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Tight Lower Bounds on the Size of a Maximum Matching in a Regular Graph

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Abstract. In this paper we study tight lower bounds on the size of a maximum matching in a regular graph. For $k \geq 3$, let G be a connected k-regular graph of order n and let $\alpha'(G)$ be the size of a maximum matching in G. We show that if k is even, then $\alpha'(G) \geq \min\left\{\left(\frac{k^2+4}{k^2+k+2}\right) \times \frac{n}{2}, \frac{n-1}{2}\right\}$, while if k is odd, then $\alpha'(G) \geq \frac{(k^3-k^2-2)\,n-2k+2}{2(k^3-3k)}$. We show that both bounds are tight.

Key words. Lower bounds, matching number, regular graph.

AMS Subject Classification: 05C70

1. Introduction

Two edges in a graph G are *independent* if they are not adjacent in G. A set of pairwise independent edges of G is called a *matching* in G, while a matching of maximum cardinality is a *maximum matching*. The number of edges in a maximum matching of G is called the *matching number* of G which we denote by $\alpha'(G)$. In this paper we study tight lower bounds on the size of a maximum matching in a connected regular graph, that is, in a graph in which every vertex has the same degree. Matchings in graphs are extensively studied in the literature (see, for example, the survey articles by Plummer [5] and Pulleyblank [6]). For a graph G and a set $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$.

For $k \geq 2$, let G be a connected k-regular graph of order n. If $k = 2$, then G is a cycle C_n and $\alpha'(G) \geq (n-1)/2$ with this bound achieved when G is an odd cycle. Hence in what follows, we assume that $k \geq 3$. When $k = 3$, Biedl et al. [2] proved that $\alpha'(G) \geq (4n-1)/9$. When $k = 4$ and $n \geq 6$, Lichiardopol [4] has shown that $\alpha'(G) \geq 7n/17$, and if $k \geq 5$, then $\alpha'(G) \geq ((3k-6)n)/(7k-13)$. In this paper, we generalize the result of Biedl et al. [2] when $k = 3$ to all values $k \geq 3$. Our results improve those of Lichiardopol [4].

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2. Preliminary Results

We need the following result of Berge [1] about the matching number of a graph, which is sometimes referred to as the Tutte-Berge formulation for the matching number.

Theorem 1 (Berge [1]). *For every graph G,*

$$\alpha'(G) = \min_{X \subseteq V(G)} \frac{1}{2} \left(|V(G)| + |X| - \mathrm{oc}(G - X) \right),$$

where $\mathrm{oc}(G - X)$ *denotes the number of odd components of* $G - X$.

We shall also need the following two observations.

Observation 1. *Every graph has an even number of vertices of odd degree.*

Observation 2. *Let* $k > 1$ *and let* G *be a graph with* $\Delta(G) \leq k$*. If*

$$\sum_{x \in V(G)} (k - d(x)) < k,$$

then $|V(G)| \geq k + 1$.

Proof. Let G be defined as in the observation and let $p = \sum_{x \in V(G)} (k - d(x))$ and $n = |V(G)|$. As the maximum degree in G is at most $n - 1$ we note that $p \geq n(k - (n - 1))$, which implies that $n(k + 1 - n) < k$. As this is equivalent to $0 < (n - 1)(n - k)$ we note that $n > k$, which implies the observation. □

3. Main Results

We shall show that:

Theorem 2. *For* $k \geq 2$ *even, if* G *is a connected* k*-regular graph of order* n*, then*

$$\alpha'(G) \geq \min \left\{ \left(\frac{k^2 + 4}{k^2 + k + 2} \right) \times \frac{n}{2}, \frac{n - 1}{2} \right\},$$

and this bound is tight.

Theorem 3. *For* $k \geq 3$ *odd, if* G *is a connected* k*-regular graph of order* n*, then*

$$\alpha'(G) \geq \frac{(k^3 - k^2 - 2)\, n - 2k + 2}{2(k^3 - 3k)},$$

and this bound is tight.

4. Proof of Theorem 2

First we present a proof of the lower bound of Theorem 2. If $k = 2$, then $\alpha'(G) = \alpha'(C_n) \geq (n-1)/2$, and the theorem holds. Hence we may assume that $k \geq 4$. Let X be a set of vertices in G such that $(n+|X|-\text{oc}(G-X))/2$ is minimum and let M be a maximum matching in G. By Theorem 1 we note that $|M| = (n+|X|-\text{oc}(G-X))/2$. If $X = \emptyset$, then $|M| \geq (n-1)/2$, and we are done, so assume that $X \neq \emptyset$.

By Observation 1, we note that there is no odd component in $G-X$ with exactly one edge into X in G, as it would have only one vertex of odd degree in $G-X$. Therefore every odd component has at least two edges into X in G. Let y_k denote the number of odd components in $G-X$ that have at least k edges into X in G and let y_2 denote the number of odd components in $G-X$ that have less than k edge into X in G. If there are $d(X, V-X)$ edges between X and $V(G)-X$, then we obtain the following, by the above,

$$k|X| \geq d(X, V-X) \geq 2y_2 + ky_k.$$

Hence, $|X| \geq y_k + 2y_2/k \geq 2y_2/k$. By Observation 2, we note that any odd component in $G-X$ with less than k edges into X in G must contain at least $k+1$ vertices. This implies that $n \geq |X| + y_2(k+1)$. We therefore obtain the following.

$$\begin{aligned}
|M| &= \frac{1}{2}(n + |X| - \text{oc}(G-X)) \\
&= \left(\frac{k^2+4}{k^2+k+2} \times \frac{n}{2}\right) + \left(\frac{k-2}{k^2+k+2} \times \frac{n}{2}\right) + \left(\frac{|X| - (y_2+y_k)}{2}\right) \\
&\geq \left(\frac{k^2+4}{k^2+k+2} \times \frac{n}{2}\right) + \left(\frac{k-2}{k^2+k+2} \times \frac{|X| + y_2(k+1)}{2}\right) + \left(\frac{\frac{2y_2}{k} - y_2}{2}\right) \\
&\geq \left(\frac{k^2+4}{k^2+k+2} \times \frac{n}{2}\right) + \left(\frac{k-2}{k^2+k+2} \times \frac{\frac{2y_2}{k} + y_2(k+1)}{2}\right) - y_2\left(\frac{k-2}{2k}\right) \\
&= \left(\frac{k^2+4}{k^2+k+2} \times \frac{n}{2}\right) + \left(\frac{y_2(k-2)}{2k(k^2+k+2)} \times \left(k^2+k+2-(k^2+k+2)\right)\right) \\
&= \frac{k^2+4}{k^2+k+2} \times \frac{n}{2}.
\end{aligned}$$

This establishes the lower bound of Theorem 2. The following proposition shows that the lower bound of Theorem 2 is tight.

Proposition 4. *For every integer $p \geq 2$ and every even integer $k \geq 4$, there exists a connected k-regular graph G_p^k of order $p(k^2+k+2)/2$ satisfying*

$$\alpha'(G_p^k) = \left(\frac{k^2+4}{k^2+k+2}\right) \times \frac{|V(G_p^k)|}{2}.$$

Proof. Let G_p^k be the connected k-regular graph of order $p(k^2+k+2)/2$ defined as follows. Let X_p be any k-regular connected multigraph of order p (i.e., we allow multiple edges). Note that $|E(X_p)| = kp/2$. For every edge $uv \in E(X_p)$, add a complete graph, C_{uv}, on $k+1$ new vertices, delete the edge uv and any edge $xy \in E(C_{uv})$, and then add the edges ux and vy. Once we have done this for all edges in X_p, we note that the resulting graph, G_p^k, is a connected k-regular graph of order $|V(G_p^k)| = |V(X_p)| + (k+1)kp/2 = p(1+k(k+1)/2) = p(k^2+k+2)/2$. Furthermore by deleting the vertices $V(X_p)$ from G_p^k we obtain $kp/2$ odd components (namely all the C_{uv}'s). Therefore, by Theorem 1,

$$\begin{aligned}
\alpha'(G_p^k) &\le \frac{1}{2}\left(|V(G_p^k)| + |V(X_p)| - \mathrm{oc}(G_p^k - V(X_p))\right) \\
&= \frac{1}{2}\left(\frac{1}{2}p(k^2+k+2) + p - \frac{1}{2}kp\right) \\
&= \frac{p}{4}\left((k^2+k+2) + 2 - k\right) \\
&= \left(\frac{k^2+4}{k^2+k+2}\right) \times \frac{p}{4}\left(k^2+k+2\right) \\
&= \left(\frac{k^2+4}{k^2+k+2}\right) \times \frac{|V(G_p^k)|}{2}.
\end{aligned}$$

However the lower bound of Theorem 2 shows that $\alpha'(G_p^k) \ge \left(\frac{k^2+4}{k^2+k+2}\right) \times \frac{|V(G_p^k)|}{2}$, and so the desired result of the proposition follows. □

We remark that the $(n-1)/2$ bound in the statement of Theorem 2 is only included as it is necessary when n is very small or $k = 2$. As an immediate consequence of Theorem 2, we have the following slightly weaker result.

Corollary 1. *For $k \ge 4$ even, if G is a connected k-regular graph of order n, then*

$$\alpha'(G) \ge \left\lfloor \left(\frac{k^2+4}{k^2+k+2}\right) \times \frac{n}{2} \right\rfloor,$$

and this bound is tight.

Proof. Observe that if $n \ge k + 3 + \dfrac{8}{k-2}$, then

$$\frac{k^2+4}{k^2+k+2} \times \frac{n}{2} \le \frac{n-1}{2},$$

while if $n < k + 3 + \dfrac{8}{k-2}$, then $\dfrac{k^2+4}{k^2+k+2} \times \dfrac{n}{2} - \dfrac{n-1}{2} \in \left(0, \dfrac{1}{2}\right)$, whence

$$\left\lfloor \frac{k^2+4}{k^2+k+2} \times \frac{n}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Hence for all $k \geq 4$ and $n \geq k+1$, we have that

$$\left\lfloor \frac{k^2+4}{k^2+k+2} \times \frac{n}{2} \right\rfloor \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

The desired result now follows from Theorem 2. □

5. Proof of Theorem 3

In order to prove Theorem 3, we shall need the following definition.

Definition 5.1. *Let $k \geq 3$ be an odd integer and let H_{k+2} be the graph with vertex set $V(H_{k+2}) = \{w_1, w_2, \ldots, w_{k+2}\}$ and containing all possible edges except the following:*

$$\left(\bigcup_{i=0}^{(k-3)/2} \{w_{2i+1}w_{2i+2}\} \right) \cup \{w_k w_{k+2}, w_{k+1}w_{k+2}\}.$$

Note that the degree of every vertex in H_{k+2} is k, except for w_{k+2} which has degree $k-1$.

We are now in a position to prove Theorem 3.

Proof of Theorem 3. Let

$$\phi(k,n) = \frac{(k^3-k^2-2)\,n - 2k + 2}{2(k^3-3k)}.$$

We wish to show that $\alpha'(G) \geq \phi(k,n)$. For a subset $X \subset V(G)$, we define

$$M_X = \frac{1}{2}\left(n + |X| - \mathrm{oc}(G-X)\right).$$

We proceed further with the following claim.

Claim. If X is an independent set of vertices in G such that every component H in $G-X$ has odd order and the number of edges between X and $V(H)$ is either one or k, then $M_X \geq \phi(k,n)$.

Proof. Let y_k denote the number of components in $G-X$ that have k edges into X in G and let V_k denote the set of all vertices that lie in such components. Let y_1 denote the number of components in $G-X$ that have one edge into X in G and let V_1 denote the set of all vertices that lie in such components. Note that the following holds.

(1) $|X| = y_k + y_1/k$. This is the case as if there are $d(X, V-X)$ edges between X and $V(G)-X$, then we note that $k|X| = d(X, V-X) = ky_k + y_1$.
(2) $n \geq |X| + y_k + (k+2)y_1$. This follows from $|V_k| \geq y_k$ and from Observations 1 and 2, which imply that $|V_1| \geq (k+2)y_1$.

(3) $y_k \geq \frac{y_1 - k}{k(k-2)}$. Note that $G - V_k$ has exactly $|X|$ distinct components. By adding to $G - V_k$ a component from $G[V_k]$ and the resulting k edges from that component into X, we decrease the number of components by at most $k - 1$. As G is connected we must therefore have $y_k(k-1) \geq |X| - 1$. Using (1), we get $y_k(k-1) \geq y_k + y_1/k - 1$, and so solving for y_k implies (3).

By (1) we note that if $y_1 = 0$, then $M_X = n/2$ which implies that the claim holds in this case. Hence we may assume that $y_1 \geq 1$. By (1), we can bound M_X as follows:

$$\begin{aligned} M_X &= \frac{1}{2}\left(n + |X| - oc(G - X)\right) \\ &\geq \frac{1}{2}\left(n + \frac{y_1}{k} + y_k - y_1 - y_k\right) \\ &= \frac{1}{2}\left(n + \frac{y_1(1-k)}{k}\right). \end{aligned}$$

Now,

$$\frac{1}{2}\left(n + \frac{y_1(1-k)}{k}\right) \geq \frac{(k^3 - k^2 - 2)\,n - 2k + 2}{2(k^3 - 3k)}$$

$$\Updownarrow$$

$$\frac{y_1(1-k)}{k} + \frac{2(k-1)}{k(k^2-3)} \geq \left(\frac{(k^3 - k^2 - 2) - (k^3 - 3k)}{k(k^2-3)}\right) n$$

$$\Updownarrow$$

$$\frac{y_1(1-k)(k^2-3) + 2(k-1)}{k(k^2-3)} \geq \left(\frac{-k^2 + 3k - 2}{k(k^2-3)}\right) n$$

$$\Updownarrow$$

$$-y_1(k-1)(k^2-3) + 2(k-1) \geq -(k-1)(k-2)n$$

$$\Updownarrow$$

$$n(k-2) \geq y_1(k^2-3) - 2$$

$$\Updownarrow$$

$$n \geq \frac{y_1(k^2-3)}{k-2} - \frac{2}{k-2}.$$

However, by (1), (2) and (3) we get the following bound on n.

$$\begin{aligned} n &\geq |X| + y_k + (k+2)y_1 \\ &\geq \left(y_k + \frac{y_1}{k}\right) + y_k + (k+2)y_1 \\ &\geq \frac{2(y_1 - k)}{k(k-2)} + \frac{y_1}{k} + (k+2)y_1 \\ &= y_1 \times \frac{2 + (k-2) + k(k-2)(k+2)}{k(k-2)} - \frac{2k}{k(k-2)} \\ &= \frac{y_1(k^2-3)}{k-2} - \frac{2}{k-2}. \end{aligned}$$

Hence we have established that

$$M_X \geq \frac{1}{2}\left(n + \frac{y_1(1-k)}{k}\right) \geq \frac{(k^3-k^2-2)\,n - 2k + 2}{2(k^3-3k)} = \phi(k,n),$$

as desired. □

We now return to the proof of Theorem 3. Let X be a set of vertices in G such that M_X is minimum. By Theorem 1, $\alpha'(G) = M_X$. We wish to show that $M_X \geq \phi(k,n)$. Suppose, to the contrary, that $M_X < \phi(k,n)$. Then, X does not satisfy the conditions of Claim 5. Hence we may assume that at least one of the following conditions holds.

Condition (a). X is not independent.
Condition (b). There are even components in $G - X$.
Condition (c). There are odd components in $G - X$ which do not have one or k edges into X.

Now define a graph Z_X and integers z_X, p_X, and r_X as follows.

– If $G - X$ has odd components which do not have one or k edges into X or even components, then let Z_X be the graph consisting of all such components. Furthermore, let

$$z_X = |V(Z_X)|.$$

– For each $x \in X$, let p_x denote the degree of x in $G[V(Z_X) \cup X]$; that is, p_x is the number of vertices in $V(Z_X) \cup X$ adjacent to x in G. Let

$$p_X = \sum_{x \in X} p_x.$$

– Let r_X denote the number of components in $G - E(G[X]) - V(Z_X)$ (i.e., we delete all edges in $G[X]$ and all vertices in Z_X).

For notational convenience, in what follows we denote Z_X, z_X, p_X, and r_X simply by Z, z, p, and r.

Let G_X be the graph obtained from $G - V(Z)$ by deleting all edges in $G[X]$, if any. Hence, G_X can be obtained from G by the following three steps.

Step (a). Delete all edges in $G[X]$, if any.

Step (b). Delete all vertices in even components in $G - X$, if such components exist.

Step (c). Delete all vertices in odd components in $G - X$ which do not have one or k edges into X, if such components exist.

Notice that in steps (b) and (c) we have deleted z vertices, while in step (c) we have deleted $\mathrm{oc}(Z)$ components. Furthermore we remark that the graph G_X consists of r components and

$$p = \sum_{x \in V(G_X)} (k - d_{G_X}(x)).$$

Now for every vertex x in G_X with $d_{G_X}(x) < k$ (necessarily, $x \in X$), add $k - d_{G_X}(x)$ copies of the subgraph H_{k+2} to G_X and in each added copy of H_{k+2}, join the vertex named w_{k+2} in H_{k+2} to x (see Definition 5.1). The resulting graph, which we call G^*, is clearly k-regular and still consists of r components. Also note that X is a subset of $V(G^*)$, as we didn't delete any vertices from X in steps (a), (b) or (c). As X satisfies the conditions of Claim 5 in the graph G^*, we know that the following holds (note that we actually apply Claim 5 to each of the r components in G^*).

$$\frac{1}{2}\left(|V(G^*)| + |X| - \mathrm{oc}(G^* - X)\right) \geq \frac{(k^3 - k^2 - 2)\,|V(G^*)| - 2r(k-1)}{2(k^3 - 3k)}.$$

Since $|V(G^*)| = p(k+2) + n - z$ and $\mathrm{oc}(G^* - X) = \mathrm{oc}(G - X) - \mathrm{oc}(Z) + p$, this implies that

$$\begin{aligned} M_X &+ \frac{1}{2}\left(p(k+2) - z - (p - \mathrm{oc}(Z))\right) \\ &\geq \phi(k,n) + \frac{(k^3 - k^2 - 2)\,(p(k+2) - z) - 2(r-1)(k-1)}{2(k^3 - 3k)}. \end{aligned}$$

Thus since $\phi(k,n) - M > 0$ (by assumption), we know that the following holds:

$$\frac{1}{2}\left(p(k+2) - z - (p - \mathrm{oc}(Z))\right) > \frac{(k^3 - k^2 - 2)\,(p(k+2) - z) - 2(r-1)(k-1)}{2(k^3 - 3k)}$$

$$\Updownarrow$$

$$p\left(k + 1 - \frac{(k+2)(k^3 - k^2 - 2)}{k^3 - 3k}\right) + \mathrm{oc}(Z) > z\left(1 - \frac{k^3 - k^2 - 2}{k^3 - 3k}\right) - 2(r-1)\left(\frac{k-1}{k^3 - 3k}\right)$$

$$\Updownarrow$$

$$p(4 - k - k^2) + \mathrm{oc}(Z)(k^3 - 3k) > z(k^2 - 3k + 2) - 2(r-1)(k-1).$$

Note that $p > 0$ since otherwise we were done by Claim 5 (it can also be seen as steps (a), (b) and (c) all decrease the degree of at least one vertex in X and $p = \sum_{x \in V(G_X)}(k - d_{G_X}(x))$). As G is connected we note that every component in G_X has at least one vertex that has degree less than k. Therefore we must have $p \geq r$, which by the above implies the following.

$$p(4 - k - k^2) + \mathrm{oc}(Z)(k^3 - 3k) > z(k^2 - 3k + 2) - 2(p-1)(k-1)$$

$$\Updownarrow$$

$$\mathrm{oc}(Z)(k^3 - 3k) > z(k^2 - 3k + 2) + p(k^2 - k - 2) + 2(k-1).$$

Note that if $\mathrm{oc}(Z) = z = p = 0$, then the last inequality is false as $0 > 2(k-1)$ is false. We now consider what happens to this last inequality when each of the steps (a), (b) and (c), respectively, are performed.

(a) When step (a) is performed, each edge in $G[X]$ contributes two to p and zero to each of z and $\mathrm{oc}(Z)$.
(b) When step (b) is performed, each even component in $G - X$ contributes at least one to p, at least two to z, and zero to $\mathrm{oc}(Z)$.

(c) When step (c) is performed, we consider the following two possibilities. Consider the case when we delete an odd component, C, of $G - X$ with less than k, but with at least two, edges into X. Then, C contributes at least two to p and one to $\mathrm{oc}(Z)$. By Observation 2, we note that $|V(C)| \geq k+1$. As $|V(C)|$ is odd we must have $|V(C)| \geq k+2$, and so C contributes at least $k+2$ to z. Therefore we increase the left-hand side above with $(k^3 - 3k)$ and we increase the right-hand side with at least $(k+2)(k^2 - 3k + 2) + 2(k^2 - k - 2) = (k^3 - 3k) + (k^2 - 3k)$. As $k \geq 3$, this means we increase the right-hand side by at least the same as the left-hand side.
Now consider the case when we delete a component, C, of $G - X$ with more than k edges into X. In this case C contributes at least $k + 1$ to p and at least one to z, which implies that again we increase the right-hand side by at least as much as the left-hand side.

As $\mathrm{oc}(Z)(k^3 - 3k) \geq z(k^2 - 3k + 2) + p(k^2 - k - 2) + 2(k - 1)$ was false before we performed any of the steps (a), (b) or (c) (i.e., when $\mathrm{oc}(Z) = z = p = 0$), then by the above it must remain false as we never increase the left-hand side by more than the right-hand side. This is the desired contradiction. Hence, $M_X \geq \phi(k, n)$ as claimed. □

The following proposition shows that the lower bound of Theorem 3 is tight.

Proposition 5. *For every integer $p \geq 1$ and every odd integer $k \geq 3$, there exists a connected k-regular graph H_p^k of order $p(k^3 - 3k) + k^2 + 2k + 1$ satisfying*

$$\alpha'(H_p^k) = \frac{(k^3 - k^2 - 2)\,|V(H_p^k)| - 2k + 2}{2(k^3 - 3k)}.$$

Proof. Let T_p^k be the bipartite graph with partite sets $V_1 = \{w_1, w_2, \dots, w_p\}$ and $V_2 = \{v_1, v_2, \dots, v_{p(k-1)+1}\}$, and where $N(w_i) = \{v_{(k-1)(i-1)+1}, v_{(k-1)(i-1)+2}, \dots, v_{(k-1)(i-1)+k}\}$ for all $i = 1, 2, \dots, p$. Note that the degree of all the $w_i \in V_1$ is k and the degree of all vertices in V_2 is one except for the vertices in $\{w_{(k-1)+1}, w_{2(k-1)+1}, \dots, w_{(p-1)(k-1)+1}\}$ which have degree two. Now for every vertex x in $V(T_p^k)$ with $d_{T_p^k}(x) < k$, add $k - d_{T_p^k}(x)$ copies of the subgraph H_{k+2} to T_p^k and in each added copy of H_{k+2}, join the vertex named w_{k+2} in H_{k+2} to x (see Definition 5.1). Let H_p^k denote the resulting graph. Then, H_p^k is a connected k-regular graph of order

$$\begin{aligned}
|V(H_p^k)| &= |V_1| + |V_2| + (|V_2|(k-1) - (p-1))(k+2) \\
&= p + (p(k-1)+1) + ((p(k-1)+1)(k-1) - (p-1))(k+2) \\
&= p(1 + (k-1) + (k-1)^2(k+2) - (k+2)) + 1 + (k-1)(k+2) + (k+2) \\
&= p(k^3 - 3k) + k^2 + 2k + 1.
\end{aligned}$$

Furthermore by deleting the vertices V_2 from H_p^k we obtain $p + (p(k-1)+1)(k-1) + (p-1)$ odd components. Therefore, by Theorem 1,

$$
\begin{aligned}
\alpha'(H_p^k) &\le \frac{1}{2}\Big(|V(H_p^k)| + |V_2| - \mathrm{oc}(H_p^k - V_2)\Big)\\
&= \frac{1}{2}\Big((p(k^3-3k)+(k+1)^2)+(p(k-1)+1)\\
&\quad -(p + (p(k-1)+1)(k-1) + (p-1))\Big)\\
&= \frac{1}{2}\Big((p(k^3-3k + (k-1) - 1 - (k-1)^2 + 1) + (k+1)^2 - 1 - (k-1)+1\Big)\\
&= \frac{1}{2}\Big(p(k^3 - k^2 - 2) + k^2 + k + 2\Big)\\
&= \left(\frac{k^3 - k^2 - 2}{k^3 - 3k} \times \frac{p(k^3 - 3k) + k^2 + 2k + 1}{2}\right)\\
&\quad + \frac{(k^3 - 3k)(k^2 + k + 2) - (k^2 + 2k + 1)(k^3 - k^2 - 2)}{2(k^3 - 3k)}\\
&= \frac{(k^3 - k^2 - 2)\,|V(H_p^k)| - 2k + 2}{2(k^3 - 3k)}.
\end{aligned}
$$

However the lower bound of Theorem 3 shows that

$$\alpha'(H_p^k) \ge \frac{(k^3 - k^2 - 2)\,|V(H_p^k)| - 2k + 2}{2(k^3 - 3k)},$$

and so the desired result of the proposition follows. □

Using a shorter proof than that of Theorem 3 it is possible to prove the following weaker result. If $k \ge 1$ is odd and G is a k-regular graph of order n, then $\alpha'(G) \ge \frac{k^2+k+2}{k^2+2k+1} \times \frac{n}{2}$. This can also be shown as a consequence of Theorem 3 as $\lceil \frac{k^2+k+2}{k^2+2k+1} \times \frac{n}{2} \rceil \le \lceil \frac{(k^3-k^2-2)\,n-2k+2}{2(k^3-3k)} \rceil$ holds for all n and k we consider. However this weaker result can be useful in some cases as for some values of n and k we actually have $\frac{k^2+k+2}{k^2+2k+1} \times \frac{n}{2} > \frac{(k^3-k^2-2)\,n-2k+2}{2(k^3-3k)}$. It is for example used in the paper [3].

6. Closing Remarks

Let G be a connected k-regular graph. As $\alpha'(G) \le \frac{n}{2}$ we note that our results imply that $\frac{\alpha'(G)}{n} \to \frac{1}{2}$ when $k \to \infty$. In comparison, the bound, $n(3k-6)/(7k-13)$, in [4] is always less than $3n/7$. For small values of k, we summarize our results in the accompanying table.

G is a connected k-regular graph						
k	3	4	5	6	7	8
$\alpha'(G) \ge$	$\frac{4n-1}{9}$	$\min\left\{\frac{5n}{11}, \frac{n-1}{2}\right\}$	$\frac{49n-4}{110}$	$\min\left\{\frac{5n}{11}, \frac{n-1}{2}\right\}$	$\frac{73n-3}{161}$	$\min\left\{\frac{17n}{37}, \frac{n-1}{2}\right\}$

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