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Total Colorings of Planar Graphs without Small Cycles*

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Abstract. Let G be a planar graph with maximum degree Δ. It is proved that if $\Delta \geq 8$ and G is free of k-cycles for some $k \in \{5, 6\}$, then the total chromatic number $\chi''(G)$ of G is $\Delta + 1$.

Key words. Planar graph, total coloring, cycle.

1. Introduction

In this paper, all graphs are finite, simple and undirected. Any undefined notation follows that of Bondy and Murty [2]. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ (or V, E, δ and Δ for simplicity) to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. A vertex of degree Δ is called the *maximum vertex* of G. Let $d(v)$ denote the degree of vertex v.

A *total-k-coloring* of a graph G is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ is the smallest integer k such that G has a total-k-coloring. It is clear that $\chi''(G) \geq \Delta + 1$. Behzad [1] and Vizing [14] conjectured that $\chi''(G) \leq \Delta + 2$ for every graph G. This conjecture was verified by Rosenfeld [10] and Vijayaditya [13] for $\Delta = 3$ and by Kostochka [7–9] for $\Delta \leq 5$. For planar graphs the conjecture was verified by Borodin [3] for $\Delta \geq 9$, and by Sanders and Zhao for $\Delta = 7$ [12]. In 1989, Sanchez-Arroyo [11] proved that for any graph G, it is NP-complete to decide if $\chi''(G) = \Delta + 1$. But for planar graphs it is often possible to determine $\chi''(G)$ precisely. Borodin et al. [4] proved that a planar graph G with maximum degree $\Delta \geq 11$ has $\chi''(G) = \Delta + 1$ and they also obtained several related results by adding girth restrictions [5]. Wang and Wu [15], and Hou et al. [6] considered planar graphs without 4-cycles and got some interesting results. In this paper, we consider planar graphs without k-cycles for some $k \in \{5, 6\}$ and get the following result.

Theorem 1. *Let G be a planar graph with maximum degree $\Delta \geq 8$. If G is free of k-cycles for some $k \in \{5, 6\}$, then $\chi''(G) = \Delta + 1$.*

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2. Proof of Theorem 1.1

Let us introduce some notations and definitions. Let $G = (V, E, F)$ be a planar graph, where F is the set of faces of G. A k-, k^+- or k^--*vertex* is a vertex of degree of k, at least k or at most k, respectively. For $f \in F$, we use $b(f)$ to denote the boundary of f and write $f = [u_1u_2...u_n]$ if $u_1, u_2, ..., u_n$ are the vertices of f in the clockwise order. The degree of a face f, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A k-, k^+- or k^--*face* is a face of degree k, at least k or at most k, respectively. For $v \in V$, let $n_k(v)$, $n_{k^+}(v)$ or $n_{k^-}(v)$ denote the number of k-faces, k^+-faces or k^--faces incident with v, respectively. Let $f_1, f_2, ..., f_k$ denote the faces incident with vertex v with $d(v) = k$ in the clockwise order. f_i is said to be the $(i-1)$th successor of f_1 and denoted by $f_1^{(i-1)+}$ for $i = 2, 3, ..., k$. We write $f_2 = f_1^+$. Simarily, we write $f_k = f_1^-$ and $f_i = f_1^{(k-i+1)-}$ for $i = 2, 3, ..., k$. Let $\delta(f)$ denote the minimum degree of vertices incident with f. The face f is said to be the special adjacent face of a 3-face f' if it is the case in Fig. 1 with $d(v) = 2$.

In the proof of Theorem 1.1, we use the technique of discharging. In the beginning, we define the initial charge function for each element in $V \cup F$. By following the rules stated in the proof of the theorem, we will redistribute the charges for the vertices and faces so that the new charges are nonnegative and the sum of the new charges is still the same as before, which leads to a contradiction to Euler's formula.

Proof of Theorem 1.1. Let $G = (V, E, F)$ be a minimal counterexample to the theorem in terms of number of vertices and edges. Then every proper subgraph of G is total $(\Delta + 1)$-colorable. It is easy to see that G is 2-connected and hence has no vertices of degree 1. Furthermore, G has the following proprieties.

(a) G contains no edge uv with $\min\{d(u), d(v)\} \leq \lfloor \frac{\Delta}{2} \rfloor$ and $d(u) + d(v) \leq \Delta + 1$.
(b) G contains no even cycle $v_1v_2...v_{2t}v_1$ such that $d(v_1) = d(v_3) = ... = d(v_{2t-1}) = 2$.
(c) G has no subgraph isomorphic to the configuration in Fig. 2 such that $d(u) = 2$ and $d(v) = 3$.
(d) If the maximum vertex $v \in V$ is adjacent to two 2-vertices, say v_1, v_2, then neither v_1 nor v_2 is incident with 3-faces.

(a) can be found in [3]. (b) and (c) can be found in [4]. Next we will show (d).

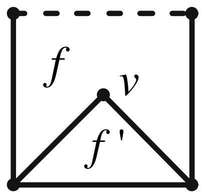

Fig. 1. Special adjacent faces

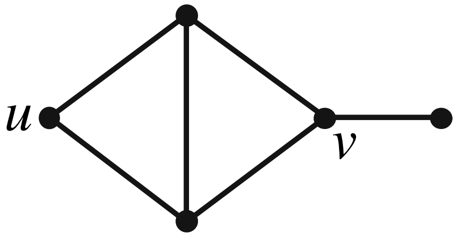

Fig. 2. Reducible Configuration

On the contrary, suppose that v_1 is incident with a 3-face vv_1u_1 and the other vertex v_2 adjacent to is u_2. Let $G' = G - u_1v_1$. By the minimality of G, G' has a proper total-$(\Delta + 1)$-coloring ϕ. Then erase the color on v_1, v_2. Assume that $\phi(vv_1) = 1$, $\phi(vu_1) = 2$ and $\phi(vv_2) = 3$. It is easy to verify that color 1 does not appear at u_1. Since otherwise, there must be a color α that does not appear at u_1. We color v_1u_1 with α. Vertices v_1 and v_2 incident or adjacent to at most four colors. So we can color v_1, v_2 properly. It follows that ϕ is extended to a total-$(\Delta + 1)$-coloring of G. Contradicting to the choice of G. If $\phi(v_2u_2) \neq 2$, then recolor vv_2 with 2, vu_1 with 1, vv_1 with 3, and color v_1u_1 with 2. Otherwise, recolor vv_1 with 3, vv_2 with 1 and color v_1u_1 with 1. Next color vertices v_1, v_2 properly. In any case, ϕ can be extended to a total-$(\Delta + 1)$-coloring of G. This contradicts the minimality of G. It completes the proof of (d).

Let G_2 be the subgraph induced by the edges incident with the 2-vertices of G. Since $\Delta \geq 8$, (a) implies that G does not contain two adjacent 2-vertices. Hence G_2 does not contain any odd cycle. It follows from (b) that G_2 is a forest. In each component T of G_2, if $|V(T)| \geq 4$, then it is possible to find a matching M in T that pairs off all the 2-vertices with some of the Δ-vertices: in T, choose a Δ-vertex u as the root of T, and match each 2-vertex v with the Δ-vertex w adjacent to v that is further from u (Note that the leaves of T are all Δ-vertices). In this case, vertex w is called the *2-master* of v and v is called the *dependent* of w. Otherwise, T is a path v_1vv_2 such that $d(v) = 2$ and v_i is adjacent to exactly one 2-vertex for $i = 1, 2$. In this case, vertex v_i is called the *special 2-master* of v and v is called the *special dependent* of v_i for $i = 1, 2$.

Note that each vertex of degree Δ can be the 2-master or special 2-master of at most one 2-vertex. Each 2-vertex is either a dependent of one maximum vertex or a special dependent of two maximum vertices. It follows from (d) that if maximum vertex v is incident with a $(2, \Delta, \Delta)$-face, then v is the special 2-master of the 2-vertex adjacent to v. There are two subsection we shall consider.

2.1. *G is free of 5-cycles*

By the choice of G, we have the following observations.

$(O_{1.1})$ Every k-vertex with $k \geq 5$ is incident with at most $(k - 2)$ 3-faces.
$(O_{1.2})$ Let v be a k-vertex with $k \geq 6$. If $n_3(v) \neq 0$, then $n_3(v) + n_4(v) \leq k - 2$.

Next we will show that $(O_{1.2})$ is true. Let f be the 3-face incident with v. If f^+ is a 4-face, then f^+ is the special adjacent face of f. In this case, if f^- is a 3-face, then

f^{2-} and f^{2+} are 6^+-faces. Otherwise, f^{2+} and f^- are 6^+-faces. If f^+ is a 3-face, then f^{2+} is a 6^+-face. In this case, if f^- is a 6^+-face, we are done. Otherwise, f^{2-} is a 6^+-face.

Since G is a planar graph, by Euler's formula, we have

$$\sum_{v\in V}(2d(v)-6)+\sum_{f\in F}(d(f)-6)=-6(|V|-|E|+|F|)=-12<0.$$

Now we define the initial charge function $w(x)$ for each $x \in V \cup F$. Let $w(v) = 2d(v) - 6$ if $v \in V$ and $w(f) = d(f) - 6$ if $f \in F$. It follows that $\sum_{x\in V\cup F} w(x) < 0$. The discharging method distributes the positive charge to neighbors so as to leave as little positive charge remaining as possible. This leads to $\sum_{x\in V\cup F} w(x) \geq 0$. A contradiction follows.

To prove the theorem, we are ready to construct a new charge $w^*(x)$ on G as follows:

- $(R_{1.1})$ Each 2-vertex receives 2 from its 2-master or receives 1 from each of its special 2-master.
- $(R_{1.2})$ From each 4-vertex to each of its incident k-face f, where $3 \leq k \leq 4$, transfer $\frac{1}{2}$.
- $(R_{1.3})$ From each 5-vertex to each of its incident k-face f, where $3 \leq k \leq 4$, transfer
  1, if $k = 3$,
  $\frac{1}{2}$, if $k = 4$.
- $(R_{1.4})$ From each 6-vertex to each of its incident k-face f, where $3 \leq k \leq 4$, transfer
  $\frac{5}{4}$, if $k = 3$,
  $\frac{1}{2}$, if $k = 4$.
- $(R_{1.5})$ From each 7^+-vertex to each of its incident k-face f, where $3 \leq k \leq 4$, transfer
  $\frac{3}{2}$, if $k = 3$,
  1, if $k = 4$.

Let $f \in F$. Clearly, $w^*(f) = w(f) \geq 0$ if $d(f) \geq 6$. Assume that $d(f) = 3$. If $\delta(f) \leq 3$, then f is incident with two 7^+-vertices. So $w^*(f) = w(f) + 2 \times \frac{3}{2} = 0$. If $\delta(f) = 4$, then f is incident with two 6^+-vertices. So $w^*(f) \geq w(f) + \frac{1}{2} + 2 \times \frac{5}{4} = 0$. If $\delta(f) \geq 5$, then $w^*(f) \geq w(f) + 3 \times 1 = 0$. Let f be a 4-face. If $\delta(f) \leq 3$, then f is incident with at least two 7^+-vertices. So $w^*(f) \geq w(f) + 2 \times 1 = 0$. Otherwise, then $w^*(f) \geq w(f) + 4 \times \frac{1}{2} = 0$.

Let v be a k-vertex. If $k = 2$, then $w^*(v) \geq w(v) + \min\{2, 1 \times 2\} = 0$. If $k = 3$, then $w^*(v) = w(v) = 0$. If $k = 4$, then $w^*(v) \geq w(v) - 4 \times \frac{1}{2} = 0$. If $k = 5$, then v is incident with at most three 3-faces by $(O_{1.1})$. So $w^*(v) \geq w(v) - 3 \times 1 - 2 \times \frac{1}{2} = 0$. Let $k = 6$. If v is incident with no 3-face, then $w^*(v) \geq w(v) - 6 \times \frac{1}{2} = 3 > 0$. Otherwise, $n_3(v) + n_4(v) \leq 4$ by $(O_{1.2})$. So $w^*(v) \geq w(v) - 4 \times \frac{5}{4} = 1 > 0$. Let $k = 7$. If v is incident with no 3-face, then $w^*(v) \geq w(v) - 7 \times 1 = 1 > 0$. Otherwise, then v is incident with at least two 6^+-faces by $(O_{1.2})$. So $w^*(v) \geq w(v) - 5 \times \frac{3}{2} = \frac{1}{2} > 0$. Let $k = 8$. If v is adjacent to no 2-vertex, then $w^*(v) \geq$

$w(v) - \max\{6 \times \frac{3}{2}, 8 \times 1\} = 1 > 0$. Otherwise, in the evaluation of the lower bound of $w^*(v)$, it suffices to consider the case that v is 2-master or special 2-master of some 2-vertex. If v is incident with no 3-face, then $w^*(v) \geq w(v) - 2 - 8 \times 1 = 0$. Otherwise, v is incident with at least two 6^+-faces by $(O_{1.2})$. If v is incident with at least three 6^+-faces, then $w^*(v) \geq w(v) - 2 - 5 \times \frac{3}{2} = \frac{1}{2} > 0$. Otherwise, v is incident with exactly two 6^+-faces. In this case, v is incident with at most four 3-faces. So $w^*(v) \geq w(v) - 2 - 4 \times \frac{3}{2} - 2 \times 1 = 0$. Let $k \geq 9$. If v is incident with no 3-face, then $w^*(v) \geq w(v) - 2 - k \times 1 = k - 8 > 0$. Otherwise, v is incident with at least two 6^+-faces by $(O_{1.2})$. If v is incident with at least three 6^+-faces, then $w^*(v) \geq w(v) - 2 - (k-3) \times \frac{3}{2} = \frac{k}{2} - \frac{7}{2} > 0$. Now suppose that v is incident with exactly two 6^+-faces. Since v is not incident with three continuous 3-faces, we have v is incident with at most $(k-3)$ 3-faces. Thus $w^*(v) \geq w(v) - 2 - (k-3) \times \frac{3}{2} - 1 = \frac{k}{2} - \frac{9}{2} \geq 0$. This completes the proof of the case that G is free of 5-cycles.

2.2. *G is free of* 6-*cycles*

Since G is a planar graph, by Euler's formula, we have

$$\sum_{v \in V}(3d(v) - 8) + \sum_{f \in F}(d(f) - 8) = -8(|V| - |E| + |F|) = -16 < 0.$$

Now we define the initial charge function $w(x)$ for each $x \in V \cup F$. Let $w(v) = 3d(v) - 8$ if $v \in V$ and $w(f) = d(f) - 8$ if $f \in F$. It follows that $\sum_{x \in V \cup F} w(x) < 0$.

To prove the theorem, we are ready to construct a new charge $w^*(x)$ on G as follows:

$(R_{2.1})$ Each 2-vertex receives 2 from its 2-master or receives 1 from each of its special 2-master.

$(R_{2.2})$ From each 3-vertex v to its incident 3-face f, transfer
- 1, if $n_3(v) = 1$;
- $\frac{1}{3}$, otherwise.

$(R_{2.3})$ From each 4-vertex to its incident k-face f, where $3 \leq k \leq 5$, transfer 1.

$(R_{2.4})$ From each 5-vertex to its incident k-face f, where $3 \leq k \leq 7$, transfer
- $\frac{5}{3}$, if $k = 3$;
- 1, if $k = 4$ or 5;
- $\frac{1}{4}$, if $k = 7$.

$(R_{2.5})$ From each 6-vertex to its incident k-face f, where $3 \leq k \leq 7$, transfer
- 2, if $k = 3$;
- 1, if $k = 4, 5$;
- $\frac{1}{4}$, if $k = 7$.

$(R_{2.6})$ From each 7-vertex to its incident k-face f, where $3 \leq k \leq 7$, transfer
- $\frac{7}{3}$, if $k = 3$;
- 2, if $k = 4$;
- 1, if $k = 5$;
- $\frac{1}{4}$, if $k = 7$.

$(R_{2.7})$ From each 8^+-vertex to its incident 3-face f, transfer
- $\frac{5}{2}$, if f is a $(2, \Delta, \Delta)$-face;

$\frac{7}{3}$, if $\delta(f) = 3$ and 3-vertex u incident with f transfers $\frac{1}{3}$ to f.
2, if $\delta(f) = 4$ or $\delta(f) = 3$ and 3-vertex u incident with f transfers 1 to f.
$\frac{5}{3}$, if $\delta(f) \geq 5$.

$(R_{2.8})$ From each 8^+-vertex to its incident k-face f, where $4 \leq k \leq 7$, transfer
2, if $k = 4$, and f is incident with two 3^--vertex.
$\frac{3}{2}$, if $k = 4$, and f is incident with at most one 3^--vertex.
1, if $k = 5$;
$\frac{1}{4}$, if $k = 7$.

Let $\gamma(x \to y)$ denote the amount transfers from element x to element y. Because G is free of 6-cycles, we have following properties.

$(O_{2.1})$ Let v be a maximum vertex of G and let u be the 2-vertex adjacent to v. If u is incident with a 3-face, then v is the special 2-master of u by (d). Hence $\gamma(v \to u) = 1$.

$(O_{2.2})$ Let v be a 7^+-vertex of G. Then $n_{5^+}(v) \geq 2$.

$(O_{2.3})$ Let f be a 3-face incident with v with $\delta(f) = 2$. Then $n_{7^+}(v) \geq 1$.

We will show $(O_{2.3})$. Suppose that f^- is the special adjacent face of f. If $d(f^-) \geq 7$, we are done. Otherwise, $d(f^-) = 4$ or $d(f^-) = 5$. If $d(f^-) = 4$, we consider the successor face f^+ of f. We first consider the case that $d(f^+) = 3$. If $d(f^{2+}) \geq 7$, then we are done. Otherwise, we have that $d(f^{2+}) = 3$. In this case, we have $d(f^{3+}) \geq 7$. Next we consider the case that $d(f^+) = 4$, then $d(f^{2-}) \geq 7$. Finally, if $d(f^+) \notin \{4, 5\}$, then $d(f^+) \geq 7$ because G is free of 6-cycles. If $d(f^-) = 5$, it follows from G is free of 6-cycles that $d(f^+) = 3$ or $d(f^+) \geq 7$. If $d(f^+) \geq 7$, we are done. Otherwise, $d(f^{2+}) \geq 7$.

$(O_{2.4})$ Let $d(v) \geq 7$ and v be incident with two 3-faces f_1, f_2. If f_1 is adjacent to f_2, then v is incident with at least one 7^+-face. Furthermore, if v is incident with exactly one 7^+-face, then either $\delta(f_1) = 2$ or $\delta(f_2) = 2$. So $\gamma(v \to f_1) + \gamma(v \to f_2) \leq 2 + \frac{5}{2} = \frac{9}{2}$ by (c).

$(O_{2.5})$ If the 3-face f is adjacent to a 5-face f', then it must be the following two cases in Fig. 3. In Fig. 3, the other faces adjacent to f are neither 4-faces nor 5-faces. This implies that if $d(f) = 3$ and $d(f^+) = 5$, then $d(f^-) \neq 4$ and $d(f^-) \neq 5$.

Now we shall show that $(O_{2.5})$ is true. Let $f = [uvw]$ be a 3-face adjacent to incident with a 5-face f'. Since G is free of 6-cycles, we have $V(f) \subseteq V(f')$. This implies that it must be two case in Fig. 3. If f' is the special adjacent face of f, without loss of generality, let $d(w) = 2$. Then f and f' are adjacent at edges uw and vw. Since uv contains in an i-cycle for any $i \in \{3, 4\}$. Thus the other face adjacent to f at uv is neither 4-face nor 5-face. Otherwise, suppose that f and f' are adjacent at the edge wv. Then for any edge $e \in \{uw, vw\}$, e contains in an i-cycle for any $i \in \{3, 4\}$. Thus the other face adjacent to f at e is neither 4-face nor 5-face.

$(O_{2.6})$ If a 4-face f is adjacent to two 3-faces, say f_1, f_2, then it must be the cases in Fig. 4.

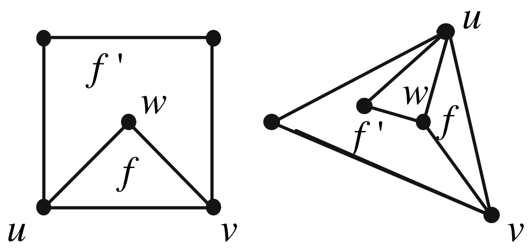

Fig. 3. the 3-face f adjacent to a 5-face f'

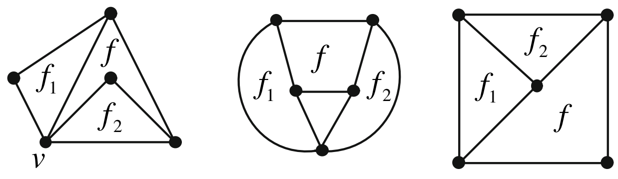

Fig. 4. the 4-face f adjacent to two 3-faces f_1 and f_2

It follows from $(O_{2.6})$ that if 7^+-vertex is incident with a 4-face f such that $\delta(f) \geq 3$ and $d(f^+) = 3$, then $d(f^-) \neq 3$.

Let $f \in F$. Clearly, $w^*(f) = w(f) \geq 0$ if $d(f) \geq 8$. Assume that f is a 3-face. If $\delta(f) = 2$, then f is incident with two 8^+-vertices. So $w^*(f) = w(f) + 2 \times \frac{5}{2} = 0$. If $\delta(f) = 3$, then f is incident with two 7^+-vertices. In this case, if 3-vertex u incident with f is incident with exactly one 3-face, then $\gamma(u \to f) = 1$. So $w^*(f) \geq w(f) + 1 + 2 \times 2 = 0$. Otherwise, $w^*(f) = w(f) + \frac{1}{3} + 2 \times \frac{7}{3} = 0$. If $\delta(f) = 4$, then $w^*(f) \geq w(f) + 1 + 2 \times 2 = 0$. If $\delta(f) \geq 5$, $w^*(f) \geq w(f) + 3 \times \frac{5}{3} = 0$. Let f be a 4-face. If $\delta(f) \geq 4$, then $w^*(f) \geq w(f) + 4 \times 1 = 0$. Otherwise, $\delta(f) \leq 3$. In this case, if f is incident with two 3^--vertex, then $w^*(f) = w(f) + 2 \times 2 = 0$. Otherwise, $w^*(f) \geq w(f) + 2 \times \frac{3}{2} + 1 = 0$. If f is a 5-face, then f is incident with at least three 5^+-vertices by (a). So $w^*(f) \geq w(f) + 3 \times 1 = 0$. If f is 7-face, then f is incident with at least four 5^+-vertices by (a). So $w^*(f) \geq w(f) + 4 \times \frac{1}{4} = 0$.

Let v be a k-vertex. If $k = 2$, then $w^*(v) = w(v) + \min\{2, 2 \times 1\} = 0$. If $k = 3$, then $w^*(v) \geq w(v) - \max\{1, 3 \times \frac{1}{3}\} = 0$. If $k = 4$, then $w^*(v) \geq w(v) - 4 \times 1 = 0$. If $k = 5$, then v is incident with at most three 3-faces because G is free of 6-cycles. So $w^*(v) \geq w(v) - 3 \times \frac{5}{3} - 2 \times 1 = 0$. If $k = 6$, then v is incident with at most four 3-faces since G is free of 6-cycles. So $w^*(v) \geq w(v) - 4 \times 2 - 2 \times 1 = 0$. Suppose that $k = 7$. If $n_{7^+}(v) \leq 1$, then by $(O_{2.2})$ and $(O_{2.4})$, we have $n_{5^+}(v) \geq 2$ and $n_3(v) \leq 3$. Thus $w^*(v) \geq w(v) - 3 \times \frac{7}{3} - 2 \times 2 - 2 \times 1 = 0$. Otherwise, $n_{7^+}(v) \geq 2$. Thus $w^*(v) \geq w(v) - 5 \times \frac{7}{3} - 2 \times \frac{1}{4} = \frac{5}{6} > 0$.

Suppose that $k = 8$. Then $n_3(v) \leq 6$ and $n_{5^+}(v) \geq 2$ by $(O_{2.2})$. If v is adjacent to no 2-vertex, then $w^*(v) \geq w(v) - 6 \times \frac{7}{3} - 2 \times 1 = 0$. Next we consider that v is adjacent to at least one 2-vertex. In the evaluation of the lower bound of $w^*(v)$, it suffices to consider the case that v is 2-master or special 2-master of the 2-vertex u. If v is incident with at least three 7^+-faces, then $w^*(v) \geq w(v) - 2 - 5 \times \frac{5}{2} - 3 \times \frac{1}{4} = \frac{3}{4} > 0$. Now we will show that $w^*(v) \geq 0$ if $n_{7^+}(v) \leq 2$.

Case 1. $n_{7^+}(v) = 0$.

If $n_3(v) + n_4(v) \leq 4$, then $w^*(v) \geq w(v) - 2 - \frac{5}{2} - 3 \times \frac{7}{3} - 4 \times 1 = \frac{1}{2} > 0$. Assume that $n_3(v) + n_4(v) \geq 5$. Since $n_{5^+}(v) \geq 2$ by $(O_{2.2})$, we have two subcases.

Subcase 1.1. $n_3(v) + n_4(v) = 5$.

Since $n_{7^+}(v) = 0$, we have v is incident with no (2, 8, 8)-face by $(O_{2.3})$. Thus the faces incident with u are 4^+-faces. It follows from $(O_{2.4})$ that $n_3(v) \leq 3$. Thus $w^*(v) \geq w(v) - 2 - 3 \times \frac{7}{3} - 2 \times 2 - 3 \times 1 = 0$.

Subcase 1.2. $n_3(v) + n_4(v) = 6$.

In this case, $n_5(v) = 2$. Let f be any 3-face incident with v. Then $\delta(f) \geq 3$ by $(O_{2.3})$ and $d(f^+) = d(f^-) = 4$ by $(O_{2.5})$. If $\delta(f) \geq 4$, then $\gamma(v \to f) \leq 2$. Otherwise, without loss of generality, let w be the 3-vertex incident with f. Then $n_3(w) = 1$ because G is free of 6-cycles. In either case, we have $\gamma(v \to f) \leq 2$. Hence $w^*(v) \geq w(v) - 2 - 6 \times 2 - 2 \times 1 = 0$.

Case 2. $n_{7^+}(v) = 1$.

In this case, $n_3(v) + n_4(v) + n_5(v) = 7$ and $n_5(v) \geq 1$ by $(O_{2.2})$. If $n_3(v) + n_4(v) \leq 4$, then $w^*(v) \geq w(v) - 2 - \frac{5}{2} - 3 \times \frac{7}{3} - 3 \times 1 - \frac{1}{4} = \frac{5}{4} > 0$. Next we consider following two subcases.

Subcase 2.1. $n_3(v) + n_4(v) = 5$.

If v is incident with a (2, 8, 8)-face, then $w^*(v) \geq w(v) - 1 - \frac{5}{2} - 4 \times \frac{7}{3} - 2 \times 1 - \frac{1}{4} = \frac{11}{12} > 0$. Otherwise, it follows from $(O_{2.4})$ that $n_3(v) \leq 4$. Thus we have $w^*(v) \geq w(v) - 2 - 4 \times \frac{7}{3} - 2 - 2 \times 1 - \frac{1}{4} = \frac{5}{12} > 0$.

Subcase 2.2. $n_3(v) + n_4(v) = 6$.

In this case, it is easy to verify that $n_4(v) \geq 1$. We first consider the case that v is incident with a 3-face. If $n_3(v) \leq 4$, then $w^*(v) \geq w(v) - 1 - \frac{5}{2} - 3 \times \frac{7}{3} - 2 \times 2 - 1 - \frac{1}{4} = \frac{1}{4} > 0$. Otherwise, v must be incident with two adjacent 3-faces, say f_1, f_2, such that either $d(f_1) = 2$ or $d(f_2) = 2$ by $(O_{2.4})$. Thus $\gamma(v \to f_1) + \gamma(v \to f_2) \leq \frac{5}{2} + 2 = \frac{9}{2}$ and $w^*(v) \geq w(v) - 1 - \frac{9}{2} - 3 \times \frac{7}{3} - 2 - 1 - \frac{1}{4} = \frac{1}{4} > 0$. Next we consider the case that v is incident with no (2, 8, 8)-face. Note that v is incident with no adjacent 3-faces. This implies that $n_3(v) \leq 4$. If $n_3(v) \leq 2$, then $w^*(v) \geq w(v) - 2 - 2 \times \frac{7}{3} - 4 \times 2 - 1 - \frac{1}{4} = \frac{1}{12} > 0$. Otherwise, $n_3(v) \geq 3$ and $n_4(v) \geq 2$. It follows from $(O_{2.5})$ that there exists a 4-face f incident with v such that f is adjacent to two 3-faces. This is impossible by $(O_{2.6})$.

Case 3. $n_{7^+}(v) = 2$.

In this case, if u is incident with a 3-face, then $w^*(v) \geq w(v) - 1 - \frac{5}{2} - 5 \times \frac{7}{3} - 2 \times \frac{1}{4} = \frac{1}{3} > 0$. Otherwise, if $n_5(v) \geq 1$, then $w^*(v) \geq w(v) - 2 - 5 \times \frac{7}{3} - 1 - 2 \times \frac{1}{4} = \frac{5}{6} > 0$. Now we will consider that $n_3(v) + n_4(v) = 6$. It follows from u is incident with no 3-face and G is free of 6-cycles that $n_4(v) \geq 1$. We only consider the following two cases.

Subcase 3.1. $n_4(v) \geq 2$.
In this case, $w^*(v) \geq w(v) - 2 - 4 \times \frac{7}{3} - 2 \times 2 - 2 \times \frac{1}{4} = \frac{1}{6} > 0$.

Subcase 3.2. $n_4(v) = 1$.
In this case, $n_3(v) = 5$. Let f be the 4-face incident with v. Since G is free of 6-cycles, v is not incident with four continuous 3-faces. Thus $\delta(f) = 2$ and f must be the case in Fig. 5.

It follows from (c) that $d(w) \geq 4$ in Fig. 5. This implies that $\gamma(v \to f) = \frac{3}{2}$. Thus $w^*(v) \geq w(v) - 2 - 5 \times \frac{7}{3} - \frac{3}{2} - 2 \times \frac{1}{4} = \frac{1}{3} > 0$. In any case, if $d(v) = 8$, we have $w^*(v) \geq 0$.

Suppose that $k = 9$. Then $n_3(v) \leq 7$ and $n_{5^+}(v) \geq 2$ by $(O_{2.2})$. If v is adjacent to no 2-vertex, then $w^*(v) \geq w(v) - 7 \times \frac{7}{3} - 2 \times 1 = \frac{2}{3} > 0$. Next we consider that v is adjacent to at least one 2-vertex. In the evaluation of the lower bound of $w^*(v)$, it suffices to consider the case that v is 2-master or special 2-master of the 2-vertex u. If v is incident with at least two 7^+-faces, then $w^*(v) \geq w(v) - 2 - \frac{5}{2} - 6 \times \frac{7}{3} - 2 \times \frac{1}{4} = 0$. Now we shall show that $w^*(v) \geq 0$ if $n_{7^+}(v) \leq 1$. We first consider the case that $n_{7^+}(v) = 0$. This implies that $n_5(v) \geq 2$ by $(O_{2.2})$ and v is incident with no $(2, 9, 9)$-face by $(O_{2.3})$. Thus $\gamma(v \to f) \leq \frac{7}{3}$ for any 3-face f incident with v. If $n_5(v) \geq 3$, then $w^*(v) \geq w(v) - 2 - 6 \times \frac{7}{3} - 3 \times 1 = 0$. Otherwise, we have $n_5(v) = 2$. This implies that $n_3(v) + n_4(v) = 7$. Let f be any 3-face incident with v. Then $\delta(f) \geq 3$ by $(O_{2.3})$ and $d(f^+) = d(f^-) = 4$ by $(O_{2.5})$. If $\delta(f) \geq 4$, then $\gamma(v \to f) \leq 2$. Otherwise, without loss of generality, let w be the 3-vertex incident with f. Then $n_3(w) = 1$ because G is free of 6-cycles. In either case, we have $\gamma(v \to f) \leq 2$. Hence $w^*(v) \geq w(v) - 2 - 7 \times 2 - 2 \times 1 = 1 > 0$. Next we consider the case that $n_{7^+}(v) = 1$. This implies that $n_3(v) + n_4(v) + n_5(v) = 8$ and

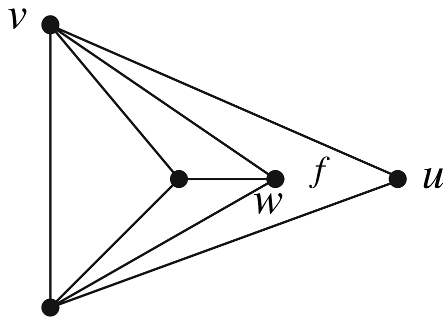

Fig. 5. configuration of f

$n_5(v) \geq 1$ by $(O_{2.2})$. If v is incident with a $(2, 9, 9)$-face, then $\gamma(v \to u) = 1$. Thus $w^*(v) \geq w(v) - 1 - \frac{5}{2} - 6 \times \frac{7}{3} - 1 - \frac{1}{4} = \frac{1}{4} > 0$. Otherwise, $\gamma(v \to f) \leq \frac{7}{3}$ for any 3-face incident with v. Since v is incident with no adjacent 3-faces by $(O_{2.4})$, we have $n_3(v) \leq 4$. Thus $w^*(v) \geq w(v) - 2 - 4 \times \frac{7}{3} - 3 \times 2 - 1 - \frac{1}{4} = \frac{5}{12} > 0$. In any case, if $d(v) = 9$, we have $w^*(v) \geq 0$.

Let $k \geq 10$. Then $n_{5^+}(v) \geq 2$. If v is incident with a $(2, k, k)$-face, then $w^*(v) \geq w(v) - 1 - \frac{5}{2} - (k-3) \times \frac{7}{3} - 2 \times 1 = \frac{2}{3}k - \frac{13}{2} > 0$. Otherwise, we have $\gamma(v \to f) \leq \frac{7}{3}$ for any 3-face incident with v. In this case, if $n_{7^+}(v) \geq 1$, then we have $w^*(v) \geq w(v) - 1 - (k-2) \times \frac{7}{3} - 1 - \frac{1}{4} = \frac{2}{3}k - \frac{79}{12} > 0$. Otherwise, we have $n_5(v) = 2$ and $n_3(v) \leq \frac{k}{2}$ by $(O_{2.4})$. Thus $w^*(v) \geq w(v) - 1 - \frac{k}{2} \times \frac{7}{3} - (\frac{k}{2} - 2) \times 2 - 2 \times 1 = \frac{5}{6}k - 8 > 0$.

This completes the proof of Theorem 1.1. □

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