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Graphs and Combinatorics

Subdivision Extendibility

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Abstract. Let H be a multigraph and G a graph containing a subgraph isomorphic to a subdivision of H, with $S \subset V(G)$ (the ground set) the image of $V(H)$ under the isomorphism. We consider connectivity and minimum degree or degree sum conditions sufficient to imply there is a spanning subgraph of G isomorphic to a subdivision of H on the same ground set S. These results generalize a number of theorems in the literature.

1. Introduction

Many results deal with finding connectivity and/or degree conditions which imply the existence of certain types of spanning subgraphs of a given graph. Exactly how large the connectivity and degree conditions must be to ensure that these subgraphs span, depends on the structure of the subgraphs. Here, we prove a theorem which generalizes many of these results.

We first note that this result can be viewed as showing the existence of a spanning subdivision of a given multigraph H. We exhibit connectivity, minimum degree and minimum degree sum conditions for G based upon the structure of H. We use these conditions to show that if there is a subdivision of H in G then there is a spanning subdivision of H in G.

For terms not defined here see [3]. Unless otherwise specified, in this paper, $G = (V, E)$ will denote a simple loopless graph with $|G| = |V(G)| = n$. A path between x and y will be called an $[x, y]$-path. If $x = y$ then P is said to be a cycle. When a path P is not a cycle we will say that P is a proper path. Note that the word "path" is used here in a more general sense than the usual since we include cycles in it's definition. For any given $u, v \in V(P)$, $[u, v]_P$ (respectively $(u, v)_P$) will denote the subpath of P including (respectively, not including) the end-vertices u and v. Let

$$\sigma_2(G) = \min\{d(x) + d(y) : xy \notin E(G)\}.$$

If λ is a graph parameter, we say that a number m is the λ-threshold of a property $\mathcal{P}$ if any graph satisfying $\lambda \geq m$ has property $\mathcal{P}$ and m is the smallest such number. Many results determine the δ- or σ_2-threshold of a given property. For example, Dirac's [5] famous Theorem states that the δ-threshold for Hamiltonicity is $\frac{n}{2}$ and Ore

[15] showed that the σ_2-threshold for Hamiltonicity is n. Though $\sigma_2(G) \geq 2\delta(G)$, it is not always true that the σ_2-threshold of a property is exactly twice the δ-threshold of this property (this is, for example, the case with the property of being k-ordered (see [10])). For the class of properties we will be dealing with here, the δ-threshold will be shown to be half of the σ_2-threshold, hence our result for the σ_2-threshold will imply the corresponding result for the δ-threshold.

In order to illustrate our approach, consider the following:

Theorem 1 [16]. *If $n \geq 3$ and $\sigma_2(G) \geq n+1$ then for any distinct vertices u and v of G, there is a spanning $[u,v]-path$ in G.*

Justesen [13] proved that if $\sigma_2(G) \geq 4k$ then there exists k vertex disjoint cycles in G. Using this, Brandt et al. [2] generalized Ore's Theorem on Hamiltonicity, showing that the condition $\sigma_2(G) \geq n$ implies, for every k $(1 \leq k \leq \frac{n}{4})$, the existence of k vertex disjoint cycles whose union spans G.

Theorem 2 [2]. *If $n \geq 4k$, $\sigma_2(G) \geq n$ and F is a collection of k disjoint cycles of maximal order, then F spans G.*

As one sees, often the proofs of these type of path or cycle results may be divided in two phases. In the first, one proves that the given conditions are sufficient to imply the existence of a particular subgraph. Second, one shows that the conditions are sufficient to imply that this subgraph may be chosen to be spanning. In [11] we generalize the existence phase. It is the extension phase that we wish to generalize here.

2. H-Extendibility: A Unification of Many Results

Throughout this paper, let H be a multigraph, possibly with loops. We say that $e \in E(H)$ is a proper edge if it is not a loop. We denote by $d_H(w)$, the degree of w in H, the number of proper edges plus two times the number of loops incident with the vertex w.

Let $e = uv$ be a proper edge of H, and w be a vertex not in $V(H)$. We say that w is insertible in uv if the edge e can be replaced by the path uwv. If we have completed the process we say w has been inserted in H. A multigraph H' is called a *subdivision of H* if one can obtain H' from H by recursively inserting vertices.

Let $\mathcal{L}(G)$ denote the set of paths in G. To an *H-subdivision* $\mathcal{S}$ in G we will associate a pair of mappings $f_{\mathcal{S}} : V(H) \to V(G)$ and $g_{\mathcal{S}} : E(H) \to \mathcal{L}(G)$ such that:

(i) $f_{\mathcal{S}}$ is injective;
(ii) for every edge $xy \in E(H)$, $g_{\mathcal{S}}(xy)$ is an $[f_{\mathcal{S}}(x), f_{\mathcal{S}}(y)]$-path in G and distinct edges of $E(H)$ map to internally disjoint paths in G.

Let $V(H) = \{w_1, \ldots, w_h\}$ and $\mathbf{S} = f_{\mathcal{S}}(V(H)) = \big(f_{\mathcal{S}}(w_1), \ldots, f_{\mathcal{S}}(w_h)\big)$. The $f_{\mathcal{S}}(w_i)$ are called the ground vertices of $\mathcal{S}$ and the paths of $g_{\mathcal{S}}(E(H))$ are called *branches* of the subdivision. We say that $\mathcal{S}$ is an H-subdivision on $\mathbf{S}$.

Definition 1. *Given a multigraph H and a graph G, we will say that G is H-extendible if when there exists an H-subdivision on a $|H|$-tuple* $\mathbf{S}$ *of distinct vertices of G, then there exists a spanning H-subdivision on* $\mathbf{S}$.

By an *independent vertex set* we mean a set $I \subseteq V(H)$ such that the subgraph induced by I has no edges. Note, this means any vertex with a loop will be excluded from an independent set. By an *independent edge set* we mean a set $E \subseteq E(H)$ such that the end-vertices of each edge have degree one in the graph induced by E. Note, since we consider a loop as adding two to the degree of a vertex, this definition excludes all loops from independent edge sets.

Let F be a subgraph of G. Let $\mathcal{I}(F)$ represent the family of all independent vertex sets of F. Let $\alpha(F)$ be the maximum order of an independent set of F, called the independence number of F. Let $\mathcal{I}_{max}(F) = \{I \in \mathcal{I}(F) : |I| = \alpha(F)\}$ be the family of all independent vertex sets of maximum order. The edge-independence number $\beta(G)$ of G is the maximum order of an independent edge set.

We will prove the following Theorem in section 4, which gives a lower bound on $\sigma_2(G)$ sufficient to imply that a given H-subdivision on $\mathbf{S}$ can be extended to an H-subdivision on $\mathbf{S}$ spanning the vertices of G. We let $h_0(H)$ be the number of vertices of degree zero and $h_1(H)$ the number of vertices of degree one in H.

Theorem 3. *Let H be a multigraph with at least two edges, G a simple* $(\max\{\alpha(H), \beta(H)\} + 1)$*-connected graph and $\mathcal{S}$ an H-subdivision on* $\mathbf{S}$. *Suppose that* $|\mathcal{S}| \geq 6|E(H)| + 3(|H| - h_1(H))$*, $|\mathcal{S}|$ is maximal, and*

$$\sigma_2(G) \geq 2\alpha(\mathcal{S}) + |G - \mathcal{S}|$$

then $|\mathcal{S}| = |G|$.

We wish to have a lower bound on $\sigma_2(G)$ expressed only in terms of parameters of H. To do so, we will need to relate $\alpha(\mathcal{S})$ to characteristics of H and $\mathcal{S}$. If $e \in E(H)$, a branch $B = g_{\mathcal{S}}(e)$ of $\mathcal{S}$ is called a loop branch, a leaf branch or an independent branch if e is respectively a loop, an edge having exactly one vertex of degree one or an independent edge. We say B is a normal branch otherwise. A branch is said to be even (respectively odd) if it has an even (respectively odd) number of vertices. Let $\xi(\mathcal{S})$ denote the sum of the number of even loop branches, even leaf branches, odd independent branches, and odd normal branches of $\mathcal{S}$.

We prove the following Lemma in section 3:

Lemma 1. *Let H be a multigraph, G be a simple graph, and $\mathcal{S}$ be an H-subdivision on* $\mathbf{S}$. *Then*

$$\alpha(\mathcal{S}) \geq \frac{|\mathcal{S}| + \xi(\mathcal{S}) - |H| + h_1(H)}{2} + h_0(H) \tag{1}$$

$$\alpha(\mathcal{S}) \leq \frac{|\mathcal{S}| + |E(H)| - |H| + h_1(H)}{2} + h_0(H). \tag{2}$$

Using this Lemma along with Theorem 3, we then find the σ_2-threshold for H-extendibility.

Theorem 4. *If H is a multigraph and G is a simple* $(\max\{\alpha(H), \beta(H)\}+1)$*-connected graph of order* $n > 11|E(H)| + 7(|H| - h_1(H))$ *such that*

$$\sigma_2(G) \geq n + |E(H)| - |H| + h_1(H) + 2h_0(H),$$

then G is H-extendible.

Our counterexample will show that the δ-threshold for H-extendibility is in fact simply half of the σ_2-threshold, establishing that the lower bound in the following Corollary is the best possible.

Corollary 1. *If H is a multigraph and G is a simple* $(\max\{\alpha(H), \beta(H)\}+1)$*-connected graph of order* $n > 11|E(H)| + 7(|H| - h_1(H))$ *such that*

$$\delta(G) \geq \frac{n + |E(H)| - |H| + h_1(H) + 2h_0(H)}{2},$$

then G is H-extendible.

Note that $\sigma_2(G) \geq n + k - 2$ implies that the graph is k-connected. Thus, if $\max\{\alpha(H), \beta(H)\} \leq |E(H)| - |H| + h_1(H) + 2h_0(H) + 1$, then the connectivity condition in Theorem 4 is redundant, and can be omitted.

We show now that the lower bounds on $\sigma_2(G)$ and $\delta(G)$ in Theorem 4 and Corollary 1, respectively, are best possible. Let H be a multigraph with $|E(H)| = \xi$, $|H| = h$, $h_1(H) = h_1$, and $h_0(H) = h_0$. Consider the split graph $G(a, b) = K_b + \overline{K_a}$ (where $+$ indicates the standard join operation) such that $a = \frac{n-(\xi-h+h_1+2h_0-1)}{2}$ and $b = \frac{n+\xi-h+h_1+2h_0-1}{2}$. Now G is a graph of order n with $\sigma_2(G) = 2\delta(G) = 2b = n + \xi - h + h_1 + 2h_0 - 1$. If we choose $\mathbf{S}$ such that for all $w \in V(H)$, $f_{\mathbf{S}}(w) \in K_b$ if and only if $d_H(w) \leq 1$, we will show that though there is an H-subdivision on $\mathbf{S}$, this subdivision cannot be spanning.

Indeed, it is easy to see that one can find an H-subdivision $\mathcal{S}$ on $\mathbf{S}$. Since $E(\overline{K_a}) = 0$, if we let $A \subset \overline{K_a}$ be the set of those vertices of $\mathbf{S}$ which are in $\overline{K_a}$, we see that $\mathcal{S} - A$ is a collection $\mathcal{P}$ of $\xi + h_0$ vertex-disjoint proper paths of $G - A$ whose end-vertices are in K_b. Note that some of these proper paths may be singletons (i.e. composed of a single vertex).

Since all the end-points of the paths of $\mathcal{P}$ are in B, for every $P \in \mathcal{P}$, $|P \cap \overline{K_a}| \leq |P \cap K_b| - 1$, hence $|\mathcal{P} \cap \overline{K_a}| \leq |\mathcal{P} \cap K_b| - |\mathcal{P}|$. Yet $|\mathcal{P}| = \xi + h_0$, so if $\mathcal{P}$ covered all $|K_b| = \frac{n+\xi-h+h_1+2h_0-1}{2}$ vertices of K_b, we would have

$$|\mathcal{P} \cap \overline{K_a}| \leq \frac{n + \xi - h + h_1 + 2h_0 - 1}{2} - (\xi + h_0) < |\overline{K_a} - A|.$$

This shows that $\mathcal{P}$ cannot cover all the vertices of the graph.

We next state several results which are corollaries of Theorem 4. Note that the lower bounds on n given in the following Theorems may be larger than those found in the original papers. The lower bounds we include here are those implied by our main Theorem, which have been made larger than necessary in order to simplify the proofs.

We will make use of the following multigraphs. Let M_k be the multigraph on two vertices having k proper edges. Let $T_{k,t} = tK_1 \cup K_{1,k-t}$ (where $0 \leq t \leq k-1$) and $E_{k,t} = tK_1 \cup (k-t)K_2$. Let C_k be a cycle of order k and L_k be the graph composed of k vertices, each having one loop.

Corollary 2 [15]. *Let G be a graph on $n \geq 18$ vertices such that $\sigma_2(G) \geq n$. Then G contains a Hamilton cycle.*

Proof. Consider Theorem 4 with the multigraph L_1. An L_1-subdivision is a cycle, and we have $|E(L_1)| = 1$, $|L_1| = 1$, and $h_1(L_1) = h_0(L_1) = \alpha(L_1) = \beta(L_1) = 0$. Hence we see that if $n \geq 18$ and $\sigma_2(G) \geq n + 1 - 1 + 0 + 0 = n$, then any cycle is extendible. Since it is easy to see that $\sigma_2(G) \geq n$ implies the existence of a cycle, we see that G has a cycle covering all the vertices of G. □

Corollary 3 [2]. *Let G be a graph on n vertices such that $\sigma_2(G) \geq n$. Then for all $1 \leq k \leq \frac{n}{18}$, G contains k vertex disjoint cycles, together covering all the vertices of G.*

Proof. Consider Theorem 4 with the multigraph L_k. An L_k-subdivision is a system of k vertex-disjoint cycles. Also $|E(L_k)| = k$, $|L_k| = k$, and $h_1(L_k) = h_0(L_k) = \alpha(L_k) = \beta(L_k) = 0$. Hence we see that if $n \geq 18k$ and $\sigma_2(G) \geq n+k-k+0+0 = n$, then any such system of k cycles is extendible. Justesen [13] showed that if $\sigma_2(G) \geq 4k$, then there exists k vertex disjoint cycles, so we may extend these cycles to cover all the vertices of G. □

Note that if we apply Theorem 4 to $E_{k,t}$ for $t \in [k]$, we get a Theorem found in [12] which generalizes results of [1] and [7]. Let A and B be sets of vertices having the same cardinality k. An (A, B)-system is the union of $A \cap B$ with a collection of $|A - B|$ disjoint paths having one vertex in $A - B$ and the other in $B - A$.

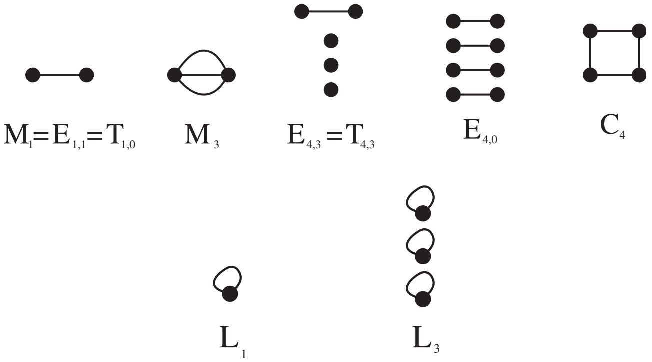

Fig. 1. A few examples of multigraphs

Corollary 4. *Let A and B be two sets of vertices of order k and G be a graph of order $n \geq 11k$ and $\sigma_2(G) \geq n+k$, then any (A, B)-system $\mathcal{P}$ can be extended to an (A, B)-system $\mathcal{P}'$ spanning the vertices of G and such that the paths of $\mathcal{P}$ and $\mathcal{P}'$ have the same end-vertices.*

Proof. We apply Theorem 4 to $E_{k,t}$, where $t = |A \cap B|$. Here $|E(E_{k,t})| = k - t$, $|E_{k,t}| = 2k - t$, $h_1(E_{k,t}) = 2(k-t)$, $h_0(E_{k,t}) = t$, $\alpha(E_{k,t}) = k$ and $\beta(E_{k,t}) = k - t$. Thus, if $n \geq 11k \geq 11(k-t) + 7(2k - t - 2k + 2t)$, and

$$\begin{aligned}\sigma_2(G) &\geq n + (k-t) - (2k-t) + 2(k-t) + 2t \\ &= n + k\end{aligned}$$

then Theorem 4 yields the result. □

Corollary 5 [14], [8], [4]. *Let G be a $\left\lceil \frac{k+1}{2} \right\rceil$-connected graph on $n \geq 18k$ vertices such that $\sigma_2(G) \geq n$. Let C be a cycle encountering a vertex sequence $\mathbf{S} = (x_1, \ldots, x_k)$ in that order. Then G has a Hamilton cycle encountering the vertices of $\mathbf{S}$ in the given order.*

Proof. Consider the cycle C_k on k vertices. We have $|E(C_k)| = k$, $|C_k| = k$, $h_1(C_k) = h_0(C_k) = 0$, and $\alpha(C_k) = \beta(C_k) = \left\lfloor \frac{k}{2} \right\rfloor$. Applying Theorem 4 we get the result. □

A collection L of t disjoint paths, s of them being singletons, with $|V(L)| = k$, is called a (k, t, s)-linear forest. A graph G is said to be strongly (k, t, s)-ordered if for every (k, t, s)-linear forest of G, there is a cycle encountering the paths of the forest in their given order and according to their given orientation. If the cycle can always be made spanning, we say that G is strongly (k, t, s)-ordered Hamiltonian.

Corollary 6 [9]. *Let s, t, k be integers with $0 \leq s < t < k$ or $s = t = k \geq 3$. If G is a strongly (k, t, s)-ordered graph on $n \geq k$ vertices with $\sigma_2(G) \geq n + k - t$ then G is strongly (k, t, s)-ordered Hamiltonian.*

Proof. Taking H to be a $(k, t, k - s - 2t)$-linear forest and applying Theorem 4 yields the result. □

Note that our corollary yields a stronger result than that found in [9], where the lower bound $\sigma_2(G) \geq n + k - t + s - 1$ is used when $s > 0$.

Theorem 1 has been extended in [1], [7], and [12] using diverse extensions and characterizations of the concept of connectivity. The path-systems used in [17] and [6] to characterize k-connectivity can both be used to extend the idea of Hamilton connectivity to higher connectivities, resulting in the interest of knowing what the σ_2-thresholds for M_k-extendibility and $T_{k,t}$-extendibility are. Theorem 4 may readily be used to find these.

Corollary 7. *Let G be a graph on $n \geq 11k + 14$ vertices such that $\sigma_2(G) \geq n + k - 2$. Let x and y be two distinct vertices of G and $\mathcal{S}$ be a system of k internally disjoint $[x, y]$-paths. Then $\mathcal{S}$ can be extended to cover all the vertices of G.*

Corollary 8. *Let k and t be integers such that $0 \leq t \leq k$ and G be a graph on $n > 11k - 4t - 7$ vertices such that $\sigma_2(G) \geq n + k - 1$. If G has a $T_{k,t}$-subdivision $\mathcal{S}$ on* $\mathbf{S}$ *then $\mathcal{S}$ can be extended to cover all the vertices of G.*

3. The Independence Number of Subdivisions

In this section we consider how the independence number of an H-subdivision relates to H and the order of the subdivision. Let us first extend the idea of insertion of a vertex into a graph.

Let $\mathcal{S}$ be an H-subdivision on $\mathbf{S}$. If, for $u, v \in V(G) - f_{\mathcal{S}}(V(H))$, P is a proper $[u, v]$-path of $G - \mathcal{S}$, we say that P is *uniformly insertible* in $\mathcal{S}$ if there exists an edge $xy \in E(\mathcal{S})$ such that $ux, vy \in E(G)$. When P *uniformly inserts* in $\mathcal{S}$ we obtain a larger H-subdivision $\mathcal{S} \triangleleft P$ on $\mathbf{S}$, namely

$$\mathcal{S} \triangleleft P = \mathcal{S} - xy \cup xu \cup P \cup vy.$$

If $R \subset V(G - \mathcal{S})$ we say that R is *insertible* in $\mathcal{S}$ if there is a sequence $\mathcal{S}_0, \ldots, \mathcal{S}_t$ of H-subdivisions on $\mathbf{S}$ where $\mathcal{S}_0 = \mathcal{S}$, $V(\mathcal{S}_t) = V(\mathcal{S}) \cup V(R)$, and for all $i \in [t]$, $\mathcal{S}_i$ is obtained by uniformly inserting a subpath of $G[R] - \mathcal{S}_{i-1}$ into $\mathcal{S}_{i-1}$: We say that $\mathcal{S}_t$ was obtained by *inserting* R into $\mathcal{S}$. Figure 2 illustrates insertion and uniform insertion.

The following fact is easy to verify:

Fact 1. *If P is a path,*

$$\alpha(P) = \begin{cases} \left\lfloor \frac{|P|+1}{2} \right\rfloor & \text{if } P \text{ is a proper path} \\ \left\lfloor \frac{|P|}{2} \right\rfloor & \text{if } P \text{ is a cycle.} \end{cases}$$

Proposition 1. *Let $\mathcal{S}$ be an H-subdivision on* $\mathbf{S}$ *and P be a proper path of $G - \mathcal{S}$. Then*

$$\left\lfloor \frac{|P|}{2} \right\rfloor \leq \alpha(\mathcal{S} \triangleleft P) - \alpha(\mathcal{S}) \leq \left\lceil \frac{|P|}{2} \right\rceil.$$

Note that when $|P|$ is even, this becomes $\alpha(\mathcal{S} \triangleleft P) - \alpha(\mathcal{S}) = \frac{|P|}{2}$.

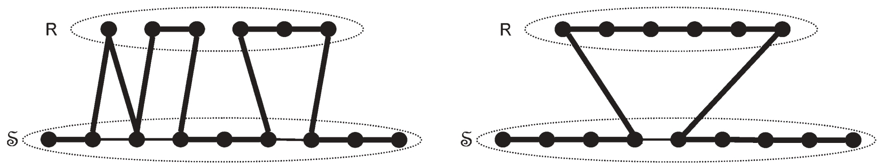

Fig. 2. On the left: R is *inserted* in $\mathcal{S}$. On the right: R is *uniformly inserted* in $\mathcal{S}$

Proof. First, we prove it when $|P| = 2$,

$$\alpha(\mathcal{S} \lhd P) = \alpha(\mathcal{S}) + 1. \tag{3}$$

The Proposition then follows by recursively inserting $\left\lfloor \frac{|P|}{2} \right\rfloor$ edges of P, then inserting a final vertex if $|P|$ is odd.

Suppose now that P is the edge uv. On one hand we have $\alpha(\mathcal{S} \lhd P) \geq \alpha(\mathcal{S}) + 1$. Indeed, take an $I \in \mathcal{I}_{max}(\mathcal{S})$; we must have $|\{x, y\} \cap I| \leq 1$, so if xy were replaced with the path (x, u, v, y), we have an independent set $I' = I \cup \{u\}$ (if $y \in I$), or $I' = I \cup \{v\}$ (if $x \in I$) in $\mathcal{S} \lhd P$.

On the other hand we have $\alpha(\mathcal{S} \lhd P) \leq \alpha(\mathcal{S}) + 1$. Indeed, take any $I' \in \mathcal{I}_{max}(\mathcal{S} \lhd P)$; we must have $|\{x, u, v, y\} \cap I'| \leq 2$, so if x and y were both in I', $I = I' - \{y\}$ would be an independent set of $\mathcal{S}$ of order $|I'| - 1$, since neither u nor v could be in I'. However, if say v were in I', then neither y nor u could be, and we get an independent set $I = I' - \{v\}$ of order $|I'| - 1$. □

Proof of Lemma 1. Let H be a multigraph and $\mathcal{S}$ be an H-subdivision on $\mathbf{S}$. Note that we may assume $h_0(H) = 0$ since isolated vertices may be added to an independent set, conserving it's independence.

Consider the graph $\mathcal{S} - S_2$ where S_2 is the set of ground vertices of degree at least two. Now $\mathcal{S} - S_2$ is a family $\mathcal{P}$ of disjoint paths, $\xi(\mathcal{S})$ of which have odd order. By Fact 1 we have

$$\begin{aligned}
\alpha(\mathcal{S} - S_2) &= \Sigma_{P \in \mathcal{P}} \left\lceil \frac{|P|}{2} \right\rceil \\
&= \Sigma_{|P| \text{ even}} \frac{|P|}{2} + \Sigma_{|P| \text{ odd}} \frac{|P| + 1}{2} \\
&= \Sigma_{P \in \mathcal{P}} \frac{|P|}{2} + \Sigma_{|P| \text{ odd}} \frac{1}{2} \\
&= \frac{|\mathcal{S}| - (|H| - h_1(H))}{2} + \frac{\xi(\mathcal{S})}{2} \\
&= \frac{|\mathcal{S}| + \xi(\mathcal{S}) - |H| + h_1(H)}{2}.
\end{aligned}$$

Since $\alpha(\mathcal{S}) \geq \alpha(S_2)$, (1) is proven.

In order to prove (2) we make use of the following definitions. For a given vertex $u \in V(\mathcal{S})$, let $\mathbf{c}(u)$, called the corona of u, be the set of edges adjacent to u in $\mathcal{S}$. A constellation C of $\mathcal{S}$ is a family of edge-disjoint coronas of vertices of $\mathcal{S}$ and $\mathcal{C}$ is the family of all possible constellations of $\mathcal{S}$. It is easy to see that

$$\alpha(\mathcal{S}) = \max_{C \in \mathcal{C}} |C|.$$

Further, if C is any constellation of $\mathcal{S}$ and $C_i = \{\mathbf{c}(u) \in C : |\mathbf{c}(u)| = i\}$, since all the coronas are disjoint,

$$E(\mathcal{S}) \geq |C_1| + 2\Sigma_{i \geq 2}|C_i|,$$

which shows that $\Sigma_{i\geq 2}|C_i| \leq \frac{E(\mathcal{S})-|C_1|}{2}$. Yet $|C| = |C_1| + \Sigma_{i\geq 2}|C_i|$, so

$$|C| \leq \frac{|E(\mathcal{S})| + |C_1|}{2}.$$

Yet $\max_{C\in\mathcal{C}} |C_1| = h_1(H)$, and further, one may verify that $|E(\mathcal{S})| = |\mathcal{S}| + |E(H)| - |H|$. Hence (2) is proven. □

Note that when $\xi(\mathcal{S}) = |E(H)|$, the Lemma implies the equality

$$\alpha(\mathcal{S}) = \frac{|\mathcal{S}| + |E(H)| - |H| + h_1(H)}{2} + h_0(H).$$

The following two Propositions will be useful.

Proposition 2. *Let $\mathcal{S}$ be an H-subdivision, R be a subgraph of $\mathcal{S}$, and ∂R be the set of vertices of R that are adjacent to $\mathcal{S} - R$. Then*

$$\alpha(\mathcal{S}) - \alpha(\mathcal{S} - R) \geq \alpha(R - \partial R).$$

More particularly, if $R = P$ is a subpath of a branch of $\mathcal{S}$, then $\alpha(\mathcal{S}) - \alpha(\mathcal{S} - P) \geq \left\lceil \frac{|P|-2}{2} \right\rceil$.

Proof. Since $E(\mathcal{S} - R, R - \partial R) = \emptyset$, the union of any independent set of $\mathcal{S} - R$ and one from $R - \partial R$, will be an independent set. □

In the following, given two subgraphs A and B of G, we denote by $E(A, B)$ the number of edges of $E(G)$ having one end-vertex in A and the other in B. Moreover, $d(A, B) = |E(A, B)|$ and $\delta(A, B) = min_{a\in A} d(a, B)$.

Proposition 3. *Let $\mathcal{S}$ be an H-subdivision on $\mathbf{S}$, R be a set of vertices not in $\mathcal{S}$, and $A = V(\mathcal{S}) \cup R$. Then,*

(a) *If $\delta(R, A) > \alpha(\mathcal{S}) + |R| - 1$ then R is insertible in $\mathcal{S}$.*
(b) *If some path P of $G[R]$ is uniformly insertible in $\mathcal{S}$, and $\delta(R, A) > \alpha(\mathcal{S}) + |R| - \left\lfloor \frac{|P|}{2} \right\rfloor - 1$ then R is insertible in $\mathcal{S}$.*

Proof. Assume the premises of (a) and let $z \in V(R)$. Since

$$d(z, \mathcal{S}) \geq \delta(R, A) - d(z, R) > \alpha(\mathcal{S}) + |R| - 1 - (|R| - 1) = \alpha(\mathcal{S})$$

we may insert z into $\mathcal{S}$, creating an H-subdivision $\mathcal{S}'$ on $\mathbf{S}$. Let $R' = R - \{z\}$. By Proposition 1, $\alpha(\mathcal{S}') \leq \alpha(\mathcal{S}) + 1$, so every vertex $w \in V(R')$ satisfies

$$\begin{aligned} d(w, A) &= d(w, V(\mathcal{S}) \cup R) \\ &> \alpha(\mathcal{S}) + |R| - 1 \\ &\geq \alpha(\mathcal{S}') + |R'| - 1 \end{aligned}$$

and we conclude the proof of (a) by induction.

Assume now the premises of (b) and let $\mathcal{S}'$ be the subdivision obtained by inserting P uniformly in $\mathcal{S}$. Now, using Proposition 1, we see that for every $z \in R' = R - V(P)$ we have

$$\begin{aligned} d(z, A) &\geq \delta(R, A) \\ &> \alpha(\mathcal{S}') + \big(\alpha(\mathcal{S}) - \alpha(\mathcal{S}')\big) + |R| - \left\lfloor \frac{|P|}{2} \right\rfloor - 1 \\ &\geq \alpha(\mathcal{S}') - \left\lceil \frac{|P|}{2} \right\rceil + |R| - 1 - \left\lfloor \frac{|P|}{2} \right\rfloor \\ &= \alpha(\mathcal{S}') + |R'| - 1, \end{aligned}$$

so using (a), we may conclude the proof of (b) by induction. □

4. Proof of Theorems 3 and 4

Before proceeding with the proofs of Theorems 3 and 4, we will exhibit a few facts that are common to both. First note that it suffices to prove both Theorems with $h_0(H) = 0$. Indeed, if J is the set of isolated vertices of H and $\mathcal{J} = f_{\mathcal{S}}(J)$ is the corresponding set of vertices in G, we define $H' = H - J$, $\mathcal{S}' = \mathcal{S} - \mathcal{J}$, and $G' = G - \mathcal{J}$. It is easy to verify that the premises of the Theorem hold for H', $\mathcal{S}'$, and G'. For instance, if $\sigma_2(G) \geq 2\alpha(\mathcal{S}) + |G - \mathcal{S}|$ then, observing that $|G - \mathcal{S}| = |G' - \mathcal{S}'|$ and $h_0 = h_0(H) = |J| = |\mathcal{J}|$, we have

$$\begin{aligned} \sigma_2(G') &\geq \sigma_2(G) - 2h_0 \\ &\geq 2(\alpha(\mathcal{S}') - h_0) + |G - \mathcal{S}| - 2h_0 \\ &\geq 2\alpha(\mathcal{S}') + |G - \mathcal{S}|. \end{aligned}$$

Hence, we assume H to be a multigraph without isolated vertices. We also assume $|E(H)| \geq 2$ since this is needed for Theorem 3 to be true, and the case $|E(H)| = 1$ of Theorem 4 is implied by Theorem 1.

Let H be a multigraph on e edges and h vertices, h_1 of these being of degree one, and let $\gamma = \max\{\alpha(H), \beta(H)\}$. Let G be a $(\gamma + 1)$-connected graph of order n. We are given $\mathcal{S}$, an H-subdivision on **S** of maximal order and let $Q = G - \mathcal{S}$. If $Q = \emptyset$ we have nothing to prove, so suppose $Q \neq \emptyset$ and let $q = |Q|$.

By Lemma 1,

$$\begin{aligned} n + e - h + h_1 &= 2\left(\frac{|\mathcal{S}| + e - h + h_1}{2}\right) + |G - \mathcal{S}| \\ &\geq 2\alpha(\mathcal{S}) + q. \end{aligned}$$

Hence, $\sigma_2(G) \geq n + e - h + h_1$ of Theorem 4 implies

$$\sigma_2(G) \geq 2\alpha(\mathcal{S}) + q. \tag{4}$$

as in Theorem 3, so we will assume this condition from here on.

Since $\mathcal{S}$ is maximal, by Proposition 3, for every $w \in V(Q)$,

$$d(w, \mathcal{S}) \leq \alpha(\mathcal{S}). \tag{5}$$

Thus, for every non-adjacent vertices $w, w' \in V(Q)$,

$$d(w, Q) + d(w', Q) \geq 2\alpha(\mathcal{S}) + q - d(w, \mathcal{S}) - d(w', \mathcal{S}) \geq q,$$

and hence

$$Q \text{ is connected.} \tag{6}$$

If F_1 and F_2 are two subgraphs of G, let $N(F_1, F_2)$ be the set of vertices of F_2 that are adjacent to at least one vertex of F_1. The above claims show that $Z_Q = N(Q, \mathcal{S})$ must be an independent set of $\mathcal{S}$.

Claim. *Suppose there exists some branch $B = [x, y]$ of $\mathcal{S}$ such that $|B \cap Z_Q| \geq 2$. Choose B and $z_1, z_2 \in B \cap Z_Q$ so that $z_2 \in (z_1, y]_B$ and $R = V\big((z_1, z_2)_B\big)$ has minimal order. Then*

(a) $|R|$ *is even,*
(b) Q *is complete,*
(c) *for all $w \in V(Q)$, $N(w, \mathcal{S} - R) = Z_Q$, a maximal independent set,*
(d) $|B \cap Z_Q| = 2$ *and* $|[x, z_1]_B|, |[z_2, y]_B| \leq 2$, *and*
(e) $|\mathcal{S}| \leq 4e + h$.

Proof. Assume there exists some branch $B = [x, y]$ such that $|B \cap Z_Q| \geq 2$. Choose B and $z_1, z_2 \in B \cap Z_Q$ so that $z_2 \in (z_1, y]_B$ and $R = V\big((z_1, z_2)_B\big)$ has minimal order. Further, let $Z_R = N(R, \mathcal{S} - R)$.

Connect z_1 and z_2 through Q and call P the connecting path in Q. Let $\mathcal{S}_1 = (\mathcal{S} - R) \cup P$. Now $\mathcal{S}_1$ is an H-subdivision on $\mathbf{S}$. Let $r = |R|$ and $R' \subset V(\mathcal{S})$ be such that $R \subseteq R'$. For all $w \in V(Q)$, the maximality of $\mathcal{S}$ assures that

$$d(w, \mathcal{S} - R') \leq \alpha(\mathcal{S} - R') \tag{7}$$

thus

$$\begin{aligned} d(w) &= d(w, Q) + d(w, R') + d(w, \mathcal{S} - R') \\ &\leq d(w, Q) + d(w, R') + \alpha(\mathcal{S} - R'). \end{aligned} \tag{8}$$

For all $u \in R$, the minimality of R implies that $uw \notin E(G)$, thus $d(u) = d(u, \mathcal{S})$, and

$$\begin{aligned} d(u, \mathcal{S}) &\geq 2\alpha(\mathcal{S}) + q - d(w) \\ &\geq \alpha(\mathcal{S} - R) + [\alpha(\mathcal{S}) - \alpha(\mathcal{S} - R)] \\ &\qquad + [q - d(w, Q)] \\ &\qquad + \max_{R \subseteq R'} \big(\alpha(\mathcal{S}) - \alpha(\mathcal{S} - R') - d(w, R')\big). \end{aligned}$$

Taking $R' = R$ and using Proposition 2 we see that

$$\alpha(\mathcal{S}) - \alpha(\mathcal{S} - R) \geq \left\lceil \frac{r-2}{2} \right\rceil, \tag{9}$$

$$q - d(w, Q) \geq 1, \text{ and} \tag{10}$$

$$\max_{R \subseteq R'} \big(\alpha(\mathcal{S}) - \alpha(\mathcal{S} - R') - d(w, R')\big) \geq \left\lceil \frac{r-2}{2} \right\rceil, \tag{11}$$

yielding

$$d(u, \mathcal{S}) \geq \alpha(\mathcal{S} - R) + r - 1. \tag{12}$$

If any of the inequalities (9), (10), or (11) were strict, inequality (12) would be strict, and using Proposition 3(a), we see that R could be inserted in $\mathcal{S}_1$, contradicting the maximality of $\mathcal{S}$. Hence these must all be equalities.

The equalities of (9) and (10) prove respectively (a) and (b). Further, if for $R' = R$, (7) were a strict inequality, (8) and therefore (12) would be strict. This shows that for all $w \in V(Q)$, $d(w, \mathcal{S} - R) = \alpha(\mathcal{S} - R)$, and since Z_Q must be an independent set, it is a maximal one, proving (c).

If (d) were not true, we could suppose without loss of generality, that $|[z_2, y]_B| \geq 3$ and let $[z_2, v]_B$ be the subpath of $[z_2, y]_B$ of order three starting at z_2. The minimality of R assures that $(z_2, v]_B \cap Z_Q = \emptyset$, so letting $R' = R \cup V([z_2, v]_B)$ we see that $\alpha(\mathcal{S}) - \alpha(\mathcal{S} - R') \geq \left\lceil \frac{|R'|-2}{2} \right\rceil = \left\lceil \frac{r+1}{2} \right\rceil$ and by the minimality of R $d(w, R') = 1$. Thus,

$$d(u, \mathcal{S}) \geq \alpha(\mathcal{S} - R) + \left\lceil \frac{r-2}{2} \right\rceil - 1 + \left\lceil \frac{r+1}{2} \right\rceil + 1 = \alpha(\mathcal{S} - R) + r$$

and hence (12) is a strict inequality, proving (d).

Suppose now that

$$|\mathcal{S}| > 4e + h. \tag{13}$$

In order to derive a contradiction to (4), we will first prove the following:

(f) for all $u \in R$, $N(u, \mathcal{S} - R) = Z_R$, a maximal independent set,
(g) $\mathcal{S}$ is a balanced bipartite graph.

Observe that no sub-path of $G[R]$ of order greater than one can be uniformly insertible in $\mathcal{S} - R$ or the inequality (12) would be sufficient to apply Proposition 3(b) and show that R is insertible in $\mathcal{S} - R$. Suppose that for all $u_i \in R$ there exists two vertices $v_i, v_i' \in N(u_i, \mathcal{S} - R)$ which are adjacent in $\mathcal{S} - R$. The $v_i v_i'$ edges must be independent for otherwise there would be a uniformly insertible sub-path in $G[R]$, and given that they are independent, R can be inserted into $\mathcal{S} - R$, one vertex at a time.

Hence there must be a $u \in R$ such that $Z_R = N(u, \mathcal{S} - R)$ is an independent set of $\mathcal{S} - R$; further (12) shows that it is maximal. This implies in turn that every $u' \in R$ satisfies $N(u', \mathcal{S} - R) = Z_R$ since any $v' \in N(u', \mathcal{S} - R) - Z_R$ would be adjacent to a $v \in Z_R$, and $[u, u']_B$ would be uniformly insertible in $\mathcal{S} - R$, which we've ruled out.

Suppose $|R| = 2$. If there were a branch $B' \neq B$ of $\mathcal{S}$ of order more than six, then the maximality of the independent set Z_Q implies that there would be distinct vertices $v, v' \in B' \cap Z_Q$ such that $|(v, v')_{B'}| = 1$, contradicting the minimality of R. Thus all branches of $\mathcal{S}$ have order no more than six, implying that $|\mathcal{S}| \leq h + 4e$, which contradicts (13).

Hence (a) shows that $|R| \geq 4$. By (c) and the minimality of R, all branches $B' \neq B$ must satisfy $|B' \cap Z_Q| \leq 1$, and

$$|B'| \leq 3 \text{ if } B' \text{ is a loop, leaf, or independent branch, and} \tag{14}$$
$$|B'| \leq 4 \text{ otherwise.} \tag{15}$$

Out of all branches $B' \neq B$ such that $B' \cap Z_Q \neq \emptyset$, let λ_1 be the number of these that are either a loop, leaf, or independent branch and λ_2 be the remaining number of these. By (14) and (15) we have

$$|\mathcal{S} - R| \leq h + 2 + 2\lambda_2 + \lambda_1. \tag{16}$$

Suppose there were a vertex $u \in \mathcal{S} - R - Z_Q - Z_R$. Since $u \notin Z_Q$, condition (4) yields (12), which we write as $d(u) \geq r - 1 + |Z_Q|$. Yet, on the other hand we have $d(u) \leq |\mathcal{S} - R| - 1$, therefore

$$r \leq |\mathcal{S} - R| - |Z_Q|. \tag{17}$$

Let τ denote the number of components of H that are trees, ξ_1 the total number of loop, leaf, and independent edges of H, and $\xi_2 = e - 1 - \xi_1$. Observe that $h \leq e + \tau$ and $\lambda_1 + \tau \leq 3\xi_1$ since a tree has at least two leaf edges. Using this, (16), (17), and the fact that $|Z_Q| \geq 2 + \lambda_2 + \lambda_1$, we find that

$$\begin{aligned}
|\mathcal{S}| &\leq 2|\mathcal{S} - R| - |Z_Q| \\
&\leq 2h + 4 + 4\lambda_2 + 2\lambda_1 - |Z_Q| \\
&\leq h + e + \tau + \lambda_1 + 3\lambda_2 + 2 \\
&\leq h + e + 3(\xi_1 + \xi_2) + 2 \\
&\leq h + e + 3(e - 1) + 2 \\
&< h + 4e,
\end{aligned}$$

which contradicts (13). Hence $\mathcal{S} - R = Z_Q \cup Z_R$.

If u was a vertex of $Z_Q \cap Z_R$, it would have to be isolated because if $v \in \mathcal{S} - R$ were adjacent to u in $\mathcal{S} - R$, then v cannot be in $Z_Q \cup Z_R$ without contradicting the fact that both sets are independent. This shows that $Z_Q \cap Z_R = \{x, y\}$ and that B cannot be adjacent to any other branch of $\mathcal{S}$. Since r is even, $|B| = r + 2$ is even, so B is balanced bipartite. Further, this also shows that $\mathcal{S} - B = \mathcal{S} - R - \{x, y\}$ is balanced bipartite since $Z_Q - \{x, y\}$ and $Z_R - \{x, y\}$ are two disjoint maximal independent sets of $\mathcal{S} - B$ whose union equals $V(\mathcal{S} - B)$. Hence (g) is proven.

Since $\mathcal{S}$ is balanced bipartite, $\alpha(\mathcal{S}) = \frac{|\mathcal{S}|}{2}$, so condition (4) becomes

$$\sigma_2(G) \geq |\mathcal{S}| + q = n. \tag{18}$$

Yet taking two non-adjacent vertices $u \in R$ and $w \in Q$, we have $d(u) \leq r - 1 + \frac{|\mathcal{S}-R|}{2}$ and $d(w) \leq q - 1 + \frac{|\mathcal{S}-R|}{2}$, hence $d(u) + d(w) \leq n - 2$, contradicting (18). Thus (e) is proven, concluding the proof of Claim 4. □

Claim. *If for every branch B^*, $|B^* \cap Z_Q| \leq 1$, then there exists two adjacent branches $B = [x, y]$ and $B' = [x', y]$, with $B \cap Z_Q = \{z\}$, $B' \cap Z_Q = \{z'\}$, $z, z' \neq y$, and $z \neq x$.*

Proof. Since $\gamma = \max\{\alpha(H), \beta(H)\}$, G is a priori $(\beta(H)+1)$-connected, hence $|Z_Q| \geq \beta(H)+1$, and since no branch may contain more than one vertex of Z_Q, there must be two adjacent branches B and B' each containing a vertex of Z_Q.

If it were not possible to find adjacent branches B and B' such that either $z \neq x$ or $z' \neq x'$, then a simple induction on the number of edges of H would show that Z_Q could only contain ground vertices. Since, a priori, G is $(\alpha(H)+1)$-connected, this would show that there has to be a branch whose end-vertices are both in Z_Q. □

With this background we are now ready to prove the theorems.

Proof of Theorem 3. We assume here that

$$|\mathcal{S}| \geq 6e + 3(h - h_1). \tag{19}$$

and let $\zeta = |Z_Q|$. Recall further that $\gamma = \max\{\alpha(H), \beta(H)\}$. Since $6e + 3(h - h_1) > 4e + h$, Claim 4**(e)** shows that any branch B of $\mathcal{S}$ satisfies $|B \cap Z_Q| \leq 1$. Since G is $(\gamma+1)$-connected, this implies that

$$\gamma + 1 \leq \zeta \leq e. \tag{20}$$

If $h = h_1$, then H would be a set of independent edges, thus $\gamma = e$, contradicting (20). Hence

$$h > h_1. \tag{21}$$

Further, Claim 4 applies, and we take B, B', x, x', z, z' and y as described in this Claim.

Let $R = (x, z)_B$, $R' = (y, z')_{B'}$, and P' be a $[z, z']$-path whose internal vertices are in Q. If $ab \in E(H)$ is the edge corresponding to branch B (that is, such that $g_{\mathcal{S}}(ab) = B$) then $\mathcal{S}' = (\mathcal{S} \cup P') - R - R'$ is an $(H - ab)$-subdivision on $\mathbf{S}$. In the following, we will show how to recover this missing branch with vertices of $\mathcal{S} - Z_Q$, and then insert the remaining vertices in the H-subdivision on $\mathbf{S}$ thus constructed, so that the maximality of $\mathcal{S}$ is contradicted.

For any $w \in V(Q)$ we have $d(w) \leq \zeta + q - 1$, so (4) shows that for any $v \in V(\mathcal{S}) - Z_Q$,

$$d(v) \geq 2\alpha(\mathcal{S}) - \zeta + 1. \tag{22}$$

Case 1. Suppose there exists $u \in R \cup \{x\}$ and $u' \in R' \cup \{y\}$ such that $uu' \in E(G)$. Then the path $[x, u)_B \cup uu' \cup [u', y)_{B'}$ recovers the missing branch. Choose u and u' so that the sum of the cardinalities of $R_1 = V\big((u, z)_B\big)$ and $R_2 = V\big((u', z')_{B'}\big)$ is minimum. If $R_1 \cup R_2 = \emptyset$, we would be done, so assume not, and without loss of generality, suppose $R_2 \neq \emptyset$ and let $u'' \in R_2$. We have thus

$$d(u, R_2) = d(u'', R_1) = 0, \text{ and by default, } uu'' \notin E(G). \tag{23}$$

Moreover, we let $R_3 = R_4 = \emptyset$ for later use.

Case 2. Suppose $E(R \cup \{x\}, R' \cup \{y\}) = \emptyset$ but $R \cup R' \neq \emptyset$ and without loss of generality, let $R \neq \emptyset$ and u be the vertex adjacent to x in $[x, z)_B$. Let $R_1 = V\big((u, z)_B\big)$ and $R_2 = V\big((y, z')_{B'}\big)$. By our assumption,

$$d(x, R_2) = d(y, R_1) = 0, \text{ and } xy \notin E(G). \tag{24}$$

Let $N = N(u, \mathcal{S}) \cap N(x, \mathcal{S}) \cap N(y, \mathcal{S})$. Since $u, x, y \in V(\mathcal{S}) - Z_Q$, by (22) we have $\mathcal{S} - N(u, \mathcal{S})$, $\mathcal{S} - N(x, \mathcal{S})$, and $\mathcal{S} - N(y, \mathcal{S})$ all have order at most $|\mathcal{S}| - \big(2\alpha(\mathcal{S}) - \zeta + 1\big)$, so

$$\begin{aligned}
|N| \quad &\geq |\mathcal{S}| - 3\Big(|\mathcal{S}| - \big(2\alpha(\mathcal{S}) - \zeta + 1\big)\Big) \\
&= 2\big(3\alpha(\mathcal{S}) - |\mathcal{S}|\big) - 3\zeta + 3 \\
&> 2\big(3\tfrac{|\mathcal{S}| - (h - h_1)}{2} - |\mathcal{S}|\big) - 3e \qquad\qquad \text{by (20) and Lemma 1} \\
&= |\mathcal{S}| - 3(h - h_1) - 3e \\
&\geq 3e.
\end{aligned}$$

This shows that for some sub-path B'' of a branch, $|N \cap B''| \geq 4$, so we may find vertices $v_1, v_2, v_3, v_4 \in V(B'')$, appearing in that order in B'', such that all but possibly v_2 and v_3 are distinct, and $v_1, v_4 \in N(u)$, $v_2 \in N(x)$, and $v_3 \in N(y)$. Since B'' has at most one vertex w of Z_Q, we may choose v_1, v_2, v_3, and v_4 so that w is neither in $R_3 = V\big((v_1, v_2)_{B''}\big)$ nor in $R_4 = V\big((v_3, v_4)_{B''}\big)$. Moreover, we choose R_3 and R_4 to have minimal cardinality, so that

$$d(x, R_3) = d(y, R_4) = 0. \tag{25}$$

Now $P = xv_2 \cup [v_2, v_3]_{B''} \cup v_3 y$ recovers branch B at the expense of B'', which in turn may be recovered using $P'' = [x'', v_1]_{B''} \cup v_1 u \cup u v_4 \cup [v_4, y'']_{B''}$, where x'' and y'' are the end-vertices of B''. The H-subdivision on $\mathbf{S}$ we thus obtain, misses the vertices of R_1, R_2, R_3 and R_4.

Case 3. Finally, $R \cup R' = \emptyset$. Let $N = N(x, \mathcal{S}) \cap N(y, \mathcal{S})$, and note that by similar arguments as seen in Case 2, we may find a sub-path of a branch B'' of $\mathcal{S}$ such that $|N \cap B''| \geq 5$. Therefore, we may find five vertices, say $x', v_1, v_2, u', y' \in V(B'')$, appearing in that order in B'', such that all but possibly v_1 and v_2 are distinct, $v_1 \in N(x)$, $v_2 \in N(y)$, $x', u', y' \notin Z_Q$, and $x'v_1, v_2u', u'y' \in E(B'')$. If $x'u' \in E(G)$ we would be done, so we assume that

$$x'u' \notin E(G). \tag{26}$$

Since $x', u', y' \notin Z_Q$, their degrees are large enough so that we may repeat the arguments made on y, u, and x in Case 2, and find a sub-path of a branch $B^{(3)}$ having four vertices v_3, v_4, v_5, v_6 such that $P'' = y'v_4 \cup [v_4, v_5]_{B^{(3)}} \cup v_5 x'$ and $P^{(3)} = v_3 u' \cup u' v_6$ are disjoint paths. Again, using P, P'' and $P^{(3)}$, we may recover our H-subdivision on $\mathbf{S}$. In this case, we let $R_1 = V\big((v_5, v_6)_{B^{(3)}}\big)$, $R_2 = V\big((v_3, v_4)_{B^{(3)}}\big)$, and $R_3 = R_4 = \emptyset$, where R_1 and R_2 are chosen to be minimal so that

$$d(x', R_2) = d(u', R_1) = 0, \text{ and } x'u' \notin E(G). \tag{27}$$

Using the respective definitions of the R_i in the above three cases, we let $R = R_1 \cup R_2 \cup R_3 \cup R_4$. We wish to insert R in the subdivisions constructed in the respective cases, so as to contradict the maximality of $\mathcal{S}$. To do so, it is sufficient to prove that R may be inserted in $\mathcal{S} - R$.

Since $R \in \mathcal{S} - Z_Q$, by (22) and (2),

$$\begin{aligned}\delta(R, \mathcal{S}) &\geq 2\alpha(\mathcal{S}) - \zeta + 1 \\ &= \alpha(\mathcal{S} - R) + 2[\alpha(\mathcal{S}) - \alpha(\mathcal{S} - R)] + \alpha(\mathcal{S} - R) - \zeta + 1 \qquad (28)\\ &= \alpha(\mathcal{S} - R) + |R| - 7 + \alpha(\mathcal{S} - R) - \zeta. \qquad (29)\end{aligned}$$

If (29) is greater than $\alpha(\mathcal{S} - R) + |R| - 1$, then Proposition 3 will yield that R is insertible. Thus, we wish to prove that

$$\alpha(\mathcal{S} - R) - \zeta > 6. \qquad (30)$$

Let $w_1 = u$ and $w_2 = u'$ in Case 1, $w_1 = x$ and $w_2 = y$ in Case 2, and $w_1 = x'$ and $w_2 = u'$ in Case 3. Now $w_1, w_2 \notin Z_Q$, so by (23), (24), (25), (26), and (27), we have $w_1 w_2 \notin E(G)$ and $d(w_1, R_2 \cup R_3) = d(w_2, R_1 \cup R_4) = 0$, thus (4) yields

$$\begin{aligned}|\mathcal{S}| - 2 - |R_2| - |R_3| \geq d(w_1) \geq 2\alpha(\mathcal{S}) - \zeta + 1, \text{ and}\\ |\mathcal{S}| - 2 - |R_1| - |R_4| \geq d(w_2) \geq 2\alpha(\mathcal{S}) - \zeta + 1.\end{aligned}$$

This implies that $2|\mathcal{S}| - |R| \geq 4\alpha(\mathcal{S}) - 2\zeta + 6$, so using (19), (20), (21) and Lemma 1, we get

$$\begin{aligned}|\mathcal{S} - R| - \zeta &\geq 4\left(\frac{|\mathcal{S}| - h + h_1}{2}\right) - 2\zeta + 6 - |\mathcal{S}| - \zeta \\ &= |\mathcal{S}| - 2h + 2h_1 - 3\zeta + 6 \\ &= 6e + 3(h - h_1) - 2h + 2h_1 - 3\zeta + 6 \\ &= 3e + h - h_1 + 6 \\ &\geq 7,\end{aligned}$$

thus (30) is shown, concluding the proof of Theorem 3. □

Proof of Theorem 4. Assume that

$$\sigma_2(G) \geq n + e - h + h_1 \qquad (31)$$

and

$$n \geq 11e + 7(h - h_1). \qquad (32)$$

Since (31) has been shown to imply the degree condition of Theorem 3, all we need to show in order to apply Theorem 4, is that

$$|\mathcal{S}| \geq 6e + 3(h - h_1). \qquad (33)$$

Suppose for some branch B of $\mathcal{S}$, $|B \cap Z_Q| \geq 2$, and let R be as in Claim 4. By (b) and (c) of Claim 4 we see that $|R| \geq |Q|$ or P would be larger than $[z_1, z_2]_B$, and the maximality of $\mathcal{S}$ would be contradicted. Yet, observing that $|\mathcal{S} - R| \geq h$, Claim 1(e) implies then that

$$\begin{aligned} n &= |Q| + |\mathcal{S}| \leq |R| + |\mathcal{S}| \\ &= 2|\mathcal{S}| - |\mathcal{S} - R| \leq 2|\mathcal{S}| - h \\ &< 2(4e + h) - h, \end{aligned}$$

contradicting (32).

Hence every branch B of $\mathcal{S}$ satisfies $|B \cap Z_Q| \leq 1$, and Claim 4 applies. Thus, we take B, B', x, x', z, z' and y to be as in Claim 4. Let u be the vertex adjacent to z in $[x, z]_B$ and u' be the vertex adjacent to z' in $[y, z']_{B'}$. If $uu' \in E(G)$ then $\mathcal{S}$ could be extended using vertices of Q. Yet both u and u' have all their adjacencies in $\mathcal{S}$, so $uu' \notin E(G)$ implies that

$$\begin{aligned} |\mathcal{S}| &> \frac{d(x') + d(y')}{2} \\ &\geq \frac{n + e - h + h_1}{2} \quad by(31) \\ &\geq \frac{11e + 7(h - h_1) + e - h + h_1}{2} \quad by(32) \\ &\geq 6e + 3(h - h_1). \end{aligned}$$

□

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