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Graphs and Combinatorics

Gauss-Bonnet Formula, Finiteness Condition, and Characterizations of Graphs Embedded in Surfaces

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Abstract. Let G be an infinite graph embedded in a closed 2-manifold, such that each open face of the embedding is homeomorphic to an open disk and is bounded by finite number of edges. For each vertex x of G, define the combinatorial curvature

$$\Phi_G(x) = 1 - \frac{d(x)}{2} + \sum_{\sigma \in F(x)} \frac{1}{|\sigma|}$$

as that of [8], where $d(x)$ is the degree of x, $F(x)$ is the multiset of all open faces σ in the embedding such that the closure $\bar{\sigma}$ contains x (the multiplicity of σ is the number of times that x is visited along $\partial\sigma$), and $|\sigma|$ is the number of sides of edges bounding the face σ. In this paper, we first show that if the absolute total curvature $\sum_{x \in V(G)} |\Phi_G(x)|$ is finite, then G has only finite number of vertices of non-vanishing curvature. Next we present a Gauss-Bonnet formula for embedded infinite graphs with finite number of accumulation points. At last, for a finite simple graph G with $3 \le d_G(x) < \infty$ and $\Phi_G(x) > 0$ for every $x \in V(G)$, we have (i) if G is embedded in a projective plane and $\#(V(G)) = n \ge 1722$, then G is isomorphic to the projective wheel P_n; (ii) if G is embedded in a sphere and $\#(V(G)) = n \ge 3444$, then G is isomorphic to the sphere annulus either A_n or B_n; and (iii) if $d_G(x) = 5$ for all $x \in V(G)$, then there are only 49 possible embedded plane graphs and 16 possible embedded projective plane graphs.

Key words. Combinatorial curvature, Gauss-Bonnet formula, Euler relation, Infinite graph, Embedding, Face cycle, Finiteness theorem.

Mathematical Subject Classification (1991). 05A16, 05C10, 05C30, 57M15, 57N05

1. Introduction

The notion of combinatorial curvature was introduced by Gromov [7] to study hyperbolic groups. Later it was modified by Ishida [9] and was defined directly for embedded plane graphs. Using this combinatorial curvature, Higuchi [8] considered discrete analogs of isoperimetric inequality and Myer's theorem of Riemannian

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geometry for embedded plane graphs. In fact, Higuchi [8] obtained certain isoperimetric constant and asked for a combinatorial analog of Myer's theorem.

Myer's theorem originally states that if a Riemannian n-manifold M has positive Ricci curvature bounded away from zero, say, Ric $\geq \frac{n-1}{r}$, then M is compact and has diameter at most πr. Higuchi's isoperimetric constant is interesting because it is quite different from that of continuous case. The difference perhaps lies in the fact that the combinatorial curvature does not approximate the curvature in the continuous case. For instance, for a closed 2-manifold with a triangulation in an Euclidean space, one easily finds that there is no logical relation in general between the combinatorial curvature and the curvature of the surface. However, the notion of discrete curvature that does approximate the curvature in the continuous case were considered by some others, for example, [1–4]. Other notions of combinatorial or discrete curvature are studied in [6, 10, 11, 13]. In particular, a discrete curvature for triangulations of 3-manifolds is introduced in [10] and is applied to characterize 3-spheres.

Since the difference in nature between the combinatorial curvature and the curvature in the continuous case, the discrete analogs of the isoperimetric inequality and the discrete analogs of Myer's theorem might be quite different from their continuous forms. The main purpose of this paper is to give one of such discrete analogs of Myer's theorem. A notable feature of our statement (certain finiteness theorem) for embedded graphs does not have similar continuous forms for Riemannian manifolds in the continuous case.

Let G be a graph (loops and multiple edges are allowed) embedded in a closed 2-manifold M. The graph G may have infinite number of vertices and edges, and may have loops and multiple edges. However, each vertex is required to have finite degree. We view the vertex set $V(G)$ as a subset of M and each edge of G as an open arc. We consider G as the union of $V(G)$ and all arcs, so that G is a subset of M. If $V(G)$ is infinite, the accumulation set

$$V'(G) := \overline{G} - G$$

may be non-empty, where $\overline{G}$ is the closure of the subset $G \subset M$. To avoid pathological cases that we have no interest, it is assumed that the embedding satisfies the following properties:

- (E1) The accumulation set $V'(G)$ is finite.
- (E2) The complement $M - \overline{G}$ is a disjoint union of connected open sets, each such open set U is homeomorphic to an open disk, and its boundary $\partial U := \overline{U} - U$ is a finite subgraph of G.

Then the punctured surface $S := M - V'(G)$ is decomposed into a (possibly infinite) collection of vertices, open arcs, and open regions. We call each open region an *open face* (or just *face*) of G and each accumulation point in $V'(G)$ an *end*. We write $\mathrm{End}(G)$ instead of $V'(G)$.

Note that the closure of a face may not be homeomorphic to a closed disk. This means that the boundary of a face may not be a cycle of G. Since each edge of G in the surface has two sides, we say that a side of an edge bounds a face σ provided that σ is exactly on that side of the edge. The number of sides of a face σ is called the

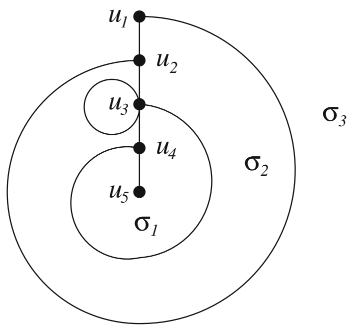


Fig. 1. An embedding with faces whose boundaries are not cycles

the *length* of σ, denoted $|\sigma|$. A face may be on both sides of an edge; if so the face has the two sides of the edge. For instance, viewing the graph in Fig. 1 embedded in a sphere, the length of the face σ_1 is 4, its closure $\bar{\sigma}_1$ is homeomorphic to a closed disk, but its boundary is not a cycle; the length of the face σ_2 is 7, its closure $\bar{\sigma}_2$ is not homeomorphic to a closed disk, and its boundary is not a cycle; the length of the face σ_3 is 2, its closure is homeomorphic to a disk, and its boundary is a cycle of length 2.

For each vertex x of G, we denote by $d_G(x)$ or just $d(x)$ the *degree* of x (the number of edges incident with x, having loops counted twice), and by $F(x)$ the multiset of faces σ such that x is contained in the closure $\bar{\sigma}$; the multiplicity of a face σ is the number of times that x is visited when one travels along the sides of σ in an orientation. For instance, in Fig. 1, the multiplicity of σ_1 is 1 at both vertices u_3 and u_5, while its multiplicity at u_4 is 2; the multiplicity of σ_2 is 1 at vertices u_1 and u_4, and its multiplicity is 2 at u_2 and 3 at u_3. Thus $1 \le d(x) = |F(x)| < \infty$ for all vertices x. We denote by $V(G)$, $E(G)$, and $F(G)$ the sets of vertices, edges, and faces of G, respectively. The *cell complex* of G is the collection

$$\Delta(G) := V(G) \cup E(G) \cup F(G).$$

Definition 1.1. Let G be a graph (finite or infinite) embedded in a closed 2-manifold M. The *combinatorial curvature* of G is the function $\Phi_G : V(G) \to \mathbb{R}$, defined for $x \in V(G)$ by

$$\Phi_G(x) = 1 - \frac{d(x)}{2} + \sum_{\sigma \in F(x)} \frac{1}{|\sigma|}. \tag{1.1}$$

The number $\Phi_G(x)$ is called the curvature of G at the vertex x.

The definition $\Phi_G(x)$ applies to graphs with loops and multiple edges and does not require (E1) and (E2). If an embedding of G satisfies (E1) and (E2), the graph G can not have loops and vertices of degree 1 (multiple edges are allowed). So we assume throughout the whole paper that G is a simple graph. When G is embedded in a closed 2-manifold M, we further assume that (E1) and (E2) are automatically satisfied. We write $\Phi(x)$ instead of $\Phi_G(x)$ whenever the graph G is clear in the context. It is observed by Higuchi [8] that $\Phi(x)$ may be used to measure the "difficulty" of tiling a plane by regular polygons at the vertex x in the following sense: each face σ incident with x forms an angle; if we regard σ as a regular $|\sigma|$-gon and assign $\pi - \frac{2\pi}{|\sigma|}$ to this angle, then

$$\sum_{\sigma \in F(x)} \left(\pi - \frac{2\pi}{|\sigma|}\right) = 2\pi\Big(1 - \Phi(x)\Big).$$

Thus, if $\Phi(x) \neq 0$, the union $\bigcup_{\sigma \in F(x)} \bar{\sigma}$ can not be embedded in the plane "conformally". However, if $\Phi(x) = 0$, $d(x) = d$, and $|\sigma| =$ const at every vertex v of σ, then $|\sigma| = \frac{2d}{d-2}$. There are three possibilities for the pair $\big(|\sigma|, d\big)$: (6, 3), (4, 4), (3, 6), which correspond to the tilings of the plane by triangles, rectangles, and hexagons, respectively. Higuchi [8] made the following finiteness conjecture.

Conjecture 1.2. (Higuchi) If $\Phi(x) > 0$ for all $x \in V(G)$, then G has finite number of vertices.

The conjecture was partly confirmed by Higuchi himself [8] for some special cases, and by Sun and Yu [12] for the case of 3-regular graphs. The conjecture was fully solved geometrically by Devos and Mohar [5] by establishing a Gauss-Bonnet inequality on polygonal surface. In this paper we are interested in finding the relation between the number of vertices with non-vanishing curvature and the finiteness of the following absolute total curvature.

Definition 1.3. *Let G be a graph (finite or infinite) embedded in a closed 2-manifold M. The absolute total curvature* of G is the sum

$$|\Phi|(G) := \sum_{x \in V(G)} |\Phi(x)|.$$

The first main result of the paper is the following theorem.

Theorem 1.4. *Let G be a simple graph embedded in a closed 2-manifold M, satisfying (E1) and (E2). Then G has finite number of vertices of non-vanishing curvature if and only if $\sum_{x \in V(G)} |\Phi(x)| < \infty$.*

It is trivial that the finiteness of the number of vertices with non-vanishing curvature implies the finiteness of the absolute total curvature. However, the converse is not obvious. The graphs in Fig. 16 are non-trivial examples regarding the Theorem 1.4. The proof of Theorem 1.4 is given in Section 3.

Definition 1.5. *Let G be a graph (finite or infinite) embedded in a closed 2-manifold M. For each end $p \in \mathrm{End}(G)$, let U_n be a sequence of open neighborhoods of p satisfying the conditions: (i) each U_n is homeomorphic to an open disk, its closure $\overline{U_n}$*

is homeomorphic to a closed disk, and $\overline{U}_n \subset U_{n-1}$*; (ii) all the boundaries* ∂U_n *are disjoint cycles of* G*; and (iii)* $\bigcap_{n=1}^{\infty} U_n = \{p\}$*. Set* $G_n = G \cap (M - U_n)$*. The* curvature *of* G *at* p *is the limit (whenever exists)*

$$\Phi_G(p) := \lim_{n\to\infty} \sum_{x \in V(\partial U_n)} \Phi_{G_n}(x). \tag{1.2}$$

The limit in (1.2) may not exist. However, if the absolute total curvature is finite, the limit does exist and is unique; see Lemma 3.2. Conversely, if the limit exits at each end, it is obvious that the absolute total curvature must be finite. The next main result of the paper is the following Gauss-Bonnet formula (or Euler's relation).

Theorem 1.6. *Let* G *be a simple graph embedded in a closed 2-manifold* M*, satisfying (E1) and (E2). If the absolute total curvature* $|\Phi|(G)$ *is finite, then*

$$\sum_{x \in \overline{V}(G)} \Phi_G(x) = \chi(S), \tag{1.3}$$

where $\overline{V}(G) := V(G) \cup \mathrm{End}(G)$*,* $S := M - \mathrm{End}(G)$ *is the punctured surface, and* $\chi(S) := \chi(M) - \#(\mathrm{End}(G))$ *is the Euler characteristic number of* S*.*

The most interesting and important problem about the combinatorial curvature is perhaps to classify embedded graphs whose curvatures satisfy certain properties. We are interested in classifying the embedded graphs with positive curvature at every vertex. To state our result on such classification of finite graphs with positive curvature everywhere, we introduce a special type of *projective wheel graphs* P_n, and two types of *sphere annulus graphs* A_n and B_n with $n \geq 3$ vertices. The vertex set of P_n is

$$V(P_n) = \{x_1, x_2, \ldots, x_n\}$$

and the edge set of P_n is given as follows:

Let $x_i = x_{n+i}$ for $1 \leq i \leq n$. For odd $n = 2s + 1$,

$$E(P_{2s+1}) = \{x_i x_{s+i},\ x_i x_{s+i+1} : 1 \leq i \leq s+1\},$$

and for even $n = 2s$,

$$E(P_{2s}) = \{x_i x_{s+i} : 1 \leq i \leq s\}.$$

The examples for P_n are demonstrated in Fig. 2.
The vertex sets of A_n and B_n are the same, having $2n$ vertices as

$$V(A_n) = V(B_n) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}.$$

The edge set of A_n is

$$E(A_n) = \{x_i x_{i+1},\ y_i y_{i+1},\ x_i y_i : 1 \leq i \leq n\},$$

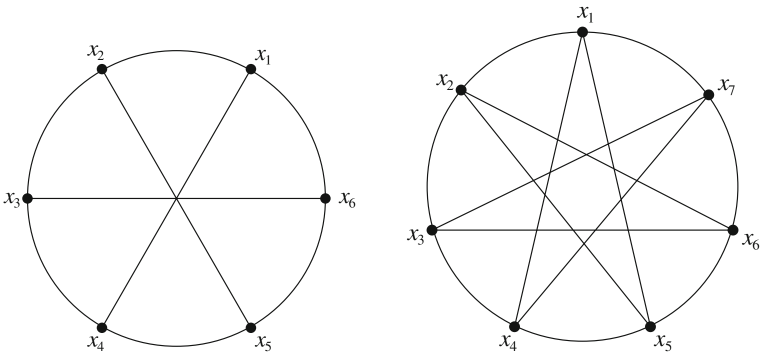


Fig. 2. The projective wheel graphs P_6 and P_7

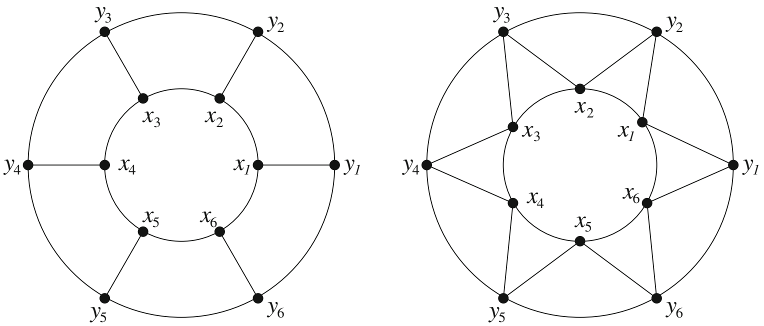


Fig. 3. The sphere annulus graphs A_6 and B_6

and the edge set of B_n is

$$E(B_n) = \{x_i x_{i+1},\ y_i y_{i+1},\ x_i y_i,\ x_i y_{i+1} : 1 \leq i \leq n\},$$

where $x_{n+1} = x_1$, $y_{n+1} = y_1$. The examples for A_n and B_n are demonstrated in Fig. 3.

The third main result of the paper is the following Theorem 1.7, a classification of finite graphs having positive curvature at every vertex and having large enough number of vertices. Devos and Mohar [5] independently obtained almost the same results by different methods, except that our bound for projective plane is 1722 rather than their bound 3444.

Theorem 1.7. *Let G be a finite simple graph embedded in a closed 2-manifold such that* $3 \leq d(x) < \infty$, $\Phi(x) > 0$ *for all* $x \in V(G)$, *and (E1) and (E2) are satisfied.*

(a) *If G is embedded in a projective plane with* $n \geq 1722$ *vertices, then G is isomorphic to the projective wheel graph* P_n.
(b) *If G is embedded in a sphere with* $n \geq 3444$ *vertices, then G is isomorphic to the sphere annulus graphs either* A_n *or* B_n.

The lower bounds 1722 and 3444 are conveniently selected so that the proof of our statement can be easily obtained. Of course, the lower bounds may be drastically lowered. Finally, for 5-regular graphs we further obtain that there are only possibly 16 such projective plane graphs and possibly 49 such plane graphs as stated in Theorem 1.8. Some of these graphs are illustrated in Fig. 13 and Fig. 14. We believe that there may be only a few of such graphs.

Theorem 1.8. *Let G be a finite* 5*-regular simple graph and* $\Phi(x) > 0$ *for all* $x \in V(G)$.

(a) *If G is embedded in a projective plane, then G has* 16 *possible cases listed in the proof, each contains at most 30 vertices.*
(b) *If G is embedded in a sphere, then G has* 49 *possible cases listed in the proof, each contains at most* 60 *vertices, and there exists a graph G having exactly* 60 *vertices.*

It is interesting and important to construct a concrete example of infinite graph whose curvature is non-negative at every vertex but there are infinitely many vertices with positive curvature. We doubt the existence of such an infinite graph. However, there exist infinite plane graphs having non-negative curvature at every vertex but having only finite number vertices of positive curvature; see Fig. 16. We end up the introduction by making the following conjecture.

Conjecture 1.9. Let G be an infinite plane graph with non-negative curvature at every vertex. Then there are only finite number of vertices with non-vanishing curvature.

2. Properties of Combinatorial Curvature

In this section we follow the book [14] on notations of graphs. We assume that G is a simple graph (no loops and multiple edges) embedded in a closed 2-manifold S, satisfying (E1) and (E2). Let x be a vertex of G. The degree of x is denoted by $d = d(x)$. Let $\sigma_1, \ldots, \sigma_d$ be faces of G incident with x, listed in increasing order of lengths $|\sigma_1| \le \cdots \le |\sigma_d|$. The *face vector* of G at the vertex x is the ordered tuple

$$f(x) = \big(|\sigma_1|, \ldots, |\sigma_d|\big).$$

A cycle of G is called a *face cycle* if it bounds a face of G.

Lemma 2.1. *If* $\Phi(x) > 0$ *at a vertex* x*, then* $1 \le d(x) \le 5$*. Moreover, the face vectors and the curvatures for each of the cases are characterized into the following patterns.*

(1) *For* $d(x) = 1$*, we have* $f(x) = (k)$ *with* $k \ge 5$ *and* $\Phi(x) = \frac{1}{2} + \frac{1}{k} > \frac{1}{2}$.
(2) *For* $d(x) = 2$*, we have* $f(x) = (3, k)$ *with* $k \ge 4$*,* $\Phi(x) = \frac{1}{3} + \frac{1}{k} > \frac{1}{3}$*;* $f(x) = (m, k)$ *with* $4 \le m \le k$*,* $\Phi(x) = \frac{1}{m} + \frac{1}{k} > \frac{1}{m} \ge \frac{1}{k}$.

(3) *For* $d(x) = 3$,

$f(x)$	$(3, 3, k)$, $k \geq 3$ $(3, 6, k)$, $k \geq 6$ $(3, 9, k), 9 \leq k \leq 17$ $(4, 4, k)$, $k \geq 4$ $(4, 7, k)$, $7 \leq k \leq 9$	$(3, 4, k)$, $k \geq 4$ $(3, 7, k)$, $7 \leq k \leq 41$ $(3, 10, k), 10 \leq k \leq 14$ $(4, 5, k)$, $5 \leq k \leq 19$ $(5, 5, k)$, $5 \leq k \leq 9$	$(3, 5, k)$, $k \geq 5$ $(3, 8, k)$, $8 \leq k \leq 23$ $(3, 11, k), 11 \leq k \leq 13$ $(4, 6, k)$, $6 \leq k \leq 11$ $(5, 6, k)$, $6 \leq k \leq 7$
$\Phi(x)$	$= 1/6 + 1/k$ $= 1/k$ $\geq 1/306$ $= 1/k$ $\geq 1/252$	$= 1/12 + 1/k$ $\geq 1/1722$ $\geq 1/210$ $\geq 1/380$ $\geq 1/90$	$= 1/30 + 1/k$ $\geq 1/552$ $\geq 1/858$ $\geq 1/132$ $\geq 1/105$

(4) *For* $d(x) = 4$,

$f(x)$	$(3, 3, 3, k)$, $k \geq 3$ $(3, 3, 5, k), 5 \leq k \leq 7$	$(3, 3, 4, k), 4 \leq k \leq 11$ $(3, 4, 4, k)$, $4 \leq k \leq 5$
$\Phi(x)$	$= 1/k$ $\geq 1/105$	$\geq 1/132$ $\geq 1/30$

(5) *For* $d(x) = 5$, *we have* $f(x) = (3, 3, 3, 3, k)$ *with* $3 \leq k \leq 5$ *and* $\Phi(x) \geq \frac{1}{30}$.

Proof. For $d(x) \geq 6$, the face vector $f(x)$ at the vertex x with the largest curvature is the vector $f(x) = (3, \dots, 3)$ and the corresponding curvature at x is

$$\Phi(x) = 1 - \frac{d}{2} + \frac{d}{3} = -\frac{d-6}{6} \leq 0.$$

The other five cases are routine enumerations. □

The following Lemma 2.2 is due to Higuchi [8].

Lemma 2.2. *If* $\Phi(x) < 0$ *at a vertex* x*, then* $\Phi(x) \leq -\varepsilon$*, where* $\varepsilon = 1/1806$.

Lemma 2.3. *If* $\Phi(x) = 0$ *at a vertex* x*, then* $3 \leq d(x) \leq 6$*. Moreover, the face vectors for each case are characterized into the following patterns.*

(1) *For* $d(x) = 3$, $f(x) = (3, 7, 42), (3, 8, 24), (3, 9, 18), (3, 10, 15), (3, 12, 12), (4, 5, 20), (4, 6, 12), (4, 8, 8), (5, 5, 10), (6, 6, 6)$.
(2) *For* $d(x) = 4$, $f(x) = (3, 3, 4, 12), (3, 3, 6, 6), (3, 4, 4, 6), (4, 4, 4, 4)$.
(3) *For* $d(x) = 5$, $f(x) = (3, 3, 3, 3, 6), (3, 3, 3, 4, 4)$.
(4) *For* $d(x) = 6$, $f(x) = (3, 3, 3, 3, 3, 3)$.

Proof. For $d = d(x) \geq 7$, the face vector with the largest curvature is $f(x) = (3, \dots, 3)$ and the curvature at x is

$$\Phi(x) = 1 - \frac{d}{2} + \frac{d}{3} = -\frac{d-6}{6} < 0.$$

The other cases are routine calculations. □

Corollary 2.4. *Let σ be a face of an embedded graph G such that $|\sigma| \geq 13$ and $\Phi(x) \geq 0$ for all $x \in V(\partial\sigma)$. Then $\partial\sigma$ is a cycle of G. Moreover, if C_1 and C_2 are cycles of lengths $|C_1| \geq 13$ and $|C_2| \geq 13$, then C_1 and C_2 are disjoint.*

Proof. Note that the boundary of a face is a cycle if and only if the face is counted once at each of its vertices. It is easily checked from the data in Lemma 2.1 and Lemma 2.3 that there are no two faces of both length at least 13 near a common vertex. So $\partial\sigma$ is a cycle if $|\partial\sigma| \geq 13$. Moreover, if C_1 and C_1 are cycles of G such that $|C_1| \geq 13$ and $|C_2| \geq 13$, then C_1 and C_2 must be disjoint. □

Contrary to the phenomenon that the combinatorial curvature is bounded away from zero whenever it is negative, the combinatorial curvature being positive does not imply that it is bounded away from zero. Actually there is no universal lower bound for positive combinatorial curvature. For example, the projective wheel graphs P_n and the sphere annulus graphs A_n and B_n have curvature $\Phi(x) = 1/n$ at every vertex x.

Lemma 2.5. *Let σ be a face of the graph G. If either $|\sigma| \geq 43$ and $\Phi(x) \geq 0$ for all $x \in V(\partial\sigma)$, or $|\sigma| \geq 42$ and $\Phi(x) > 0$ for all $x \in V(\partial\sigma)$, then the $\partial\sigma$ is a cycle of G and*

$$\sum_{x \in V(\partial\sigma)} \Phi(x) \geq 1.$$

Proof. It follows directly from Corollary 2.4 that the boundary $\partial\sigma$ is a cycle of G. Write $k = |\sigma|$. By Lemma 2.1 and Lemma 2.3, the face vector $f(x)$ at each vertex $x \in \partial\sigma$ has the following possible patterns:

$$(k),\ (m, k),\ (3, 3, k),\ (3, 4, k),\ (3, 5, k),\ (3, 6, k),\ (4, 4, k),\ (3, 3, 3, k).$$

Note that $\Phi(x) \geq \frac{1}{k}$ for each of these patterns. Hence $\sum_{x \in \partial\sigma} \Phi(x) \geq 1$. □

Corollary 2.6. *Let $\sigma_1, \ldots, \sigma_n$ be faces of the graph G. If $\Phi(x) \geq 0$ for all $x \in \partial\sigma_1 \cup \cdots \cup \partial\sigma_n$ and $|\sigma_i| \geq 43$ for $1 \leq i \leq n$, then*

$$\sum_{x \in V(\partial\sigma_1 \cup \cdots \cup \partial\sigma_n)} \Phi(x) \geq n.$$

Moreover, the boundaries $\partial\sigma_1, \ldots, \partial\sigma_n$ are actually disjoint cycles of G.

Proof. Since $\Phi(x) \geq 0$ for every vertex x of G and $|\sigma_i| \geq 43$ for all i, then by Corollary 2.4 the boundaries $\partial\sigma_1, \ldots, \partial\sigma_n$ are disjoint cycles. The inequality follows immediately from Lemma 2.5. □

Lemma 2.7. *Let x be a vertex of the graph G such that $0 < \Phi(x) < 1/1722$. Then x is on a cycle C of G, bounding a face of length at least 43.*

Proof. We first show that there is a face at x whose length is at least 43. Suppose this is not true. Then all faces at x have length at most 42. Since $\Phi(x) > 0$, by Lemma 2.1, we have

$$\Phi(x) \geq \min\left\{\frac{1}{k}, \frac{1}{6}, \frac{1}{12}, \frac{1}{30}, \frac{1}{90}, \frac{1}{105}, \frac{1}{132}, \frac{1}{252}, \frac{1}{210}, \frac{1}{306}, \frac{1}{380}, \frac{1}{552}, \frac{1}{585}, \frac{1}{1722}\right\} = \frac{1}{1722},$$

where $k \leq 42$. This is contradictory to $\Phi(x) < \frac{1}{1722}$. Thus x must be on the boundary of a face whose length is at least 43. Since there are no two faces of length at least 12 at any vertex by Lemma 2.1, the boundary of any face of length at least 43 is actually a cycle of G. □

3. The Proof of Theorem 1.4 and Theorem 1.6

Let D be the unit open disk of $\mathbb{R}^2$ with the center at the origin. The boundary of D, denoted by ∂D, is the unit circle. Let A be an open subset of the surface S. The *boundary* of A is the set $\partial A = \overline{A} - A$, which is not necessarily homeomorphic to a circle. We say that a pair $(A, \partial A)$ is *homeomorphic* to $(D, \partial D)$ if there is a homeomorphism $\phi : \overline{A} \to \overline{D}$ such that $\phi(A) = D$ and $\phi(\partial A) = \partial D$.

First Proof of Theorem 1.4. We shall show that if $\sum_{x \in V(G)} |\Phi(x)| < \infty$, there are finite number of vertices having non-vanishing curvature. By Lemma 2.2, if $\Phi(x) < 0$ then $\Phi(x) \leq -1/1806$. So the number of vertices with negative curvature must be finite. Suppose there are infinitely many vertices of positive curvature. These infinitely many vertices must have an accumulation point p in the closed 2-manifold S.

Let v_n be a sequence of vertices convergent to p and $\Phi(v_n) > 0$ for all n. Since $\sum_{n=1}^{\infty} |\Phi(v_n)| < \infty$, then $\Phi(v_n) \to 0 (n \to \infty)$. Since negative curvature is bounded away from zero, there is a positive integer N such that $0 \leq \Phi(v_n) < 1/1722$ for all $n \geq N$.

Let U_1 be an open neighborhood of p such that ∂U_1 is a cycle of G and $\Phi(x) \geq 0$ for all $x \in V(\overline{U}_1)$. We then have

$$\sum_{x \in V(\overline{U}_1)} \Phi(x) < \infty. \tag{3.1}$$

Let v_{n_1} be a vertex in U_1 with $n_1 > N$. Then $0 < \Phi(v_{n_1}) < 1/1722$. By Lemma 2.7, there is a face σ_1 at v_{n_1} such that $|\sigma_1| \geq 43$. Clearly, $\bar{\sigma}_1 \subset \overline{U}_1$, so $\Phi(x) \geq 0$ for all $x \in V(\partial\sigma_1)$. Hence by Lemma 2.5, $\sum_{x \in V(\partial\sigma_1)} \Phi(x) \geq 1$.

Let U_2 be an open neighborhood of p such that $\overline{U}_2 \subset U_1$ and $\overline{U}_2$ is disjoint from $\overline{\sigma}_1$. Let v_{n_2} be a vertex in U_2 with $n_2 > n_1$. Again $0 < \Phi(v_{n_2}) < 1/1722$; and by Lemma 2.7, there is a face σ_2 at v_{n_2} such that $|\sigma_2| \geq 43$. Clearly, $\overline{\sigma}_2 \subset \overline{U}_2$, so $\Phi(x) \geq 0$ for all $x \in V(\sigma_2)$. By Lemma 2.5, $\sum_{x \in V(\partial\sigma_2)} \Phi(x) \geq 1$.

Continuing this procedure we obtain a sequence $\sigma_m (m = 1, 2, \ldots)$ of faces whose closures $\bar{\sigma}_m$ are disjoint. Thus

$$\sum_{x \in V(\overline{U}_1)} \Phi(x) \geq \sum_{x \in V(\bigcup_{m=1}^{\infty} \partial \sigma_m)} \Phi(x) = \sum_{m=1}^{\infty} \sum_{x \in V(\partial \sigma_m)} \Phi(x) = \infty.$$

This is contradictory to (3.1). □

Second Proof of Theorem 1.4. We only need to show that if the absolute total curvature $|\Phi|(G)$ is finite, then G has finite number of vertices of non-vanishing curvature. Suppose this is not true, that is, there is a graph G such that $|\Phi|(G) < \infty$, but G has infinitely many vertices of non-vanishing curvature. Let $\varepsilon = 1/1806$ and

$$V_\varepsilon = \{x \in V(G) : 0 < |\Phi(x)| < \varepsilon\}.$$

If $\Phi(x) < 0$, then $|\Phi(x)| \geq \varepsilon$ by Lemma 2.2. Thus $\Phi(x) > 0$ for all $x \in V_\varepsilon$. Since $\sum_{\Phi(x) \neq 0} |\Phi(x)| < \infty$, the complement $V - V_\varepsilon$ must be a finite set. In other words V_ε is an infinite set. Since Lemma 2.7 and $\varepsilon < \frac{1}{1722}$, there is a face σ_v of length at least 43 at each vertex $v \in V_\varepsilon$.

Note that the number of vertices having negative curvature is finite. The set of these finite number of vertices can intersect only finite number of closed faces in $\{\overline{\sigma}_v : v \in V_\varepsilon\}$. In other words, there are infinitely many closed faces in $\{\overline{\sigma}_v : v \in V_\varepsilon\}$ whose vertices have non-negative curvature. By Lemma 2.5, $\sum_{x \in V(\partial \overline{\sigma}_v)} \Phi(x) \geq 1$ for each of these closed faces $\overline{\sigma}_v$ with non-negative curvature. On the other hand, since $|\overline{\sigma}_v| \geq 43$, then by Corollary 2.4 the faces $\overline{\sigma}_v$ with non-negative curvature are disjoint. Thus $\sum_{x \in V(G)} |\Phi(x)| = \infty$. This is contradictory to (3.1). □

Lemma 3.1. *Let G be an infinite graph embedded in a closed 2-manifold M, and let p be an end of G. Let W be an open neighborhood of p such that $(W, \partial W)$ is homeomorphic to $(D, \partial D)$. Then W contains an open neighborhood U of p such that $(U, \partial U)$ is homeomorphic to $(D, \partial D)$ and ∂U is a cycle of G.*

Proof. Let $\Delta(G)$ denote the cell complex whose cells are vertices, open edges, and open faces of G on the closed 2-manifold M. Let $\Delta(G, W)$ be the subcomplex of $\Delta(G)$, generated by the cells intersecting the boundary ∂W, that is,

$$\Delta(G, W) = \{\sigma \in \Delta(G) : \sigma \subseteq \tau,\ \tau \cap \partial W \neq \emptyset\}.$$

Then the union $|\Delta(G, W)| = \bigcup_{\sigma \in \Delta(G,W)} \sigma$ is a closed subset of M, and $p \notin |\Delta(G, W)|$. So the relative complement $W - |\Delta(G, W)|$ is an open neighborhood of p. Let U be the connected component of $W - |\Delta(G, W)|$ that contains p. Thus U is an open neighborhood of p, and $(U, \partial U)$ is homeomorphic to $(D, \partial D)$. The boundary $\partial U = \overline{U} - U$ must be a cycle of G by the construction. □

Lemma 3.2. *Let G be a graph embedded in a closed 2-manifold M. If $|\Phi|(G)$ is finite, then for each end $p \in \mathrm{End}(G)$ the curvature $\Phi_G(p)$ exists and does not depend on the graph sequence G_n in Definition 1.5.*

Proof. Let U_n and W_n be two sequences of open neighborhoods of the end p, satisfying the conditions in Definition 1.5. That is, $(U_n, \partial U_n)$ and $(W_n, \partial W_n)$ are homeomorphic to $(D, \partial D)$; $\overline{U}_n \subset U_{n-1}$, $\overline{W}_n \subset W_{n-1}$, $\bigcap_n U_n = \bigcap_n W_n = \{p\}$; ∂U_n and ∂W_n are cycles of G. Let $G_n = G \cap (M - U_n)$ and $H_n = G \cap (M - W_n)$.

For a fixed m and an arbitrary n such that $\overline{U}_n \subset W_m$, we define a cylinder surface $S_{m,n} := \overline{W}_m - U_n$ and a finite graph $G_{m,n} := G \cap S_{m,n}$ embedded in $S_{m,n}$; see Fig. 4. Since the Euler characteristic of $S_{m,n}$ is zero, we have

$$\begin{aligned} 0 = \sum_{x \in V(G_{m,n})} \Phi_{G_{m,n}}(x) &= \sum_{x \in V(\partial U_n)} \Phi_{G_{m,n}}(x) \\ &+ \sum_{x \in V(\partial W_m)} \Phi_{G_{m,n}}(x) + \sum_{\substack{x \in V(G_{m,n}) \\ x \notin V(\partial W_m \cup \partial U_n)}} \Phi_{G_{m,n}}(x) \\ &= \alpha(m,n) + \beta(m,n) + \gamma(m,n), \end{aligned}$$

where $\alpha(m,n)$, $\beta(m,n)$, and $\gamma(m,n)$ are defined as the last three sums of the above right-hand side, respectively. Notice that

$$\Phi_{G_{m,n}}(x) = \begin{cases} \Phi_{G_n}(x) & \text{if } x \in V(\partial U_n), \\ \Phi_{G \cap \overline{W}_m}(x) & \text{if } x \in V(\partial W_m), \\ \Phi_G(x) & \text{if } x \in V(G_{m,n}) - V(\partial U_n \cup \partial W_m). \end{cases}$$

Applying Theorem 1.4, we see that the number of vertices with non-vanishing curvature is finite. We may choose m large enough so that $\overline{W}_m$ contains no vertices of non-vanishing curvature. It then follows that when n is large enough, $\alpha(m,n)$ is independent of m, so we may write $\alpha(m,n) = \alpha(n)$; $\beta(m,n)$ is independent of n, so we may write $\beta(m,n) = \beta(m)$; and $\gamma(m,n) = 0$. Thus for large enough fixed m we have $\alpha(n) = -\beta(m)$ for large n. This means that the sequence $\alpha(n)$ is eventually

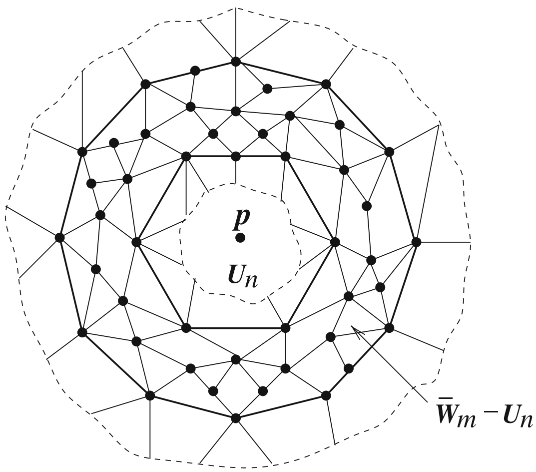


Fig. 4. The surface $S_{m,n}$ and the graph $G_{m,n}$

a constant α. Subsequently, the sequence $\beta(m)$ is eventually the constant $-\alpha$. Of course $\lim_{n\to\infty}\alpha(n)$ exists and

$$\lim_{n\to\infty}\alpha(n)=\lim_{n\to\infty}\sum_{x\in V(\partial U_n)}\Phi_{G_n}(x)=\alpha.$$

Analogously, for a fixed n and an arbitrary m' such that $\overline{W}_{m'}\subset U_n$, let $S'_{n,m'}=\overline{U}_n-W_{m'}$ and $G'_{n,m'}=G\cap S'_{n,m'}$; we shall obtain sequences $\alpha'(m')$ and $\beta'(n)$ such that for large enough fixed n, $\alpha'(m')=-\beta'(n)$ for large m'. Similarly, the sequences $\alpha'(m')$ and $\beta'(n)$ are eventually constants α' and $-\alpha'$, respectively. Of course, $\lim_{m'\to\infty}\alpha'(m')$ exists and

$$\lim_{m'\to\infty}\alpha'(m')=\lim_{m'\to\infty}\sum_{x\in V(\partial W_{m'})}\Phi_{G_{m'}}(x)=\alpha'.$$

Now since there are only finite number of vertices with non-vanishing curvature, then for large enough n we have $\sum_{x\in V(\partial U_n)}\Phi_G(x)=0$. Note that

$$\sum_{x\in V(\partial U_n)}\Phi_G(x)=\sum_{x\in V(\partial U_n)}\Big(\Phi_{G_{m,n}}(x)+\Phi_{G'_{n,m'}}(x)+\Phi_{\partial U_n}(x)\Big)$$

and $\sum_{x\in V(\partial U_n)}\Phi_{\partial U_n}(x)=\chi(\partial U_n)=0$. It follows that $\alpha(n)+\beta'(n)=0$ for large enough n. This means that $\alpha-\alpha'=0$. Hence the limit in question is unique. □

Proof of Theorem 1.6. For each end $p\in\mathrm{End}(G)$, let $U_n(p)$ be a sequence of open neighborhoods of p such that $(U_n,\partial U_n)$ is homeomorphic to $(D,\partial D)$, $\overline{U}_n\subset U_{n-1}$, ∂U_n is a cycle of G, and $\bigcap_n U_n=\{p\}$. We may further assume that $\overline{U}_n(p)$ for all $p\in\mathrm{End}(G)$ are disjoint. Let

$$S_n:=M-\bigcup_{p\in\mathrm{End}(G)}U_n(p)\quad\text{and}\quad G_n=G\cap S_n.$$

Then S_n is a 2-manifold with boundary $\bigcup_{p\in\mathrm{End}(G)}\partial U_n(p)$. Let $\partial G=G\cap\partial S_n$. It is clear that

$$\chi(M)-\#(\mathrm{End}(G))=\chi(M)-\chi\Bigg(\bigcup_{p\in\mathrm{End}(G)}\overline{U}_n(p)\Bigg)=\chi(S_n).$$

Let $S_n(p)=M-U_n(p)$, $G_n(p)=G\cap S_n(p)$, and $\partial G_n(p)=G\cap\partial U_n(p)$. Then $\partial G_n=\bigcup_{p\in\mathrm{End}(G)}\partial G_n(p)$. Applying the Euler formula to the surface S_n with the embedded finite graph G_n, we have

$$\begin{aligned}\chi(S_n)&=\sum_{x\in V(G_n)}\Phi_{G_n}(x)\\&=\sum_{x\in V(\partial G_n)}\Phi_{G_n}(x)+\sum_{x\in V(G_n)-V(\partial G_n)}\Phi_{G_n}(x)\\&=\sum_{p\in\mathrm{End}(G)}\sum_{x\in V(\partial G_n(p))}\Phi_{G_n(p)}(x)+\sum_{x\in V(G_n)-V(\partial G_n)}\Phi_G(x).\end{aligned}$$

Let n tend to ∞ and make use of definition of $\Phi_G(p)$. We obtain

$$\chi(M) - \#(\mathrm{End}(G)) = \sum_{p \in \mathrm{End}(G)} \Phi_G(p) + \sum_{x \in V(G)} \Phi_G(x).$$

$\square$

4. Proof of Theorem 1.7 and Theorem 1.8

We first prove the the statement of Theorem 1.7 for the case of projective plane. To do this we need the following lemma.

Lemma 4.1. *Let G be a finite simple graph embedded in a projective plane such that* $3 \leq d(x) < \infty$ *and* $\Phi(x) > 0$ *for all* $x \in V(G)$. *If* $\#(V(G)) \geq 1722$, *then all vertices of G are contained in a cycle bounding a face of length* $\#(V(G))$.

Proof. We first claim that there is a face σ of G whose length is at least 43. If there is one vertex x such that $\Phi(x) < \frac{1}{1722}$, then by Lemma 2.7, there is a cycle C of length at least 43 such that C bounds a face σ.

If, on the other hand, $\Phi(x) \geq \frac{1}{1722}$ for all $x \in V(G)$, we must have $\Phi(x) = \frac{1}{1722}$ for all $x \in V(G)$ and there are exactly 1722 vertices. Then by Lemma 2.1 that either there is one vertex of G which is on a cycle of length 1722, or every vertex of G has the face vector $(3, 7, 41)$. In the former case we have had a face of length 1722 which is larger than 43. In the latter case every vertex of G has three faces of lengths 3, 7, and 41, respectively. Take a vertex x and draw its faces as in Fig. 5. Since every vertex has the face vector $(3, 7, 41)$ and the vertex u, one neighbor vertex of x, has had faces of lengths 3 and 7, then the vertex u must have a face of length 41. Thus the edge uv must be on the boundary of a face of length 41. It follows that the vertex v has two faces of length 41, where v is another neighbor vertex of x. This is contradictory to Corollary 2.4.

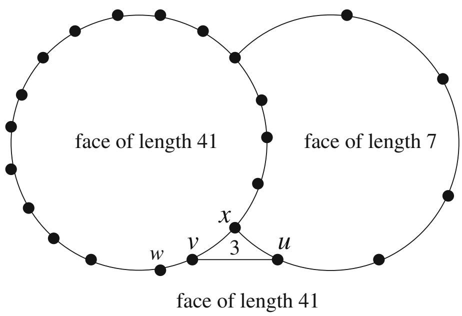


Fig. 5. Impossibility of face vector (3, 7, 41) at every vertex of G

Now we have seen that the graph G has a face σ of length at least 43. By Corollary 2.4 the boundary $\partial\sigma$ must be a cycle of G. Thus by Lemma 2.5 we have

$$1 = \sum_{x \in V(G)} \Phi_G(x) \geq \sum_{x \in V(\partial\sigma)} \Phi(x) \geq 1.$$

It forces that $V(G) = V(\partial\sigma)$. So all vertices of G are on a face cycle of G. □

THE PROOF OF THEOREM 1.7 PART (a):

Since $n \geq 1722$, it follows from Lemma 4.1 that all vertices of G are on a cycle bounding a face σ of length n.

CASE 1: *There exists a vertex of degree* 4.

Let x_1 be a vertex whose degree is 4. Then there are two edges incident with x_1 other than the edges on the cycle $\partial\sigma$; we label the two edges by 1 and 2 as shown in Fig. 6. Let the edge with label 1 be incident with a vertex y_0. We label the vertices on the cycle $\partial\sigma$ starting from x_1 in counterclockwise by $x_1, x_2, x_3, \ldots, x_i$ and label the vertices on the same cycle starting from y_0 in counterclockwise by $y_0, y_1, y_2, \ldots, y_j$, respectively. Then $i + j + 1 = n$; see Fig. 6.

Recall from Lemma 2.1 that the face vector at any vertex of degree 4 on the cycle $\partial\sigma$ is $(3, 3, 3, k)$ with $k \geq 1722$. Since $d(x_1) = 4$ and G has had the face σ of length larger than or equal to 1722, the edges x_1y_0 and y_0y_1 must bound a triangular face. It forces that x_1y_1 is an edge (label 2) of G. Thus x_1 is adjacent to y_0 and y_1. Similarly, since the edges y_1x_1 and x_1x_2 must bound a triangular face, it forces that y_1x_2 (label 3) is an edge of G. Hence $d(y_1) = 4$. By the same token the edges x_2y_1 and y_1y_2 must bound a triangular face so that x_2y_2 (label 4) is also an edge of G. Therefore x_2 is adjacent to y_1 and y_2. Now $d(y_2) = 4$; the edges y_2x_2 and x_2x_3 must bound a triangular face so that y_2x_3 (label 5) is an edge of G; and the edges x_3y_2 and y_2y_3 must bound a triangular face so that x_3y_3 is an edge (label 6) of G. We thus conclude that x_3 is adjacent to y_2 and y_3. Continuing this procedure we see that x_r is adjacent to y_{r-1} and y_r, $1 \leq r \leq \min\{i, j\}$. We divide the situation into three subcases.

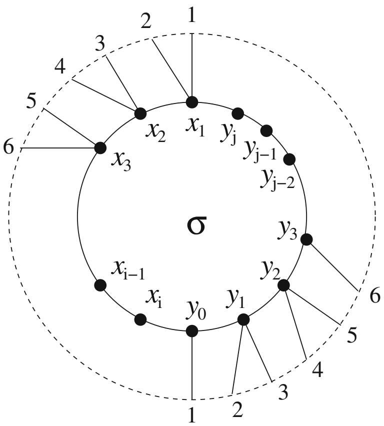


Fig. 6.

Case 1.1: $i < j$.

The edges y_0x_1 and x_1y_j must bound a triangular face so that y_0y_j is also an edge of G. Then $d(y_0) = 5$. This is a contradiction; see Fig. 7(a).

Case 1.2: $i > j$.

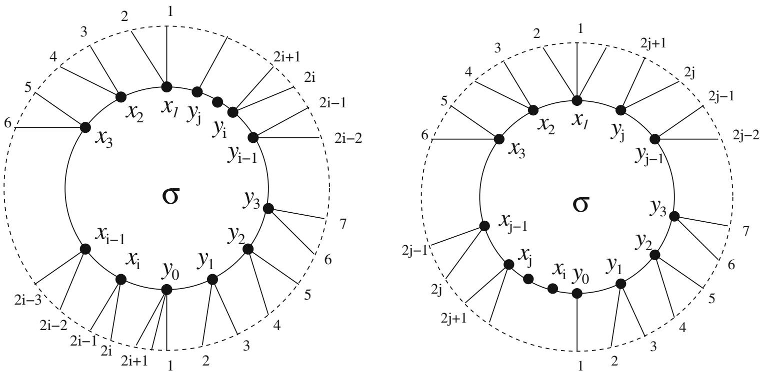
The edges x_1y_j and y_jx_j must bound a triangular face so that x_1x_j is an edge of G. Thus $d(x_1) = 5$. This is a contradiction; see Fig. 7(b).

Case 1.3: $i = j = s$.

It follows that the graph G is isomorphic to the graph P_{2s+1}; see Fig. 8(a).

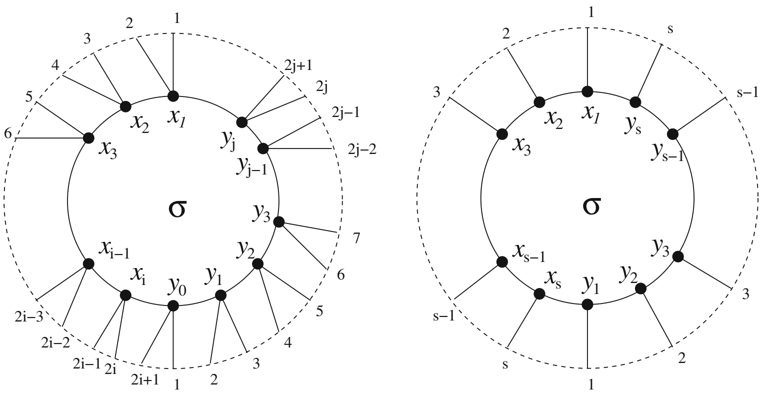
Case 2: *There is no vertex of degree* 4.

This means that every vertex of G has degree 3. Then every vertex on the cycle $\partial\sigma$ is adjacent to another vertex on the same cycle. It forces that the number of vertices on the cycle must be even. Let $n = 2s$ and let the vertices $x_1, x_2, \ldots, x_s$ be adjacent to the vertices $y_1, y_2, \ldots, y_s$, respectively. Since the graph G is embedded



(a) $i < j$ **(b)** $i > j$

Fig. 7. The impossibility cases of $i \neq j$



(a) P_{2s+1} **(b)** P_{2s}

Fig. 8. The projective plane graph isomorphic to P_n

in the projective plane, it forces that the order of $y_1, y_2, \ldots, y_s$ are the same as that of $x_1, x_2, \ldots, x_s$; see Fig. 8(b). □

Now we prove the statement of Theorem 1.7 for the case of sphere.

Lemma 4.2. *Let C be a face cycle of length at least 43 of a graph G embedded in a sphere such that $3 \le d(x) < \infty$ and $\Phi(x) > 0$ for all vertices $x \in C$. Let u and v be two vertices of C. If u and v are not adjacent in C, then u and v are not adjacent in G.*

Proof. Suppose this is not true; that is, there are two vertices u and v of C such that u and v are not adjacent in C but adjacent in G. Given an orientation of C so that the vertices of C are cyclically ordered. Let u' and u'' be the vertices that are adjacent with u in C; and let v' and v'' be the vertices that are adjacent with v in C; see Fig. 9(a). We may have $u'' = v'$. The degree $d_G(u)$ and $d_G(v)$ must be either 3 or 4, and can not be 5 by Lemma 2.1.

Case 1: $d_G(u) = 4$.

Let w be a vertex adjacent to u but w is not on C. Then w is located either in the region R_1 or in the region R_2.

Case 1.1: *The vertex w is located in the region R_1*. See Fig. 10(a).

By Lemma 2.1, the face vector at u must be $(3, 3, 3, k)$ with $k \ge 43$. So the face bounding the edges uv and uw must be a triangle. This forces that v and w must be adjacent; see Fig. 10b. By the same token, the face bounding the edges uu' and uv must be a triangle. This forces that u' and v must be adjacent, or $u' = v''$. In the former case we have $d_G(v) = 5$; this is impossible because there is no face of length at least 43 at a vertex of degree 5; see Fig. 10(c). In the latter case we have $d_G(u') = 2$, which is not allowed.

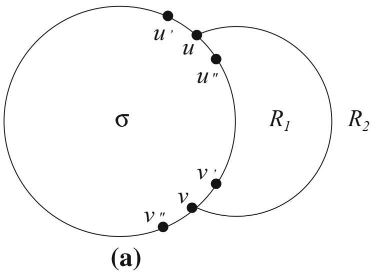
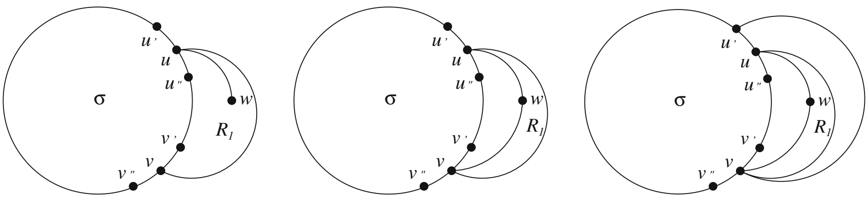
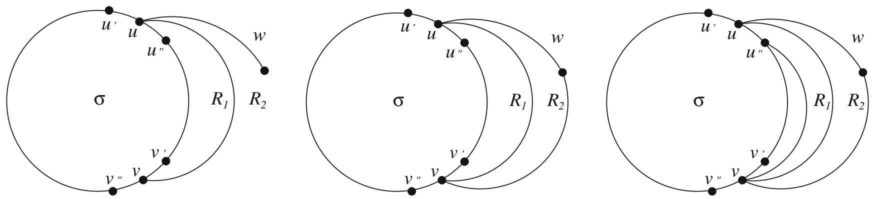


Fig. 9. u and v are not adjacent in C but adjacent in G



(a) w is located in R_1 **(b)** vw is an edge **(c)** $u'w$ is an edge

Fig. 10.



(a) w is located in R_2 **(b)** vw is an edge **(c)** $u''w$ is an edge

Fig. 11.

Case 1.2: *The vertex w is located in the region R_2.* See Fig. 11(a).

Similarly, by Lemma 2.1, the face vector at u must be $(3, 3, 3, k)$ with $k \geq 43$. Then the face bounding the edges uv and uw must be a triangle. So v and w must be adjacent; Fig. 11(b). By the same token, the face bounding the edges uu'' and uv must be a triangle. This forces that u'' and v must be adjacent, or $u'' = v'$. In the former case we have $d_G(v) = 5$; this is impossible by Lemma 2.1. In the latter case we have $d_G(u'') = 2$, which is not allowed.

Case 2: $d_G(u) = 3$.

By symmetry in Case 1 we may assume $d_G(v) = 3$. We may further assume that u and v are such a pair of adjacent vertices so that the distance $d_C(u, v)$ in C is minimal. By Lemma 2.1, the face vector at u is one of $(3, 3, k)$, $(3, 4, k)$, $(3, 5, k)$, $(3, 6, k)$, and $(4, 4, k)$ with $k \geq 43$. Note that $u'' \neq v'$; otherwise $d_G(u'') = d_G(v') = 2$, which is not allowed. Similarly, $u' \neq v''$. If there is a triangular face at u, then v is adjacent to either u' or u''. In either case we have $d_G(v) = 4$, which contradicts $d_G(v) = 3$. If the face vector at u is $(4, 4, k)$, then u'' must be adjacent to v', and u' is adjacent to v''. Hence one of the distances $d_G(u', v'')$ and $d_G(u'', v')$ must be shorter than $d_G(u, v)$, which is contradictory to the minimality of $d_G(u, v)$ or absence of vertices of degree 2 or absence of vertices of degree 2; see Fig. 9(b). □

Lemma 4.3. *Let G be a finite graph embedded in a sphere such that* $3 \leq d(x) < \infty$ *and* $\Phi(x) > 0$ *for all* $x \in V(G)$. *If* $\#(V(G)) \geq 3444$, *then all vertices of G are exactly located on two disjoint face cycles of length at least* 43.

Proof. Since the Euler characteristic of a sphere is 2, there is at least one vertex whose curvature is less than or equal to $\frac{1}{1722}$.

Case 1: *There exists one vertex x_1 such that* $\Phi(x_1) < \frac{1}{1722}$.

By Lemma 2.7 there is a face σ incident with x_1 and $|\sigma| \geq 43$. By Lemma 2.5 we have $\sum_{x \in V(\partial\sigma)} \Phi(x) \geq 1$.

Case 1.1: *The length* $|\sigma| < 1722$.

In this case we have $|V(G) - V(\partial\sigma)| > 1722$. Since $\sum_{x \notin V(\partial\sigma)} \Phi(x) \leq 1$, there is another vertex $y_1 \notin V(\partial\sigma)$ such that $\Phi(y_1) < \frac{1}{1722}$. Similarly, by Lemma 2.5 there is a face τ near y_1 such that $|\tau| \geq 43$, and by Lemma 2.5 we have $\sum_{x \in V(\partial\tau)} \Phi(x) \geq 1$.

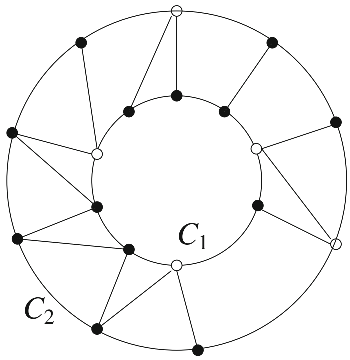


Fig. 12. The negative curvature at the white vertices

The two cycles $\partial\sigma$ and $\partial\tau$ must be disjoint by Lemma 2.4. It then follows that

$$2 = \sum_{x\in V(G)} \Phi(x) \geq \sum_{x\in V(\partial\sigma\cup\partial\tau)} \Phi(x) \geq 2.$$

Hence $V(G) = V(\partial\sigma \cup \partial\tau)$ is on exactly two cycles of G.

Case 1.2: *The length* $|\partial\sigma| = k \geq 1722$.

Let a_3, a_4, a_5, a_6, b, and c be the numbers of vertices on the cycle $\partial\sigma$ whose face vectors are $(3, 3, k)$, $(3, 4, k)$, $(3, 5, k)$, $(3, 6, k)$, $(4, 4, k)$, and $(3, 3, 3, k)$, respectively. Let G_1 be the subgraph of G induced by the vertex set $V(G) - V(\partial\sigma)$, that is, G_1 is obtained by deleting the vertices of $\partial\sigma$ and all edges incident with vertices of $\partial\sigma$. Obviously, G_1 is a plane graph with a face σ_1 that contains σ; the boundary $\partial\sigma_1$ may not be a cycle of G. Note that

$$\sum_{x\in V(\partial\sigma)} \Phi(x) = 1 + \frac{a_3}{6} + \frac{a_4}{12} + \frac{a_5}{30} \leq 2.$$

We then have $a_3 \leq 6$, $a_4 \leq 12$, and $a_5 \leq 30$. Since each vertex with the face vector $(3, 3, k)$ may result three vertices of $\partial\sigma$ corresponding to one vertex of $\partial\sigma_1$, the number of vertices of $\partial\sigma_1$ may be reduced by $2a_3$. Similarly, each vertex with the face vector $(3, 4, k)$ may result two vertices of $\partial\sigma$ corresponding to one vertex of $\partial\sigma_1$, the number of vertices of $\partial\sigma_1$ may be further reduced by a_4. However, the vertices of $\partial\sigma$ with the face vector $(3, 5, k)$ or $(3, 6, k)$ or $(3, 3, 3, k)$ can be arranged to correspond to distinct vertices of $\partial\sigma_1$. It follows that

$$|\partial\sigma_1| \geq |\partial\sigma| - 2a_3 - a_4 \geq 1722 - 2\cdot 6 - 12 = 1698.$$

If $a_3 \geq 1$ or $a_4 \geq 1$, then $\sum_{x\in V(\partial\sigma_1)} \Phi(x) \leq 1 - \frac{1}{12} = \frac{11}{12}$; it follows that there is at least one vertex $y \in V(\partial\sigma_1)$ such that $\Phi(y) \leq \frac{11}{12\cdot 1689} < \frac{1}{1722}$. By Lemma 2.7 there is a face τ of length at least 43 incident with the vertex y; and by Lemma 2.5 we have $\sum_{x\in V(\partial\tau)} \Phi(x) \geq 1$. It then forces that $V(\partial\tau) = V(G) - V(\partial\sigma)$; and subsequently, $V(\partial\tau) = V(G_1)$ and $G_1 = \partial\sigma_1$.

If $a_3 = a_4 = 0$, then $|\partial\sigma_1| \geq |\partial\sigma| \geq 1722$. We claim that $V(G) = V(\partial\sigma_1 \cup \partial\sigma)$, and if so, $\partial\sigma_1$ is obviously a face cycle of length at least 1722. Suppose $\#(V(G)) >$

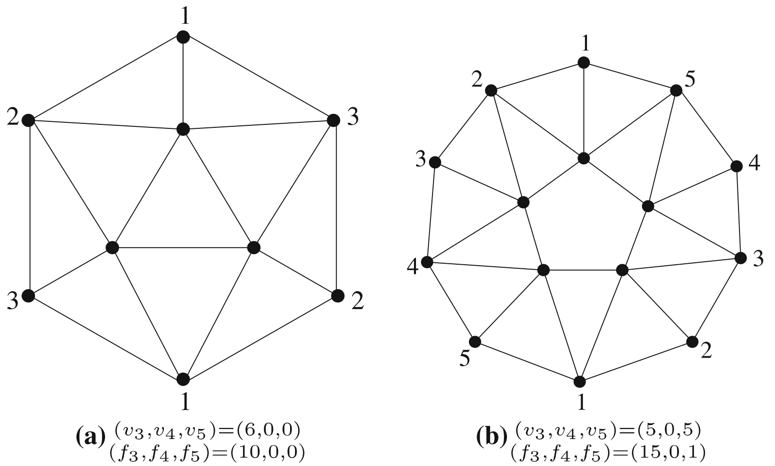


Fig. 13. 5-regular projective graphs with positive curvature at every vertex

$\#(V(\partial\sigma_1 \cup \partial\sigma))$, we have $\#(V(G_1)) > \#(\partial\sigma) \geq 1722$. Then there is a vertex $y \in V(G_1)$ such that $\Phi(y) < \frac{1}{1722}$. By Lemma 2.7 there is a face τ of length at least 43 incident with the vertex y, and by Lemma 2.5 we have $\sum_{x\in V(\partial\tau)} \Phi(x) \geq 1$. Hence $\sum_{x\in V(\partial\tau\cup\partial\sigma)} \Phi(x) \geq 2$. This forces that $V(\partial\tau) = V(G) - V(\partial\sigma)$; so we have $\partial\tau = \partial\sigma_1$ and $V(G) = V(\partial\sigma_1 \cup \partial\sigma)$ which is a contradiction.

Case 2: $\Phi(x) \geq \frac{1}{1722}$ *for all* $x \in V(G)$.

In this case we must have $\Phi(x) = \frac{1}{1722}$ for all $x \in V(G)$ and $|V(G)| = 3444$. Then the face vector at each vertex takes one of the forms (3, 6, 1722), (4, 4, 1722), (3, 3, 3, 1722), and (3, 7, 41). If there is a vertex x whose face vector is (3, 7, 41), we draw the faces at x as in Fig. 5 on the plane. Then the face vector at the vertex v must be (3, 7, 41) and the edge uv is bounding a face of length 41. Thus there are two faces of length 41 at the vertex u; this is a contradiction. So there is no vertex with the face vector (3, 7, 41). It follows that every vertex is on a face of length 1722. Take a face σ of length 1722 and a vertex v not on the boundary $\partial\sigma$. There is a face τ of length 1722 at v. Then $\partial\sigma$ and $\partial\tau$ are disjoint by Corollary 2.4, and by Corollary 2.6 we have $\sum_{x\in V(\partial\sigma\cup\partial\tau)} \Phi(x) \geq 2$. It follows that $V(G) = V(\partial\sigma \cup \partial\tau)$ and all vertices of G are on two face cycles $\partial\sigma$ and $\partial\tau$ of equal length 1722. □

PROOF OF THEOREM 1.7 PART (b):

Let C_1 and C_2 be two face cycles of length at least 43 and $V(G) = V(C_1 \cup C_2)$. Let $B(C_1, C_2)$ be the bipartite graph whose vertex set is $V(C_1) \cup V(C_2)$ and the edge set is $E(G) - E(C_1) - E(C_2)$. Note that the degree of every vertex of G is either 3 or 4 by Lemma 2.1. It follows that the degree of every vertex of the bipartite graph $B(C_1, C_2)$ is either 1 or 2. Hence $B(C_1, C_2)$ is a disjoint union of cycles and paths.

If there exists a cycle C in $B(C_1, C_2)$, since G is a plane graph and $B(C_1, C_2)$ is in the annulus between the face cycles C_1 and C_2, the cycle C must be a Hamilton cycle and all edges of $B(C_1, C_2)$ are on the Hamilton cycle C. Thus the cycles C_1 and C_2 have the same length n, and the graph G is isomorphic to the sphere annulus graph B_n.

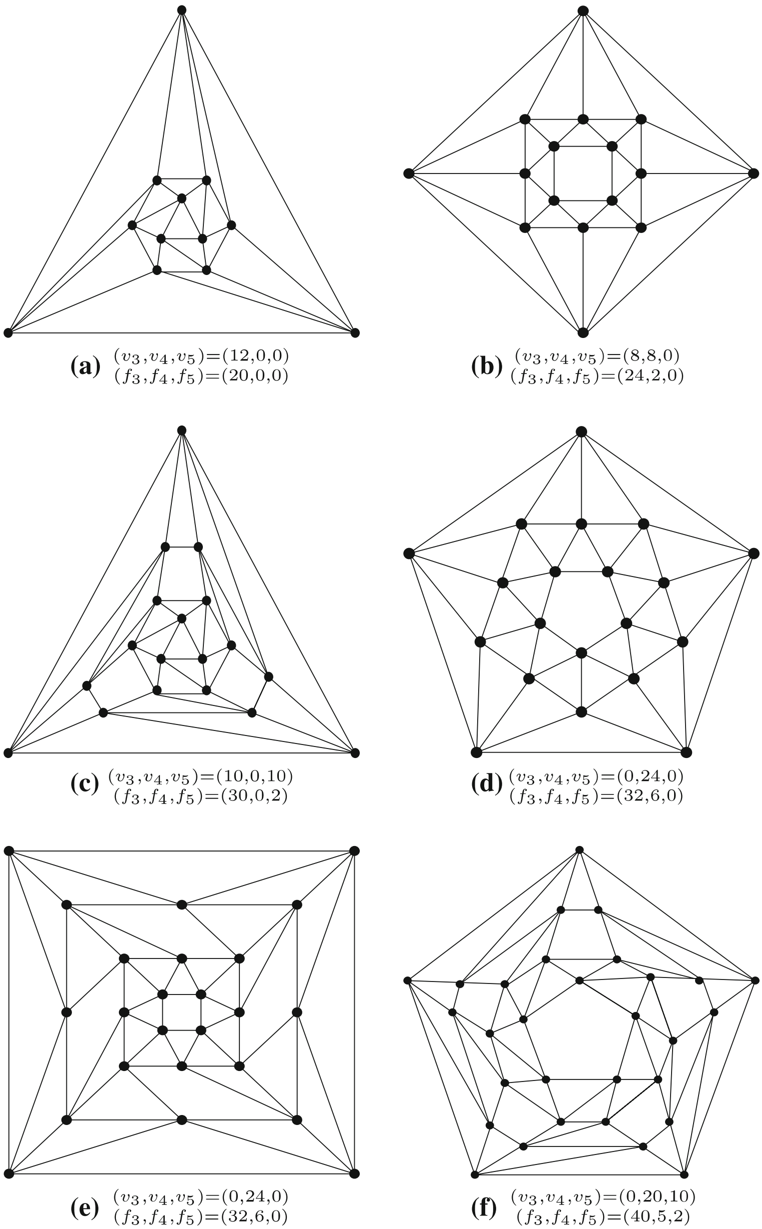


Fig. 14. 5-regular plane graphs with positive curvature at every vertex

If, on the other hand, there is no cycle in $B(C_1, C_2)$, that is, all connected components are paths, we claim that all the paths are single edges. Suppose there are paths of length larger than 1, say, $P = z_0 z_1 \cdots z_l$ with $l \geq 2$, where the degree of the beginning and ending vertices z_0 and z_l are 1 in $B(C_1, C_2)$. Then the curvature

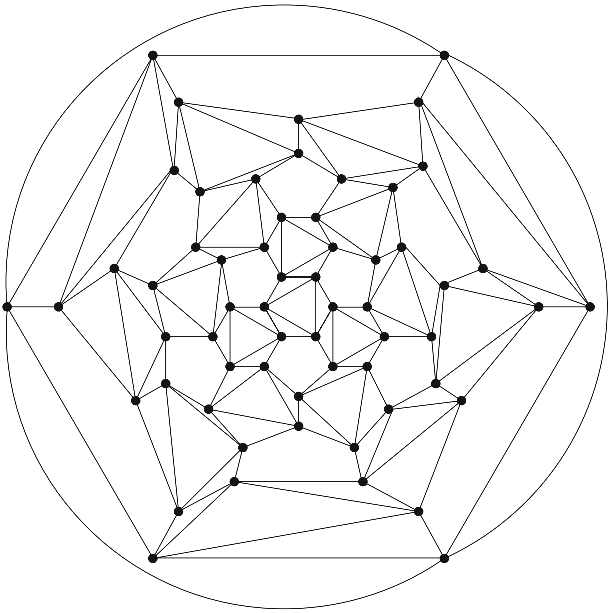


Fig. 15. $(v_3, v_4, v_5) = (0, 0, 60)$, $(f_3, f_4, f_5) = (80, 0, 12)$; the 5-regular plane graph of maximal number of vertices with positive curvature everywhere

of G at the vertices z_1 and z_{l-1} are negative for their degrees are 4 and their face vectors are of type $(3, r, s, t)$, where $r \geq 3$, $s \geq 4$, and $t \geq 43$. This is a contradiction. See Fig. 12 for example, where the white vertices have negative curvature. Thus all the paths in $B(C_1, C_2)$ have length one, that is, all the paths are single edges. Hence the cycles C_1 and C_2 have the same length n, and the graph G is isomorphic to the sphere annulus graph A_n. □

Proof of Theorem 1.8. Let G be a 5-regular and finite simple graph embedded in a closed 2-manifold M with positive curvature at every vertex. By Lemma 2.1 the face vector at any vertex x is $(3, 3, 3, 3, k)$ with $k = 3, 4, 5$. Let v, e, and f be the number of vertices, edges, and faces of G, respectively. We denote by v_k the number of vertices whose face vector is $(3, 3, 3, 3, k)$ and by f_k the number of faces whose length is k, where $k = 3, 4, 5$.

Since $5v = 2e$ and the Euler relation $v - e + f = \chi(M)$, we have $f = 3v/2 + \chi(M)$, where $\chi(M) = 1$ if M is a projective plane and $\chi(M) = 2$ if M is a sphere. Note that there are no two faces of length larger than or equal to 4 at any vertex. The boundaries for faces of length larger than or equal to 4 must be disjoint. Using the relation among the vertices, edges, and faces, we obtain

$$v_4 = 4f_4, \quad v_5 = 5f_5, \quad 5v = 2e = 3f_3 + 4f_4 + 5f_5.$$

Since $v = v_3 + v_4 + v_5$ and $f = f_3 + f_4 + f_5$, we further obtain, writing simply χ for $\chi(M)$,

$$10v = 5v_4 + 8v_5 + 60\chi, \quad 2v + 8v_3 + 3v_4 = 60\chi, \quad 10v_3 + 5v_4 + 2v_5 = 60\chi.$$

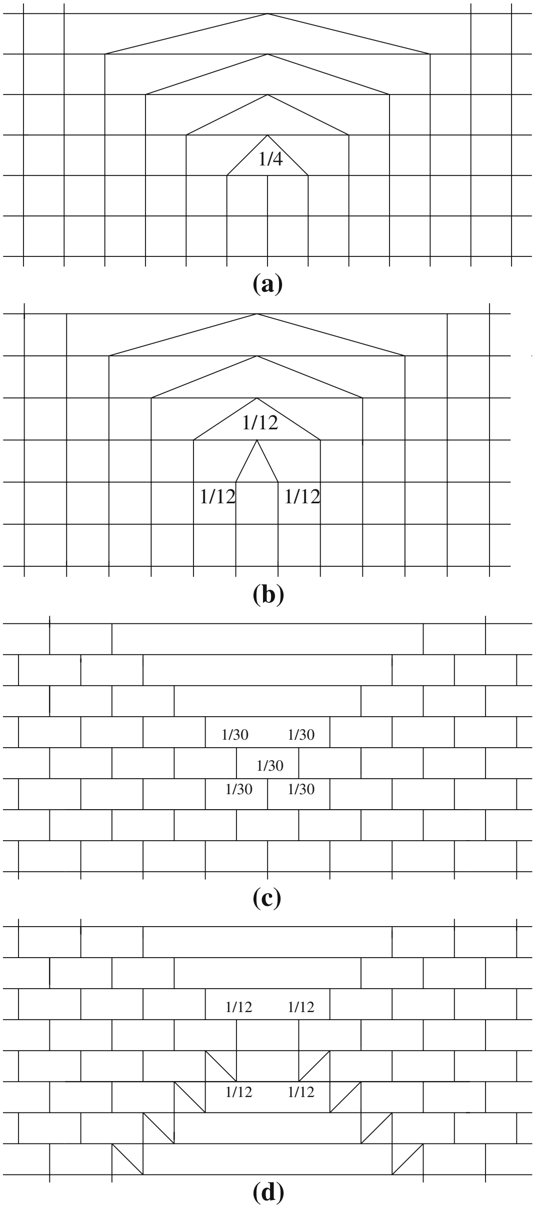


Fig. 16. Infinite plane graphs with non-negative curvature at every vertex and positive curvature at finite number of vertices

We thus have the following inequality relations

$$6\chi \leq v \leq 30\chi, \quad v_3 \leq 6\chi, \quad v_4 \leq 12\chi.$$

Applying $f = f_3 + f_4 + f_5$ again, we have $f = 3f_4 + 6f_5 + 10\chi$. This implies $f_3 \geq 10\chi$.

Now, when M is a projective plane, the vectors (v_3, v_4, v_5) and (f_3, f_4, f_5) have the possibilities of the table.

v_3	(v_3, v_4, v_5)	(f_3, f_4, f_5)
0	(0, 12, 0)	(16, 3, 0)
	(0, 8, 10)	(24, 2, 2)
	(0, 4, 20)	(32, 1, 4)
	(0, 0, 30)	(40, 0, 6)
1	(1, 8, 5)	(19, 2, 1)
	(1, 4, 15)	(27, 1, 3)
	(1, 0, 25)	(35, 0, 5)
2	(2, 8, 0)	(14, 2, 0)
	(2, 4, 10)	(22, 1, 2)
	(2, 0, 20)	(30, 0, 4)
3	(3, 4, 5)	(17, 1, 1)
	(3, 0, 15)	(25, 0, 3)
4	(4, 4, 0)	(12, 1, 0)
	(4, 0, 10)	(20, 0, 2)
5	(5, 0, 5)	(15, 0, 1)
6	(6, 0, 0)	(10, 0, 0)

When M is a sphere, the possibilities for the vectors (v_3, v_4, v_5) and (f_3, f_4, f_5) are listed in the following table.

v_3	(v_3, v_4, v_5)	(f_3, f_4, f_5)
0	(0, 24, 0)	(32, 6, 0)
	(0, 20, 10)	(40, 5, 2)
	(0, 16, 20)	(48, 4, 4)
	(0, 12, 30)	(56, 3, 6)
	(0, 8, 40)	(64, 2, 8)
	(0, 4, 50)	(72, 1, 10)
	(0, 0, 60)	(80, 0, 12)
1	(1, 20, 5)	(35, 5, 1)
	(1, 16, 15)	(43, 4, 3)
	(1, 12, 25)	(51, 3, 5)
	(1, 8, 35)	(59, 2, 7)
	(1, 4, 45)	(67, 1, 9)
	(1, 0, 55)	(75, 0, 11)
2	(2, 20, 0)	(30, 5, 0)
	(2, 16, 10)	(38, 4, 2)
	(2, 12, 20)	(46, 3, 4)
	(2, 8, 30)	(54, 2, 6)
	(2, 4, 40)	(62, 1, 8)
	(2, 0, 50)	(70, 0, 10)
3	(3, 16, 5)	(33, 4, 1)
	(3, 12, 15)	(41, 3, 3)
	(3, 8, 25)	(49, 2, 5)
	(3, 4, 35)	(57, 1, 7)
	(3, 0, 45)	(65, 0, 9)
4	(4, 16, 0)	(28, 4, 0)
	(4, 12, 10)	(36, 3, 2)
	(4, 8, 20)	(44, 2, 4)
	(4, 4, 30)	(52, 1, 6)
	(4, 0, 40)	(60, 0, 8)
5	(5, 12, 5)	(31, 3, 1)
	(5, 8, 15)	(39, 2, 3)
	(5, 4, 25)	(47, 1, 5)
	(5, 0, 35)	(55, 0, 7)
6	(6, 12, 0)	(26, 3, 0)
	(6, 8, 10)	(34, 2, 2)
	(6, 4, 20)	(42, 1, 4)
	(6, 0, 30)	(50, 0, 6)
7	(7, 8, 5)	(29, 2, 1)
	(7, 4, 15)	(37, 1, 3)
	(7, 0, 25)	(45, 0, 5)
8	(8, 8, 0)	(24, 2, 0)
	(8, 4, 10)	(32, 1, 2)
	(8, 0, 20)	(40, 0, 4)
9	(9, 4, 5)	(27, 1, 1)
	(9, 0, 15)	(35, 0, 3)
10	(10, 4, 0)	(22, 1, 0)
	(10, 0, 10)	(30, 0, 2)
11	(11, 0, 5)	(25, 0, 1)
12	(12, 0, 0)	(20, 0, 0)

□

At the end we illustrate some 5-regular projective graphs (Fig. 13) and 5-regular plane graphs (Fig. 14). The upper bound of 60 vertices for 5-regular plane graphs can be reached; see Fig. 15. However, it is not clear whether the upper bound of 30 vertices for 5-regular projective graphs can be reached.

Figure 16 demonstrates some examples of plane infinite graphs with nonnegative curvature at every vertex but having only finite number of vertices of positive curvature. These graphs confirm Theorem 1.4 and support Conjecture 1.9. More such graphs can be constructed by pinching with these sample graphs in different directions.

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