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Equitable Specialized Block-Colourings for Steiner Triple Systems

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Abstract. We continue the study of specialized block-colourings of Steiner triple systems initiated in [2] in which the triples through any element are coloured according to a given partition π of the replication number. Such colourings are equitable if π is an equitable partition (i.e., the difference between any two parts of π is at most one). Our main results deal with colourings according to equitable partitions into two, and three parts, respectively.

Key words. Steiner triple systems, block-colourings, equitable.

1. Introduction

A *Steiner triple system of order* v (STS(v)) is a pair $(V, \mathcal{B})$ where V is v-set of *elements* and $\mathcal{B}$ is a family of 3-subsets of V called *triples* such that every 2-subset of V is contained in exactly one triple of $\mathcal{B}$. It is well known that an STS(v) exists if and only if $v \equiv 1$ or 3 (*mod* 6) [1]. Every element of an STS(V) is contained in $r = \frac{v-1}{2}$ triples; r is called the *replication number*.

An STS(v) $(V, \mathcal{B})$ is *cyclic* if it admits an automorphism α consisting of a single cycle of length v which preserves $\mathcal{B}$.

A *block-colouring* of an STS(v) $(V, \mathcal{B})$ is a mapping $\phi : \mathcal{B} \to C$ where C is a set of colours. A k-block-colouring (or simply a k-colouring) is a block-colouring using k colours; each of the k colours *must* be used. For each $i = 1, \dots, k$, the subset $\mathcal{B}_i$ of $\mathcal{B}$ containing the blocks coloured with colour i is a *colour class*.

For a partition $\pi = \{\pi_1, \pi_2, \dots, \pi_s\}$ of the replication number r, a k-colouring of *type* π is a colouring of triples such that for each element $v \in V$, the triples containing v are partitioned according to π, that is, there are π_1 triples of one colour, π_2 triples of a different colour, and so on.

For an STS(v) $S = (V, \mathcal{B})$ and a partition π of r, we define the *colour spectrum* $\Omega_\pi(S) = \{k$: there exists a k-block-colouring of type π of $S\}$, and also define $\Omega_\pi(v) = \cup\Omega_\pi(S)$ where π is a partition of r into s parts, $s > 1$, and where the union is taken over the set of all STS(v).

From now on, we will simply write k-colouring instead of k-block-colouring since only block-colourings will be considered.

The *upper* π*-chromatic index* $\bar{\chi}'_\pi(S)$ and the *lower* π*-chromatic index* $\chi'_\pi(S)$ of S is defined as

$\chi'_\pi(S) = min\ \Omega_\pi(S)$, $\quad \bar{\chi}'_\pi(S) = max\ \Omega_\pi(S)$, and similarly,
$\chi'_\pi(v) = min\ \Omega_\pi(v)$, $\quad \bar{\chi}'_\pi(v) = max\ \Omega_\pi(v)$.

A partition $\pi = (\pi_1, \ldots, \pi_s)$ of r is *equitable* if $|\pi_i - \pi_j| \leq 1$ for all $i, j = 1, \ldots, s$, $i \neq j$. A colouring of type π is *equitable* if π is an equitable partition.

The organization of this paper is as follows. In Section 2 we prove some general results concerning equitable colourings when all parts of the partition π are equal. Section 3 deals with equitable colourings when π consists of two parts. In particular, it is shown that in this case a colouring, if it exists, must use exactly two colours. Section 4 treats the case when π has three parts.

2. Some Uniform Colourings

In this section we consider equitable colourings of type $\pi = (t, t, \ldots, t)$ where π is a partition of r into s parts, $s > 1$, thus $s.t = \frac{v-1}{2}$, so if $v = 6n + 1$ then s must divide $3n$ while if $v = 6n + 3$ then s must divide $3n + 1$. The following theorem gives an upper bound on the upper π-chromatic index.

Theorem 1. *If an STS(v) $S = (V, \mathcal{B})$ admits a π-colouring with π as above, then any colour class contains at least $\frac{(v+s-1)(v-1)}{6s^2}$ triples, and*

$$\bar{\chi}'_\pi(S) \leq s^2 - 1.$$

Proof. Let $c \in C$ be a colour, and let $x \in V$ be an element incident with triples of colour c. There are $\frac{v-1}{2s}$ triples of colour c incident with x; thus there are at least $1+2\frac{v-1}{2s} = \frac{v+s-1}{s}$ elements in V incident with triples of colour c, i.e. $|V(c)| \geq \frac{v+s-1}{s}$, where $V(c) = \{x : x \in V, c \in C(x)\}$ where c is a colour and $C(x)$ is the set of colours used on triples incident with x. It follows that there are at least $\frac{1}{3}\frac{v+s-1}{s}\frac{v-1}{2s}$ triples of colour c. The first part of the statement follows. Clearly,

$$\Sigma_{c \in C}|V(c)| = sv \tag{1}$$

In an h-colouring of type π, we get from (1) that $h\frac{v+s-1}{s} \leq sv$ which yields $h \leq \lfloor \frac{s^2 v}{v+s-1} \rfloor$. Assuming h is the maximum number of colours in a colouring of type π, we get

$$\bar{\chi}'_\pi(S) \leq \lfloor \frac{s^2 v}{v+s-1} \rfloor \tag{2}$$

Thus for any value of v, we have $\bar{\chi}'_\pi(S) \leq s^2 - 1$. □

3. Equitable Bicolourings

An equitable bicolouring is a colouring of type $\pi = (s, t)$ where $|t - s| \leq 1$; the prefix "bi" refers to the fact that the partition π has exactly two parts. There are only two partitions that satisfy this condition, namely $\pi_1 = (\frac{v-1}{4}, \frac{v-1}{4})$, and $\pi_2 = (\frac{v-3}{4}, \frac{v+1}{4})$.

If an STS(v) has an equitable bicolouring of type π_1 then necessarily $v \equiv 1$ or 9 (*mod* 12). Similarly, if an STS(v) has an equitable bicolouring of type π_2, then $v \equiv 3$ or 7 (*mod* 12).

Lemma 1. *If $S = (V, \mathcal{B})$ is an STS with a colouring of type π_2 then $\bar{\chi}'_{\pi_2}(S) \leq 3$.*

Proof. Assume that there exists a 4-colouring of type π_2. For each colour c we have $|V(c)| \geq 1 + 2\frac{v-3}{4} = \frac{v-1}{2}$. Moreover, if c is a colour such that there is $x \in V$ incident with $\frac{v+1}{4}$ triples of colour c then we can improve the bound to $|V(c)| \geq 1 + 2\frac{v+1}{4} = \frac{v+3}{2}$. Let us call such a colour *rich*. Clearly, there has to be a rich colour as each element x is incident with $\frac{v+1}{4}$ triples of the same colour. Suppose by contradiction that there is only one rich colour, say, c'. Then by (1)

$$2v = \Sigma_{c\in C}|V(c)| = |V(c')| + \Sigma_{c\in C, c\neq c'}|V(c)| \geq v + 3\frac{v-1}{2} = \frac{5v-3}{2},$$

a contradiction for $v > 3$. Thus there are at least two rich colours, say, c', c''. By (1),

$$\begin{aligned}2v &= \Sigma_{c\in C}|V(c)| = |V(c')| + |V(c'')| + \Sigma_{c\in C, c\neq c', c''}|V(c)| \\ &\geq 2\frac{v+3}{2} + 2\frac{v-1}{2} = 2v + 2\end{aligned}$$

a contradiction. The proof of the lemma is complete. □

Theorem 2. *Let $S = (V, \mathcal{B})$, $v > 7$, be an STS which is π_i-colourable, $i \in \{1, 2\}$. Then $\chi'_{\pi_i}(S) = 3$ if and only if $i = 2$, and S contains a sub-STS of order $\frac{v-1}{2}$. Otherwise, $\chi'_{\pi_i}(S) = \bar{\chi}'_{\pi_i}(S) = 2$.*

Proof. Let $S = (V, \mathcal{B})$ be an STS with a subsystem S' on D where $D \subset V$, $|D| = \frac{v-1}{2}$, $D = A \cup B$, with $|A| = \frac{v-3}{4}$ and $|B| = \frac{v+1}{4}$. Let $K = V \setminus D$. Clearly, all other triples of S have two elements in K and one element in D. Colour all triples of S' with colour 1. Consider a 1-factorization F of the complete graph on the elements of K; F consists of $\frac{v-1}{2}$ 1-factors. Assign to each element u of D one of those 1-factors; a triple $\{u, x, y\}$ will be in S if and only if $\{x, y\}$ is an edge of this 1-factor. The triple $\{u, x, y\}$ will be coloured with colour 2 if $u \in A$ and with colour 3 if $u \in B$. It is easy to check that this 3-colouring is of type π_2. By Theorem 1 and Lemma 1, $\bar{\chi}'_{\pi_i}(S) \leq 3$. Hence we only need to show that there is no 3-colouring of S of type π_i where $i = 1$ or 2, except when S contains a sub-STS($\frac{v-1}{2}$). Our proof will be by contradiction. Assume that S has a 3-colouring of type π_i. Each element of V is incident with triples of two colours. Let $\mathcal{P}$ be a partition of V into three sets A, B and C such that each element of A is incident with triples of colour 1 and 2, each element of B is incident with triples of colour 1 and 3, and each element of C is incident with triples of colour 2 and 3. Let $|A| = a, |B| = b, |C| = c$. We may assume w.l.o.g. that $a \leq b \leq c$.

Now we state a series of rather straightforward claims which will be frequently used.

Claim 1. There is no triple $T \in \mathcal{B}$ such that $A \cap T \neq \emptyset$, $B \cap T \neq \emptyset$, and $C \cap T \neq \emptyset$. Furthermore, if $A \cap T \neq \emptyset$ and $B \cap T \neq \emptyset$ then T is of colour 1; if $A \cap T \neq \emptyset$ and $C \cap T \neq \emptyset$ then T is of colour 2; and if $B \cap T \neq \emptyset$ and $C \cap T \neq \emptyset$ then T is of colour 3.

Proof. The statement follows easily from this simple observation: Assume $T \cap D \neq \emptyset$ where $D \in \{A, B, C\}$. Each element of D is incident with triples coloured by the same two colours. Therefore T has to be coloured by one of two colours. □

Claim 2. Exactly one of the numbers a, b, c is odd.

Proof. As $v = a + b + c$, at least one of the three numbers has to be odd. Assume by contradiction that a and b are odd. Then there is an odd number of pairs $\{x, y\}$ where $x \in A$, $y \in B$. Each of those pairs is contained in a triple $T \in \mathcal{B}$ where $A \cap T \neq \emptyset$ and $B \cap T \neq \emptyset$. If , in addition, $C \cap T = \emptyset$ then T covers exactly two of those pairs. Therefore there has to be a triple T in $\mathcal{B}$ such that $A \cap T \neq \emptyset$, $B \cap T \neq \emptyset$, $C \cap T \neq \emptyset$. However, this contradicts Claim 1. □

Claim 3. For the partition π_1, $a + b \geq \frac{v+1}{2}$, while for the partition π_2, $a + b \geq \frac{v-1}{2}$.

Proof. Let $x \in V$ be an element incident with a triple of colour 1. Then, for the partition π_1, x is incident with $\frac{v-1}{4}$ triples of colour 1, while for the partition π_2, x is incident with at least $\frac{v-3}{4}$ triples of color 1. Those triples of colour 1 incident with x contain exactly $\frac{v+1}{2}$ elements (at least $\frac{v-1}{2}$ elements, respectively), and the statement follows. □

Claim 4. $\binom{v}{2} \geq \frac{3}{2}(ab + ac + bc)$.

Proof. There are $ab + ac + bc$ pairs $Q = \{x, y\}$ of elements of V such that x and y belong to different parts of the partition $\mathcal{P}$. Each pair Q is covered by a triple T containing two elements from the same part of $\mathcal{P}$; moreover, such a triple contains two such pairs Q. Therefore, the number $\binom{a}{2} + \binom{b}{2} + \binom{c}{2}$ of pairs $\{z, v\}$ where z and v are from the same part of $\mathcal{P}$, has to be at least $\frac{1}{2}(ab + ac + bc)$. The claim now follows by a simple rearrangement of this inequality. □

We are now ready to prove Theorem 2. Let d be the number of pairs $\{x, y\}$ where either both $x, y \in B$ or both $x, y \in C$, and $\{x, y\}$ is contained in a triple of colour

3. By Claim 1, all pairs $\{x, y\}, x \in B, y \in C$ are contained in triples of colour 3. Therefore

$$d \geq \frac{bc}{2} \tag{3}$$

To derive an upper bound on d, we will count the number of pairs $Q = \{x, y\}$ so that either $Q \subset B$ or $Q \subset C$, and Q is contained in a triple of colour 1 or 2. There are in total $e = \frac{1}{3}.\frac{(v+a)(v-1)}{4} = \frac{1}{3}.(\frac{1}{2}\binom{v}{2} + \frac{1}{4}.a(v-1))$ triples of colour 1 or 2 for the partition π_1. Indeed, each element in A is incident with $\frac{v-1}{2}$ of those triples while each element in $B \cup C$ is incident with $\frac{v-1}{4}$ of them. By a similar reasoning we get that there are $e \geq \frac{1}{3}(a\frac{v-1}{2} + (b+c)\frac{v-3}{4}) = \frac{1}{3}(\frac{1}{2}\binom{v}{2} + \frac{1}{4}a(v-1) - \frac{b+c}{2})$ triples of colour 1 or 2 for the partition π_2. Further, $\frac{ab+ac}{2}$ of those e triples of colour 1 or 2 contain elements in two parts of $\mathcal{P}$ (either they contain elements in A and B or elements in A and C). Let t be the number of triples T which are of colour 1 or 2 and $T \subset A$. Then there are $\binom{a}{2} - 3t$ triples of colour 1 or 2 of type AAB or AAC which in turn implies that there are $e - (\binom{a}{2} - 3t)$ triples T of colour 1 or 2 of type ABB or ACC. Further, there are $e - \frac{ab+ac}{2} - t$ triples T of colour 1 or 2 such that $T \subset B$ or $T \subset C$. They cover additional $3(e - \frac{ab+ac}{2} - t)$ pairs of elements. Thus, in aggregate,

$$d = \binom{b}{2} + \binom{c}{2} - \left(\frac{ab+ac}{2} - \binom{a}{2} + 3t\right) - 3\left(e - \frac{ab+ac}{2} - t\right) \tag{4}$$

Combining (3) and (4) and rearranging leads to

$$\binom{a}{2} + \binom{b}{2} + \binom{c}{2} + ab + ac + bc - 3e \geq \frac{3}{2}bc$$

that is,

$$\binom{v}{2} - 3e \geq \frac{3}{2}bc \tag{5}$$

Substituting for e we get, for the partition π_1,

$$\binom{v}{2} - \frac{1}{2}a(v-1) \geq 3bc,$$

that is,

$$(v-1)(v-a) \geq 6bc \tag{6}$$

and, for the partition π_2,

$$\binom{v}{2} - \frac{1}{2}a(v-1) + b + c \geq 3bc$$

that is,

$$(v+1)(v-a) \geq 6bc \tag{7}$$

To show that (6) is not satisfied for any $v \geq 9$, $v \equiv 1$ or 9 (mod 12), we prove that $(v-1)(v-a) < 6bc$. Let a be a fixed number. Consider two cases.

Assume first that $a \geq \frac{v+3}{4}$. Since $v - a = b + c$ is a fixed number as well, we get that $min\ 6bc$ is attained at $b = a$ and $c = v - 2a$ (recall that by our assumption, $a \leq b \leq c$). Therefore it suffices to verify that $(v-1)(v-a) < 6a(v-2a)$, that is, to show that $f(a) = 6a(v-2a) - (v-1)(v-a) > 0$ for all a, $\frac{v+3}{4} \leq a \leq \frac{v}{3}$. We have $f(\frac{v}{3}) = 2v.\frac{v}{3} - (v-1)\frac{2}{3}v = \frac{2}{3}v > 0$, and $f(\frac{v+3}{4}) = \frac{3}{2}(v+3)\frac{v-3}{2} - (v-1)\frac{3v-3}{4} = \frac{3}{4}(2v-10) > 0$ which completes the proof in this case because $f(a)$ is a parabola opening down.

Let now $1 \leq a \leq \frac{v-1}{4}$. Then, by Claim 3, $b \geq \frac{v+1}{2} - a$, and $min\ 6bc$ is attained at $b = \frac{v+1}{2} - a$, $c = v - (a+b) = \frac{v-1}{2}$.Thus, we need to prove that in this case $(v-1)(v-a) < 6(\frac{v+1}{2} - a)\frac{v-1}{2}$. It is easy to check that the inequality is satisfied for all $a \leq \frac{v+3}{4}$. This completes the proof of Theorem 2 for the partition π_1.

In order to prove that (7) is not satisfied for any $v > 7$, $v \equiv 3$ or 7 (mod 12), we show that $(v+1)(v-a) < 6bc$. Let a be a fixed number. Consider two cases.

I. $1 \leq a \leq \frac{v-3}{4}$. Then, by Claim 3, $b \geq \frac{v-1}{2} - a$, and $min\ 6bc$ is attained at $b = \frac{v-1}{2} - a, c = v - (a+b) = \frac{v+1}{2}$.Therefore, it needs to be shown that $(v+1)(v-a) < 6(\frac{v-1}{2} - a)\frac{v+1}{2}$. This inequality is true for all $1 \leq a < \frac{v-3}{4}$. So, assume now $a = \frac{v-3}{4}$. It is easy to see that $(v+1)(v-a) < 6bc$ is satisfied for $a = \frac{v-3}{4}, b = \frac{v-1}{2} - a + 1, c = v - (a+b) = \frac{v-1}{2}$, hence it is satisfied for all $b \geq \frac{v-1}{2} - a + 1$. Therefore we are left with the case $a = \frac{v-3}{4}, b = \frac{v-1}{2} - a = \frac{v+1}{4}, c = v - (a+b) = \frac{v+1}{2}$. We show that in this case S has to contain a sub-STS($\frac{v-1}{2}$).

Each of these $a + b = \frac{v-1}{2}$ elements is incident with at least $\frac{v-3}{4}$ triples of color 1; therefore there are at least $\alpha = \frac{1}{3}\frac{v-1}{2}\frac{v-3}{4}$ triples of colour 1. Further, there are at least $\beta = \frac{ac}{2} = \frac{(v-3)(v+1)}{16}$ triples of colour 2 (cf. Claim 1), and at least $\gamma = \frac{bc}{2} = \frac{(v+1)^2}{16}$ triples of colour 3. Since $\alpha + \beta + \gamma = \frac{1}{3}\binom{v}{2} = |\mathcal{B}|$, the number of triples of colour 1, 2 and 3 is α, β and γ, respectively. Thus there is no triple $T \subset C$, and triples of colour 1 form a sub-STS on $A \cup B$, as given in the statement of the theorem. We are now done with the proof for the case $1 \leq a \leq \frac{v-3}{4}$. We point out that in this case we proved the statement for all $v \equiv 3$ or 7 (mod 12).

II. $\frac{v+1}{4} \leq a \leq \frac{v}{3}$. Then $min\ 6bc$ is attained at $b = a$ and $c = v - 2a$. Thus, to prove the nonexistence of a 3-colouring in this case, we need to show that

$$(v+1)(v-a) < 6a(v-2a) \tag{8}$$

Similarly as in the case of the partition π_1, set $f(a) = 6a(v-2a) - (v+1)(v-a)$. We have to consider two subcases.

IIa. $\frac{v+5}{4} < a \leq \frac{v}{3} - 1$ and $v > 39$; $a = \frac{v+5}{4}$ and $v = 39$. We have $f(\frac{v}{3} - 1) = 6(\frac{v}{3} - 1)(\frac{v}{3} + 2) - (v+1)(\frac{2v}{3} + 1) = \frac{v}{3} - 13 > 0$ for $v > 39$. On the other hand, $f(\frac{v+5}{4}) = 6.\frac{v+5}{4}\frac{v-5}{2} - (v+1)\frac{3v-5}{4} = \frac{2v-70}{4} > 0$ for $v > 35$. As $f(a)$ is a parabola opening down, $f(\frac{v}{3} - 1) > 0$ and $f(\frac{v+5}{4}) > 0$ implies that (8) is satisfied for all a, $\frac{v+5}{4} < a \leq \frac{v}{3} - 1$, and $v > 39$, and for $a = \frac{v+5}{4}$, $v \geq 39$.

IIb. So we are left with two cases. Assume first that $a > \frac{v}{3} - 1$. If $v \equiv 3$ (mod 12) then $a = b = c = \frac{v}{3}$. However, then a, b, and c are all odd which contradicts Claim

2. For $v \equiv 7 \ (mod\ 12)$ we have $a = b = \frac{v-1}{3}$, and $c = \frac{v+2}{3}$ which contradicts Claim 4.

Now let $a = \frac{v+1}{4}$. Then for $b \geq \frac{v+5}{4}$, the minimum $min\ 6bc$ is attained for $b = \frac{v+5}{4}$, and, consequently, $c = v-(a+b) = \frac{v-3}{2}$. If we substitute into (7), we obtain that the inequality is satisfied only for $v \leq 11$. Finally, consider the last case when $a = b = \frac{v+1}{4}$, and $c = v-(a+b) = \frac{v-1}{2}$. By Claim 1, there are at least $\alpha = \frac{ac}{2}$ triples of colour 2. Further, each element in $A \cup B$ is incident with at least $\frac{v-3}{4}$ triples of colour 1, hence the number of triples of colour 1 is at least $\beta = \frac{1}{3}\frac{v+1}{2}\frac{v-3}{4}$. Thus the total number e of triples of colour 1 or colour 2 is $e \geq \alpha+\beta \geq \frac{(v+1)(v-1)}{16}+\frac{1}{3}\frac{(v+1)(v-3)}{8}$. Substitute into (5) to obtain $\binom{v}{2} - (\frac{3(v+1)(v-1)}{16} + \frac{(v+1)(v-3)}{8}) \geq \frac{3}{2}\frac{(v+1)(v-1)}{8}$, which in turn implies $v(v-1) \geq (v+1)\frac{3v-3+3v-3+2(v-3)}{8}$, that is, $v(v-1) \geq (v+1)\frac{8v-8}{8} - \frac{4}{8}(v+1)$, and finally, $\frac{v+1}{2} \geq v-1$. Clearly, this holds only for $v \leq 3$. Thus we have proved that for all $v \geq 11$, the inequality (8) is satisfied for $a = \frac{v+1}{4}$ and for $a > \frac{v}{3} - 1$.

By combining now the results of I, IIa and IIb we get that the theorem is proved for all $v > 39$, $v \equiv 3$ or $7 \ (mod\ 12)$.

To finish the proof, the nonexistence of a 3-colouring of $\mathcal{B}$ of type π_2 needs to be proved for

$$\frac{v+5}{4} \leq a \leq \frac{v}{3} - 1, \quad v < 39; \quad \frac{v+5}{4} < a \leq \frac{v}{3} - 1, \quad v = 39. \tag{9}$$

We have to consider orders $v = 15, 19, 27, 31$, and 39. It is easy to check that for $v = 15, 19$ there is no value of a satisfying (9). For $v = 27, 31$ and 39, the only value of a that satisfies (9) is $a = \lfloor \frac{v}{3} - 1 \rfloor$. Set $a = b$, and $c = v - (a + b)$. By a routine calculation, we obtain in those three cases $\binom{27}{2} < \frac{3}{2}(8\times 8+8\times 11+8\times 11)$, $\binom{31}{2} < \frac{3}{2}(9 \times 9 + 9 \times 13 + 9 \times 13)$, and $\binom{39}{2} < \frac{3}{2}(12 \times 12 + 12 \times 15 + 12 \times 15)$, contradicting Claim 4. To see that Claim 4 is not satisfied for other values of b as well it is sufficient to note that for the function $f(x, y, z) = xy + xz + yz$ we have $f(x, y, z) > f(x', y', z')$ for $x = x', y + z = y' + z'$, and $|z - y| < |z' - y'|$. The proof of Theorem 2 is now complete. □

The following theorem contains a complete characterization of the spectra for equitable bicolourings of all admissible orders.

Theorem 3. $\Omega_{\pi_2}(7) = \{3\}$. *For* $v > 7, v \equiv 1$ *or* $3 \ (mod\ 6)$, $\Omega_\pi = \{2\}$ *where* $\pi = \pi_1$ *when* $v \equiv 1$ *or* $9 \ (mod\ 12)$, *and* $\pi = \pi_2$ *when* $v \equiv 3$ *or* $7 \ (mod\ 12)$.

Proof. According to [2], the unique STS(7) has a k-colouring of type $(1, 2)$ if and only if $k = 3$. For $v > 7$, in view of Theorem 1 we only have to show that there exists an STS(v) admitting an equitable bicolouring with 2 colours. When $v \equiv 3$ or $9 \ (mod\ 12)$, this is an easy consequence of the existence of Kirkman triple systems (cf. [1]): when $v \equiv 9 \ (mod\ 12)$, colour the triples of $\frac{v-1}{4}$ parallel classes with colour 1, and those of the remaining $\frac{v-1}{4}$ parallel classes with colour 2 (when $v \equiv 3 \ (mod\ 12)$, the corresponding numbers of parallel classes are $\frac{v-3}{4}$ and $\frac{v+1}{4}$, respectively).

When $v \equiv 1$ (*mod* 12), we obtain an STS(v) with an equitable bicolouring by using cyclic STS(v). The triples of any cyclic STS(v) in this case are partitioned into $\frac{v-1}{6}$ orbits under the induced action of the cyclic group (which acts on elements), and each orbit of triples consists of v triples; the collection of triples of any such orbit may be viewed as a 3-configuration (cf. [1]). Choosing any $\frac{v-1}{12}$ orbits and colouring its triples with colour 1, and the triples of the remaining $\frac{v-1}{12}$ orbits with colour 2 yields an equitable bicolouring.

Finally, we deal with the case when $v \equiv 7$ (*mod* 12). The number of orbits of a cyclic STS(v) in this case is odd. If we can show that one orbit Q of a cyclic STS(v) can be partitioned into two $\{1, 2\}$-configurations (cf. [2]), say C_1 and C_2, then colouring the triples of $\frac{v-7}{12}$ orbits (other than Q) together with the triples of C_1 with colour 1, and the triples of the remaining $\frac{v-7}{12}$ orbits together with the triples of C_2 with colour 2 will have produced an equitable bicolouring of our STS(v). Thus it remains to be shown that such a partitionable orbit can always be found. First we observe that if an orbit of triples is determined by a base triple $\{0, 1, 2t\}$ $(mod\ v)$, $t < \frac{v}{2}$, such a partition is possible: e.g., let C_1 consist of the $\frac{v+1}{2}$ triples $\{2i, 2i + 1, 2i + 2t\}$, $i = 0, 1, \ldots, \frac{v-1}{2}$, and let C_2 consist of the remaining triples of this orbit. Both C_1 and C_2 are easily seen to be $\{1, 2\}$-configurations. By [3], for every $v \geq 91$ there exists a cyclic STS(v) containing an orbit of triples determined by the base triple $\{0, 1, 4\}(mod\ v)$. Such cyclic STS(v) can also be shown to exist for $v = 31, 43, 55, 67$, and 79. There is no such cyclic STS(19), however, there is a cyclic STS(19) with an orbit determined by the base triple $\{0, 1, 8\}$, and so our proof is complete. □

A particular STS may admit several partitions into two $\frac{v-1}{4}$-configurations, and thus several colourings of type π_1. As an illustration, consider the two nonisomorphic STS(13). The cyclic STS(13) admits 14 partitions into two 3-configurations. One of these partitions is the one described in the proof of Theorem 3, and is preserved by the full automorphism group of this STS (of order 39). The other 13 partitions are permuted under the automorphism group of the STS, and have only the identity as an automorphism. Thus there are two inequivalent colourings of type π_1 of the cyclic STS(13).

The noncyclic STS(13) also admits 14 partitions into two 3-configurations, however, these fall into 5 classes under the (full) automorphism group of the noncyclic STS(13) (which is of order 6). The five classes contain 6, 3, 3, 1, and 1 partitions, respectively (with automorphism groups of orders 1, 2, 2, 6, and 6, respectively). Thus there are five inequivalent colourings of type π_1 of the noncyclic STS(13).

Whether there exist STSs without any colouring of type π_1 (or of type π_2, for that matter) remains an open question.

4. Equitable Tricolourings

In this section we deal with colourings of type σ where σ is a partition having exactly three parts, with the difference between any two parts not exceeding one. We call such colourings *equitable tricolourings*. Only two types of equitable tricolourings of STSs may exist: those of type $\sigma_1 = (\frac{v-1}{6}, \frac{v-1}{6}, \frac{v-1}{6})$, and those of type

$\sigma_2 = (\frac{v-3}{6}, \frac{v-3}{6}, \frac{v+3}{6})$. Moreover, equitable tricolourings of type σ_1 may exist only if $v \equiv 1$ $(mod\ 6)$, and those of type σ_2 may exist only if $v \equiv 3$ $(mod\ 6)$.

First we deal with equitable tricolourings for STSs of small order.

Lemma 2.

(i) $\Omega_{\sigma_1}(7) = \{7\}$
(ii) $\Omega_{\sigma_2}(9) = \{3, 4, 5, 6, 7\}$
(iii) $\Omega_{\sigma_1}(13) = \{4, 5\}$.

Proof. For (i) and (iii), see [2] ((i) is trivial). For (ii), if $\mathcal{R} = \{R_1, R_2, R_3, R_4\}$ are the four parallel classes of the (unique) STS(9), colour the triples of R_1 and R_2 with colour 1, those of R_3 with colour 2, and those of R_4 with colour 3; this yields a 3-colouring of type $\sigma_2 = (1, 1, 2)$. Then recolour consecutively the triples of R_3 and R_4 until all 6 of their triples have an individual colour, to obtain a 4−, 5−, 6−, and 7−-colouring of type σ_2, respectively. Assuming the existence of an 8-colouring of type σ_2 implies that there can be no colour class with more than 5 triples, but then there must be an element whose all four triples incident with it have mutually different colours. □

Lemma 3. *If an STS(v) $S = (V, \mathcal{B})$ has a colouring of type σ_2 then $\bar{\chi}'_{\sigma_2}(S) \leq 8$.*

Proof. Assume that S has a 9-colouring of type σ_2. Similarly as in the proof of Lemma 1, for each colour we have $|V(c)| \geq 1 + 2\frac{v-3}{6} = \frac{v}{3}$, and for any rich colour c, we have $|V(c)| \geq 1 + 2\frac{v+3}{6} = \frac{v}{3} + 2$. Let c' be a rich colour, then, from (1) (cf. Section 3), we get

$$3v = \Sigma_{c \in C}|V(c)| = |V(c')| + \Sigma_{c \in C, c \neq c'}|V(c)| \geq \frac{v}{3} + 2 + 8\frac{v}{3} = 3v + 2,$$

a contradiction. □

Theorem 4. *If an STS(v) $S = (V, \mathcal{B})$ has a colouring of type σ_i, $i \in \{1, 2\}$ then*

$$3 \leq \chi'_{\sigma_i}(S) \leq \bar{\chi}'_{\sigma_i}(S) \leq 7.$$

Proof. The lower bound is trivial. By Theorem 1 and Lemma 3, $\bar{\chi}'_{\sigma_i}(S) \leq 8$. Suppose, by contradiction, that there is an 8-colouring ϕ of an STS(v) of type σ_i. Let $C = \{1, 2, \ldots, 8\}$ be the set of colours used in σ_i, and for $x \in V$, let $C(x) \subset C$ be the set of colours used on triples incident with x. As σ_i is a partition into three parts, $|C(x)| = 3$ for each $x \in V$. Now we define a family $\mathcal{A}$ of subsets of C. A set A, $A \subset C$, belongs to $\mathcal{A}$ if there is an element $x \in V$ such that $A = C(x)$. Finally, for $A \in \mathcal{A}$ put $V(A) = \{x : x \in V, C(x) = A\}$, and for $c \in C$, put $V(c) = \{x : c \in C(x)\}$. We start with a series of more or less trivial observations on $\mathcal{A}$, $V(A)$, and $V(c)$.

Obviously,

Claim 1. The family $\{V(A) : A \in \mathcal{A}\}$ forms a partition of V.

Further,

Claim 2. (i) $|A| = 3$ for each $A \in \mathcal{A}$; (ii) if $A, A' \in \mathcal{A}$ then $A \cap A' \neq \emptyset$; (iii) for each colour $c \in C$, $|V(c)| \geq \frac{v+2}{3}$ for the partition σ_1, and $|V(c)| \geq \frac{v}{3}$ for the partition σ_2; (iv) for each $c \in C$ there is $A \in \mathcal{A}$ such that $c \notin A$.

Proof. The first part of the statement is trivial. Let $x \in V(A), x' \in V(A')$, and let $B \in \mathcal{B}$ be a triple containing the pair $\{x, x'\}$. Then the colour of $B \in A \cap A'$. This proves part (ii). As to (iii), consider an element $x \in V$ that is incident with a triple coloured with colour c. Then, for the partition σ_1, x is incident with $\frac{v-1}{6}$ triples of colour c, and for the partition σ_2, x is incident with at least $\frac{v-3}{6}$ triples of colour c. This in turn implies that there are, for the partition σ_1, at least $2\frac{v-1}{6} + 1 = \frac{v+2}{3}$ elements in V incident with triples of colour c, and for the partition σ_2, at least $2\frac{v-3}{6} + 1 = \frac{v}{3}$ elements in V incident with triples of colour c. To prove (iv), suppose that there is a colour $c' \in C$ such that $c' \in A$ for all $A \in \mathcal{A}$, that is, for each element $x \in V$, x is incident with a triple of colour c'. As each element in V is incident with triples of three colours, we get, applying part (iii) and (1): $3v = \Sigma_{c \in C}|V(c)| = |V(C')| + \Sigma_{c \in C, c \neq c'}|V(c)| \geq v + 7\frac{v}{3} = \frac{10}{3}v$, a contradiction. This completes the proof. □

Claim 3. For any three colours c_1, c_2, c_3, there is a pair $i, j \in \{1, 2, 3\}$ such that $V(c_i) \cap V(c_j) \neq \emptyset$.

Proof. Suppose by contradiction that there are three colours such that the sets $V(c_1), V(c_2), V(c_3)$ are mutually disjoint. Then we have $|V(c_1) \cup V(c_2) \cup V(c_3)| = \Sigma_{1 \leq i \leq 3}|V(c_i)|$, and, for the partition σ_1, by Claim 2,(iii), $\Sigma_{1 \leq i \leq 3}|V(c_i)| \geq 3\frac{v+2}{3}$, a contradiction. Consider now the partition σ_2. By Claim 2,(iii), $|V(c)| \geq \frac{v}{3}$ for each colour $c \in C$, therefore $|V(c_i)| = \frac{v}{3}$ for each $i = 1, 2, 3$. Further, by (iii), there are at least $\frac{1}{3}\frac{v}{3}\frac{v-3}{6} = \frac{1}{3}\binom{\frac{v}{3}}{2}$ triples of each colour c_i, $1 \leq i \leq 3$. Therefore if elements x, y belong to $V(c_i)$ for some i, $1 \leq i \leq 3$, then the pair $\{x, y\}$ belongs to a triple B of colour c_i. Consequently, $B \subset V(c_i)$, that is $\mathcal{B}$ induces on $V(c_i)$ a sub-STS. This in turn implies that for each triple $B = \{x, y, z\} \in \mathcal{B}$, either $B \subset V(c_i)$ for some i, $1 \leq i \leq 3$, or $B \cap V(c_i) \neq \emptyset$ for each i, $1 \leq i \leq 3$. Now we show that if c is a colour such that $c \notin \{c_1, c_2, c_3\}$ then $V(c) \geq \frac{v-3}{2}$. Indeed, if $c \in C(x)$ where x is an element of (say) $V(c_1)$, then x is incident with at least $\frac{v-3}{6}$ triples of colour c; hence there are at least $\frac{v-3}{6}$ elements in both $V(c_2)$ and $V(c_3)$ contained in those triples of colour c that are incident with x. By the same token we get that there are at least $\frac{v-3}{6}$ elements in $V(c_1)$ contained in triples of colour c. Therefore $|V(c)| \geq 3\frac{v-3}{6}$. By (1) we get

$$3v = \Sigma_{c \in C}|V(c)| = \Sigma_{1 \leq i \leq 3}|V(c_i)| + \Sigma_{c \notin \{c_1, c_2, c_3\}}|V(c)| \geq$$
$$3\frac{v}{3} + 5\frac{v-3}{2} = \frac{7}{2}v - \frac{15}{2}$$

which implies $v \leq 15$. However, there is no STS of order $v = \frac{15}{3} = 5$. Thus, as $v \equiv 3$ (mod 6), we are left just with $v = 9$. However, there is no 8-colouring of type $\sigma_2 = (1, 1, 2)$ by Lemma 2. □

Let us introduce some more notation. For each colour c, let $\mathcal{A}(c) = \{A : A \in \mathcal{A}, c \in A\}$.

Claim 4. For each $c \in C$, $|\bigcup_{A \in \mathcal{A}(c)} A| \geq 7$. In particular, $|\mathcal{A}(c)| \geq 4$ for all $c \in C$.

Proof. Assume first that there is a colour c such that $|\mathcal{A}(c)| = 1$, say, $|\mathcal{A}(1)| = 1$, and that $A' = \{1, 2, 3\} \in \mathcal{A}$. By Claim 2,(ii), for each $A \in \mathcal{A}$, we have $A \cap A' \neq \emptyset$, that is, $A \cap \{2, 3\} \neq \emptyset$. Further, for each $i \neq j$, $4 \leq i, j \leq 8$, there is $A \in \mathcal{A}$ such that $\{i, j\} \subset A$; otherwise, the colours $1, i, j$ would contradict Claim 3. Assume, w.l.o.g., that $\{2, 4, 5\}$ in $\mathcal{A}$. Then in order not to contradict Claim 2,(ii), $\{2, 6, 7\}$, $\{2, 6, 8\}$, $\{2, 7, 8\}$ have to be in $\mathcal{A}$ as well, and, for the same reason, $\{2, i, j\} \in \mathcal{A}$ for all $4 \leq i \neq j \leq 8$. With this in hand, and by Claim 2,(ii), it is easy to see that for each $A \in \mathcal{A}$, one has $2 \in A$, contradicting Claim 2,(iv). Thus $|\mathcal{A}(c)| > 1$. Assume now that there is a colour $c \in C$ such that $|A^*| = |\bigcup_{A \in \mathcal{A}(c)} A| < 7$. In order not to contradict Claim 3, for each two colours $i, j \notin A^*$, $i \neq j$, there is $A' \in \mathcal{A}$ such that $i, j \in A'$. As A' has to intersect each set in $\mathcal{A}(c)$ and $|\mathcal{A}(c)| > 1$, there has to be a colour $c' \neq c$ such that $|\mathcal{A}(c)| \leq |\mathcal{A}(c')|$, and c' is the only colour with this property. It is now easy to check that in order not to contradict Claim 2,(ii), it must be that $c' \in A$ for each $A \in \mathcal{A}$. However, this again contradicts Claim 2,(iv). Thus for each colour c, $|\bigcup_{A \in \mathcal{A}(c)} A| \geq 7$. This in turn implies $|\mathcal{A}(c)| \geq 3$. Suppose that there is a colour c such that $|\mathcal{A}(c)| = 3$. Then, since $|\bigcup_{A \in \mathcal{A}(c)} A| \geq 7$, we get that if $A, A' \in \mathcal{A}(c)$ then $A \cap A' = \{c\}$. Let c' be the colour such that $c' \notin A^*$. Then for any $B \in \mathcal{A}$, $c' \in B$, there would have to be $A \in \mathcal{A}(c)$ so that $B \cap A = \emptyset$, a contradiction. This completes the proof of the claim. □

Now we are ready to prove the theorem. We use graph theory language to prove this part. Consider a graph $G = (V, E)$ where $V = C - \{1\}$, and $\{i, j\} \in E$ if there is $A \in \mathcal{A}(1)$ such that $A = \{1, i, j\}$. This graph will be called the graph corresponding to $\mathcal{A}(1)$ (and in general, the graph corresponding to $\mathcal{A}(c)$ for colour c). We need to distinguish three cases with respect to the size of a maximum matching in G. Assume first that a maximum matching in G is of size 1. As $|\mathcal{A}(1)| \geq 4$ (cf. Claim 4), we have $|E| \geq 4$. Hence there is a vertex u in G such that all edges of G are incident with u. In other words, there is a colour $c' \neq 1$ such that $\mathcal{A}(1) \subset \mathcal{A}(c')$, or, in other words yet, if $A \in \mathcal{A}(1)$, then $A = \{1, c', x\}$. However, together with Claim 2(ii), this implies that for each $A \in \mathcal{A}$, we have $c' \in A$ which contradicts Claim 2(iv).

Further, suppose that the size of a maximum matching in G equals 2. This means that there are two independent edges in G, say, $\{2, 3\}$ and $\{4, 5\}$ (i.e., $\{1, 2, 3\}$ and $\{1, 4, 5\}$ are in $\mathcal{A}$). As $|\bigcup_{A \in \mathcal{A}(1)} A| \geq 7$ (cf. Claim 4), there are vertices u, w in G, $u, w \notin \{2, 3, 4, 5\}$ such that both u, w are incident with at least one edge of G. Let, say, $\{u, w\} = \{6, 7\}$. Let $A = \{6, a, b\} \in \mathcal{A}$. Then $\{a, b\} \subset \{1, 2, 3, 4, 5\}$. Indeed, if $\{7, 8\} \cap \{a, b\} \neq \emptyset$ then either A would not intersect one of the sets $\{1, 2, 3\}$, $\{1, 4, 5\}$ or there would be a matching of size 3 in G. By the same argument, if $A = \{7, a, b\}$ then

$\{a, b\} \subset \{1, 2, 3, 4, 5\}$. In addition, there are at most four sets $A \in \mathcal{A}$ such that $|A \cap \{1, 6, 7\}| = 2$. To see this, it suffices to note that, e.g., if $\{1, 2, 6\} \in \mathcal{A}$ then $\{1, 3, 7\} \notin \mathcal{A}$ since otherwise there would be a matching of size 3 in G. By Claim 4, $|\mathcal{A}(6)| \geq 4$ and $|\mathcal{A}(7)| \geq 4$. Therefore there is an $A' \in \mathcal{A}$ such that $1 \notin A'$, and $A' \cap \{6, 7\} \neq \emptyset$. Assume w.l.o.g. that $A' = \{2, 4, 6\}$ (A' has to intersect both $\{1, 2, 3\}$ and $\{1, 4, 5\}$). Then, to satisfy Claim 2(ii), $\mathcal{A}(7) \subset \{\{1, 2, 7\}, \{1, 4, 7\}, \{2, 4, 7\}, \{2, 5, 7\}, \{3, 4, 7\}\}$. However, if both $\{2, 5, 7\}, \{3, 4, 7\}$ were in $\mathcal{A}(7)$ then to satisfy the Claim 2(ii), we would have to have $\mathcal{A}(6) \subset \{\{2, 4, 6\}, \{3, 5, 6\}\}$ which contradicts $|\mathcal{A}(6)| \geq 4$. Hence we may assume w.l.o.g. that $\mathcal{A}(7) = \{\{1, 2, 7\}, \{1, 4, 7\}, \{2, 4, 7\}, \{2, 5, 7\}\}$. Then, again to satisfy Claim2(ii), we would have to have $\mathcal{A}(6) \subset \{\{2, 4, 6\}, \{1, 2, 6\}\}$, a contradiction.

Thus if there is a colour c such that the graph corresponding to the $\mathcal{A}(c)$ graph has a maximum matching of size < 3, we are done. Finally, suppose that for each colour $c \in C$ the corresponding graph G contains a matching of size 3. Let c' be the (unique) colour that is not covered by a maximum matching M of size 3 in the graph corresponding to $\mathcal{A}(1)$. Let M' be a maximum matching of size 3 in the graph G' corresponding to $\mathcal{A}(c')$. Then M' contains an edge e which does not cover the colour 1. However, then the set $A \in \mathcal{A}(c')$ that corresponds to the edge e does not intersect at least one of the three sets in $\mathcal{A}(1)$ forming the maximum matching M which violates Claim 2(ii). The proof is now complete. □

Lemma 4. *An STS(v) with a* 3-*colouring of type* σ_1 *exists for all* $v \equiv 1$ (*mod* 18).

Proof. If $v = 18t + 1$, the number of orbits of triples in any cyclic STS(v) is divisible by 3 and, in fact, equals $3t$. It suffices to colour the triples of any t orbits with colour 1, the triples of next t orbits with colour 2, and of the remaining t orbits with colour 3. □

Lemma 5. *An STS(v) with a* 3-*colouring of type* σ_2 *exists for all* $v \equiv 3$ (*mod* 6).

Proof. This is an easy consequence of the existence of Kirkman triple systems: colour the triples of any $\frac{v-3}{6}$ parallel classes with colour 1, those of the next $\frac{v-3}{6}$ parallel classes with colour 2, and those of the remaining $\frac{v+3}{6}$ parallel classes with colour 3. □

When $v \equiv 7$ or 13 (*mod* 18), one can improve the lower bound given in Theorem 4.

Lemma 6. *Let* $v \equiv 7$ *or* 13 (*mod* 18)*, then there exists no 3-colouring of type* σ_1 *for any STS(v).*

Proof. If an STS(v) $S = (V, \mathcal{B})$ has a 3-colouring of type σ_1, then each of the three colour classes must contain the same number of triples. Indeed, if σ is an equitable partition of $r = \frac{v-1}{2}$ into s equal parts and we consider an s-colouring of S then s has to divide $\frac{v(v-1)}{6}$, as each colour class contains the same number of triples. However, this is not satisfied when $s = 3$ and $v \equiv 7$ or 13 (*mod* 18). □

Lemma 7. *For all* $v \equiv 3$ *or* 9 $(mod\ 18)$*,* $\{3, 4, 5\} \subset \Omega_{\sigma_2}(v)$.

Proof. We use a special case of the well known generalized direct product (cf. [1]).Let $(V, \mathcal{B})$ be an STS(v), let $X = V \times \{1, 2, 3\}$, and let $(X, \mathcal{G}, \mathcal{C})$ be a resolvable transversal design TD(3, v) with $\mathcal{G} = \{V \times \{i\}, i = 1, 2, 3\}$. Such a resolvable TD is well known to exist for all $v \geq 7$ (cf. [1]). Construct an STS($3v$) by putting on each group $V \times \{i\}$ an STS(v) $(V \times \{i\}, \mathcal{B}_i)$, and adjoining the set of triples $\mathcal{C}$ of the TD. Colour now any $\frac{v-1}{2}$ parallel classes of $\mathcal{C}$ with colour 1, the remaining $\frac{v+1}{2}$ parallel classes of $\mathcal{C}$ with colour 2, and then colour the triples of $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ with colour 3, or colour the triples of $\mathcal{B}_1, \mathcal{B}_2$ with colour 3 and those of $\mathcal{B}_3$ with colour 4, or colour the triples of $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ with colours 3, 4, and 5, respectively. In either case, a colouring of type σ_2 is obtained. □

Lemma 8. *For all* $v \equiv 1$ *or* 7 $(mod\ 18)$ *there exists a* 7*-colouring of type* σ_1.

Proof. We use a variant of Wilson's fundamental construction (cf. [1]). Let $(V, \mathcal{B})$ be the following STS(7): $V = \{0, a_i, b_i, c_i : i = 1, 2\}$, $\mathcal{B} = \{\{0, a_1, a_2\}, \{0, b_1, b_2\}, \{0, c_1, c_2\}, \{a_1, b_1, c_1\}, \{a_1, b_2, c_2\}, \{a_2, b_1, c_2\}, \{a_2, b_2, c_1\}\}$; the last four triples form a Pasch configuration. Give now every element, except the element 0, weight $3k$ or $3k + 1$ according as $v = 18k + 1$ or $v = 18k + 7$. This replaces the elements $a_1, a_2, b_1, b_2, c_1, c_2$ with sets of elements $A_1, A_2, B_1, B_2, C_1, C_2$, each of cardinality $3k$ (or $3k + 1$). Put now on the sets $W_1 = \{0\} \cup A_1 \cup A_2$, $W_2 = \{0\} \cup B_1 \cup B_2$, $W_3 = \{0\} \cup C_1 \cup C_2$ an STS($6k+1$) (or STS($6k+3$), respectively), say, $(W_i, \mathcal{D}_i)$, $i = 1, 2, 3$. Put on each of the sets $A_1 \cup B_1 \cup C_1$, $A_1 \cup B_2 \cup C_2$, $A_2 \cup B_1 \cup C_2$, $A_2 \cup B_2 \cup C_1$ a transversal design TD(3, $3k$) (or TD(3, $3k + 1$), respectively) - the latter is just equivalent to a latin square of order $3k$ or $3k + 1$. Let $\mathcal{E}_j$, $j = 1, 2, 3, 4$ be the sets of triples of these respective four transversal designs. Then $(\bigcup_{i=1}^{3} W_i, \bigcup_{j=1}^{4} \mathcal{E}_j)$ is an STS($18k + 1$) (or STS($18k + 7$), respectively). Colour now the triples of the four transversal designs above with colours 1, 2, 3, 4, respectively; colour the triples of $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ with colours 5, 6, and 7, respectively. It is straightforward to verify that this yields a 7-colouring of our STS. □

We summarize the results in this section as follows.

Theorem 5.

(i) *An STS*(v) *with an equitable tricolouring with* 3 *colours exists if and only if* $v \equiv 3$ $(mod\ 6)$ *or* $v \equiv 1$ $(mod\ 18)$.

(ii) *An STS*(v) *with an equitable tricolouring with* 4 *colours* (5 *colours, respectively*) *exists if* $v \equiv 3$ *or* 9 $(mod\ 18)$.

(iii) *An STS*(v) *with an equitable tricolouring with* 7 *colours exists if* $v \equiv 1$ *or* 7 $(mod\ 18)$.

Proof. Combine Lemmas 4 to 8. □

Several questions concerning the existence of STSs with equitable tricolourings remain open. In particular, does there exist, for all $v \equiv 13$ $(mod\ 18)$, $v > 13$, an

STS(v) with an equitable tricolouring? Does there exist an STS(v) with an equitable tricolouring with 6 colours? Does there exist an STS(v) without an equitable tricolouring?

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