Graphs and Combinatorics (2006) 22:297–309
Digital Object Identifier (DOI) 10.1007/s00373-006-0663-2

Graphs and Combinatorics

Sub-Ramsey Numbers for Arithmetic Progressions

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Abstract. Let the integers $1, \ldots, n$ be assigned colors. Szemerédi's theorem implies that if there is a dense color class then there is an arithmetic progression of length three in that color. We study the conditions on the color classes forcing totally multicolored arithmetic progressions of length 3.

Let $f(n)$ be the smallest integer k such that there is a coloring of $\{1, \ldots, n\}$ without totally multicolored arithmetic progressions of length three and such that each color appears on at most k integers. We provide an exact value for $f(n)$ when n is sufficiently large, and all extremal colorings. In particular, we show that $f(n) = 8n/17 + O(1)$. This completely answers a question of Alon, Caro and Tuza.

Key words. Sub-Ramsey, Arithmetic progressions, Bounded colorings

1. Introduction

In this paper we investigate colorings of sets of natural numbers. We say that a subset is *monochromatic* if all of its elements have the same color and we say that it is *rainbow* if all of its elements have distinct colors. A famous result of van der Waerden [5] can be reformulated in the following way.

Theorem 1. *For each pair of positive integers k and r there exists a positive integer M such that any coloring of integers $1, \ldots, M$ with r colors yields a monochromatic arithmetic progression of length k.*

This theorem was generalized by the following very strong statement of Szemerédi [4].

Theorem 2. *For every natural number k and positive real number δ there exists a natural number M such that every subset of $\{1, \ldots, M\}$ of cardinality at least δM contains an arithmetic progression of length k.*

This means that "large" color classes force monochromatic arithmetic progressions. In this paper we invesigate conditions on the color classes which force a totally multicolored arithmetic progression of length three.

Assume that the integers in $\{1, \dots, n\}$ are colored by r colors. Can we always find an arithmetic progression of length k so that all of its elements are colored with distinct colors? We call such colored arithmetic progressions *rainbow* AP(k).

The answer to this question is "No", for $r \le \lfloor \log_3 n + 1 \rfloor$. The following coloring c of $\{1, \dots, n\}$, given by Jungić, et al. [3], demonstrates this fact. Let $c(i) = \max\{q : i \text{ is divisible by } 3^q\}$. This coloring has no rainbow arithmetic progressions of length 3 or more.

It is an open question to determine certain conditions which force the existence of rainbow arithmetic progressions. There are two natural approaches which can be studied. First, one can fix the number of colors and require that each color class is not "too small". Second, one can require that each color class is not "too big" to guarantee some rainbow arithmetic progression.

The first approach for AP(3) and three colors, among others, was studied in [3] and completely resolved by Fon-Der-Flaass and the first author as follows.

Theorem 3 ([2]). *Let* $[n]$ *be colored in three colors, each color class has size larger than* $(n+4)/6$. *Then there is a rainbow* AP(3). *Moreover, for each* $n = 6k - 4$ *there is a coloring of* $[n]$ *in three colors with the smallest color class of size* k *and with no rainbow* AP(3).

The second approach was introduced and developed by Alon, et al. [1]. It was called "Sub-Ramsey numbers for arithmetic progressions" as a way to investigate the problem provided that the size of the largest color class is bounded. Specifically, a coloring of $[n]$ was called a **sub-k-coloring** if every color appears on at most k integers. For a given m and a given k, the *Sub-Ramsey number*, $\mathrm{sr}(m, k)$, is defined to be the minimum n_0 such that any sub-k-coloring of $[n]$, $n > n_0$ contains a rainbow AP(m). When $m = 3$, i.e., when the desired rainbow arithmetic progressions are of size three, the following bounds were proved in [1].

Theorem 4. *As* k *grows,* $2k < \mathrm{sr}(3, k) \le (4.5 + o(1))k$.

In that paper it was suggested that the lower bound is close to the correct order of magnitude for $\mathrm{sr}(3, k)$. Here, we show that the truth is away from both the lower and upper bounds. In theorem 6, we compute tight bounds for $\mathrm{sr}(3, k)$ in a dual form. In particular, theorem 6 implies the following:

Theorem 5. *For any* $k \ge 1$, $(17/8)k - 4 \le \mathrm{sr}(3, k) \le (17/8)k + 10$.

Moreover, for k large enough, we determine the value of $\mathrm{sr}(3, k)$ exactly.

2. Main Results

Definition 1. We define $f(n)$ to be the smallest integer k such that there is a coloring of $[n]$ with the largest color class of size k and with no rainbow AP(3).

The following proposition allows us to determine $\mathrm{sr}(3, k)$ from $f(n)$:

Proposition 1. *The value* $\mathrm{sr}(3, k)$ *is the largest value of* n *such that* $k \geq f(n)$.

Proof. Since there exists a k-bounded coloring of $[\mathrm{sr}(3, k)]$ with no rainbow AP(3), $f(\mathrm{sr}(3, k)) \leq k$. Assume that $f(\mathrm{sr}(3, k)+1) \leq k$, then there is a k-bounded coloring of $[\mathrm{sr}(3, k) + 1]$ with no rainbow AP(3), a contradiction. □

For the rest of the paper, we analyze the function $f(n)$. Theorem 6 immediately implies the conclusion we draw in theorem 5.

We find an extremal coloring c_0 with no rainbow AP(3) and with largest color class of the smallest possible size.

Construction.

$$c_0(i) = \begin{cases} \mathbf{G}, & \text{if } i \equiv 0 \pmod{17}, \\ \mathbf{R}, & \text{if } i \equiv \pm 1, \pm 2, \pm 4, \pm 8 \pmod{17}, \\ \mathbf{B}, & \text{if } i \equiv \pm 3, \pm 5, \pm 6, \pm 7 \pmod{17}. \end{cases}$$

Let $q(I)$ be the size of the largest color class of c_0 in the interval I and

$$Q(n) = \min\{q(I) : I \text{ has length } n\}.$$

It can be easily verified that $Q(n) = \lceil 8(n-1)/17 \rceil + \epsilon$, where $\epsilon = \begin{cases} 1, & n \equiv 3, 5 \pmod{17}, \\ 0, & \text{otherwise.} \end{cases}$

Theorem 6. *Let* $n_0 = 2600$. *If* $n \geq n_0$ *then*

$$f(n) = Q(n).$$

Any extremal coloring of $\{1, \ldots, n\}$ *is colored identically to a subinterval of* $\mathbb{Z}$ *colored by* c_0. *Moreover, for any* $n \geq 1$,

$$Q(n) - 4 \leq f(n) \leq Q(n).$$

Corollary 1.

$$\left\lceil \frac{8(n-1)}{17} \right\rceil \leq f(n) \leq \left\lceil \frac{8(n-1)}{17} \right\rceil + 1,$$

for $n \geq 2600$. *Moreover*

$$\frac{8(n-1)}{17} - 4 \leq f(n) \leq \frac{8(n-1)}{17} + 2,$$

for $n \geq 1$.

Remark 1. We did not try to optimize the constant n_0. A more careful analysis of the proof results in a smaller number. We believe that in fact $f(n) = Q(n)$ for all values of n and this must be a coloring of some subinterval of $\mathbb{Z}$ for all but a very small number of values of n.

3. Definitions and Notations, Outline of the Proof

Let $[n] = \{1, \dots, n\}$. For convenience, sometimes we shall use the closed interval notation $[1, n]$ for $[n]$. Let $c : [n] \to \{\mathbf{R}, \mathbf{G}, \mathbf{B}\}$. We say that a color $X \in \{\mathbf{R}, \mathbf{G}, \mathbf{B}\}$ is **solitary** if there is no $x \in [n-1]$ such that $c(x) = c(x+1) = X$. For a set $S \subseteq [n]$, we denote by $r(S)$, $g(S)$, $b(S)$ the number of elements in S colored **R**, **G**, **B** respectively. We write $|\mathbf{R}| = r([n])$, $|\mathbf{G}| = g([n])$, $|\mathbf{B}| = b([n])$. If all elements of S have the same color X, we write $c(S) = X$.

We say that the interval $[x, x+i]$ is **X-X-interval** if $c(x) = c(x+i+1) = X$ and $c(x+j) \neq X$ for all $1 \leq j \leq i$, note that the left X is included in the interval but the right one is not. For a color X, we define a set $N(X)$ of **neighbors of X** as follows $N(X) = \{i \in [n] : c(i+1) = X \quad \text{or} \quad c(i-1) = X\}$. For a sequence of colors $A_0, A_1, \dots, A_k, \quad A_i \in \{\mathbf{R}, \mathbf{G}, \mathbf{B}\}$, we say that a coloring c **contains** $A_0 A_1 \cdots A_k$ in the interval I if there is an integer $x \in I$, such that $x + k \in I$ and $c(x+i) = A_i$, $i = 0, \dots, k$. Sometimes we shall simply say that I contains $A_0 A_1 \cdots A_k$. We use subintervals of $[1, n]$ or subsets of $[n]$ wherever convenient.

In order to prove our upper bound on $f(n)$, we consider an arbitrary coloring of $[n]$ with no rainbow AP(3) and first reduce the analysis to the case of three colors only. We show that there must be a solitary color, say **G**. Moreover we show that each number in the neighbor set of **G** must have the same color, say **R**. I.e., each integer colored **G** is surrounded by two integers colored **R**. Therefore the interval $[1, n]$ can be split into **G-G** intervals and perhaps some initial and terminal intervals containing no **G**. Next, we show that either each **G-G** interval has many integers colored **R**, thus arriving at a conclusion that $|\mathbf{R}| \geq Q(n)$ or that there are not too many integers colored **G** and either $|\mathbf{R}|$ or $|\mathbf{B}|$ is at least $(n - |\mathbf{G}|)/2 \geq Q(n)$.

We present the proof in the section 4, and all necessary technical lemmas in sections 5, 6.

4. Proof of Theorem 6

Let c be a coloring of $[n]$ with no rainbow AP(3). We shall conclude that one of the color classes has size at least $Q(n)$. By lemma 2, we can assume that c uses three colors, say **R**, **G**, **B**. Lemma 4 implies an existence of a solitary color, without loss of generality **G**. If there are only two numbers of color **G**, then either **R** or **B** has size at least $(n-2)/2 \geq 8(n-1)/17 + 3 > Q(n)$, for $n \geq n_0$ and $(n-2)/2 \geq Q(n) - 3$ for $n \geq 1$. Otherwise, by lemma 5, we can assume that the neighbor set of **G** is colored **R**. We can also assume that there are two consecutive numbers colored **B** in $[n]$; otherwise, the cardinality of **R** is at least $(n-2)/2 > Q(n)$, for $n \geq n_0$ and $(n-2)/2 > Q(n) - 3$ for $n \geq 1$.

Since **G** is a solitary color and **R** is the color of its neighborhood, we see that c looks as follows:

$$* * \cdots * * \mathbf{RGR} * * \cdots * * \mathbf{RGR} * * \cdots * * \mathbf{RGR} * * \cdots * * \mathbf{RGR} * * \cdots * *,$$

where $* \in \{\mathbf{R}, \mathbf{B}\}$. Furthermore, there is a **BB** somewhere in $[n]$.

Case 1. All **G-G**-intervals contain **BB**.
Lemma 8 proves that the smallest length of a **G-G** interval containing **BB** is 15 and there is no such interval of length 16. Assume first that there is such an interval of length 15. Then lemma 9 shows that this coloring must be very specific, in particular, it is defined up to translation on all integers except, perhaps, every 15^{th} one. So, in that case, lemma 9 gives that $|\mathbf{R}| \geq 8(n-1)/15 - 1 \geq 8(n-1)/17 + 3 > Q(n)$ for $n \geq n_0$ and $8(n-1)/15 - 1 > Q(n) - 3$ for $n \geq 1$. If the smallest **G-G** interval has length 17 then lemma 10 says that the coloring of $[n]$ must be a translation of c_0 for all integers except, perhaps, every 17^{th} one. In this case $|\mathbf{R}| = Q(n)$. Finally, if all intervals have length at least 18, lemma 8 proves that in fact, the smallest interval has length 21. Then $|\mathbf{G}| \leq n/21 + 1$. Thus either $|\mathbf{B}|$ or $|\mathbf{R}|$ is at least $(n - |\mathbf{G}|)/2 \geq (10n - 11)/21 > Q(n)$ for $n \geq n_0$ and $(10n - 11)/21 > Q(n) - 3$ for $n \geq 1$.

Case 2. There is a **G-G**-interval containing no **BB**.

We split interval $[1, n]$ and find a lower bound on the number of integers colored **R** in each of those subintervals. There are two subcases we shall treat. In case 2.1, the initial subinterval contains at least three **G**s, and we use our structural lemmas. Otherwise, we have case 2.2, in which we apply case 1 to a special subinterval. We shall define the following special subintervals.

$\circ$ I_1 is the longest initial segment of $[n]$ containing no **BB** and ending with **G**, $I_1 = [1, l]$,

$\circ I_2$ is an interval following I_1, containing no **BB** except for the last two positions which are colored **BB**,

$\circ I_3 = [n] - I_1 - I_2$,

$\circ I_0 \subseteq I_1$ is the longest initial segment of $[1, n]$ containing no **G**,

$\circ I_2' = [l+1, 2l-1]$, $I_2'' = I_2 \setminus I_2'$.

$\circ I_t$ is the longest terminal subinterval of $[n]$ containing no **BB**.

Case 2.1. Let $g(I_1) \geq 3$. Let g_i be the number of **G-G** intervals of length i in $I_1 \setminus \{l\}$. Lemma 6(b) and 6(c) claims that there is no **GRG** or **GRRG** in $[n]$. Thus each **G-G** interval in $[n]$ has length at least 4 and $g_i = 0$ for $i \leq 3$. In particular, $|I_1| = |I_0| + \sum_{i \geq 4} i g_i + 1$.

Since I_1 contains no **BB** we have

$$r(I_1) \geq |I_0|/2 + \sum_{i \geq 4} (i/2 + 1) g_i. \tag{1}$$

Lemma 11 states that $I_2' \subseteq I_2$ and $r(I_2') \geq r(I_1)$. Since I_2'' does not contain any **BB** except at the last two positions, $r(I_2'') \geq |I_2''|/2 - 1$. Thus

$$r(I_2) = r(I_2') + r(I_2'') \geq r(I_1) + |I_2''|/2 - 1. \tag{2}$$

Finally, by lemma 12,

$$r(I_3) \geq (|I_3| - 3)/4. \tag{3}$$

We can summarize (1), (2) and (3) as follows.

$$\begin{aligned}
|\mathbf{R}| &= r(I_1) + r(I_2') + r(I_2'') + r(I_3)\\
&\geq 2r(I_1) + |I_2''|/2 - 1 + (|I_3| - 3)/4\\
&= 2r(I_1) + |I_2''|/2 - 1 + (n - |I_1| - |I_2'| - |I_2''| - 3)/4\\
&\geq (n-3)/4 - 1 + 2r(I_1) - |I_1|/2\\
&\geq (n-7)/4 + 2\left[|I_0|/2 + \sum_{i\geq 4} g_i(i/2+1)\right] - (1/2)\left[|I_0| + \sum_{i\geq 4} i g_i + 1\right]\\
&\geq (n-9)/4 + |I_0|/2 + \sum_{i\geq 4} g_i(i/2+2)\\
&\geq (n-9)/4 + 4\sum_{i\geq 4} g_i\\
&= (n-9)/4 + 4(g(I_1) - 1).
\end{aligned}$$

Lemma 12 implies that $g(I_1) - 1 = |\mathbf{G}| - 1 - g(I_2 \cup I_3) \geq |\mathbf{G}| - 3$. So,

$$|\mathbf{R}| \geq (n-9)/4 + 4(|\mathbf{G}| - 3).$$

Let $M = \max\{|\mathbf{R}|, |\mathbf{B}|\}$. By definition, it is the case that $|\mathbf{R}| \leq M$ and $|\mathbf{G}| \geq n - 2M$. As a result,

$$M \geq |\mathbf{R}| \geq (n-9)/4 + 4(|\mathbf{G}| - 3) \geq (n-9)/4 + 4(n - 2M - 3).$$

Thus

$$M \geq \frac{17n - 57}{36} \geq 8(n-1)/17 + 3 \geq Q(n),$$

for $n \geq n_0$. We also have that $M \geq (17n - 57)/36 \geq 8(n-1)/17 > Q(n) - 3$ for all values of $n \geq 1$.[1]

Case 2.2. Let $g(I_1) \leq 2$. By symmetry, we can also assume that $g(I_t) \leq 2$, otherwise we can apply the previous calculation to the coloring defined as $c'(i) = c(n+1-i)$, $i \in [n]$. Let $J = [n] \setminus (I_1 \cup I_t)$. If J contains no **G** then $g([n]) \leq 4$ and either $|\mathbf{R}|$ or $|\mathbf{B}|$ is at least $(n-4)/2 \geq 8(n-1)/17 + 3 \geq Q(n)$ for $n \geq n_0$, moreover $(n-4)/2 \geq Q(n) - 3$ for all $n \geq 1$.

If there is at least one **G** in J then we conclude that all **G-G** intervals in $J \cup \{l\}$ contain **BB** by lemma 7 and that $r(I_1) \geq |I_1|/2$ and $r(I_t) \geq |I_t|/2$. As in case 1, we observe that if J contains a **G-G** interval of length 15 then $|R| \geq 8(n-1)/15 - 1 \geq$

[1] Note that this is the only time we need the value of 2600 for n_0, in all other calculations, a smaller bound of 900 is sufficient.

$8(n-1)/17+3 \geq Q(n)$, for $n \geq n_0$. In addition, if $J \cup \{l\}$ contains a **G-G** interval of length 17 then lemma 10 gives that the coloring must be a translation of c_0 except, perhaps on every 17^{th} position. In this case, $|\mathbf{R}| \geq Q(n)$. Otherwise, the length of each **G-G** interval is at least 21. This follows from lemma 8. In that case, $g(J) \leq |J|/21+1$. Thus $|\mathbf{G}| \leq g(J)+4 \leq (n-4)/21+5$. Therefore either $|\mathbf{R}|$ or $|\mathbf{B}|$ is at least $(n-|\mathbf{G}|)/2 \geq (10n-51)/21 \geq 8(n-1)/17+3 > Q(n)$, for $n \geq n_0$, moreover $|\mathbf{R}| \geq 8(n-1)/17-1 \geq Q(n)-4$ for all $n \geq 1$.

This case concludes the proof of the theorem. □

5. General Lemmas for Colorings with No Rainbow AP(3)s

Lemma 1. *The coloring c_0 does not have any rainbow* AP(3)*s.*

Proof. Consider AP(3) at positions $i < j < k$ with $c(j) = \mathbf{G}$. Then $j = 0 \pmod{17}$ and then $i = -k \pmod{17}$. Therefore, by construction, $c(i) = c(k)$ and this AP(3) is not rainbow. □

Now, let us have AP(3) at positions $i < j < k$ such that $c(i) = \mathbf{G}$. Then, since $i = 0 \pmod{17}$ we have $k = 2j \pmod{17}$. We claim that $c(j) = c(k)$ in this case simply by multiplying the numbers in corresponding congruence classes by two as follows:

x	1	2	4	8	3	5	6	7
$2x \pmod{17}$	2	4	8	−1	6	−7	−5	−3

Therefore, in this case we see that this AP(3) is not rainbow and there is no rainbow AP(3) in our coloring.

Lemma 2. *Let c be a coloring of $[n]$ with no rainbow* AP(3)*, $n \geq 21$ and every color class of size at most m, $(n+4)/6 \leq m < (n-4)/2$. Then there is a coloring c' of $[n]$ with no rainbow* AP(3)*, in three colors with each color class being the union of some color classes of c and such each color class of c' has size at most m.*

Proof. Let $A_1, A_2, \ldots$ be the color classes of c. Note first that if c' is formed by merging color classes of c then c' does not have rainbow AP(3)s. If there were a rainbow AP(3) in c', then it must be a rainbow AP(3) in c, a contradiction.

Assume first that there are two color classes A_1 and A_2 of sizes more than $(n+4)/6$. Consider $S = [n] - A_1 - A_2$. Let the color classes of c' be A_1, A_2, S. If $|S| > (n+4)/6$ then the new color are all of sizes at least $(n+4)/6$, thus there is a rainbow AP(3) in c' by Theorem 3, a contradiction. Otherwise, $|S| \leq (n+4)/6 \leq m$ and all color classes in c' have sizes at most m.

Now, assume that there is exactly one color class of size more than $(n+4)/6$, say A_1. Let $T = A_2 \cup A_3 \cup \cdots \cup A_q$ such that $|T| > (n+4)/6$ but $|T \setminus A_q| \leq (n+4)/6$. Then, we see that $|T| \leq (n+4)/3$. Therefore, $n - |T| - |A_1| \geq n - (n+4)/3 - m > (n+4)/6$. If we make the new color classes $A_1, T, [n] \setminus (T \cup A_1)$, then by Theorem 3, there is a rainbow AP(3) in c', a contradiction.

Finally, if each color class has cardinality less than $(n+4)/6$ then we choose color classes of c' greedily. Let $B_1 = A_1 \cup A_2 \cup \cdots A_q$ and $B_2 = A_{q+1} \cup \cdots A_r$ be two new color classes such that $(n+4)/6 < |B_i| \leq (n+4)/3$, $i = 1, 2$. Let $B_3 = [n] \setminus (B_1 \cup B_2)$. Then $|B_3| \geq (n-8)/3$. If $n \geq 21$, then $|B_3| > (n+4)/6$ and we again apply Theorem 3 to get a rainbow AP(3) in c' a contradiction. □

Lemma 3 ([2]). *Let c be a coloring of $[n]$ in three colors with no rainbow* AP(3). *Let there be integers x and z, $1 \leq x < z < n$ such that $c(x) = c(x+1) = X$ and $c(z) = c(z+1) = Z$, $X \neq Z$. Then there is w, $x < w < z$ such that ($c(w) = X$, $c(w+1) = Z$) or ($c(w) = Z$, $c(w+1) = X$).*

Lemma 4. *Let c be a coloring of $[n]$ in three colors with no rainbow* AP(3). *Then there is a solitary color.*

Proof. Assume the opposite. Let c be a coloring of $[n]$ with colors **R**, **G**, **B** and such that each color appears on consecutive positions somewhere in $[n]$. In particular, there are numbers $1 \leq x < y < z < n$ such that, without loss of generality, $c(x) = c(x+1) = \mathbf{R}$, $c(y) = c(y+1) = \mathbf{G}$, and $c(z) = c(z+1) = \mathbf{B}$, and such that there are no two consecutive integers colored **BB** or **RR** in the interval $[x+1, z]$.

By lemma 3, there is a w, with $x < w < z$, such that ($c(w) = \mathbf{R}$ and $c(w+1) = \mathbf{B}$) or ($c(w) = \mathbf{B}$ and $c(w+1) = \mathbf{R}$). Assume without loss of generality that $x < w < y$ and that w is closest to y satisfying this property, and $c(w) = \mathbf{R}$, $c(w+1) = \mathbf{B}$. Note that $w+1 < y-1$, otherwise $\{w, w+1, w+2\}$ will be a rainbow AP(3). But now $c(w+2) = \mathbf{B}$ otherwise we shall contradict the choice of w. Therefore, we have $c(w+1) = c(w+2) = \mathbf{B}$, a contradiction. □

Lemma 5. *Let c be a coloring of $[n]$ in three colors* **R**,**G**,**B** *with no rainbow* AP(3). *Let color* **G** *be solitary. Then, either the neighbor set of* **G** *is monochromatic or there are at most two numbers x, y with $c(x) = c(y) =$* **G**.

Proof. Note first that if $c(x) = \mathbf{G}$, for some $x \in \{2, \ldots, n-1\}$ then $c(x-1) = c(x+1) \in \{\mathbf{B}, \mathbf{R}\}$. Now, assume that there are two integers x, y, $1 \leq x < y \leq n$, such that $c(x) = c(y) = \mathbf{G}$ but $c(z) \neq \mathbf{G}$ for all $x < z < y$ and such that $c(x+1) = \mathbf{R}$ and $c(y-1) = \mathbf{B}$. Assume that there are at least three integers colored **G**. Then, it is easy to see that we may assume that $x \geq 2$ or $y \leq n-2$. Let y be at most $n-2$, without loss of generality.

If $y+x$ is odd then $c((y+x+1)/2) = \mathbf{R}$ and $c((y+x+1)/2) = \mathbf{B}$ which follows from considering the AP(3) $\{x+1, (x+y+1)/2, y\}$ and $\{x, (x+y+1)/2, y+1\}$, respectively, a contradiction.

If $y+x$ is even and $c(y+2) = \mathbf{B}$, we have $c(x+2) = \mathbf{R}$. Then $c((x+y+2)/2) = \mathbf{R}$ and $c((x+y+2)/2) = \mathbf{B}$ from the AP(3) $\{x+2, (x+y+2)/2, y\}$, and the AP(3) $\{x, (x+y+2)/2, y+2\}$, a contradiction.

If $y+x$ is even and $c(y+2) = \mathbf{G}$, consider the largest w, $x < w < y$ such that $c(w) = c(w+1) = \mathbf{R}$. Then one of $w+y$ and $w+1+y$ is even. Assume, without loss of generality, that $w+y$ is even. Then $(w+y)/2$ and $(w+y+2)/2$ will have to have color **R** because of AP(3)s $\{w, (w + y)/2, y\}$ and $\{w, (w + y + 2)/2, y + 2\}$, a contradiction to maximality of w. □

6. Lemmas Specific to the Main Theorem

In all of the following lemmas we consider a coloring c of $[n]$ in three colors **R**, **G**, **B** with a solitary color **G** having all neighbors of color **R**. We also assume that this coloring has two consecutive integers colored **B**. The intervals I_1, I_2, I_3 are defined as in the proof of the theorem in section 4.

Lemma 6.
(a) *If* $x \in [1, n - 1]$ *and* $c(x), c(x + 1) \in \{\mathbf{G}, \mathbf{B}\}$ *then* $c(x) = c(x + 1) = \mathbf{B}$.
(b) $[1, n]$ *does not contain* **GRG**
(c) $[1, n]$ *does not contain* **GRRG**.
(d) *If* $x \in [1, n - 2]$ *and* $c(x), c(x + 2) \in \{\mathbf{G}, \mathbf{B}\}$ *then* $c(x) = c(x + 2) = \mathbf{B}$.

Proof. (a) Note that having $c(x) = c(x + 1) = \mathbf{G}$ is impossible since **G** is a solitary color. Having exactly one integer x or $x + 1$ of color **G** and another of color **B** is impossible since the neighbors of **G** are colored with **R**.

(b) Without loss of generality, we may assume that there are integers $w, y \in [n]$, $y > w$ and such that $w, w+1, w+2$ is colored **GRG** and y is the least integer such that $c(y) = c(y + 1) = \mathbf{B}$. If y has the same parity as w then the AP(3) $\{w, (w + y)/2, y\}$ and $\{w + 2, (w + 2 + y)/2, y\}$ imply that $c((w + y)/2) = c((w + 2 + y)/2) = \mathbf{B}$. If $y + 1$ has the same parity as w then the AP(3) $\{w, (w + y + 1)/2, y + 1\}$ and $\{w+2, (w+2+y+1)/2, y+1\}$ imply that $c((w+y+1)/2) = c((w+2+y+1)/2) = \mathbf{B}$. This is a contradiction to the minimality of y.

(c) Without loss of generality, we may assume that there are integers $y, w \in [n]$ such that $w, w + 1, w + 2, w + 3$ is colored **GRRG** and that y is the least integer such that $c(y) = c(y + 1) = \mathbf{B}$. If $w + y$ is even, then consider the following AP(3)s: $\{w, (w+y)/2, y\}$ and $\{w+2, (w+2+y)/2, y\}$. It follows that $c((w+y)/2) = c((w+y+2)/2) = \mathbf{B}$. Since $y > (w+y)/2 > w$, we have a contradiction to the minimality of y. If $w+y$ is odd, the consider the following AP(3)s: $\{w, (w+y+1)/2, y+1\}$ and $\{w+3, (w+3+y)/2, y\}$. It follows that $c((w+y+1)/2) = c((w+y+3)/2) = \mathbf{B}$. Since $y > (w + y + 1)/2 > w$, we have a contradiction to the minimality of y.

(d) Note that $c(x) = c(x+2) = \mathbf{G}$ is impossible because of b). If $\{c(x), c(x+2)\} = \{\mathbf{B}, \mathbf{G}\}$ then, since $c(x + 1) = \mathbf{R}$, $\{x, x + 1, x + 2\}$ is a rainbow AP(3). □

Lemma 7. *Let* $x < y$, $c(x) = c(y) = \mathbf{G}$ *and both intervals* $[1, x]$ *and* $[y, n]$ *contain* **BB***s. Then* $[x, y]$ *contains* **BB**.

Proof. Let w be the largest number such that $w < x$ and $c(w) = c(w + 1) = \mathbf{B}$. Let z be the smallest number such that $z > y$ and $c(z) = c(z - 1) = \mathbf{B}$. Assume without loss of generality that $x - w \le z - y$. By considering the AP(3)s $\{w, x, 2x - w\}$ and $\{w + 1, x, 2x - w - 1\}$, we have that $c(2x - w - 1), c(2x - w) \in \{\mathbf{B}, \mathbf{G}\}$, and using lemma 6 a), we have $c(2x - w - 1) = c(2x - w) = \mathbf{B}$. If $x < 2x - w - 1 < y$, then we are done. Otherwise, $2x - w - 1 > y$ and $2x - w - 1 - y < z - y$, a contradiction to the choice of z. □

Lemma 8. *Let I be a* **G-G** *interval with at least one* **B**. *Then for each such I of length at most* 21*, I must be colored as in Table* 1.

Proof. Let $I = [0, k-1]$; i.e., $c(0) = c(k) = \mathbf{G}$. Because there is no rainbow AP(3), we must have that $c(x) = c(2x)$ for all $x < k/2$ and $c(2x-k) = c(x)$ for all $x > n/2$. Since the neighbor set of **G** is **R**, $c(1) = c(k-1) = \mathbf{R}$. With these conditions we can exhibit all possible colorings of I. The ones with at least one **B** are listed in table 1, for $1 \le k \le 21$. □

Now we present the main structural lemma.

Lemma 9. *Let* $[x, x+14]$ *be a* **G-G** *interval containing* **BB**. *Then*

$$c(z) = \begin{cases} \mathbf{R}, & \text{if } (z-x) \equiv \pm 1, \pm 2, \pm 4, \pm 7 \pmod{15}, \\ \mathbf{B}, & \text{if } (z-x) \equiv \pm 3, \pm 5, \pm 6 \pmod{15}. \end{cases}$$

Proof. To simplify our calculations, we shift the indices so that considered **G-G** interval is [0, 14] and the whole interval being colored is $[1-x, n-x]$. The lemma 8 shows that the coloring of [0, 15] must be as follows:

GRRBRBBRRBBRBRRG.

In particular, we have that

$$c(2i) = c(i), \quad c(2i-1) = c(7+i), \quad i \in \{1, \dots, 7\}. \tag{4}$$

Table 1. Colorings of **G**-**G** intervals of lengths at most 21 containing **B**. Here, $x, y \in \{\mathbf{R}, \mathbf{B}\}$

Interval Length	Coloring
6	**G R R B R R**
9	**G R R B R R B R R**
10	**G R R R R B R R R R**
12	**G R R B R R B R R B R R**
14	**G R R R R R R B R R R R R R**
15	**G R R** x **R** y x **R R** x y **R** x **R R**
17	**G R R B R B B B R R B B B R B R R**
18	**G R R B R R B R R B R R B R R B R R**
20	**G R R R R B R R R R B R R R R B R R R R**
21	**G R R** x **R R** x y **R** x **R R** x **R** y x **R R** x **R R**

Let $A = [-w+1, z-1]$ be the largest interval having the coloring c as in the statement of the lemma. I.e., for each $y \in A$

$$c(y) = \begin{cases} \mathbf{R}, & \text{if } y \equiv \pm 1, \pm 2, \pm 4, \pm 7 \pmod{15}, \\ \mathbf{B}, & \text{if } y \equiv \pm 3, \pm 5, \pm 6 \pmod{15} \end{cases}$$

Let $z = 15k + i$, $0 < i < 15$. If $z \leq n - x$, we shall show that z must be colored as in c, thus contradicting the maximality of $[-w+1, z-1]$. By symmetry, it will be the case that if $-w \geq 1 - x$ then $-w$ must be colored as in c, again contradicting the maximality of $[-w+1, z-1]$. Therefore we shall conclude that $A = [-w+1, z-1] = [1-x, n-x]$.

First we show that $c(z) \neq \mathbf{G}$ if $i \neq 0$. Assume that $c(z) = c(15k+i) = \mathbf{G}$. If $i \in \{4, 5, 6, 7, 8, 10, 11, 12, 13, 14\}$ then either $c(z-1) = \mathbf{B}$ or ($c(z-2) = \mathbf{B}$ and $c(z-1) = \mathbf{R}$). We arrive at a contradiction since the neighbors of **G** are colored **R** and we can not have three consecutive numbers colored **BRG**. For $i \in \{1, 2, 3, 9\}$ we consider the following AP(3)s: $\{15k-3, 15k-1, 15k+1\}$, $\{15k-6, 15k-2, 15k+2\}$, $\{15k-5, 15k-1, 15k+3\}$, $\{15k+5, 15k+7, 15k+9\}$. Note that the first two terms in each of these four AP(3)s have distinct colors from the set $\{\mathbf{R}, \mathbf{B}\}$, thus the last terms can not be colored with **G**.

Next we show that $c(15k+i) = c(i)$.

Case 1. k is even, i is even.
Consider AP(3) $\{0, (15k+i)/2, 15k+i\}$. Since $c((15k+i)/2) = c(15(k/2)+i/2) = c(15(k/2)+i) = c(i)$, we have that $c(15k+i) = c(i)$.

Case 2. k is odd, i is odd.
Consider AP(3) $\{0, (15k+i)/2, 15k+i\}$. Since $c((15k+i)/2) = c(15((k-1)/2) + (15+i)/2) = c(15((k-1)/2) + 15 + i) = c(i)$, we have that $c(15k+i) = c(i)$.

Case 3. k is odd, i is even.
Consider AP(3) $\{15, (15(k+1)+i)/2, 15k+i\}$. Since $c((15(k+1)+i)/2) = c(15((k+1)/2)+i/2) = c(15((k+1)/2)+i) = c(i)$, we have that $c(15k+i) = c(i)$.

Case 4. k is even, i is odd.
Consider AP(3) $\{15, (15k+i+15)/2, 15k+i\}$. Since $c((15k+i+15)/2) = c(15(k/2)+(i+15)/2) = c(15(k/2)+(i+15)) = c(i+15) = c(i)$, we have that $c(15k+i) = c(i)$.

□

Lemma 10. *Let* $[x, x+16]$ *be a* **G-G** *interval. Then*

$$c(z) = \begin{cases} \mathbf{R}, & \text{if } (z-x) \equiv \pm 1, \pm 2, \pm 4, \pm 8 \pmod{17}, \\ \mathbf{B}, & \text{if } (z-x) \equiv \pm 3, \pm 5, \pm 6, \pm 7 \pmod{17}. \end{cases}$$

Remark. The proof is almost identical to the proof of the previous lemma and can be easily mimicked by replacing 15 with 17 and modifying corresponding indices.

Lemma 11. $|I_1| \le |I_2| + 1$ *and* $r(I_2') \ge r(I_1)$.

Proof. Assume first that $|I_1| \ge |I_2| + 2$. Let $I_1 = [1, l]$ and $I_2 = [l + 1, b + 1]$. Recall that $c(l) = \mathbf{G}$ and $c(b) = c(b + 1) = \mathbf{B}$. The following AP(3)s: $\{2l - b, l, b\}$ and $\{2l - b - 1, l, b + 1\}$ and lem 6(a) imply that $c(2l - b) = c(2l - b - 1) = \mathbf{B}$, a contradiction to the fact that I_1 does not contain **BB**. To prove the second statement, consider $\{x, l, 2l - x\}$, where $x \in I_1$ and $c(x) = \mathbf{R}$. Since $2l - x \in I_2'$ and $c(2l - x) \ne \mathbf{G}$, we have $c(2l - x) = \mathbf{R}$. Therefore, for each $x \in I_1$ such that $c(x) = \mathbf{R}$ there is a unique $y \in I_2'$ such that $c(y) = \mathbf{R}$. □

Lemma 12. *If* $g(I_1) \ge 3$ *then* $g(I_2 \cup I_3) \le 2$ *and* $r(I_3) \ge (|I_3| - 3)/4$.

Proof. Assume that $I_2 \cup I_3$ contains at least three integers colored **G**. Since $g(I_2) = 0$ by definition of I_2, we have $g(I_3) \ge 3$. We know that I_1 contains at least three **G**s as well. Then, there are $x, x' \in I_1$ and $y, y' \in I_3$, such that $c(x) = c(x') = c(y) = c(y') = \mathbf{G}$, x and x' are of the same parity and y and y' are of the same parity. Let $x < x'$ and $y < y'$. Let b be the smallest integer such that $c(b) = c(b + 1) = \mathbf{B}$. Note that $x' < b < y$.

Claim. $n < 2b+2-x'$. Assume not, then $2b+2-x' \in [1, n]$, thus considering AP(3)s $\{x', b, 2b-x'\}$ and $\{x', b+1, 2b+2-x'\}$ we see that $c(2b-x') = c(2b+2-x') = \mathbf{B}$. Now, the AP(3)s $\{x, b-(x'-x)/2, 2b-x'\}$ and $\{x, b+1-(x'-x)/2, 2b+2-x'\}$ show that $c(b - (x' - x)/2) = c(b + 1 - (x' - x)/2) = \mathbf{B}$. This contradiction to minimality of b proves the claim.

Let z be the largest number such that $z < y$ and $c(z) = c(z + 1) = \mathbf{B}$. Observe that $2z-y \ge 2b-y \ge n-2+x'-y+1 = n-(y-x')-1$. Since $x' \ge 4$ and $y \le n$, we have that $2z-y \ge n-n+4-1 \ge 3$. Therefore we can consider the following AP(3)*s*: $\{2z-y, z, y\}$, $\{2z-y+2, z+1, y\}$, which imply that $c(2z-y) = c(2z-y+2) = \mathbf{B}$. Then $\{2z - y, z + (y' - y)/2, y'\}$, $\{2z - y + 2, z + 1 + (y' - y)/2, y'\}$ give us that $c(z + (y' - y)/2) = c(z + 1 + (y' - y)/2) = \mathbf{B}$, contradicting maximality of z.

This proves that there are at most two integers colored **G** in I_3. In order to prove the second statement of the lemma we show that I_3 does not contain **BBBB**.

Assume that there is $y \in I_3$ such that $y + 3 \in I_3$ and $y, y + 1, y + 2, y + 3$ is colored **BBBB**. Assume that y and $y + 2$ have the same parity as x' (otherwise take $y + 1$ and $y + 3$). Then $\{x', (y + x')/2, y\}$ and $\{x, (y + x')/2 + 1, y + 2\}$ imply that $c((y + x')/2) = c((y + x')/2 + 1) = \mathbf{B}$. Using the claim, we have that $y \le n - 3 < 2b+2-x'-3$. Thus $(y+x')/2 < (2b-1)/2 < b$, a contradiction to the minimality of b. □

Acknowledgments. We would like to thank Zsolt Tuza for bringing this problem to our attention. We also thank an anonymous referee for helpful comments.

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Received: June 3, 2004
Final version received: December 22, 2005