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A Degree Sum Condition Concerning the Connectivity and the Independence Number of a Graph

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Abstract. Let G be a graph and $S \subset V(G)$. We denote by $\alpha(S)$ the maximum number of pairwise nonadjacent vertices in S. For $x, y \in V(G)$, the local connectivity $\kappa(x, y)$ is defined to be the maximum number of internally-disjoint paths connecting x and y in G. We define $\kappa(S) = \min\{\kappa(x, y) : x, y \in S, x \neq y\}$. In this paper, we show that if $\kappa(S) \geq 3$ and $\sum_{i=1}^{4} d_G(x_i) \geq |V(G)| + \kappa(S) + \alpha(S) - 1$ for every independent set $\{x_1, x_2, x_3, x_4\} \subset S$, then G contains a cycle passing through S. This degree condition is sharp and this gives a new degree sum condition for a 3-connected graph to be hamiltonian.

Key words. Degree sum, Connectivity, Independence number, Cyclable.

1. Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. Let G be a graph and $S \subset V(G)$. We denote the degree of a vertex x in G by $d_G(x)$. We denote by $G[S]$ the subgraph induced by S. For $X \subset S$, X is called *an independent set of S* if $G[X]$ has no edges. We define $\alpha(S)$ as the maximum cardinality of an independent set of S.

For $x, y \in V(G)$, the local connectivity $\kappa(x, y)$ is defined to be the maximum number of internally-disjoint paths connecting x and y in G. We define $\kappa(S) = \min\{\kappa(x, y) : x, y \in S, x \neq y\}$. If $\alpha(S) \geq k$, let

$$\sigma_k(S) = \min \left\{ \sum_{x \in X} d_G(x) \colon X \text{ is an independent set of } S \text{ with } |X| = k \right\};$$

otherwise $\sigma_k(S) = +\infty$. If $S = V(G)$, we simply write α, κ and σ_k instead of $\alpha(S)$, $\kappa(S)$ and $\sigma_k(S)$, respectively.

In 1960, Ore introduced a degree sum condition for a graph to be hamiltonian.

Theorem 1 (Ore [8]). *Let G be a graph of order $n \geq 3$. If $\sigma_2 \geq n$, then G is hamiltonian.*

On the other hand, in 1972, Chvátal and Erdős showed the relationship between the connectivity, the independence number and hamiltonicity.

Theorem 2 (Chvátal and Erdős [5]). *Let G be a graph of order at least 3. If $\alpha \leq \kappa$, then G is hamiltonian.*

For $S \subset V(G)$, we say that S is *cyclable* in G if G contains a cycle passing through S. In 1992, Shi gave a degree condition for a set of vertices to be cyclable.

Theorem 3 (Shi [10]). *Let G be a 2-connected graph on n vertices, and $S \subset V(G)$. If $\sigma_2(S) \geq n$, then S is cyclable in G.*

In 1989, Bauer, Broersma, Li and Veldman gave a σ_3 condition with the connectivity.

Theorem 4 (Bauer et al. [1]). *Let G be a 2-connected graph on n vertices. If $\sigma_3 \geq n+\kappa$, then G is hamiltonian.*

In 1997, Broersma, H. Li, J. Li, Tian and Veldman showed a generalizatin of Theorems 2 and 4. They defined another notion of connectivity of S as follows. If $G[S]$ is not complete, let $\kappa'(S)$ be the minimum cardinality of a set of vertices of G separating two vertices of S. If $G[S]$ is complete, let $\kappa'(S) = |S| - 1$. We define $\delta(S) := \min\{d_G(x) : x \in S\}$.

Theorem 5 (Broersma et al. [4]). *Let G be a 2-connected graph and $S \subset V(G)$. If $\alpha(S) \leq \kappa'(S)$, then S is cyclable in G.*

Theorem 6 (Broersma et al. [4]). *Let G be a 2-connected graph on n vertices, and $S \subset V(G)$. If $\sigma_3(S) \geq n + \min\{\kappa'(S), \delta(S)\}$, then S is cyclable in G.*

By the definitions of $\kappa(S)$, $\kappa'(S)$ and $\delta(S)$, the following proposition is obvious.

Proposition 7. *If $G[S]$ is not a complete graph, then $\kappa(S) \leq \min\{\kappa'(S), \delta(S)\}$.*

There exists a pair of a graph G and $S \subset V(G)$ satisfying $\kappa(S) < \min\{\kappa'(S), \delta(S)\}$. For two graphs G and H, $G + H$ means the graph which is obtained from the join of G and H. Let r, s and t be positive integers with $r < s < t$ and we consider the graph G_1 obtained from $(rK_1 + sK_1) \cup K_t$ by joining each vertex of K_t and a vertex u of sK_1. Let v, w be distinct vertices of rK_1 and $S := \{u, v, w\}$. Then $\kappa(S) = r$, since there exist only r internally-disjoint paths connecting u and v. On the other hand, we must remove $V(sK_1)$ to separate v and w, and hence $\kappa'(S) = s$. Therefore $\kappa(S) = r < s = \kappa'(S) = \delta(S)$.

By this proposition and by Theorem 5, we obtain the following theorem.

Theorem 8. *Let G be a 2-connected graph and $S \subset V(G)$. If $\alpha(S) \leq \kappa(S)$, then S is cyclable in G.*

In the light of Theorem 8 we deduce the following lemma which shows the intimate relationship between $\kappa(S)$ and $\kappa'(S)$ under the condition $\alpha(S) \geq \kappa(S) + 1$.

Lemma 9. *Let G be a graph and $S \subset V(G)$ with $\alpha(S) \geq \kappa(S) + 1$. Then there exists $T \subset V(G)$ such that $|T| = \kappa(S)$ and T separates two vertices of S. In particular, $\kappa(S) = \kappa'(S)$.*

Proof. Let u and v be vertices in S such that $\kappa(u, v) = \kappa(S)$. If $uv \notin E(G)$, then by Menger's theorem, there exists $T \subset V(G) - \{u, v\}$ with $|T| = \kappa(S)$ which separates u and v.

Suppose that $uv \in E(G)$. Then $G - uv$ has $T \subset V(G) - \{u, v\}$ with $|T| = \kappa(S) - 1$ which separates u and v. If $S - (T \cup \{u, v\}) \neq \emptyset$, then $T \cup \{u\}$ or $T \cup \{v\}$ is a desired separating set. Thus, we may assume that $S \subset T \cup \{u, v\}$. Then since $uv \in E(G)$, $\alpha(S) \leq |S| - 1 \leq |T| + 1 = \kappa(S)$, contradicting the assumption. □

By considering Theorem 8 and Lemma 9, the proof of Theorem 6 can be used to prove the following result. By Proposition 7, Theorem 10 is stronger than Theorem 6.

Theorem 10. *Let G be a graph on n vertices, and $S \subset V(G)$ with $\kappa(S) \geq 2$. If $\sigma_3(S) \geq n + \kappa(S)$, then S is cyclable in G.*

In 2000, Harkat-Benhamdine, Li and Tian gave a σ_4 condition with the independence number.

Theorem 11 (Harkat-Benhamdine et al. [6]). *Let G be a 3-connected graph on n vertices, and $S \subset V(G)$. If $\sigma_4(S) \geq n + 2\alpha(S) - 2$, then S is cyclable in G.*

In this paper, we give a σ_4 condition with the connectivity and the independence number.

Theorem 12. *Let G be a graph on n vertices, and $S \subset V(G)$ with $\kappa(S) \geq 3$. If $\sigma_4(S) \geq n + \kappa(S) + \alpha(S) - 1$, then S is cyclable in G.*

Theorem 12 is best possible. Let m be a positive integer, and let $G_2 = K_m + (m+1)K_1$ and $S = (m+1)K_1$. Then $\sigma_4(S) = 4m = (2m+1) + m + (m+1) - 2 = |V(G_2)| + \kappa(S) + \alpha(S) - 2$ and S is not cyclable in G_2.

By combining Theorems 8 and 12, we obtain Theorem 11. In fact, by considering the following graphs, Theorem 12 is stronger than Theorems 10 and 11 for a graph G and $S \subset V(G)$ with $\kappa(S) \geq 3$. Let k, n, m be positive integers with $3 \leq k \leq m - 2$ and $\frac{n+k-1}{3} \leq m \leq \frac{n-1}{2}$. We consider the graph G_3 obtained from $(K_m + ((m-2)K_1 \cup K_2)) \cup K_{n-2m}$ by joining k vertices of K_m and each vertex of K_{n-2m}.

Let $S = (m-2)K_1 \cup K_2 \cup K_{n-2m}$. Then $|V(G_3)| = n, \kappa(S) = k, \alpha(S) = m, \sigma_3(S) = (n-2m+k-1)+2m = |V(G_3)|+\kappa(S)-1$ and $\sigma_4(S) = (n-2m+k-1)+3m = n+m+k-1 = |V(G_3)|+\kappa(S)+\alpha(S)-1 < n+2m-2 = |V(G_3)|+2\alpha(S)-2$.

If $S = V(G)$, then we obtain the following as a corollary of Theorem 12.

Corollary 13. *Let G be a 3-connected graph on n vertices. If $\sigma_4 \geq n+\kappa+\alpha-1$, then G is hamiltonian.*

2. Notes

Motivated by Theorem 8, we consider to improve Theorem 11 to Theorem 12. Similarly, one might suspect that if $\sigma_4(S) \geq |V(G)| + 2\kappa(S)$ then S is cyclable in G. However, this is not true. Let k, m, n be positive integers $3 \leq k \leq m$ and $\frac{n+k-1}{3} \leq m \leq \frac{n-1}{2}$. We consider the graph G_4 obtained from $(K_m + mK_1) \cup K_{n-2m}$ by joining k vertices of K_m and each vertex of K_{n-2m}. Let $S = mK_1 \cup K_{n-2m}$. Then $|V(G_4)| = n$, $\kappa(S) = k$, $\sigma_4(S) = n+k+m-1 \geq |V(G_4)| + 2\kappa(S)$, but G_4 does not contain a cycle passing through S. When $S = V(G)$, the graph G_4 contains no hamiltonian cycle although $\sigma \geq |V(G_4)| + 2\kappa$. Therefore $\sigma_4 \geq |V(G)| + 2\kappa$ does not imply a hamiltonicity of a graph G. However, in 2005, Lu, Liu and Tian showed that any longest cycle is dominating under the condition $\sigma_4 \geq |V(G)|+2\kappa$. A cycle C of a graph G is said to be *a dominating cycle* if $V(G-C)$ is an independent set of G.

Theorem 14 (Lu et al. [7]). *Let G be a 3-connected graph on n vertices. If $\sigma_4 \geq n+2\kappa$, then each longest cycle of G is a dominating cycle.*

On the other hand, Ota (1995) gave another degree condition concerning cyclability.

Theorem 15 (Ota [9]). *Let G be a graph on n vertices, and $S \subset V(G)$. Suppose that $\kappa(S) \geq 2$. If for any l ($l \geq \kappa(S)$),*

$$\sigma_{l+1}(S) \geq n + l^2 - l,$$

then S is cyclable in G.

By proving the following proposition, we show that the assumption of Theorem 15 is weaker than that of Theorem 11. Hence, Theorem 15 implies Theorem 11.

Proposition 16. *Let G be a 3-connected graph on n vertices and $S \subset V(G)$. If $\sigma_4(S) \geq n + 2\alpha(S) - 2$, then $\sigma_{l+1}(S) \geq n + l^2 - l$ for any l ($3 \leq l \leq \alpha(S) - 1$).*

Proof. Since $\sigma_{l+1}(S) \geq \frac{l+1}{4}\sigma_4(S) \geq \frac{l+1}{4}(n+2\alpha(S)-2)$, we shall prove that

$$\frac{l+1}{4}(n+2\alpha(S)-2) \geq n + l^2 - l.$$

Because the above inequality is a quadratic function on l, it suffices to prove that it holds for $l = 3$ and $l = \alpha(S) - 1$. Since $3 \le l \le \alpha(S) - 1$, note that $\alpha(S) \ge 4$.

Case 1. $l = 3$.

In this case, $\frac{l+1}{4}(n + 2\alpha(S) - 2) \ge n + 6 = n + l^2 - l$. Therefore this completes Case 1.

Case 2. $l = \alpha(S) - 1$.

Suppose that $\frac{l+1}{4}(n + 2\alpha(S) - 2) < n + l^2 - l$. By the assumption of Case 2, this implies $(\alpha(S) - 4)(n - 2\alpha(S) + 2) < 0$. Since $\alpha(S) \ge 4$, we have $n < 2\alpha(S) - 2$.

On the other hand, let X be an independent set of S with $|X| = \alpha(S)$ and choose $x \in X$ so that $d_G(x)$ is as large as possible. Since $|V(G) - X| \ge |N_G(x)| \ge \frac{1}{4}\sigma_4(S)$ by the degree condition, we obtain $n - \alpha(S) \ge \frac{1}{4}(n + 2\alpha(S) - 2)$, and this implies $n \ge 2\alpha(S) - \frac{2}{3}$, a contradiction. This completes the proof. □

However, Theorem 15 does not imply Theorem 12. Let k, r, m be integers such that $k \ge 5$, $r \ge 4$ and $m = 4(r - 1)$. We consider the graph G_5 obtained from $(K_1 + kK_1) \cup (K_{k+m} + mK_1)$ by joining each vertex of kK_1 and $(k + m - r)$ vertices of K_{k+m}, and let $S = K_1 \cup kK_1 \cup mK_1$. Then $|V(G_5)| = 2k + 2m + 1$, $\kappa(S) = k$ and $\alpha(S) = k + m$. Since

$$\begin{aligned}\sigma_4(S) &= \min\{k + 3(k + m),\ 4(k + m - r + 1)\}\\ &= 4k + 3m\\ &= |V(G_5)| + \kappa(S) + \alpha(S) - 1,\end{aligned}$$

the assumption of Theorem 12 holds. However, since $k \ge 5$ and $r \ge 4$,

$$\begin{aligned}&|V(G_5)| + (\alpha(S) - 1)^2 - (\alpha(S) - 1) - \sigma_{\alpha(S)}\\ &= (2k + 2m + 1) + (k + m - 1)(k + m - 2) - \{k(k + m - r + 1) + m(k + m)\}\\ &= kr - 2k - m + 3\\ &= (k - 4)(r - 2) - 1 > 0.\end{aligned}$$

Hence the assumption of Theorem 15 does not hold for $l = \alpha(S) - 1$.

3. Proof of Theorem 12

For standard graph-theoretic terminology not explained in this paper, we refer the reader to [3]. Let G be a graph and H be a subgraph of G, and let $x \in V(G)$ and $X \subset V(G)$. We denote by $N_G(x)$ and $N_G(X)$ the neighborhood in G of x and the set of vertices in $V(G - X)$ which are adjacent to some vertex in X, respectively. We define $N_H(x) := N_G(x) \cap V(H)$ and $d_H(x) := |N_H(x)|$. Furthermore, we define $N_H(X) := N_G(X) \cap V(H)$. If there is no fear of confusion, we often identify H with its vertex set $V(H)$. For example, we often write $G - H$ instead of $G - V(H)$.

Let C be a cycle in G. We give an orientation to C and write the oriented cycle C by $\overrightarrow{C}$. For $x, y \in V(C)$, we denote an xy-path on $\overrightarrow{C}$ by $x\overrightarrow{C}y$, and write the reverse sequence of $x\overrightarrow{C}y$ by $y\overleftarrow{C}x$. For $x \in V(C)$, we denote the h-th successor and the h-th predecessor of x on $\overrightarrow{C}$ by x^{+h} and x^{-h}, respectively. For $X \subset V(C)$, we define $X^{+h} := \{x^{+h} : x \in X\}$ and $X^{-h} := \{x^{-h} : x \in X\}$. We often write x^+, x^-, X^+ and X^- for x^{+1}, x^{-1}, X^{+1} and X^{-1}, respectively.

For a subgraph H, a path P is called an H*-path* if both end vertices of P are contained in H and all internal vertices and all edges of P are not contained in H.

Let G be a graph and H be a subgraph of G and let $x \in V(G-H)$. A subgraph F is called *an* (x, H)*-fan*, if $F = \bigcup_{i=1}^{m} P_i$ such that P_i is a path connecting x and a vertex of H and $V(P_i) \cap V(P_j) = \{x\}$.

Proof of Theorem 12. Let G be a graph and S a subset of $V(G)$ satisfying the assumption of Theorem 12. Let C be a cycle in G. If C contains all vertices of S, then there is nothing to prove. By Theorem 8, we may assume $\alpha(S) \geq \kappa(S) + 1$ and $S \cap V(G-C) \neq \emptyset$, say $x_0 \in S \cap V(G-C)$. By Lemma 9, there exists $T \subset V(G)$ such that $|T| = \kappa(S)$ and T separates two vertices of S. Choose a cycle C, x_0 and an (x_0, C)-fan F so that

(C1) $|V(C) \cap S|$ is as large as possible;
(C2) $x_0 \notin T$ if possible, subject to (C1);
(C3) $|V(C) \cap V(F)|$ is as large as possible, subject to (C2);
(C4) $|V(F)|$ is as small as possible, subject to (C3).

By (C3), note that $|V(C) \cap V(F)| \geq \kappa(S) \geq 3$. Let P_i be a path of F connecting x_0 and u_i, where $u_i \in V(C) \cap V(F)$ $(1 \leq i \leq m)$. Let $x_i \in S$ be the first vertex from u_i along $\overrightarrow{C}$ for each $i = 1, 2, \ldots, m$. By (C1), $u_j \notin V(u_i\overrightarrow{C}x_i)$ and hence $x_i \neq x_j$ for $i \neq j$. Let $X := \{x_1, x_2, \ldots, x_m\}$ and H be a component of $G-C$ such that $x_0 \in V(H)$.

Claim 1. $N_H(x_i) = \emptyset$ for $1 \leq i \leq m$.

Proof. Suppose that $N_H(x_i) \neq \emptyset$. Then there exists a $(C \cup F)$-path Q connecting x_i and $v \in V(F)$. If $v \in V(P_j)$ $(j \neq i)$, then $C' = vQx_i\overrightarrow{C}u_iP_ix_0P_jv$ is a cycle containing $(V(C) \cap S) \cup \{x_0\}$, contradicting (C1).

Therefore we may assume that $v \in V(P_i)$. Let $C' = vQx_i\overrightarrow{C}u_iP_iv$ and $F' = \left(F - x_0P_iu_i\right) \cup x_0P_iv$. Then C' is a cycle with $V(C') \cap S = V(C) \cap S$, and F' is an (x_0, C')-fan with $|V(C) \cap V(F)| = |V(C') \cap V(F')|$ and $|V(F')| < |V(F)|$. This contradicts (C4). Hence $N_H(x_i) = \emptyset$ for $1 \leq i \leq m$. □

By (C1), we obtain the following claim.

Claim 2. For $1 \leq i \neq j \leq m$, the following statements hold.

(i) For any $v \in V(u_j^+\overrightarrow{C}x_j)$, there exists no C-path connecting x_i and v.

(ii) For any $w_1 \in V(x_i^+\overrightarrow{C}u_j)$ and $w_2 \in V(x_i^+\overrightarrow{C}w_1^-)$ with $V(w_2^+\overrightarrow{C}w_1^-) \cap S = \emptyset$, if there exists a C-path connecting x_i and w_1, then there exists no C-path connecting x_j and w_2.

By Claims 1 and 2 (i), $X \cup \{x_0\}$ is an independent set in $G[S]$, and hence $|X| \leq \alpha(S) - 1$. By (C3), $d_C(x_0) \leq |X|$. Therefore we have

$$d_C(x_0) \leq \alpha(S) - 1. \tag{1}$$

Let $x_1, x_2, x_3 \in X$ be three distinct vertices such that x_1, x_2 and x_3 appear in the consecutive order along $\overrightarrow{C}$, where the indices are taken modulo 3. Let $D_i := u_i^+ \overrightarrow{C} x_i^-$, $C_i := x_i \overrightarrow{C} u_{i+1}$, $W_i := \{w \in V(C_i) : w^+ \in N_{C_i}(x_i) \text{ and } w^- \in N_{C_i}(x_{i+1})\}$ for each $i = 1, 2, 3$ and let $W := W_1 \cup W_2 \cup W_3$. Note that $x_0, x_1, x_2, x_3 \notin W$.

Claim 3. $W \subset S$. Moreover, if $x_0 \in T$, then $W \subset T$.

Proof. Let $w \in W$. Without loss of generality, we may assume that $w \in W_1$. Then $x_1 w^+ \overrightarrow{C} u_2 P_2 x_0 P_1 u_1 \overleftarrow{C} x_2 w^- \overleftarrow{C} x_1$ is a cycle containing $((V(C) \cap S) \cup \{x_0\}) - \{w\}$. By (C1), we have $w \in S$. Therefore $W \subset S$. Moreover, if $x_0 \in T$, then $w \in T$ by (C2). Hence $W \subset T$. □

By Claim 2 (i), we obtain

$$d_{D_i}(x_j) = 0 \quad \text{for } 1 \leq i \neq j \leq 3 \tag{2}$$

and hence

$$\sum_{j=1}^{3} d_{D_i}(x_j) \leq |V(D_i)| \quad \text{for } 1 \leq i \leq 3.$$

By Claim 2 (ii), $N_{C_i}(x_i)^- \cap N_{C_i}(x_{i+2}) = \emptyset$ and $N_{C_i}(x_{i+1})^+ \cap N_{C_i}(x_{i+2}) = \emptyset$ for $i = 1, 2, 3$. Clearly, $N_{C_i}(x_i)^- \cap N_{C_i}(x_{i+1})^+ = W_i$ and $N_{C_i}(x_i)^- \cup N_{C_i}(x_{i+1})^+ \cup N_{C_i}(x_{i+2}) \subset V(C_i) \cup \{u_{i+1}^+\}$. Therefore for $i = 1, 2, 3$,

$$d_{C_i}(x_1) + d_{C_i}(x_2) + d_{C_i}(x_3) \leq |C_i| + 1 + |W_i|.$$

Thus, we deduce

$$\begin{aligned} d_C(x_1) + d_C(x_2) + d_C(x_3) &= \sum_{i=1}^{3} \sum_{j=1}^{3} (d_{D_i}(x_j) + d_{C_i}(x_j)) \\ &\leq \sum_{i=1}^{3} (|V(D_i)| + |V(C_i)| + 1 + |W_i|) \\ &= |V(C)| + |W| + 3. \end{aligned} \tag{3}$$

By Claim 2 (i), $N_{G-C-H}(x_i) \cap N_{G-C-H}(x_j) = \emptyset$ for $1 \leq i \neq j \leq 3$. Therefore by Claim 1,

$$\begin{aligned} &d_{G-C}(x_0) + d_{G-C}(x_1) + d_{G-C}(x_2) + d_{G-C}(x_3) \\ &\leq |V(H) - \{x_0\}| + |V(G - C - H)| = |V(G - C)| - 1. \end{aligned} \tag{4}$$

Claim 4. $|X| \geq \kappa(S) + 1$.

Proof. By Claim 3, $W \subset S$. We prove that W is an independent set. Assume that there exist $w_1 \in W_i$ and $w_2 \in W_j$ with $w_1 w_2 \in E(G)$. Suppose first that $i = j$. Without loss of generality, we may assume that $i = j = 1$, and w_1 and w_2 appear in this order along $\overrightarrow{C_1}$. Then $C' = x_1 w_1^+ \overrightarrow{C} w_2^- x_2 \overrightarrow{C} u_1 P_1 x_0 P_2 u_2 \overleftarrow{C} w_2 w_1 \overleftarrow{C} x_1$ is a cycle such that $|V(C') \cap S| > |V(C) \cap S|$, contradicting (C1). We may now assume that $i \neq j$. Without loss of generality, we may assume that $i = 1$ and $j = 2$. Then $x_1 w_1^+ \overrightarrow{C} u_2 P_2 x_0 P_1 u_1 \overleftarrow{C} w_2^+ x_2 \overrightarrow{C} w_2 w_1 \overleftarrow{C} x_1$ is a cycle containing $(V(C) \cap S) \cup \{x_0\}$, a contradiction. Hence W is an independent set in $G[S]$. By Claim 2, $W \cup X \cup \{x_0\}$ is an independent set in $G[S]$. Since $x_0, x_1, x_2, x_3 \notin W$, we obtain $\alpha(S) \geq |W \cup X \cup \{x_0\}| \geq |W| + 4$, and hence $|W| \leq \alpha(S) - 4$.

By the inequality (3), we deduce

$$\begin{aligned} d_C(x_1) + d_C(x_2) + d_C(x_3) &\leq |V(C)| + |W| + 3 \\ &\leq |V(C)| + (\alpha(S) - 4) + 3 \\ &= |V(C)| + \alpha(S) - 1. \end{aligned}$$

Thus, it follows from the inequality (4) that $d_G(x_0) + d_G(x_1) + d_G(x_2) + d_G(x_3) \leq d_C(x_0) + n + \alpha(S) - 2$. Since $\sigma_4(S) \geq n + \kappa(S) + \alpha(S) - 1$, we have $d_C(x_0) \geq \kappa(S) + 1$. Hence $|X| \geq \kappa(S) + 1$. □

Let $U_1, U_2, \ldots, U_p$ be the components of $G - T$. We show that $|\{U_i : X \cap U_i \neq \emptyset\}| \leq 2$. Suppose that $|\{U_i : X \cap U_i \neq \emptyset\}| \geq 3$. Without loss of generality, we may assume that $x_i \in X \cap U_i$ for $i = 1, 2, 3$. By Claims 1 and 2 (i), we have

$$d_G(x_i) \leq |U_i| + |T| - |(U_i \cup T) \cap (V(H) \cup X)|.$$

Thus, by Claim 4, we obtain

$$\begin{aligned} & d_G(x_1) + d_G(x_2) + d_G(x_3) \\ & \leq \sum_{i=1}^{3} |U_i| + 3|T| - \sum_{i=1}^{3} |(U_i \cup T) \cap (V(H) \cup X)| \\ & = n + 2|T| - \sum_{i=4}^{p} |U_i| - \sum_{i=1}^{3} |(U_i \cup T) \cap (V(H) \cup X)| \\ & \leq n + 2\kappa(S) - (|V(H)| + |X|) \\ & \leq n + \kappa(S) - |V(H)| - 1 \\ & \leq n + \kappa(S) - d_H(x_0) - 2. \end{aligned}$$

By the inequality (1), $d_G(x_0) + d_G(x_1) + d_G(x_2) + d_G(x_3) \leq n + \kappa(S) + \alpha(S) - 3$, a contradiction. Hence, without loss of generality, we may assume that $X \cap \bigcup_{h=3}^{p} U_h = \emptyset$ and $|X \cap U_1| \geq |X \cap U_2|$.

Claim 5. $|W| \geq \kappa(S) - 2$.

Proof. Suppose that $|W| \leq \kappa(S) - 3$. By the inequality (3), we obtain

$$d_C(x_1) + d_C(x_2) + d_C(x_3) \leq |V(C)| + \kappa(S).$$

Hence the inequalities (1) and (4) yield

$$\begin{aligned} &d_G(x_0) + d_G(x_1) + d_G(x_2) + d_G(x_3) \\ &\leq |V(C)| + \kappa(S) + \alpha(S) - 1 + |V(G - C)| - 1 \\ &\leq n + \kappa(S) + \alpha(S) - 2, \end{aligned}$$

a contradiction. □

Claim 6. $x_0 \notin T$ or $|X \cap T| \leq 1$.

Proof. Suppose that $x_0 \in T$ and $|X \cap T| \geq 2$. By Claim 3, $W \subset T$. Since $|X \cap T| \geq 2$, we may assume that x_1, x_2 and x_3 are chosen so that $x_1, x_2 \in X \cap T$. Since $x_0, x_1, x_2 \in T - W$, we obtain $|W| \leq \kappa(S) - 3$, a contradiction. □

Claim 7. $|X \cap U_1| \geq 2$.

Proof. First we prove that $|X - T| \geq 2$. Suppose that $|X - T| \leq 1$. Then by Claim 4, note that $T \subset X$ and $|X - T| = 1$. Since $G[V(C) \cup V(H)] - T$ is connected, we have $(C \cup H) - T \subset U_1$ and $\bigcup_{h=2}^{p} U_h \subset G - (C \cup H)$. Since T is a separating set of S, $U_i \cap S \neq \emptyset$ for some i, $2 \leq i \leq p$. Thus, $\kappa(S) \geq 3$ implies that $|N_C(U_i) \cap T| \geq 3$, that is, $|N_C(U_i) \cap X| \geq 3$. This contradicts Claim 2 (i). Therefore $|X - T| \geq 2$.

Suppose that $|X \cap U_1| \leq 1$, that is, $|X \cap U_1| = |X \cap U_2| = 1$ and $|X - T| = 2$. By symetry, we can assume that x_1 and x_2 are chosen so that $x_1 \in X \cap U_1$ and $x_2 \in X \cap U_2$. By Claim 4, we have $|X \cap T| = |X| - |X - T| \geq (\kappa(S) + 1) - 2 = \kappa(S) - 1 \geq 2$. Also, we have $|T - X| = |T| - |X \cap T| \leq \kappa(S) - \big(\kappa(S) - 1\big) = 1$. Let Q_1 be an x_1x_2-path in $x_1 \overrightarrow{C} u_{\tau(1)} P_{\tau(1)} x_0 P_2 u_2 \overrightarrow{C} x_2$ and Q_2 be an x_2x_1-path in $x_2 \overrightarrow{C} u_{\tau(2)} P_{\tau(2)} x_0 P_1 u_1 \overrightarrow{C} x_1$, where $\tau(i)$ is an integer with $V(x_i^+ \overrightarrow{C} u_{\tau(i)}) \cap X = \emptyset$. Since $x_1 \in U_1$ and $x_2 \in U_2$, we have $Q_1 \cap T \neq \emptyset$ and $Q_2 \cap T \neq \emptyset$. Moreover, since $Q_1 \cap X = Q_2 \cap X = \{x_1, x_2\}$, we have $Q_1 \cap (T - X) \neq \emptyset$ and $Q_2 \cap (T - X) \neq \emptyset$. Since $|T - X| \leq 1$, we have $x_0 \in T$, which contradicts Claim 6. □

Without loss of generality, we can assume $x_1, x_2 \in X \cap U_1$ and $x_3 \in X$. Since $x_1, x_2 \in U_1$, we have $N_{D_i}(x_i) \subset V(D_i) \cap (U_1 \cup T)$ for $i = 1, 2$. Therefore by the inequality (2), we obtain

$$d_{D_i}(x_1) + d_{D_i}(x_2) \leq |V(D_i) \cap U_1| + |V(D_i) \cap T| \quad \text{for } i = 1, 2. \tag{5}$$

Let $A_i := \{z \in V(C) \cap U_2 \colon z^+ \in N_C(x_i)\}$ for $i = 1, 2, 3$, and let $B_1 := \{z \in V(C) \cap U_2 \colon z^- \in N_C(x_1)\}$.

Claim 8. $X \subset U_1 \cup T$.

Proof. Suppose that $X \cap U_2 \neq \emptyset$. We may assume that $x_3 \in X \cap U_2$. By Claim 2 (ii), we obtain the following statements.

(I) $N_{C_1}(x_1)^-$ and $N_{C_1}(x_2)$ are disjoint, and $N_{C_1}(x_1)^- \cup N_{C_1}(x_2) \subseteq V(C_1) \cap (U_1 \cup T \cup A_1 \cup \bigcup_{h=3}^{p} U_h)$.
(II) $N_{C_2}(x_2)^-$ and $N_{C_2}(x_1)$ are disjoint, and $N_{C_2}(x_2)^- \cup N_{C_2}(x_1) \subseteq V(C_2) \cap (U_1 \cup T \cup A_2 \cup \bigcup_{h=3}^{p} U_h)$.
(III) $N_{C_3}(x_1)^+$ and $N_{C_3}(x_2)$ are disjoint, and $N_{C_3}(x_1)^+ \cup N_{C_3}(x_2) \subseteq (V(C_3) \cap (U_1 \cup T \cup B_1 \cup \bigcup_{h=3}^{p} U_h)) \cup \{u_1^+\}$.

Let $A := (V(C_1) \cap A_1) \cup (V(C_2) \cap A_2) \cup (V(C_3) \cap B_1)$. By (I)–(III) and by the inequalities (2) and (5), we obtain

$$d_C(x_1) + d_C(x_2) \leq \sum_{h \neq 2} |V(C) \cap U_h| + |V(C) \cap T| + |A| + 1.$$

On the other hand, by Claim 2 (ii), x_3 is not adjacent to any vertex of A. Thus, we have

$$d_C(x_3) \leq |V(C) \cap U_2| + |V(C) \cap T| - |A| - 1,$$

since $x_3 \notin A$. Thus $d_C(x_1) + d_C(x_2) + d_C(x_3) \leq |V(C)| + |V(C) \cap T| \leq |V(C)| + \kappa(S)$. Therefore, by the inequalities (1) and (4),

$$\begin{aligned} &d_G(x_0) + d_G(x_1) + d_G(x_2) + d_G(x_3) \\ &\leq |V(C)| + \kappa(S) + \alpha(S) - 1 + |V(G - C)| - 1 \\ &= n + \kappa(S) + \alpha(S) - 2, \end{aligned}$$

a contradiction. □

Claim 9. $x_0 \in U_1$.

Proof. By Claims 4 and 8, there exist $|X| \geq \kappa(S) + 1$ paths connecting x_0 and each vertex in $X \subset U_1 \cup T$, and hence $x_0 \in U_1 \cup T$. Suppose that $x_0 \in T$. Note that $W \subset T$ by Claim 3. By (C2), $V(G - C) \cap U_2 \cap S = \emptyset$, otherwise we can choose a vertex in $V(G - C) \cap U_2 \cap S$ instead of x_0. Let $y \in V(C) \cap U_2 \cap S$. Then by Claim 8, there exist $t_1, t_2 \in V(C) \cap T$ such that $y \in V(t_1^+ \overrightarrow{C} t_2^-)$ and $V(t_1^+ \overrightarrow{C} t_2^-) \subset U_2$, because $x_1, x_2 \in X \cap U_1$.

By Claim 8, $X \cap U_1 = X - T$. By Claim 6, $|X \cap T| \leq 1$, and hence $|X - T| = |X| - |X \cap T| \geq \kappa(S) \geq 3$. Thus we have $|X \cap U_1| \geq 3$. Therefore we may assume that $x_3 \in X \cap U_1$. Since $x_1, x_2 \in X \cap U_1$ and $t_1^+, t_2^- \in U_2$, we have $t_1, t_2 \in T - W$. Thus, $|W| \leq |T - \{x_0, t_1, t_2\}| = \kappa(S) - 3$, contradicting Claim 5. □

By Claims 1 and 2 (i), $N_H(x_i) = \emptyset$ for $i = 1, 2$ and $N_{G-C}(x_1) \cap N_{G-C}(x_2) = \emptyset$. Therefore $d_{G-C}(x_1) + d_{G-C}(x_2) \leq |V(G - C - H) \cap (U_1 \cup T)|$. By Claim 9, we have $d_{G-C}(x_0) \leq |V(H) \cap (U_1 \cup T)| - 1$. Thus,

$$\begin{aligned} &d_{G-C}(x_0) + d_{G-C}(x_1) + d_{G-C}(x_2) \\ &\leq |V(G - C) \cap U_1| + |V(G - C) \cap T| - 1. \end{aligned} \tag{6}$$

Let $y_0 \in U_2 \cap S$. Then

$$d_G(y_0) \leq |U_2| + |T| - 1 = |U_2| + \kappa(S) - 1, \tag{7}$$

and $y_0 \notin N_G(x_0) \cup N_G(x_1) \cup N_G(x_2)$ by Claim 9. Let $C'_1 := C_1 = x_1 \overrightarrow{C} u_2$ and $C'_2 := C_2 \cup D_3 \cup C_3 = x_2 \overrightarrow{C} u_1$.

Based on the results of the previous claims, the proof is completed by considering two cases for the cardinality of $X \cap U_1$: $|X \cap U_1| = 2$ and $|X \cap U_1| = 3$.

Case 1. $|X \cap U_1| = 2$.

By the definition of A_i and Claim 2 (i), for $i = 1, 2$, $A_i^+ \subset T$ and $A_i^+ \cap X = \emptyset$, and hence $A_i^+ \subset T - X$. Moreover, by Claims 4 and 8 and by the assumption of Case 1, $\kappa(S) - 1 \leq |X \cap T|$. Hence we have $|V(C'_1) \cap A_1| + |V(C'_2) \cap A_2| \leq |T - X| = |T| - |X \cap T| \leq 1$.

By Claim 2 (ii), we obtain the following statements.

(I) $N_{C'_1}(x_1)^-$ and $N_{C'_1}(x_2)$ are disjoint, and $N_{C'_1}(x_1)^- \cup N_{C'_1}(x_2) \subseteq V(C'_1) \cap (U_1 \cup T \cup A_1 \cup \bigcup_{h=3}^{p} U_h)$.

(II) $N_{C'_2}(x_2)^-$ and $N_{C'_2}(x_1)$ are disjoint, and $N_{C'_2}(x_2)^- \cup N_{C'_2}(x_1) \subseteq V(C'_2) \cap (U_1 \cup T \cup A_2 \cup \bigcup_{h=3}^{p} U_h)$.

By (I) and (II) and by the inequality (5), we have

$$\begin{aligned} & d_C(x_1) + d_C(x_2) \\ & \leq \sum_{h \neq 2} |V(C) \cap U_h| + |V(C) \cap T| + |V(C'_1) \cap A_1| + |V(C'_2) \cap A_2| \\ & \leq \sum_{h \neq 2} |V(C) \cap U_h| + |V(C) \cap T| + 1. \end{aligned}$$

Combining with the inequalities (1) and (6), we obtain $d_G(x_0) + d_G(x_1) + d_G(x_2) \leq \sum_{h \neq 2} |U_h| + |T| + \alpha(S) - 1$. Then by the inequality (7), we have $d_G(x_0) + d_G(y_0) + d_G(x_1) + d_G(x_2) \leq |V(G)| + \kappa(S) + \alpha(S) - 2$, a contradiction. □

Case 2. $|X \cap U_1| \geq 3$.

We may assume that $x_3 \in X \cap U_1$. For each $z \in A_i$, we define $\tilde{z}$ to be the vertex satisfying $\tilde{z} \in V(C) \cap T$ and $V(\tilde{z}^+ \overrightarrow{C} z) \subset U_2$. Since $x_i \in U_1$, note that $\tilde{z} \in V(x_i^+ \overrightarrow{C} z^-)$ for $i = 1, 2, 3$. Let $\tilde{A}_i = \{\tilde{z} \colon z \in A_i\}$ for $i = 1, 2, 3$.

Claim 10. Let $z \in A_i$. If $|X \cap U_1 \cap V(z^+ \overrightarrow{C} u_i)| \geq 2$, then $V(\tilde{z}^+ \overrightarrow{C} z) \cap S = \emptyset$.

Proof. By symmetry, we may assume that there exists $z_3 \in A_3$ such that $|X \cap U_1 \cap V(z_3^+ \overrightarrow{C} u_3)| \geq 2$ and $V(\tilde{z}_3^+ \overrightarrow{C} z_3) \cap S \neq \emptyset$. Let $y_3 \in V(\tilde{z}_3^+ \overrightarrow{C} z_3) \cap S$. Choose y_3 so that $|V(y_3 \overrightarrow{C} z_3)|$ is as small as possible. Then note that $y_3 \in U_2$. Since $|X \cap U_1 \cap V(z_3^+ \overrightarrow{C} u_3)| \geq 2$, we may assume that $x_1, x_2 \in X \cap U_1 \cap V(z_3^+ \overrightarrow{C} u_3)$. We partition C_3 into F_1, F_2, F_3 so that $F_1 := x_3 \overrightarrow{C} \tilde{z}_3$ $F_2 := \tilde{z}_3^+ \overrightarrow{C} z_3$ and $F_3 := z_3^+ \overrightarrow{C} u_1$. Note that $V(F_2) \subset U_2$ and x_i has no neighbors in U_2 for $i = 1, 2$.

By Claim 2 (ii), we obtain the following statements.

(I) $N_{C_1}(x_1)^-$ and $N_{C_1}(x_2)$ are disjoint, and $N_{C_1}(x_1)^- \cup N_{C_1}(x_2) \subseteq V(C_1) \cap (U_1 \cup T \cup A_1 \cup \bigcup_{h=3}^{p} U_h)$.
(II) $N_{C_2}(x_2)^-$ and $N_{C_2}(x_1)$ are disjoint, and $N_{C_2}(x_2)^- \cup N_{C_2}(x_1) \subseteq V(C_2) \cap (U_1 \cup T \cup A_2 \cup \bigcup_{h=3}^{p} U_h)$.
(III) $N_{F_1}(x_2)^-$ and $N_{F_1}(x_1)$ are disjoint, and $N_{F_1}(x_2)^- \cup N_{F_1}(x_1) \subseteq V(F_1) \cap (U_1 \cup T \cup A_2 \cup \bigcup_{h=3}^{p} U_h)$.
(IV) $N_{F_2}(x_i) = \emptyset$ for $i = 1, 2$.
(V) $N_{F_3}(x_1)^+$ and $N_{F_3}(x_2)$ are disjoint, and $N_{F_3}(x_1)^+ \cup N_{F_3}(x_2) \subseteq (V(F_3) \cap (U_1 \cup T \cup B_1 \cup \bigcup_{h=3}^{p} U_h)) \cup \{u_1^+\}$.

Let $A' := (V(C_1) \cap A_1) \cup (V(C_2) \cap A_2) \cup (V(F_1) \cap A_2) \cup (V(F_3) \cap B_1)$. By (I)–(V) and by the inequalities (2) and (5), we obtain

$$d_C(x_1) + d_C(x_2) \leq \sum_{h \neq 2} |V(C) \cap U_h| + |V(C) \cap V(T)| + |A'| + 1.$$

Suppose that $A' \cap N_C(y_3) \neq \emptyset$, say $z \in A' \cap N_C(y_3)$. Let

$$C' = \begin{cases} x_1 z^+ \overrightarrow{C} u_3 P_3 x_0 P_1 u_1 \overleftarrow{C} z_3^+ x_3 \overrightarrow{C} y_3 z \overleftarrow{C} x_1 & \text{if } z \in V(C_1) \cap A_1, \\ x_2 z^+ \overrightarrow{C} u_3 P_3 x_0 P_1 u_2 \overleftarrow{C} z_3^+ x_3 \overrightarrow{C} y_3 z \overleftarrow{C} x_2 & \text{if } z \in V(C_2) \cap A_2, \\ x_2 \overrightarrow{C} u_3 P_3 x_0 P_2 u_2 \overleftarrow{C} z_3^+ x_3 \overrightarrow{C} z y_3 \overleftarrow{C} z^+ x_2 & \text{if } z \in V(F_1) \cap A_2, \\ x_1 \overrightarrow{C} u_3 P_3 x_0 P_1 u_1 \overleftarrow{C} z y_3 \overleftarrow{C} x_3 z_3^+ \overrightarrow{C} z^- x_1 & \text{if } z \in V(F_3) \cap B_1. \end{cases}$$

Note that by the choice of y_3, there are no vertices of S between y_3 and z_3. Then C' is a cycle containing $(V(C) \cap S) \cup \{x_0\}$, a contradiction. Hence $A' \cap N_C(y_3) = \emptyset$. Moreover, by the definition of A', we have $y_3 \notin A'$. Therefore we obtain

$$d_C(y_3) \leq |V(C) \cap U_2| + |V(C) \cap T| - |A'| - 1,$$

which implies

$$\begin{aligned} d_C(x_1) + d_C(x_2) + d_C(y_3) &\leq |V(C)| + |V(C) \cap T| \\ &\leq |V(C)| + \kappa(S). \end{aligned} \tag{8}$$

By Claim 2 (ii), $N_{G-C}(x_i) \cap N_{G-C}(y_3) = \emptyset$ for $i = 1, 2$. On the other hand, by a similar argument as in the proof of Claim 1, we obtain $N_H(y_3) = \emptyset$. Hence

$$d_{G-C}(x_0) + d_{G-C}(x_1) + d_{G-C}(x_2) + d_{G-C}(y_3) \leq |V(G - C)| - 1. \tag{9}$$

Therefore, by the inequalities (1), (8) and (9),

$$\begin{aligned} &d_G(x_0) + d_G(x_1) + d_G(x_2) + d_G(y_3) \\ &\leq |V(C)| + \kappa(S) + \alpha(S) - 1 + |V(G - C)| - 1 \\ &\leq n + \kappa(S) + \alpha(S) - 2, \end{aligned}$$

a contradiction. □

Claim 11. Let $z \in A_i$. If $|X \cap U_1 \cap V(z^+ \overrightarrow{C} u_i)| \geq 2$, then $\tilde{z} \notin N_C(x_i)^- \cup N_C(x_j)$ for any $x_j \in X \cap U_1 \cap V(z^+ \overrightarrow{C} u_i)$.

Proof. By Claim 10, $V(\tilde{z}^+ \overrightarrow{C} z) \cap S = \emptyset$. Hence, by Claim 2 (ii), we have $\tilde{z} \notin N_C(x_j)$. On the other hand, since $x_i \in U_1$ and $\tilde{z}^+ \in U_2$, we have $\tilde{z} \notin N_C(x_i)^-$. Thus, we obtain $\tilde{z} \notin N_C(x_i)^- \cup N_C(x_j)$. □

Case 2.1. $|X \cap U_1| = 3$.

By Claims 4 and 8, we have $|T - X| = |T| - |T \cap X| = |T| - (|X| - |X \cap U_1|) \leq \kappa(S) - (\kappa(S) + 1 - 3) = 2$. Therefore there exists an index i such that $V(C_i) \cap (T - X) = \emptyset$. By symmetry, we may assume that $i = 3$. Then by the definition of A_2, $V(C_3) \cap A_2 = \emptyset$. Recall that $\tilde{A}_i \subset T$. By Claims 2 (ii) and 11, we obtain

(I) $N_{C_1}(x_1)^-$ and $N_{C_1}(x_2)$ are disjoint, and $N_{C_1}(x_1)^- \cup N_{C_1}(x_2) \subseteq V(C_1) \cap (U_1 \cup (T - \tilde{A}_1) \cup A_1 \cup \bigcup_{h=3}^{p} U_h)$.
(II) $N_{C_2}(x_2)^-$ and $N_{C_2}(x_1)$ are disjoint, and $N_{C_2}(x_2)^- \cup N_{C_2}(x_1) \subseteq V(C_2) \cap (U_1 \cup (T - \tilde{A}_2) \cup A_2 \cup \bigcup_{h=3}^{p} U_h)$.
(III) $N_{C_3}(x_2)^-$ and $N_{C_3}(x_1)$ are disjoint, and $N_{C_3}(x_2)^- \cup N_{C_3}(x_1) \subseteq V(C_3) \cap (U_1 \cup T)$.

By (I)–(III) and by the inequalities (2) and (5), we have

$$d_C(x_1) + d_C(x_2) \leq \sum_{h \neq 2} |V(C) \cap U_h| + |V(C) \cap T|.$$

By the inequalities (1), (6) and (7), $d_G(x_0) + d_G(y_0) + d_G(x_1) + d_G(x_2) \leq |V(G)| + \kappa(S) + \alpha(S) - 3$, a contradiction. □

Case 2.2. $|X \cap U_1| \geq 4$.

Since $|X \cap U_1| \geq 4$, we can choose $x_1, x_2 \in X \cap U_1$ so that $V(x_1^+ \overrightarrow{C} x_2^-) \cap X \cap U_1 \neq \emptyset$ and $V(x_2^+ \overrightarrow{C} x_1^-) \cap X \cap U_1 \neq \emptyset$. By Claims 2 (ii) and 11, we obtain

(I) $N_{C'_1}(x_1)^-$ and $N_{C'_1}(x_2)$ are disjoint, and $N_{C'_1}(x_1)^- \cup N_{C'_1}(x_2) \subseteq V(C'_1) \cap (U_1 \cup (T - \tilde{A}_1) \cup A_1 \cup \bigcup_{h=3}^{p} U_h)$.
(II) $N_{C'_2}(x_2)^-$ and $N_{C'_2}(x_1)$ are disjoint, and $N_{C'_2}(x_2)^- \cup N_{C'_2}(x_1) \subseteq V(C'_2) \cap (U_1 \cup (T - \tilde{A}_2) \cup A_2 \cup \bigcup_{h=3}^{p} U_h)$.

By (I) and (II) and by the inequality (5), we obtain

$$d_C(x_1) + d_C(x_2) \leq \sum_{h \neq 2} |V(C) \cap U_h| + |V(C) \cap T|.$$

By the inequalities (1), (6) and (7), $d_G(x_0) + d_G(y_0) + d_G(x_1) + d_G(x_2) \leq |V(G)| + \kappa(S) + \alpha(S) - 3$, a contradiction. □

4. Conclusions

In [2], Bondy showed that if $\sigma_2(G) \geq n$ then $\alpha(G) \leq \kappa(G)$. A similar proof yields the following by using Theorem 8 and Lemma 9. The degree condition is best possible by considering the graph $G_2 = K_m + (m+1)K_1$ and the vertex set $S = (m+1)K_1$.

Theorem 17. *Let G be a graph on n vertices, and $S \subset V(G)$. If $\sigma_2(S) \geq n + \kappa(S) - \alpha(S) + 1$, then S is cyclable in G.*

We compare the value of the degree sum condition of Theorems 17, 10 and 12. That is

$$\begin{aligned} &\text{in Theorem 17,} \quad && \sigma_2 \geq n + \kappa(S) - \alpha(S) + 1, \\ &\text{in Theorem 10,} \quad && \sigma_3 \geq n + \kappa(S), \\ \text{and } &\text{in Theorem 12,} \quad && \sigma_4 \geq n + \kappa(S) + \alpha(S) - 1. \end{aligned}$$

In these theorems, the change from a $\sigma_2(S)$ condition to a $\sigma_3(S)$ condition causes the value which is needed to guarantee cyclability for S to increase by $\alpha(S) - 1$, and again, the change from a $\sigma_3(S)$ condition to a $\sigma_4(S)$ condition causes the value to increase by $\alpha(S) - 1$. It suggests that the "$\alpha(S) - 1$ rule" holds. Therefore, by this rule, we pose the following problem.

Problem 18. *Let G be a graph on n vertices, and $S \subset V(G)$ with $\kappa(S) \geq 4$. If $\sigma_5(S) \geq n + \kappa(S) + 2\alpha(S) - 2$, then S is cyclable in G.*

Moreover, the following statement may hold. When $k = 1, 2, 3$, this is true by Theorem 17, 10 and 12, respectivily.

Problem 19. *Let G be a graph on n vertices, and $S \subset V(G)$ with $\kappa(S) \geq k \geq 1$. If $\sigma_{k+1}(S) \geq n + \kappa(S) + (k-2)(\alpha(S) - 1)$, then S is cyclable in G.*

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