ldots i_q 1}^2 + a_{i_1 i_2\ldots i_q 2}^2}{2} < \frac{1}{8}\left(1 - \frac{1}{9^{l-q}}\right)a_{i_1 i_2\ldots i_q}^2,$$

then

$$3\lambda(N) \leq a_{i_1+1}a_{i_1+2} + a_{i_1}a_{i_1+1} + \left(\frac{1}{2}a_{i_1-1}^2 - \sum_{1\leq i_2,\ldots,i_l\leq 3} \rho_{i_1-1 i_2\ldots i_l}^2\right)$$
$$+ a_{i_1 i_2+1}a_{i_1 i_2+2} + a_{i_1 i_2 i_3+1}a_{i_1 i_2 i_3+2} + \cdots + a_{i_1\ldots i_{q-1} i_q+1}a_{i_1\ldots i_{q-1} i_q+2}$$
$$+ \frac{1}{8}\left(1 - \frac{1}{9^{l-q}}\right)a_{i_1 i_2\ldots i_q}^2 + \sum_{1\leq i_{q+1},\ldots,i_l\leq 3} \rho_{i_1 i_2\ldots i_l}^2.$$

Case 3. Only one of $a_{i_1 i_2\ldots i_q 1}$, $a_{i_1 i_2\ldots i_q 2}$, and $a_{i_1 i_2\ldots i_q 3}$ is positive. Without loss of generality, assume that $a_{i_1 i_2\ldots i_q 2} = a_{i_1 i_2\ldots i_q 3} = 0$ and $a_{i_1 i_2\ldots i_q 1} = a_{i_1 i_2\ldots i_q}$. Applying induction assumption to $a_{i_1 i_2\ldots i_q 1}$ and noting that $a_{i_1 i_2\ldots i_q 2} \cdot a_{i_1 i_2\ldots i_q 3} = 0$ and $a_{i_1 i_2\ldots i_q 1}^2 = a_{i_1 i_2\ldots i_q}^2$, we have

$$3\lambda(N) \leq a_{i_1+1}a_{i_1+2} + a_{i_1}a_{i_1+1} + \left(\frac{1}{2}a_{i_1-1}^2 - \sum_{1\leq i_2,\ldots,i_l\leq 3} \rho_{i_1-1 i_2\ldots i_l}^2\right)$$
$$+ a_{i_1 i_2+1}a_{i_1 i_2+2} + a_{i_1 i_2 i_3+1}a_{i_1 i_2 i_3+2} + \cdots + a_{i_1\ldots i_{q-1} i_q+1}a_{i_1\ldots i_{q-1} i_q+2}$$
$$+ \frac{1}{8}\left(1 - \frac{1}{9^{l-q-1}}\right)a_{i_1 i_2\ldots i_q}^2 + \sum_{1\leq i_{q+2},\ldots,i_l\leq 3} \rho_{i_1 i_2\ldots i_l}^2$$
$$\leq a_{i_1+1}a_{i_1+2} + a_{i_1}a_{i_1+1} + \left(\frac{1}{2}a_{i_1-1}^2 - \sum_{1\leq i_2,\ldots,i_l\leq 3} \rho_{i_1-1 i_2\ldots i_l}^2\right)$$
$$+ a_{i_1 i_2+1}a_{i_1 i_2+2} + a_{i_1 i_2 i_3+1}a_{i_1 i_2 i_3+2} + \cdots + a_{i_1\ldots i_{q-1} i_q+1}a_{i_1\ldots i_{q-1} i_q+2}$$
$$+ \frac{1}{8}\left(1 - \frac{1}{9^{l-q}}\right)a_{i_1 i_2\ldots i_q}^2 + \sum_{1\leq i_{q+1},\ldots,i_l\leq 3} \rho_{i_1 i_2\ldots i_l}^2.$$

The proof of Claim 3.4 is completed. □

Acknowledgements. I am thankful to an anonymous referee for the helpful comments.

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Received: January 18, 2008
Final version received: December 27, 2008