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A Sharp Upper Bound for the Number of Spanning Trees of a Graph

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Abstract. Let $G = (V, E)$ be a simple graph with n vertices, e edges and d_1 be the highest degree. Further let $\lambda_i,\ i = 1, 2, \ldots, n$ be the non-increasing eigenvalues of the Laplacian matrix of the graph G. In this paper, we obtain the following result: For connected graph G, $\lambda_2 = \lambda_3 = \ldots = \lambda_{n-1}$ if and only if G is a complete graph or a star graph or a (d_1, d_1) complete bipartite graph.

Also we establish the following upper bound for the number of spanning trees of G on n, e and d_1 only:

$$t(G) \leq \left(\frac{2e - d_1 - 1}{n - 2}\right)^{n-2}.$$

The equality holds if and only if G is a star graph or a complete graph. Earlier bounds by Grimmett [5], Grone and Merris [6], Nosal [11], and Kelmans [2] were sharp for complete graphs only. Also our bound depends on n, e and d_1 only.

Key words. Graph, spanning trees, Laplacian matrix.

1. Introduction

Let G = (V,E) be a simple graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and the cardinality of edge set e. Assume that the vertices are ordered such that $d_1 \geq d_2 \geq \ldots \geq d_n$, where d_i is the degree of v_i for $i = 1, 2, \ldots, n$. The number of spanning trees of G is denoted by $t(G)$. Let $A(G)$ be the (0, 1)-adjacency matrix of G and $D(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Geršgorin's theorem, it follows that its eigenvalues are non-negative real numbers. Moreover since its rows sum is equal to 0, 0 is the smallest eigenvalue of $L(G)$. It is known that the multiplicity of 0 as the eigenvalue of $L(G)$ is equal to the number of connected components of G. So a graph G is connected if and only if the second smallest Laplacian eigenvalue is strictly greater than 0. Throughout this paper let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0$ be the eigenvalues of $L(G)$. When more than

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one graph is under discussion, we may write $\lambda_i(G)$ instead of λ_i. The number of spanning trees of G is given by the following formula:

$$t(G) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i. \tag{1}$$

Now we give some known upper bounds on $t(G)$:

(1) Grimmett [5].

$$t(G) \le \frac{1}{n} \left(\frac{2e}{n-1} \right)^{n-1}. \tag{2}$$

(2) Grone and Merris [6].

$$t(G) \le \left(\frac{n}{n-1} \right)^{n-1} \left(\frac{\prod_{i=1}^{n} d_i}{2e} \right). \tag{3}$$

(3) Nosal [11].

$$t(G) \le n^{n-2} \left(\frac{r}{n-1} \right)^{n-1}. \tag{4}$$

(4) Kelmans ([2], p. 222).

$$t(G) \le n^{n-2} \left(1 - \frac{2}{n} \right)^{e}. \tag{5}$$

The third bound only applies to regular graphs of degree r. The first three bounds are sharp for complete graphs only. In Section 2 we characterize the graphs. In Section 3 we obtain an upper bound on the product of the degrees of a graph. In Section 4 we establish the upper bound on the number of spanning trees applying the results in Section 2 and Section 3. One of the upper bound is as follows:

$$t(G) \le \left(\frac{2e - d_1 - 1}{n-2} \right)^{n-2},$$

which is sharp for a star graph or a complete graph.

2. Characterization on Graphs

In this section we characterize the graphs. Already we have seen that graph characterization from Laplacian eigenvalues in [3, 4]. It is well known that the largest Laplacian eigenvalue is less than or equal to n. Grone and Zimmermann [8] found the following lower bound for the multiplicity of the eigenvalue n.

Lemma 2.1 [8]. *Let $G = (V, E)$ be a simple graph with n vertices. Then the multiplicity of n as an eigenvalue of $L(G)$ is at least $s - 1$ if and only if G contains a complete s-partite graph on V as a subgraph.*

Merris [7] proved that $\lambda_1 \geq d_1 + 1$ if G has at least one edge. For a connected graph G on $n > 1$ vertices, $\lambda_1 = d_1 + 1$ holds if and only if $d_1 = n - 1$. Now we define a graph S_n of order n as follows: there is at least one vertex of degree $n - 1$ or in other words it is a super-graph of $K_{1,n-1}$. We rewrite the above statement in the following Lemma 2.2.

Lemma 2.2. *Let G be a simple connected graph with at least one edge and d_1 be the highest degree. If λ_1 is the largest eigenvalue of $L(G)$, then $\lambda_1 \geq d_1 + 1$ and the equality holds if and only if G is an S_n graph.*

Lemma 2.3 [10]. *Let G be a simple graph with n vertices. If $\lambda_i,\ i = 1, 2, \ldots, n$ are the non-increasing Laplacian eigenvalues of $L(G)$, then the non-increasing Laplacian eigenvalues of $L(G^c)$ are $n - \lambda_{n-i},\ i = 1, 2, \ldots, n - 1$ and 0.*

Lemma 2.4 [1]. *Let G be a simple connected graph. Then*

$$\lambda_1 \leq \max_i \{d_i + d_j\},$$

with equality holds if and only if G is a regular bipartite graph or a semiregular bipartite graph.

Lemma 2.5 [10]. *Let G be a simple graph and $G \neq K_n$. If λ_{n-1} is the second smallest eigenvalue of $L(G)$, then $\lambda_{n-1} \leq d_n$, where d_n is the smallest degree of G.*

Theorem 2.6. *Let G be a simple graph of order n with at least one edge. Then $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1}$ if and only if G is a complete graph.*

Proof. If G is a complete graph then $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1}$ holds.

Conversely, let $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1}$. If G is not a complete graph then $\lambda_{n-1} \leq d_n < d_1 + 1 \leq \lambda_1$ (by Lemma 2.5 and Lemma 2.2), a contradiction. □

Lemma 2.7 [9]. *Let G be a connected graph with $n \geq 3$ vertices. Then $\lambda_2 \geq d_2$ with equality if G is a $r \times s$ complete bipartite graph $K_{r,s}$ or a tree T_n with degree sequence $\pi(T_n) = (\frac{n}{2}, \frac{n}{2}, 1, \ldots, 1)$, where $n \geq 4$ is even.*

Theorem 2.8. *Let G be a simple connected graph with $n \geq 3$ vertices. Then $\lambda_2 = \lambda_3 = \ldots = \lambda_{n-1}$ if and only if G is a complete graph or a star graph or a (d_1, d_1) complete bipartite graph.*

Proof. If G is a complete graph or a star graph or a (d_1, d_1) complete bipartite graph, then $\lambda_2 = \lambda_3 = \ldots = \lambda_{n-1}$ holds.

Conversely, let $\lambda_2 = \lambda_3 = \ldots = \lambda_{n-1}$. We have to prove that G is a complete graph or a star graph or a (d_1, d_1) complete bipartite graph. Two cases arise (i) $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1}$, (ii) $\lambda_1 \neq \lambda_2 = \lambda_3 = \ldots = \lambda_{n-1}$.

Case (i) : $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1}$. By Theorem 2.6, we get that G is a complete graph.

Case (ii) : $\lambda_1 \neq \lambda_2 = \lambda_3 = \ldots = \lambda_{n-1}$. In this case G is not a complete graph. Using this result and Lemma 2.5, we get $\lambda_2 = \lambda_3 = \ldots = \lambda_{n-1} \leq d_n$. From Lemma 2.7 we get $\lambda_2 \geq d_2$. Therefore $d_2 \leq \lambda_2 = \lambda_3 = \ldots = \lambda_{n-1} \leq d_n$. Since $d_2 \geq d_n$, we get that $\lambda_2 = \lambda_3 = \ldots = \lambda_{n-1} = d_2 = d_3 = \ldots = d_n$. Now, $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i$, that is, $\lambda_1 = d_1 + d_2$. By Lemma 2.4, G is a regular bipartite graph or a semiregular bipartite graph.

When G is a regular bipartite graph, $d_1 = d_2 = \ldots = d_n$. Now we consider the complement graph G^c of G. In the complement graph G^c, there are all the vertices of degree $n - d_1 - 1$. Let $\lambda_1^{'} \geq \lambda_2^{'} \geq \ldots \geq \lambda_n^{'} = 0$ be the eigenvalues of $L(G^c)$. Therefore

$$\lambda_1^{'} = \lambda_2^{'} = \ldots = \lambda_{n-2}^{'} = n - d_1, \ \lambda_{n-1}^{'} = n - 2d_1, \quad \lambda_n^{'} = 0.$$

If possible, let G^c be a connected graph. By Lemma 2.2 we conclude that G^c is an S_{n-d_1} graph because there is at least one edge and $\lambda_1^{'} = (n - d_1 - 1) + 1$, where $n - d_1 - 1$ is the highest degree of G^c. Since G^c is a graph of order n and S_{n-d_2} is a graph of order $n - d_2$, there is a contradiction. Therefore G^c is a disconnected graph. Hence $\lambda_{n-1}^{'} = n - 2d_1 = 0$, that is, $n = 2d_1$. Hence G is a (d_1, d_1) complete bipartite graph.

When G is a semiregular bipartite graph, $d_1 \neq d_2 = d_3 = \ldots = d_n$. Since G is connected and (d_1, d_2) semiregular bipartite graph, G must be a star graph. □

Theorem 2.9. *Let G be a simple connected graph with* $n \geq 3$ *vertices. Then* $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-2}$ *if and only if G is a complete graph or a graph* $K_n - e$*, where e is any edge.*

Proof. If G is a complete graph or a graph $K_n - e$, then $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-2}$ holds.

Conversely, let $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-2}$. We have to prove that G is a complete graph or a $K_n - e$ graph.

Let us consider the complement graph G^c of a graph G. Also, let $\lambda_1^{'} \geq \lambda_2^{'} \geq \ldots \geq \lambda_n^{'} = 0$ be the eigenvalues of $L(G^c)$. Therefore $\lambda_2^{'} = \lambda_3^{'} = \ldots = \lambda_{n-1}^{'}$.

If possible, let G^c be a connected graph. Using Theorem 2.8 we conclude that G^c is a complete graph or a star graph or a $(d_1^{'}, d_1^{'})$ complete bipartite graph, where $d_1^{'}$ is the highest degree of G^c. So G is a disconnected graph, a contradiction. Hence G^c is a disconnected graph, that is, $\lambda_{n-1}^{'} = 0 = \lambda_{n-2}^{'} = \ldots = \lambda_2^{'}$. Two cases arise (i) $\lambda_1^{'} = 0$, (ii) $\lambda_1^{'} \neq 0$.

Case (i) : $\lambda_1^{'} = 0$. In this case all the eigenvalues of $L(G^c)$ are 0, that is, G is a complete graph.

Case (ii) : $\lambda_1' \neq 0$. In this case G^c has exactly one edge because there is only one non-zero eigenvalue. Therefore G is a graph $K_n - e$. □

3. Maximizing the Product of the Degrees of a Graph

In this section we find the upper bounds on the product of the degrees of a connected graph in terms of only n and e. First we find the upper bound on $\prod_{i=1}^{n} d_i$ by A.M.≥ G.M. and this bound is in the following Theorem 3.1.

Theorem 3.1. *Let G be a simple connected graph with n vertices and e edges. Then*

$$\prod_{i=1}^{n} d_i \leq \left(\frac{2e}{n}\right)^n . \tag{6}$$

Moreover, the equality holds in (6) if and only if $d_1 = d_2 = \ldots = d_n = \frac{2e}{n}$.

Theorem 3.2. *Let G be a simple connected graph with n vertices and e edges. Then*

$$\prod_{s=1}^{n} d_s \leq k^{(k+1)n-2e}(k+1)^{2e-kn}, \quad \text{where } k = \left[\frac{2e}{n}\right]. \tag{7}$$

Moreover, the equality holds in (7) if and only if the difference between any two vertex degrees of graph G is at most one.

Proof. Let us consider two vertices v_i of degree d_i and v_j of degree d_j, where $d_i \geq d_j$. Also let $\prod_{i=1}^{n} d_i$ be maximum.

If possible, let $d_i - d_j \geq 2$. Therefore there exists a vertex v_k, which is adjacent to v_i, but not v_j. If we remove edge $v_k v_i$ and add edge between the vertices v_k and v_j, then the new degree sequence of the new graph is $d_1^*, d_2^*, \ldots, d_n^*$; where $d_i^* = d_i - 1,\ d_j^* = d_j + 1,\ d_t^* = d_t,\ t = 1, 2, \ldots, n;\ t \neq i, j$.

Therefore

$$\begin{aligned}\prod_{s=1}^{n} d_s^* &= \prod_{s=1}^{n} d_s \frac{(d_i - 1)(d_j + 1)}{d_i d_j}\\ &= \prod_{s=1}^{n} d_s \left(1 - \frac{1}{d_i}\right)\left(1 + \frac{1}{d_j}\right)\\ &= \prod_{s=1}^{n} d_s \left(1 - \frac{1}{d_i} + \frac{1}{d_j} - \frac{1}{d_i d_j}\right).\end{aligned} \tag{8}$$

Now,

$$\frac{1}{d_j} - \frac{1}{d_i} - \frac{1}{d_i d_j} = \frac{1}{d_j} - \frac{1}{d_i}\left(\frac{d_j + 1}{d_j}\right) \geq \frac{1}{d_j} - \frac{1}{(d_j + 2)}\left(\frac{d_j + 1}{d_j}\right) = \frac{1}{d_j(d_j + 2)} > 0.$$

Using this result in (8), we get $\prod_{s=1}^{n} d_s^* > \prod_{s=1}^{n} d_s$, a contradiction as $\prod_{s=1}^{n} d_s$ is maximum. Since v_i and v_j are arbitrary, therefore the difference of any two vertex degrees of graph G is at most one. So, some of the vertices have degree k and the remaining vertices (if any exists) have degree $k+1$, where $k = \left[\frac{2e}{n}\right]$. Therefore $2e-kn$ number of vertices have degree $k+1$. Hence

$$\prod_{s=1}^{n} d_s \leq k^{(k+1)n-2e}(k+1)^{2e-kn}, \qquad \text{where } k = \left[\frac{2e}{n}\right].$$

Now suppose that the equality holds in (7). Therefore some of the vertices have degree k and the remaining vertices (if any exists) have degree $k+1$, where $k = \left[\frac{2e}{n}\right]$. Hence the difference between any two vertex degrees of graph G is at most one.

Conversely, let the difference between any two vertex degrees of graph G be at most one. Then we can easily see that the equality holds in (7). □

4. Upper Bound on the Number of Spanning Trees

We use the results of Sections 2 and 3 to derive two upper bounds on the number of spanning trees of a connected graph in terms of n, e, and d_1 only. Also using the Theorem 3.1 we conclude that the bound given by (3) reduces to the bound (2).

Theorem 4.1. *Let G be a simple connected graph with n vertices, e edges and d_1 be the highest degree. Then the number $t(G)$ of spanning trees of G satisfies*

$$t(G) \leq \left(\frac{2e-d_1-1}{n-2}\right)^{n-2}. \tag{9}$$

Equality holds if and only if G is a star graph or a complete graph.

Proof. We have

$$\begin{aligned} t(G) &= \frac{1}{n}\prod_{i=1}^{n-1} \lambda_i \\ &= \frac{1}{n}\lambda_1 \prod_{i=2}^{n-1} \lambda_i \\ &\leq \prod_{i=2}^{n-1} \lambda_i, \qquad \text{as } \lambda_1 \leq n \\ &\leq \left(\frac{\sum_{i=2}^{n-1} \lambda_i}{n-2}\right)^{n-2} \\ &\leq \left(\frac{2e-d_1-1}{n-2}\right)^{n-2}, \qquad \text{as } \textstyle\sum_{i=2}^{n-1} \lambda_i = 2e-\lambda_1 \leq 2e-d_1-1. \end{aligned}$$

Now suppose that equality holds in (9). Then all inequalities in the above argument must be equalities. Therefore

$$\lambda_1 = n,\ \lambda_2 = \lambda_3 = \ldots = \lambda_{n-1}, \quad \text{and} \quad \lambda_1 = d_1 + 1.$$

By Lemma 2.1 and $\lambda_1 = n$, we conclude that G is a super graph of a complete bipartite graph.

By Theorem 2.8 and $\lambda_2 = \lambda_3 = \ldots = \lambda_{n-1}$, we get that G is a complete graph or a star graph or a (d_1, d_1) complete bipartite graph.

By Lemma 2.2 and $\lambda_1 = d_1 + 1$, G is only an S_n graph.

Hence G is a complete graph or a star graph.

Conversely, it is easy to verify that equality in (9) holds for a complete graph or a star graph. □

Remark. Our result (9) is sharp for a star graph or a complete graph, but (2), (3) and (4) are sharp for complete graph only. Another thing is that (9) is perhaps a bit more useful than (3), since it depends only on n, e and d_1 rather than the full-degree sequence.

Theorem 4.2. *Let G be a simple connected graph with n vertices and e edges. Then the number $t(G)$ of spanning trees of G is given by*

$$t(G) \leq \left(\frac{n}{n-1}\right)^{n-1} \frac{\left(k^{(k+1)n-2e}(k+1)^{2e-kn}\right)}{2e}, \tag{10}$$

and the equality holds in (10) if and only if G is a complete graph.

Proof. Using Theorem 3.2 and from (3), we get

$$t(G) \leq \left(\frac{n}{n-1}\right)^{n-1} \frac{\left(k^{(k+1)n-2e}(k+1)^{2e-kn}\right)}{2e},$$

where $k = \left[\frac{2e}{n}\right]$.

Now suppose that equality holds in (10). Therefore equality holds in (3) and (7).

If equality holds in (3) then G is a complete graph.

By Theorem 3.2, if the equality holds in (7) then the difference between any two vertex degrees of graph G is at most one. Hence G is a complete graph.

Conversely, it is easy to verify that equality in (10) holds for a complete graph. □

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