Graphs and Combinatorics (2008) 24:47–52
Digital Object Identifier (DOI) 10.1007/s00373-008-0763-2

Graphs and Combinatorics

Hamilton Cycle Rich 2-factorizations of Complete Multipartite Graphs

L. McCauley, C. A. Rodger

Department of Mathematics and Statistics, 221 Parker Hall, Auburn University, AL 36849-5310. e-mail: mccaulj@auburn.edu; rodgec1@auburn.edu

Abstract. For any two 2-regular spanning subgraphs G and H of the complete multipartite graph K, necessary and sufficient conditions are found for the existence of a 2-factorization of K in which

1. the first and second 2-factors are isomorphic to G and H respectively, and
2. each other 2-factor is a hamilton cycle

in the case where K has an odd number of vertices.

1. Introduction

One of the challenging problems over the past 30 years has been the Oberwolfach problem and its natural generalizations. The original problem requires one to find a 2-factorization of K_n in which all the cycles have the same length; this problem was solved over a decade ago [2, 8]. A much studied generalization of this problem is to simply require that each of the 2-factors be isomorphic to each other. To solve this would be an amazing feat, as so many possible 2-factors exist. Some progress has been made, including a complete solution when $n \leq 17$ [1], and in many cases where each 2-factor contains just two cycle lengths (see [5] for a survey of results).

Another direction in which research has developed is to allow a small number of the 2-factors to be anything, but then stipulate that the remaining 2-factors be hamilton cycles. Extending a result of Buchanan [6], in 2004 Bryant [4] found necessary and sufficient conditions for the existence of 2-factorizations of K_n and of $K_n - I$, where $K_n - I$ is the complete graph on n vertices with a 1-factor I removed, in which the cycle lengths in up to three of the 2-factors are freely specified, and all remaining 2-factors are hamilton cycles. Independently, Rodger [10] used a similar observation to settle the existence of 2-factorizations of all complete multipartite graphs, and of all complete multipartite graphs with a 1-factor removed, in which one 2-factor is freely specified and the rest of the 2-factors are hamilton cycles. One can think of this as the existence of a hamilton decomposition of the graph formed from $K(m, p)$ (the complete multipartite graph with m vertices in each of p parts) or from $K(m, p) - I$ by removing any 2-factor. Thought of in this way, the result has a relative in the world of matchings, where Plantholt [9] showed that the removal of any set of x edges from K_{2x+1} results in a graph whose edges can be partioned into $2x$ matchings ($2x + 1$ matchings are needed if fewer edges are removed).

In this paper, we extend the result of Rodger, finding necessary and sufficient conditions for the existence of a hamilton decomposition of the graph $K(m, p)$ by removing the edges of any two 2-factors. More formally, for any two 2-regular graphs G and H of order mp, when m is odd we find necessary and sufficient conditions for the existence of a 2-factorization, $\{F_1, F_2, \ldots, F_{\lfloor m(p-1)\rfloor/2}\}$, of $K(m, p)$ such that $G \cong F_1$, $H \cong F_2$, and F_i is a hamilton cycle for $3 \leq i \leq \lfloor m(p-1)\rfloor/2$.

2. Preliminaries

Before we can get to the results, some notation, lemmas, and theorems must be introduced. In this paper we use Z_n to denote the vertex set of a graph on n vertices. This allows us to define the *difference* of the edge $\{i, j\}$ is to be $d(i, j) = \min\{j - i, n - (j - i)\}$ where $i < j$; thus $d(i, j) > 0$. Let $\langle d_1, d_2, \ldots, d_x\rangle_n$ be the subgraph induced by the edges with differences in $\{d_1, d_2, \ldots, d_x\}$. Bermond et al proved the following useful result that shows when the edges of two differences can be used to form two edge-disjoint hamilton cycles. If A is a set of positive integers, let $gcd(A)$ denote the greatest common divisor of the elements of A.

Theorem 1 [3]. *Let s, t, n be positive integers with $s < t < n/2$. If $gcd(\{s, t, n\}) = 1$ then the graph $\langle s, t\rangle_n$ has a hamilton cycle decomposition.*

The next lemma was proven separately by both Bryant and Rodger. It provides a key method used to prove our results.

Lemma 1 [4, 10]. *Let $n \geq 5$ and let F' be any 2-regular graph of order n. If $gcd(\{x, n\}) = 1$ then the subgraph $\langle x, 2x\rangle_n$ of K_n has a 2-factorization $\{F, H\}$ such that H is a hamilton cycle and $F' \cong F$.*

Now, we will introduce some specific results that will be used to clear up some of the cases we will encounter. Presented first is the result from Bryant's paper previously alluded to; one might also see the related results in [1, 7].

Theorem 2 [3]. *Let $n \geq 7$ be odd and let F_1', F_2', and F_3' be any three 2-regular graphs of order n. Then there exists a 2-factorization $\{F_1, F_2, \ldots, F_{(n-1)/2}\}$ of K_n in which $F_1 \cong F_1'$, $F_2 \cong F_2'$, $F_3 \cong F_3'$, and F_i is a hamilton cycle for $4 \leq i \leq (n-1)/2$, except that when $(n, F_1', F_2', F_3') \in \{(7, C_3 \cup C_4, C_3 \cup C_4, C_7), (9, C_3 \cup C_3 \cup C_3, C_3 \cup C_3 \cup C_3, C_3 \cup C_3 \cup C_3), (9, C_3 \cup C_3 \cup C_3, C_3 \cup C_3 \cup C_3, C_3 \cup C_6), (9, C_3 \cup C_3 \cup C_3, C_3 \cup C_3 \cup C_3, C_4 \cup C_5)\}$ no such two factorization exists.*

Next we present Rodger's result.

Theorem 3 [10]. *Let $p \geq 3$ and $m \geq 1$. Let H be any 2-factor in $K(m, p)$. There exists a partition of the edge set of $K(m, p)$, one set in which induces a graph isomorphic to H, if $m(p-1)$ is odd then one set induces a 1-factor, and each other set induces a hamilton cycle.*

3. Results

Theorem 4. *Let m be odd, and suppose that G and H are any two 2-factors of $K(m, p)$ (so necessarily $m(p-1) \geq 4$). There exists a 2-factorization $\{F_1, F_2, \ldots, F_{\lfloor m(p-1)\rfloor/2}\}$ of $K(m, p)$ such that $F_1 \cong G$, $F_2 \cong H$, and F_i is a hamilton cycle for $3 \leq i \leq \lfloor m(p-1)\rfloor/2$, if and only if*

1. *p is odd, and*
2. *$(m, p, G, H) \notin \{(1, 7, C_3 \cup C_4, C_3 \cup C_4), (3, 3, C_3 \cup C_3 \cup C_3, C_3 \cup C_3 \cup C_3), (3, 3, C_3 \cup C_3 \cup C_3, C_3 \cup C_6), (3, 3, C_3 \cup C_3 \cup C_3, C_4 \cup C_5)\}$.*

Proof. If $K(m, p)$ is to have a 2-factorization, all vertices must have even degree, so $m(p-1)$ must be even, so the first condition is necessary since we are assuming that m is odd. Once one observes that the edges removed from K_9 to form $K(3, 3)$ can be thought of as the edges in $C_3 \cup C_3 \cup C_3$, Theorem 2 clearly proves the four cases described in the second condition cannot be obtained. So we now turn to a proof of the sufficiency.

Since $K(m, p)$ is an $m(p-1)$-regular graph, and since it is assumed to contain at least two 2-factors, we know that $m(p-1) \geq 4$. So, since we also know that p is odd, clearly $p \geq 3$.

Notice that if we let the j^{th} part of $K(m, p)$ be $\{ip + j \mid i \in Z_m\}$ for $j \in Z_p$ then the edges of $K(m, p)$ are the same as the edges of K_{mp} with edges of difference ip, $1 \leq i \leq \lfloor m/2 \rfloor$ removed. Therefore we will partition the edges of $K(m, p)$ by their differences, namely by the differences in the difference set $D = \{1, 2, \ldots, \lfloor (mp)/2 \rfloor\} \backslash \{ip \mid 1 \leq i \leq \lfloor m/2 \rfloor\}$. We now consider several cases in turn.

Case 1. Suppose $mp \geq 21$. Then $\{1, 2, 4, 8\} \subset D$. By Lemma 1, $\langle 1, 2\rangle_{mp}$ and $\langle 4, 8\rangle_{mp}$ each have a 2-factorization consisting of any 2-factor and a hamilton cycle; so we can choose the two 2-factors to be isomorphic to G and H respectively. It remains to partition the remaining edges into sets that induce hamilton cycles. We consider 4 subcases in turn.

Case 1a. Suppose that $p \geq 9$. By pairing all except possibly the last of the differences in $D \backslash \{1, 2, 4, 8\} = D'$ in increasing order (that is, form pairs $\{3, 5\}, \{6, 7\}, \ldots$) we produce pairs of the form either $\{d, d+1\}$ or $\{d, d+2\}$, for some $d \in D'$. Since $gcd(\{mp, (d+1)-d\}) = gcd(\{mp, 1\}) = 1$, it follows that $gcd(\{d, d+1, mp\}) = 1$. Also, since mp is odd, $gcd(\{mp, (d+2)-d\}) = gcd(\{mp, 2\}) = 1$ means that $gcd(\{mp, d+2, d\}) = 1$. Also, if $|D| = m(p-1)/2$ is odd, then the last difference, $(mp-1)/2$, is not paired, but since $gcd(\{mp, (mp-1)/2\}) = 1$, the edges with difference $(mp-1)/2$ form a hamilton cycle. Therefore, by Theorem 1, there exists a hamilton cycle decomposition of the subgraph induced by the remaining edges.

Case 1b. Suppose that $p = 7$. If $m = 3$ then the result follows from Theorem 2, since we can choose each component in F_3' to be a 3-cycle, then remove these edges to form the independent vertices in the parts of $K(3, 7)$. In all other cases (so $mp > 21$), first apply Theorem 1 to each of the pairs $\{3, 5\}$, $\{6, 10\}$, and $\{9, 11\}$ in turn (these exist since $mp > 21$). Then pair the remaining differences in increasing order as before to obtain the required hamilton cycles.

Case 1c. Suppose that $p = 5$. If $mp \geq 35$ then apply Theorem 1 to each of the pairs $\{3, 7\}$, $\{6, 14\}$, $\{12, 13\}$, and $\{9, 11\}$ in turn. Pair the remaining differences in order and proceed as in Case 1a.

If $mp = 25$, apply Theorem 1 to each of the pairs $\{3, 6\}$, $\{7, 9\}$, and $\{11, 12\}$ in turn.

Case 1d. Suppose that $p = 3$. Pair the remaining differences in order and proceed as in Case 1a.

Case 2. Suppose $mp \leq 20$ and $(m, p) \neq (5, 3)$. If $m = 1$ then $K(1, p)$ is just the complete graph K_p, so the result follows from Theorem 2. If $m = 3$ then $p \in \{3, 5\}$ so the result follows from Theorem 2, since when $m = 3$, the edges one removes from K_{mp} to form $K(m, p)$ induce the 2-factor consisting of p 3-cycles; consider this to be the third specified 2-factor.

Case 3. Suppose $(m, p) = (5, 3)$. This case takes substantial effort. It is too small to be able to apply Lemma 1 twice and be left with a difference that induces a hamilton cycle. The set of available differences is $\{1, 2, 4, 5, 7\}$, and Lemma 1 could be applied to the graphs $\langle 1, 2\rangle_{15}$ and $\langle 4, 8\rangle_{15}$ (since difference 7 is the same as difference 8), but that leaves difference 5 that induces five 3-cycles. So we do apply Lemma 1 to $\langle 4, 8\rangle_{15}$ to obtain F_1, then obtain F_2 from $\langle 1, 2, 5\rangle_{15}$ in such a way that the edges left over form two hamilton cycles. We consider the various possible cycle lengths, $c_1, c_2, \ldots, c_x$ of the x components of F_2 in turn, written as $l = (c_1, c_2, \ldots, c_x)$.

We begin with the cases in which all the cycle lengths in F_2 are divisible by 3. To construct the required cycles, we always include the hamilton cycle $\langle 2\rangle_{15}$, then swap edges in $\langle 1\rangle_{15}$ with edges in $\langle 5\rangle_{15}$ to fuse components in $\langle 5\rangle_{15}$. In each case, we begin with l, then describe how to form F_2.

$(3, 3, 3, 3, 3)$: $\langle 1\rangle_{15}$ and $\langle 2\rangle_{15}$ are hamilton cycles, and difference 5 induces F_2.

$(3, 3, 3, 6)$: Swap edges $\{0, 1\}$ and $\{5, 6\}$ in $\langle 1\rangle_{15}$ with edges $\{0, 5\}$ and $\{1, 6\}$ in $\langle 5\rangle_{15}$ to produce the hamilton cycle $(0, 5, 4, 3, 2, 1, 6, 7, \ldots, 14)$ and the graph consisting of the cycles $(0, 1, 11, 6, 5, 10)$, $(2, 7, 12)$, $(3, 8, 13)$, and $(4, 9, 14)$ respectively. The next few cases proceed similarly, so we simply present the edges to be swapped.

$(3, 3, 9)$: Swap edges $\{0, 1\}$, $\{5, 6\}$, $\{6, 7\}$, and $\{11, 12\}$ in $\langle 1\rangle_{15}$ with edges $\{0, 5\}$, $\{1, 6\}$, $\{6, 11\}$, and $\{7, 12\}$ in $\langle 5\rangle_{15}$ (so just switch two more edges from the $(3, 3, 3, 6)$ case).

$(3, 12)$: Swap edges $\{0, 1\}$, $\{2, 3\}$, $\{5, 6\}$, $\{6, 7\}$, $\{7, 8\}$, and $\{11, 12\}$ in $\langle 1\rangle_{15}$ with edges $\{0, 5\}$, $\{1, 6\}$, $\{2, 7\}$, $\{3, 8\}$, $\{6, 11\}$, and $\{7, 12\}$ in $\langle 5\rangle_{15}$ (so just switch two more edges from the $(3, 3, 9)$ case).

(3, 6, 6): Swap edges {0, 1}, {5, 6}, {7, 8}, and {12, 13} in $\langle 1\rangle_{15}$ with edges {0, 5}, {1, 6}, {7, 12}, and {8, 13} in $\langle 5\rangle_{15}$ (so just switch two more edges from the (3, 3, 3, 6) case).

(6, 9): Swap edges {0, 1}, {3, 4}, {5, 6}, {6, 7}, {8, 9}, and {11, 12} in $\langle 1\rangle_{15}$ with edges {0, 5}, {1, 6}, {3, 8}, {4, 9}, {6, 11}, and {7, 12} in $\langle 5\rangle_{15}$ (so just switch two more edges from the (3, 3, 9) case).

All but one of the remaining cases are obtained by producing F_2 using Lemma 1, then switching edges between the resulting hamilton cycle and $\langle 5\rangle_{15}$ to obtain 2 hamilton cycles. Since it is more complicated to describe, we simply provide the resulting decompositions of $\langle 1, 2, 5\rangle_{15}$.

(3, 4, 4, 4): (0, 1, 14, 13), (2, 3, 5, 4), (6, 7, 8), (9, 10, 12, 11),
(0, 5, 10, 8, 3, 13, 12, 7, 2, 1, 11, 6, 4, 9, 14),
(0, 2, 12, 14, 4, 3, 1, 6, 5, 7, 9, 8, 13, 11, 10).
(3, 3, 4, 5) : (4, 5, 6)(11, 12, 13), (7, 8, 10, 9), (0, 2, 3, 1, 14),
(0, 10, 12, 2, 7, 5, 3, 8, 6, 1, 11, 9, 4, 14, 13),
(0, 5, 10, 11, 6, 7, 12, 14, 9, 8, 13, 3, 4, 2, 1).
(5, 5, 5): (0, 2, 3, 1, 14), (4, 6, 8, 7, 5), (9, 10, 12, 13, 11),
(0, 5, 10, 11, 6, 1, 2, 12, 7, 9, 14, 4, 3, 8, 13),
(0, 10, 8, 9, 4, 2, 7, 6, 5, 3, 13, 14, 12, 11, 1).
(4, 5, 6): (10, 11, 13, 12), (5, 6, 8, 9, 7), (0, 2, 4, 3, 1, 14),
(0, 1, 11, 9, 4, 6, 7, 2, 12, 14, 13, 3, 8, 10, 5),
(0, 10, 9, 14, 4, 5, 3, 2, 1, 6, 11, 12, 7, 8, 13).
(4, 4, 7): (6, 8, 9, 7), (10, 12, 13, 11), (0, 2, 4, 5, 3, 1, 14)
(0, 5, 7, 2, 12, 14, 4, 6, 1, 11, 9, 10, 8, 3, 13),
(0, 10, 5, 6, 11, 12, 7, 8, 13, 14, 9, 4, 3, 2, 1).
(3, 5, 7): (11, 12, 13), (6, 7, 9, 10, 8), (0, 2, 4, 5, 3, 1, 14),
(0, 5, 7, 2, 1, 6, 11, 10, 12, 14, 4, 9, 8, 3, 13),
(0, 10, 5, 6, 4, 3, 2, 12, 7, 8, 13, 14, 9, 11, 1).
(3, 4, 8): (0, 5, 10), (1, 2, 7, 6), (3, 4, 14, 9, 11, 12, 13, 8),
(0, 1, 3, 2, 4, 5, 6, 11, 10, 9, 8, 7, 12, 14, 13),
(0, 2, 12, 10, 8, 6, 4, 9, 7, 5, 3, 13, 11, 1, 14).
(5, 10): (0, 2, 4, 6, 8, 7, 5, 3, 1, 14), (9, 11, 13, 12, 10),
(0, 5, 4, 14, 9, 8, 13, 3, 2, 12, 7, 6, 1, 11, 10),
(0, 1, 2, 7, 9, 4, 3, 8, 10, 5, 6, 11, 12, 14, 13).
(4, 11): (0, 2, 4, 6, 8, 9, 7, 5, 3, 1, 14), (10, 12, 13, 11),
(0, 5, 4, 14, 12, 2, 7, 6, 1, 11, 9, 10, 8, 3, 13),
(0, 1, 2, 3, 4, 9, 14, 13, 8, 7, 12, 11, 6, 5, 10). □

References

1. Adams, P., Bryant, D.: Two-factorisations of complete graphs of orders fifteen and seventeen. Australas. J. Combin. **35**, 113–118 (2006)
2. Alspach, B., Schellenberg, P., Stinson, D., Wagner, D.: The Oberwolfach problem and factors of uniform odd length cycles. J. Combin. Theory, Ser A **52**, 20–43 (1989)

3. Bermond, J., Favaron, O., Maheo, M.: Hamilton decomposition of Cayley graphs of degree 4. J. Combin. Theory, Ser. B **46**, 142–153 (1989)
4. Bryant, D.: Hamilton cycle rich 2-factorizations of complete graphs. J. Combin. Des. **12**, 147–155 (2004)
5. Bryant, D., Rodger, C.A.: Cycle decompositions, CRC Handbook of Combinatorial Designs. In: Colbourn, C.J., Dinitz, J.H., (eds.) 2nd edition, CRC Press, 2007
6. Buchanan, H. III: Ph.D. Dissertation, University of West Virginia, 1996
7. Franek, F., Rosa, A.: Two-factorizations of small complete graphs. J. Statist. Plann. Inference **86**, 435–442 (2000)
8. Hoffman, D.G., Schellenberg, P.: The existence of a C_k-factorization of $K_{2n} - F$, Discrete Math. **97**, 243–250 (1991)
9. Plantholt, M.: The chromatic index of graphs with a spanning star. J. Graph Theory **5**, 45–53 (1981)
10. Rodger, C.A.: Hamilton decomposable graphs with specified leaves. Graphs Combin. **20**, 541–543 (2004)

Received: February 2, 2006
Final version received: October 2, 2007