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Competition Hypergraphs of Products of Digraphs

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Abstract. If $D = (V, A)$ is a digraph, its *competition hypergraph* $\mathcal{CH}(D)$ has vertex set V and $e \subseteq V$ is an edge of $\mathcal{CH}(D)$ iff $|e| \geq 2$ and there is a vertex $v \in V$, such that $e = \{w \in V | (w, v) \in A\}$. For several products $D_1 \circ D_2$ of digraphs D_1 and D_2, we investigate the relations between the competition hypergraphs of the factors D_1, D_2 and the competition hypergraph of their product $D_1 \circ D_2$.

Key words. Hypergraph, Competition graph, Product of digraphs.

1. Introduction and Definitions

All hypergraphs $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ and digraphs $D = (V(D), A(D))$ considered here may have isolated vertices but no multiple edges and arcs, respectively. Moreover, in digraphs loops are forbidden.

In 1968 Cohen [2] introduced the *competition graph* $C(D)$ associated with a digraph $D = (V, A)$ representing a food web of an ecosystem. $C(D) = (V, E)$ is the graph with the same vertex set as D (corresponding to the species) and

$$E = \{\{u, v\} \mid u \neq v \ \wedge \ \exists\, w \in V : (u, w) \in A \ \wedge \ (v, w) \in A\},$$

i.e. $\{u, v\} \in E$ iff u and v compete for a common prey $w \in V$.

Surveys of the large literature around competition graphs can be found in Roberts [6], Kim [4] and Lundgren [5].

In [7] it is shown that in many cases competition hypergraphs yield a more detailed description of the predation relations among the species in $D = (V, A)$ than competition graphs. If $D = (V, A)$ is a digraph its *competition hypergraph* $\mathcal{CH}(D) = (V, \mathcal{E})$ has the vertex set V and $e \subseteq V$ is an edge of $\mathcal{CH}(D)$ iff $|e| \geq 2$ and there is a vertex $v \in V$, such that $e = \{w \in V \mid (w, v) \in A\}$. In this case we say $v \in V = V(D)$ *corresponds to* $e \in \mathcal{E}$ and vice versa.

In our paper [7] we dealt with competition hypergraphs without loops, that way we followed the most usual definition of competition graphs. In the case of digraphs D possessing vertices with only one predecessor, a competition hypergraph *with loops* contains a more detailed information on D. For that reason, we also include competition hypergraphs with loops in our investigations of competition hypergraphs of products of digraphs and modify the notion of a competition hypergraph.

If $D = (V, A)$ is a digraph its *l-competition hypergraph (competition hypergraph with loops)* $\mathcal{CH}^l(D) = (V, \mathcal{E}^l)$ has the vertex set V and $e \subseteq V$ is an edge of $\mathcal{CH}(D)$ iff $|e| \neq \emptyset$ and there is a vertex $v \in V$, such that $e = \{w \in V \mid (w, v) \in A\}$.

For the sake of brevity, in the following we often use the term *competition hypergraph* (sometimes in connection with the notation $\mathcal{CH}^{(l)}(D)$) for the competition hypergraph $\mathcal{CH}(D)$ as well as for the l-competition hypergraph $\mathcal{CH}^l(D)$.

In standard terminology concerning digraphs we follow Bang–Jensen and Gutin [1]. With $d_D^-(v), d_D^+(v), N_D^-(v)$ and $N_D^+(v)$ we denote the *in–degree, out–degree, in–neighbourhood* and *out–neighbourhood* of a vertex v in a digraph D, respectively.

For five products $D_1 \circ D_2$ (*Cartesian product* $D_1 \times D_2$, *Cartesian sum* $D_1 + D_2$, *normal product* $D_1 * D_2$, *lexicographic product* $D_1 \cdot D_2$ and *disjunction* $D_1 \vee D_2$) of digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ we investigate the construction of the competition hypergraph $\mathcal{CH}^{(l)}(D_1 \circ D_2) = (V, \mathcal{E}_\circ^{(l)})$ from $\mathcal{CH}^{(l)}(D_1) = (V_1, \mathcal{E}_1^{(l)})$, $\mathcal{CH}^{(l)}(D_2) = (V_2, \mathcal{E}_2^{(l)})$ and vice versa.

The products considered here have always the vertex set $V := V_1 \times V_2$; using the notation $\widetilde{A} := \{((a, b), (a', b')) \mid a, a' \in V_1 \wedge b, b' \in V_2\}$ their arc sets are defined as follows:

$$
\begin{aligned}
&A(D_1 \times D_2) := \{((a, b), (a', b')) \in \widetilde{A} \mid (a, a') \in A_1 \wedge (b, b') \in A_2\},\\
&A(D_1 + D_2) := \{((a, b), (a', b')) \in \widetilde{A} \mid ((a, a') \in A_1 \wedge b = b') \vee (a = a' \wedge (b, b') \in A_2)\},\\
&A(D_1 * D_2) := A(D_1 \times D_2) \cup A(D_1 + D_2),\\
&A(D_1 \cdot D_2) := \{((a, b), (a', b')) \in \widetilde{A} \mid (a, a') \in A_1 \vee (a = a' \wedge (b, b') \in A_2)\},\\
&A(D_1 \vee D_2) := \{((a, b), (a', b')) \in \widetilde{A} \mid (a, a') \in A_1 \vee (b, b') \in A_2\}.
\end{aligned}
$$

It follows immediately that $A(D_1 + D_2) \subseteq A(D_1 * D_2) \subseteq A(D_1 \cdot D_2) \subseteq A(D_1 \vee D_2)$ and $A(D_1 \times D_2) \subseteq A(D_1 * D_2)$. Except the lexicographic product all these products are commutative in the sense that $D_1 \circ D_2 \simeq D_2 \circ D_1$, where $\circ \in \{\times, +, *, \vee\}$.

Usually we arrange the vertices of $V = V_1 \times V_2$ according to the places of an (r, s)-matrix, where $r := |V_1|$ and $s := |V_2|$. Then, for each $\circ \in \{+, *, \cdot, \vee\}$, the subdigraph of $D_1 \circ D_2$ generated by the vertices of a column and a row of this matrix scheme is isomorphic to D_1 and D_2, respectively.

The factor decomposition of product graphs is an interesting question (cf. Imrich and Klavzar [3]). Related to this problem the question arises, whether or not $\mathcal{CH}^{(l)}(D_1 \circ D_2)$ can be obtained from $\mathcal{CH}^{(l)}(D_1)$ and $\mathcal{CH}^{(l)}(D_2)$ and vice versa (cf. Theorems 1-3 and Propositions 1-2).

But there is yet another point of view: In general, it is impossible to reconstruct the digraph D from its competition hypergraph $\mathcal{CH}^{(l)}(D)$, since $\mathcal{CH}^{(l)}(D)$ does not contain the complete information on D. Up to now in the literature no results concering this reconstruction problem are known. All the more it is interesting that under certain conditions $D_1 \circ D_2$ and even D_1 and D_2 can be reconstructed from $\mathcal{CH}^{(l)}(D_1 \circ D_2)$ (cf. Corollaries 1-3).

2. Determination of $\mathcal{CH}^{(l)}(D_1 \circ D_2)$ from $\mathcal{CH}^{(l)}(D_1)$ and $\mathcal{CH}^{(l)}(D_2)$

In the following let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be digraphs. By $N_1^-(v)$, $N_2^-(v)$ and $N_\circ^-(v)$ we denote the set of all predecessors of a vertex v in D_1, D_2 and $D_1 \circ D_2$, respectively, where $\circ \in \{\times, +, *, \cdot, \vee\}$.

Theorem 1. *The l-competition hypergraph $\mathcal{CH}^l(D_1 \times D_2) = (V, \mathcal{E}_\times^l)$ of the Cartesian product can be obtained from the l-competition hypergraphs $\mathcal{CH}^l(D_1) = (V_1, \mathcal{E}_1^l)$ and $\mathcal{CH}^l(D_2) = (V_2, \mathcal{E}_2^l)$ of D_1 and D_2: $\mathcal{E}_\times^l = \{e_1 \times e_2 \,|\, e_1 \in \mathcal{E}_1^l \wedge e_2 \in \mathcal{E}_2^l\}$.*

Proof. Choose $(i, j) \in V = V_1 \times V_2$ such that $N_\times^-((i, j)) \neq \emptyset$. Then $\emptyset \neq N_1^-(i) \in \mathcal{E}_1^l$ and $\emptyset \neq N_2^-(j) \in \mathcal{E}_2^l$. Obviously, with $e_1 = N_1^-(i)$ and $e_2 = N_2^-(j)$ we obtain $N_\times^-((i, j)) = e_1 \times e_2$. On the other hand, for $e_1 \in \mathcal{E}_1^l$ and $e_2 \in \mathcal{E}_2^l$ there are $i \in V_1$ and $j \in V_2$ such that $e_1 = N_1^-(i)$ and $e_2 = N_2^-(j)$, i.e. $e_1 \times e_2 = N_\times^-((i, j)) \in \mathcal{E}_\times^l$. □

Theorem 2. *The l-competition hypergraph $\mathcal{CH}^l(D_1 \vee D_2) = (V, \mathcal{E}_\vee^l)$ of the disjunction can be obtained from the l-competition hypergraphs $\mathcal{CH}^l(D_1) = (V_1, \mathcal{E}_1^l)$ and $\mathcal{CH}^l(D_2) = (V_2, \mathcal{E}_2^l)$ of D_1 and D_2, if for each of the following conditions is known whether it is true or not:*

$$(a) \quad \exists\, v_2 \in V_2 : N_2^-(v_2) = \emptyset \quad \textit{and} \quad (b) \quad \exists\, v_1 \in V_1 : N_1^-(v_1) = \emptyset.$$

In general, $\mathcal{CH}^l(D_1 \vee D_2)$ cannot be obtained from $\mathcal{CH}^l(D_1)$ and $\mathcal{CH}^l(D_2)$ without the extra information on points (a) and (b).

Proof. Let $e \in \mathcal{E}_\vee^l$. Then there is a vertex $(i, j) \in V = V_1 \times V_2$ such that $e = N_\vee^-((i, j))$. Considering a vertex $(i', j') \in e$ we obtain $i' \in N_1^-(i) \in \mathcal{E}_1^l$ or $j' \in N_2^-(j) \in \mathcal{E}_2^l$.

(Note that in case $N_1^-(i) = \emptyset$ we have $N_1^-(i) \notin \mathcal{E}_1^l$; analogously $N_2^-(j) \notin \mathcal{E}_2^l$ for $N_2^-(j) = \emptyset$).

Clearly, $e = N_\vee^-((i, j)) = (N_1^-(i) \times V_2) \cup (V_1 \times N_2^-(j))$.

First, consider $N_1^-(i) = \emptyset$.

Because of $\emptyset \neq e \in \mathcal{E}_\vee^l$ we obtain $\emptyset \neq N_2^-(j) \in \mathcal{E}_2^l$ and $e = V_1 \times N_2^-(j)$.

Analogously, from $N_2^-(j) = \emptyset$ it follows $\emptyset \neq N_1^-(i) \in \mathcal{E}_1^l$ and $e = N_1^-(i) \times V_2 \in \mathcal{E}_\vee^l$.

Let $\mathcal{A} := \{e_1 \times V_2 \,|\, e_1 \in \mathcal{E}_1^l\}$, $\mathcal{B} := \{V_1 \times e_2 \,|\, e_2 \in \mathcal{E}_2^l\}$ and $\mathcal{C} := \{(e_1 \times V_2) \cup (V_1 \times e_2) \,|\, e_1 \in \mathcal{E}_1^l \wedge e_2 \in \mathcal{E}_2^l\}$.

Then (a) is equivalent to $\mathcal{A} \subseteq \mathcal{E}_\vee^l$, (b) is equivalent to $\mathcal{B} \subseteq \mathcal{E}_\vee^l$ and, finally, $\mathcal{C}$ contains all hyperedges $N_\vee^-((i, j)) \in \mathcal{E}_\vee^l$ with $\emptyset \neq N_1^-(i) \in \mathcal{E}_1^l$ and $\emptyset \neq N_2^-(j) \in \mathcal{E}_2^l$.

Consequently, every hyperedge of $\mathcal{CH}^l(D_1 \vee D_2)$ is contained in exactly one of the sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, respectively, and therefore the edge set of $\mathcal{CH}^l(D_1 \vee D_2)$ is

$$\mathcal{E}^l_\vee = \begin{cases} \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}, & \text{if } \quad (a) \;\wedge\; \quad (b) \\ \mathcal{A} \cup \mathcal{C}, & \text{if } \quad (a) \;\wedge \sim (b) \\ \mathcal{B} \cup \mathcal{C}, & \text{if } \sim (a) \wedge \quad (b) \\ \mathcal{C}, & \text{if } \sim (a) \wedge \sim (b). \end{cases}$$

□

Proposition 1. *In general, $\mathcal{CH}(D_1 + D_2) = (V, \mathcal{E}_+)$, $\mathcal{CH}(D_1 * D_2) = (V, \mathcal{E}_*)$ and $\mathcal{CH}(D_1 \cdot D_2) = (V, \mathcal{E}_\cdot)$ cannot be obtained from $\mathcal{CH}^l(D_1)$ and $\mathcal{CH}^l(D_2)$ (less than ever $\mathcal{CH}^l(D_1 \circ D_2)$, for $\circ \in \{+, *, \cdot\}$).*

Proof. Consider the digraphs $D_1 = (V_1, A_1)$, $D_1' = (V_1, A_1')$ and $D_2 = (V_2, A_2)$ (cf. Fig. 1) with $V_1 = \{1, 2, 3, 4\}$, $V_2 = \{1, 2, 3\}$, $A_1 = \{(1, 2), (3, 2), (4, 3)\}$, $A_1' = \{(1, 4), (3, 4), (4, 2)\}$ and $A_2 = \{(1, 3), (2, 3)\}$, respectively.

Then $\mathcal{E}(\mathcal{CH}^l(D_1)) = \{\{1, 3\}, \{4\}\} = \mathcal{E}(\mathcal{CH}^l(D_1'))$. On the other hand:

- $\mathcal{CH}(D_1 + D_2) \not\cong \mathcal{CH}(D_1' + D_2)$ (cf. Fig. 1)
  (Note that in $\mathcal{CH}(D_1 + D_2)$ the only hyperedge of cardinality 3, i.e. $\{(3, 1), (3, 2), (4, 3)\}$, is adjacent to the hyperedges $\{(1, 1), (3, 1)\}$ and $\{(1, 2), (3, 2)\}$, but in $\mathcal{CH}(D_1' + D_2)$ the hyperedge $\{(2, 1), (2, 2), (4, 3)\}$ is not adjacent to other hyperedges).
- $\mathcal{CH}(D_1 * D_2) \not\cong \mathcal{CH}(D_1' * D_2)$ (cf. Fig. 2)
  (Note that in $\mathcal{CH}(D_1 * D_2)$ the only hyperedge of cardinality 5, i.e. $\{(3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$, is adjacent to three hyperedges of cardinality 2 ($\{(1, 1), (3, 1)\}$, $\{(1, 2), (3, 2)\}$ and $\{(4, 1), (4, 2)\}$), but in $\mathcal{CH}(D_1' * D_2)$ the hyperedge $\{(2, 1), (4, 1), (2, 2), (4, 2), (4, 3)\}$ is not adjacent to any hyperedge of cardinality 2).
- $\mathcal{CH}(D_1 \cdot D_2) \not\cong \mathcal{CH}(D_1' \cdot D_2)$ (cf. Fig. 3)
  (Note that in $\mathcal{CH}(D_1 \cdot D_2)$ the only hyperedge of cardinality 5, i.e. $\{(3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$, contains a hyperedge of cardinality 2 ($\{(4, 1), (4, 2)\}$), but in $\mathcal{CH}(D_1' \cdot D_2)$ the hyperedge $\{(2, 1), (4, 1), (2, 2), (4, 2), (4, 3)\}$ does not contain any hyperedge of cardinality 2). □

3. Reconstruction of $\mathcal{CH}^{(l)}(D_1)$ and $\mathcal{CH}^{(l)}(D_2)$ from $\mathcal{CH}^{(l)}(D_1 \circ D_2)$

In the following, for a set $e = \{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\} \subseteq V_1 \times V_2$ we define $\pi_1(e) := \{i_1, \ldots, i_k\}$ and $\pi_2(e) := \{j_1, \ldots, j_k\}$, respectively, i.e. π_i denotes the projection of vertices of $\mathcal{CH}^{(l)}(D_1 \circ D_2)$ onto their ith component, for $i \in \{1, 2\}$.

For the competition hypergraphs (without loops) of $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ and their products $D_1 \circ D_2$ we verify

Theorem 3. *For all products $D_1 \circ D_2$ ($\circ \in \{+, *, \cdot, \vee\}$) the competition hypergraphs $\mathcal{CH}(D_1)$ and $\mathcal{CH}(D_2)$ can be obtained from $\mathcal{CH}(D_1 \circ D_2)$.*

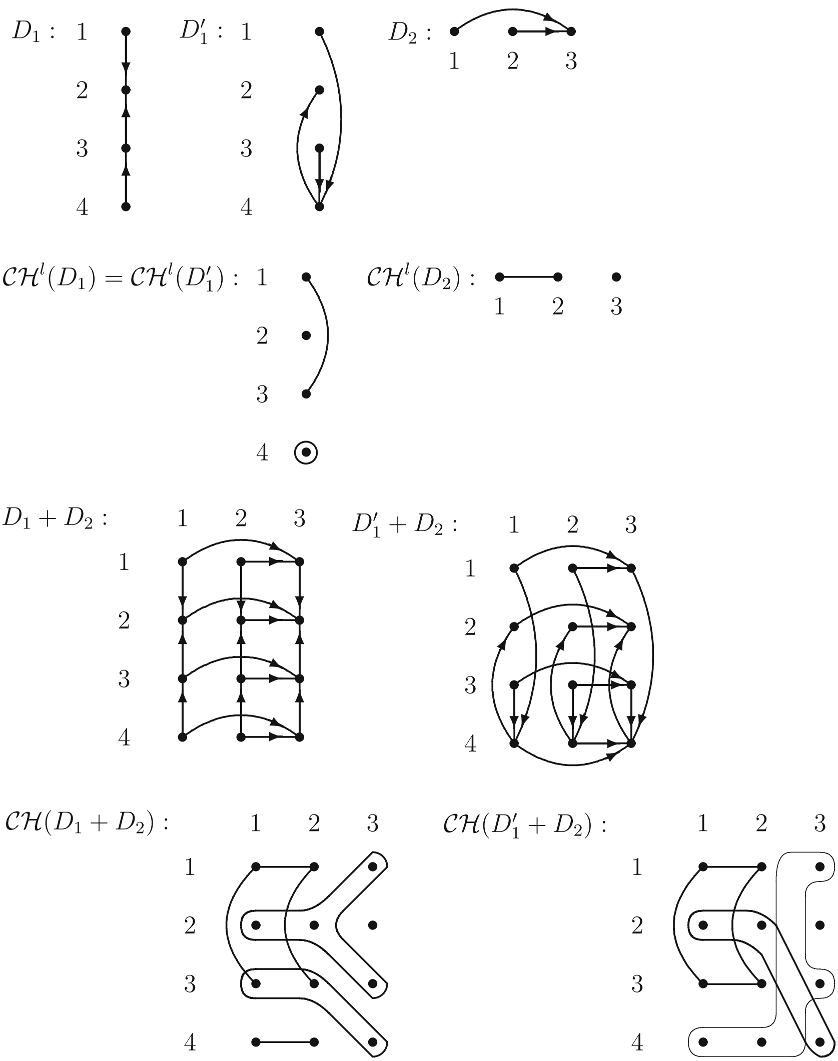
D_1 :
D'_1 :
D_2 :
$\mathcal{CH}^l(D_1) = \mathcal{CH}^l(D'_1)$:
$\mathcal{CH}^l(D_2)$:
$D_1 + D_2$:
$D'_1 + D_2$:
$\mathcal{CH}(D_1 + D_2)$:
$\mathcal{CH}(D'_1 + D_2)$:

Fig. 1.

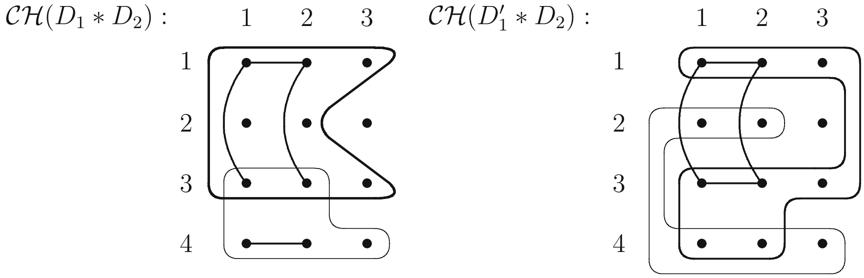
$\mathcal{CH}(D_1 * D_2)$:
$\mathcal{CH}(D'_1 * D_2)$:

Fig. 2.

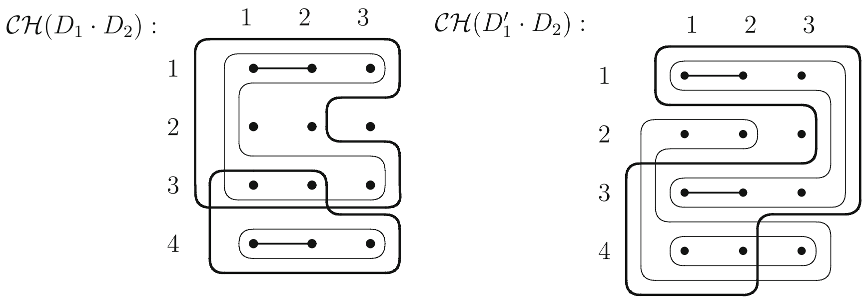

Fig. 3.

For the Cartesian product $D_1 \times D_2$ this is true by using the extra assumption $A_1 = A_2 = \emptyset$ or $A_1 \neq \emptyset \neq A_2$.

Extending the proof of the Theorem it can be shown

Proposition 2. *An analogous proposition holds for $\mathcal{CH}^l(D_1)$, $\mathcal{CH}^l(D_2)$ and $\mathcal{CH}^l(D_1 \circ D_2)(\circ \in \{\times, *, \cdot, \vee\})$.*

For the Cartesian sum $D_1 + D_2$ this is true by using the extra assumption

(1) $\mathcal{E}_+^l = \emptyset$ **or**
(2) $(\forall e \in \mathcal{E}_+^l : |\pi_1(e)| = 1) \wedge (\exists\, e \in \mathcal{E}_+^l : |\pi_2(e)| \geq 2)$ **or**
(3) $(\forall e \in \mathcal{E}_+^l : |\pi_2(e)| = 1) \wedge (\exists\, e \in \mathcal{E}_+^l : |\pi_1(e)| \geq 2)$ **or**
(4) $\exists\, e \in \mathcal{E}_+^l : |\pi_1(e)| \geq 3 \wedge |\pi_2(e)| \geq 3.$

Now we prove Theorem 3 and Proposition 2 for each $\circ \in \{\times, +, *, \cdot, \vee\}$ in the subsections 3.1 to 3.5.

3.1. The Cartesian Product $D_1 \times D_2$

Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ fulfill either $A_1 = A_2 = \emptyset$ or $A_1 \neq \emptyset \neq A_2$.

Obviously, if the arcsets of D_1 and D_2 are empty, then $\mathcal{E}(\mathcal{CH}(D_1 \times D_2)) = \emptyset$ as well as $\mathcal{E}(\mathcal{CH}^l(D_1 \times D_2)) = \emptyset$. Consequently, let $A_1 \neq \emptyset \neq A_2$.

Proof (Proposition 2). We construct $\mathcal{CH}^l(D_1) = (V_1, \mathcal{E}_1^l)$ and $\mathcal{CH}^l(D_2) = (V_2, \mathcal{E}_2^l)$ from $\mathcal{CH}^l(D_1 \times D_2) = (V, \mathcal{E}_\times^l)$.

For each $e = \{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\} \in \mathcal{E}_\times^l$ there exist $i \in V_1$ and $j \in V_2$ such that $N_1^-(i) = \pi_1(e) = \{i_1, \ldots, i_k\} \in \mathcal{E}_1^l$ and $N_2^-(j) = \pi_2(e) = \{j_1, \ldots, j_k\} \in \mathcal{E}_2^l$. (Obviously, neither $i_1, \ldots, i_k$ nor $j_1, \ldots, j_k$ have to be pairwise distinct. Moreover, note that in general the vertices i and j are not uniquely determined by $\mathcal{CH}^l(D_1 \times D_2)$.)

Since $A_1 \neq \emptyset \neq A_2$ every $e_1 \in \mathcal{E}_1^l$ and every $e_2 \in \mathcal{E}_2^l$ appears as $\pi_1(e)$ and $\pi_2(e)$ of some $e \in \mathcal{E}_\times^l$, respectively. Consequently, $\mathcal{E}_1^l = \{\pi_1(e) \mid e \in \mathcal{E}_\times^l\}$ and $\mathcal{E}_2^l = \{\pi_2(e) \mid e \in \mathcal{E}_\times^l\}$. □

From the above note concerning $i \in V_1$ and $j \in V_2$ it follows

Proposition 3. *In general, from $\mathcal{CH}^l(D_1 \times D_2)$ the digraphs D_1 and D_2 cannot be obtained.*

Proof (Theorem 3). The restriction of $\mathcal{E}_1^l$ and $\mathcal{E}_2^l$ to sets $\pi_1(e)$ and $\pi_2(e)$ of cardinality greater than 1 proves the Theorem, since hyperedges $e \in \mathcal{E}_\times^l \backslash \mathcal{E}_\times$, i.e. hyperedges of $\mathcal{CH}^l(D_1 \times D_2)$ of cardinality 1, are trivially not needed. □

3.2. The Cartesian Sum $D_1 + D_2$

At first we consider the Theorem, i.e. competition hypergraphs without loops.

Proof (Theorem 3). Since every hyperedge $e \in \mathcal{E}_+$ is the set of all predecessors $N_+^-((i,j))$ of a vertex $(i,j) \in V_1 \times V_2$, we have $e = \{\{i, j_1\}, \ldots, \{i, j_k\}, \{i_1, j\}, \ldots, \{i_l, j\}\}$, where $i, i_1, \ldots, i_l$ as well as $j, j_1, \ldots, j_k$ are pairwise distinct. Therefore, if $l \geq 2$ then $\pi_1(e)\backslash\{i\} = \{i_1, \ldots, i_l\} = N_1^-(i) \in \mathcal{E}_1$ and if $k \geq 2$ then $\pi_2(e)\backslash\{j\} = \{j_1, \ldots, j_k\} = N_2^-(j) \in \mathcal{E}_2$.

The question arises, whether or not the vertices $i \in \pi_1(e)$ and $j \in \pi_2(e)$ are uniquely determined? (Note that in some cases this determination will not be necessary).

(a) $|\pi_1(e)| = 1$.
Then $l = 0$, $\pi_1(e) = \{i\}$, $N_1^-(i) = \emptyset$, $k \geq 2$ and $\pi_2(e) = \{j_1, \ldots, j_k\} = N_2^-(j) \in \mathcal{E}_2$ with an (unknown) $j \in V_2\backslash\{j_1, \ldots, j_k\}$.

(b) $|\pi_2(e)| = 1$.
Then $k = 0$, $\pi_2(e) = \{j\}$, $N_2^-(j) = \emptyset$, $l \geq 2$ and $\pi_1(e) = \{i_1, \ldots, i_l\} = N_1^-(i) \in \mathcal{E}_1$ with an (unknown) $i \in V_1\backslash\{i_1, \ldots, i_l\}$.

(c) $|\pi_1(e)| \geq 2 \ \wedge \ |\pi_2(e)| \geq 2$.

(c1) $|e| = 2$.
Because of $e = \{(a,b), (a',b')\}$ with $a \neq a'$ and $b \neq b'$ there are two possibilities: $e = N_+^-((a,b'))$ or $e = N_+^-((a',b))$. Since all hyperedges in competition hypergraphs without loops contain at least two vertices, it follows $N_1^-(a) = \{a'\} \notin \mathcal{E}_1 \ \wedge \ N_2^-(b') = \{b\} \notin \mathcal{E}_2$ or $N_1^-(a') = \{a\} \notin \mathcal{E}_1 \ \wedge \ N_2^-(b) = \{b'\} \notin \mathcal{E}_2$.
Therefore, from case (c1) there result no hyperedges of $\mathcal{CH}(D_1)$ and $\mathcal{CH}(D_2)$, respectively.

(c2) $|e| \geq 3$.
Since $e = \{\{i, j_1\}, \ldots, \{i, j_k\}, \{i_1, j\}, \ldots, \{i_l, j\}\}$, we obtain $k \geq 2$ or $l \geq 2$, i.e. at least two vertices in e have the same first or second components. In the case $k \geq 2$ let i be the first component which appears in several vertices of e. Then $\pi_1(e)\backslash\{i\} = \{i_1, \ldots, i_l\} = N_1^-(i) \neq \emptyset$ and for $l \geq 2$ obviously $\pi_1(e)\backslash\{i\} \in \mathcal{E}_1$. Deleting the second component j of the vertices $(i_1, j), \ldots, (i_l, j) \in e$ in $\pi_2(e)$, we obtain $\pi_2(e)\backslash\{j\} = \{j_1, \ldots, j_k\} = N_2^-(j) \in \mathcal{E}_2$.
The case $k = 1 \ \wedge \ l \geq 2$ can be considered analogously.

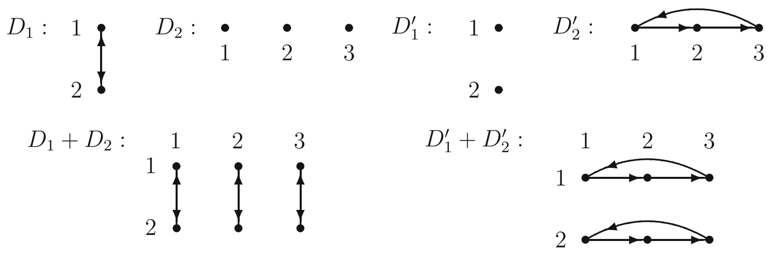

Fig. 4.

Evidently, this way we obtain all hyperedges of $\mathcal{CH}(D_1)$ and $\mathcal{CH}(D_2)$, respectively, and the Theorem holds for the Cartesian sum. $\square$

Before verifying the Proposition, we give some examples explaining the use of the additional suppositions (1)–(4).

Note that $\mathcal{E}_+ \neq \emptyset$ is equivalent to $\exists\, e \in \mathcal{E}^l_+ : |e| \geq 2$ as well as to $\sim (\forall e \in \mathcal{E}^l_+ : |\pi_1(e)| = |\pi_2(e)| = 1)$.

In general, in the case $\mathcal{E}^l_+ \neq \emptyset \wedge \mathcal{E}_+ = \emptyset$ the determination of $\mathcal{CH}^l(D_1)$ and $\mathcal{CH}^l(D_2)$ from $\mathcal{CH}^l(D_1 + D_2)$ is impossible:

Let $D_1 := C_2$, $D_2 := (\{1, 2, 3\}, \emptyset)$ and $D'_1 := (\{1, 2\}, \emptyset)$, $D'_2 := C_3$ (cf. Fig. 4; the (very simple) l-competition hypergraphs are omitted).

Obviously, $\mathcal{E}^l(D_1 + D_2) = \mathcal{E}^l(D'_1 + D'_2) = \{\{v\} \mid v \in V_1 \times V_2\}$.

On the other hand we have $\mathcal{E}^l(D_1) = \{\{1\}, \{2\}\} \neq \emptyset = \mathcal{E}^l(D'_1)$ and $\mathcal{E}^l(D_2) = \emptyset \neq \{\{1\}, \{2\}, \{3\}\} = \mathcal{E}^l(D'_2)$.

The next example shows digraphs D_1, D_2 with $\mathcal{E}_+ \neq \emptyset$ and D'_2 such that $\mathcal{CH}^l(D_1 + D_2) = \mathcal{CH}^l(D_1 + D'_2)$ but $\mathcal{CH}^l(D_2) \not\cong \mathcal{CH}^l(D'_2)$ (cf. Fig. 5):

Let $D_1 := C_2$, $D_2 := (V_2 = \{1, 2, 3, 4\}, A_2 = \{(1, 2), (3, 4), (4, 3), (4, 1)\})$ and $D'_2 := (V'_2 = \{1, 2, 3, 4\}, A'_2 = \{(1, 2), (1, 4), (4, 3), (4, 1)\})$. Then $D_2 \not\cong D'_2$, $\mathcal{CH}^l(D_2) \not\cong \mathcal{CH}^l(D'_2)$, $D_1 + D_2 \not\cong D_1 + D'_2$, but $\mathcal{CH}^l(D_1 + D_2) = \mathcal{CH}^l(D_1 + D'_2)$. Note that for $\mathcal{E}^l_+$ none of the conditions (1)–(4) is valid, since $|\pi_1(e)| = |\pi_2(e)| = 2$ for all $e \in \mathcal{E}^l_+$.

Proof (Proposition 2).
Case 1: $\mathcal{E}^l_+ = \emptyset$.

Obviously, $A(D_1 + D_2) = \emptyset = A(D_1) = A(D_2) = \mathcal{E}^l(D_1) = \mathcal{E}^l(D_2)$.

(Note that the analogous implication $\mathcal{E}^l_\circ = \emptyset \Rightarrow A(D_1 \circ D_2) = \emptyset = \ldots$ holds for all $\circ \in \{+, *, \cdot, \vee\}$.)

Case 2: $(\forall e \in \mathcal{E}^l_+ : |\pi_1(e)| = 1) \;\wedge\; (\exists\, e \in \mathcal{E}^l_+ : |\pi_2(e)| \geq 2)$.

Let $e \in \mathcal{E}^l_+$ with $|\pi_2(e)| \geq 2$, i.e. $e = \{(i, j_1), \ldots, (i, j_k)\} = N^-_+((i, j))$ with $k \geq 2$ and suitable $i \in V_1$, $j \in V_2$ and $j_1, \ldots, j_k \in V_2$. Then $N^-_2(j) = \{j_1, \ldots, j_k\} = \pi_2(e)$. (Note that for given $e \in \mathcal{E}^l_+$ in general the determination of the vertex j will be impossible. This implies that the digraph $D_2 = (V_2, A_2)$ itself cannot be obtained from $\mathcal{CH}^l(D_1 + D_2)$).

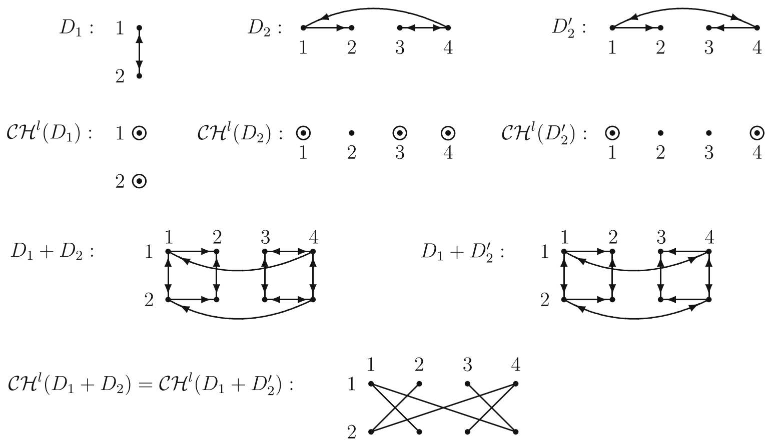

Fig. 5.

The assumption $\exists\, i' \in V_1\ \exists\, l \geq 1\ \exists\, i_1', \ldots, i_l' : N_1^-(i') = \{i_1', \ldots, i_l'\} \neq \emptyset$ would lead to $e' = N_+^-((i', j)) = \{(i_1', j), \ldots, (i_l', j), (i', j_1), \ldots, (i', j_k)\}$ with $|\pi_1(e')| \geq 2$, a contradiction.

Therefore, $\mathcal{E}_1^l = \emptyset$ and $\mathcal{E}_2^l = \{\pi_2(e) \mid e \in \mathcal{E}_+^l\}$.

Case 3: $(\forall e \in \mathcal{E}_+^l : |\pi_2(e)| = 1)\ \wedge\ (\exists\, e \in \mathcal{E}_+^l : |\pi_1(e)| \geq 2)$.

This can be treated analogously to Case 2.

Case 4: $\exists\, e \in \mathcal{E}_+^l : |\pi_1(e)| \geq 3\ \wedge\ |\pi_2(e)| \geq 3$.

Let e be such a hyperedge, $i \in V_1$ with $|\{(i, j') \mid j' \in V_2\} \cap e| \geq 2$ and $j \in V_2$ with $|\{(i', j) \mid i' \in V_1\} \cap e| \geq 2$.

Then $e = N_+^-((i, j))$ and therefore $N_1^-(i) = \{i_1, \ldots, i_l\} = \pi_1(e)\backslash\{i\}$ and $N_2^-(j) = \{j_1, \ldots, j_k\} = \pi_2(e)\backslash\{j\}$.

For each $x \in V_1$ let $e^x := \{(x, j_1), \ldots, (x, j_k), (x_1, j), \ldots, (x_{l_x}, j)\} \in \mathcal{E}_+^l$ with $l_x \geq 0$. Obviously, $e^x = N_+^-((x, j))$ and $N_1^-(x) = \{x_1, \ldots, x_{l_x}\} = \pi_1(e^x)\backslash\{x\}$. This way we obtain $D_1 = (V_1, A_1)$ as well as $\mathcal{E}_1^l = \{N_1^-(x) \mid x \in V_1\ \wedge\ N_1^-(x) \neq \emptyset\}$.

Analogously, for each $y \in V_2$ let $e^y := \{(i_1, y), \ldots, (i_l, y), (i, y_1), \ldots, (i, y_{k_y})\} \in \mathcal{E}_+^l$ with $k_y \geq 0$. Then $e^y = N_+^-((i, y))$ and $N_2^-(y) = \{y_1, \ldots, y_{k_y}\} = \pi_2(e^y)\backslash\{y\}$. □

From Cases 1 and 4 of the above proof it follows:

Corollary 1. *If (1) or (4) is valid, then from $\mathcal{CH}^l(D_1 + D_2)$ the digraphs D_1 and D_2 themselves can be obtained.*

3.3. The Normal Product $D_1 * D_2$

In case of the normal product we can strengthen Theorem 3 and Proposition 2 to the following

Corollary 2. *Suppose there is an* $e \in \mathcal{E}_*$ *with* $|\pi_1(e)| \geq 2 \wedge |\pi_2(e)| \geq 2$. *(This is equivalent to* $A_1 \neq \emptyset \wedge A_2 \neq \emptyset$.*) Then from* $\mathcal{CH}(D_1 * D_2)$ *the digraphs* D_1 *and* D_2 *themselves and, therefore, the l-competition hypergraphs* $\mathcal{CH}^l(D_1)$ *and* $\mathcal{CH}^l(D_2)$ *can be obtained.*

Proof (Theorem 3, Proposition 2 and Corollary 2).
Case 1: $A_1 = \emptyset \vee A_2 = \emptyset$.

We show how to obtain the hyperedges of $\mathcal{CH}(D_1)$, $\mathcal{CH}(D_2)$, $\mathcal{CH}^l(D_1)$ and $\mathcal{CH}^l(D_2)$, since only Theorem 3 and Proposition 2 have to be verified in Case 1. Because of $A_1 = \emptyset$ or $A_2 = \emptyset$ obviously $\forall e \in \mathcal{E}_* : |\pi_1(e)| = 1$ or $\forall e \in \mathcal{E}_* : |\pi_2(e)| = 1$ is valid.

If $\forall e \in \mathcal{E}_* : |\pi_1(e)| = 1$, then $\mathcal{E}_1 = \emptyset$ and $\mathcal{E}_2 = \{\pi_2(e) \,|\, e \in \mathcal{E}_*\}$; the analogue holds in the situation $\forall e \in \mathcal{E}_* : |\pi_2(e)| = 1$ as well as for $\mathcal{E}^l_*, \mathcal{E}^l_1, \mathcal{E}^l_2$ instead of $\mathcal{E}_*, \mathcal{E}_1, \mathcal{E}_2$. (Obviously, this includes $\mathcal{E}_1 = \mathcal{E}_2 = \emptyset$ and $\mathcal{E}^l_1 = \mathcal{E}^l_2 = \emptyset$ if $\mathcal{E}_* = \emptyset$ and $\mathcal{E}^l_* = \emptyset$, respectively.)

Case 2: $A_1 \neq \emptyset \wedge A_2 \neq \emptyset$**.**

It suffices to demonstrate Corollary 2.

Let $e = \{(i, j_1), \ldots, (i, j_k), (i_1, j), \ldots, (i_l, j), (i_1, j_1), (i_1, j_2), \ldots, (i_1, j_k), \ldots, (i_l, j_1), (i_l, j_2), \ldots, (i_l, j_k)\} \in \mathcal{E}_*$ with $|\pi_1(e)| \geq 2 \wedge |\pi_2(e)| \geq 2$.

(a) Because of $l = |\pi_1(e)| - 1 \geq 1$ and $k = |\pi_2(e)| - 1 \geq 1$, the vertices $i \in V_1$ and $j \in V_2$ with $N^-_*((i, j)) = e$ can be identified as the only vertices which occur exactly k and l times in $\pi_1(e)$ and $\pi_2(e)$, respectively.
Moreover, $\pi_1(e) \backslash \{i\} = \{i_1, \ldots, i_l\} = N^-_1(i)$ and $\pi_2(e) \backslash \{j\} = \{j_1, \ldots, j_k\} = N^-_2(j)$.

(b) Obviously, for every $x \in V_1$ with $N^-_1(x) \neq \emptyset$ in $N^-_*((x, j))$ there are at least 3 vertices: (x, j_1), (x', j), (x', j_1), where $x' \in N^-_1(x)$. Therefore $N^-_*((x, j)) \in \mathcal{E}_*$.
Analogously, for each $y \in V_2$ with $N^-_2(y) \neq \emptyset$ we get $N^-_*((i, y)) \in \mathcal{E}_*$.

(c) Note that if $x \in V_1$ with $N^-_1(x) = \emptyset$, then $N^-_*((x, j)) = \{(x, j_1), \ldots, (x, j_k)\}$; i.e. $N^-_*((x, j)) \in \mathcal{E}_*$ if and only if $k \geq 2$.
Analogously, for every $y \in V_2$ with $N_2(y) = \emptyset$ it follows $N^-_*((i, y)) \in \mathcal{E}_*$ if and only if $l \geq 2$.

Because of (b), for all vertices of D_1 and D_2, respectively, with positive indegree we get their sets of predecessors applying the procedure described in (a) to all hyperedges $e \in \mathcal{E}_*$ with $|\pi_1(e)| \geq 2$ and $|\pi_2(e)| \geq 2$. (In general, for a vertex $v_1 \in V_1$ and $v_2 \in V_2$, respectively, with positive indegree, procedure (a) will produce its set of predecessors more than once.) Trivially, each vertex for which (a) does not provide a set of predecessors has indegree 0 (cf. (c)).

Thus we obtain the digraphs D_1 and D_2 and – of course – their l-competition hypergraphs $\mathcal{CH}^l(D_1)$ and $\mathcal{CH}^l(D_2)$.

Note that we did not need hyperedges $e \in \mathcal{E}^l_* \backslash \mathcal{E}_*$, i.e. hyperedges of cardinality 1. □

3.4. The Lexicographic Product $D_1 \cdot D_2$

At first we discuss which types of hyperedges can occur in $\mathcal{CH}^l(D_1 \cdot D_2)$:

(a) $\mathbf{e} = \mathbf{e_1} \times \mathbf{V_2}$ with $e_1 \in \mathcal{E}_1^l$.
Such hyperedges exist if and only if there is a vertex $j \in V_2$ with $N_2^-(j) = \emptyset$: choosing a vertex $i \in V_1$ with $\pi_1(e) = N_1^-(i) = e_1 \subset V_1$ we get $e = N_\cdot^-((i, j))$. Therefore, each $j \in V_2$ appears exactly $d_1^-(i) = |e_1|$-times as second component of a vertex in e. (Note that in general the vertex $i \in V_1 \backslash e_1$ as well as $j \in V_2$ is not uniquely determined by e or e_1.)

(b) $\mathbf{e} = \{\mathbf{i}\} \times \mathbf{e_2}$ with $e_2 \in \mathcal{E}_2^l$.
Obviously, in this case the vertex $i \in V_1$ has $N_1^-(i) = \emptyset$. If $j \in V_2$ and $e_2 = N_2^-(j) = \pi_2(e) \subset V_2$, then $e = N_\cdot^-((i, j))$ and $|e| = |e_2| = |N_2^-(j)| = d_2^-(j)$. (Also in this case in general the vertex $j \in V_2 \backslash e_2$ is not uniquely determined by e or e_2.)

(c) $\mathbf{e} = (\mathbf{e_1} \times \mathbf{V_2}) \cup (\{\mathbf{i}\} \times \mathbf{e_2})$ with $e_1 \in \mathcal{E}_1^l$, $e_2 \in \mathcal{E}_2^l$.
Again, $i \in V_1$ has $N_1^-(i) = e_1$ and $e_2 = N_2^-(j)$ for a certain (in general unknown) vertex $j \in V_2 \backslash e_2$, i.e. $e = N_\cdot^-((i, j))$. But in contrast to (a) and (b) now we have $e_1 = N_1^-(i) \subset \pi_1(e) = N_1^-(i) \cup \{i\}$ and $e_2 = N_2^-(j) \subset \pi_2(e) = V_2$.

Proof (Proposition 2). In case of $\mathcal{E}_\cdot^l = \emptyset$ both $\mathcal{E}_1^l$ and $\mathcal{E}_2^l$ are empty, too (therefore $D_1 = (V_1, \emptyset)$ as well as $D_2 = (V_2, \emptyset)$).

So let $\mathcal{E}_\cdot^l \neq \emptyset$ and for an arbitrary hyperedge $e \in \mathcal{E}_\cdot^l$ and $i \in \pi_1(e)$ we introduce the notation $\pi_2^i(e) := \{j \mid (i, j) \in e\}$. Now let

$$\mathcal{A} := \{e \in \mathcal{E}_\cdot^l \mid \forall i \in \pi_1(e) : \pi_2^i(e) = V_2\}, \mathcal{B} := \{e \in \mathcal{E}_\cdot^l \mid \forall i \in \pi_1(e) : \pi_2^i(e) \subset V_2\} \quad \text{and}$$
$$\mathcal{C} := \{e \in \mathcal{E}_\cdot^l \mid (\exists\, i \in \pi_1(e) : \pi_2^i(e) = V_2) \wedge (\exists\, i' \in \pi_1(e) : \pi_2^{i'}(e) \subset V_2)\}.$$

Then $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ contain exactly the type (a), (b) and (c) hyperedges, respectively. Consequently, the edge set $\mathcal{E}_1^l$ of $\mathcal{CH}_1^l$ consists of all hyperedges

$$e_1 := \{i \in V_1 \mid \pi_2^i(e) = V_2\},$$

where $e \in \mathcal{A} \cup \mathcal{C}$.

(If $\mathcal{A} \neq \emptyset$, i.e. $\exists\, j \in V_2 : N_2^-(j) = \emptyset$, $\mathcal{E}_1^l$ can be obtained simply by $\mathcal{E}_1^l = \{\pi_1(e) \mid e \in \mathcal{A}\}$.)

Obviously, each type (b) hyperedge $e \in \mathcal{B}$ fulfils $|\pi_1(e)| = 1$. So we obtain $\mathcal{B} = \{e \in \mathcal{E}_\cdot^l \mid \pi_2(e) \subset V_2\}$. If $\mathcal{B} \neq \emptyset$, i.e. $\exists\, i \in V_1 : N_1^-(i) = \emptyset$, we get $\mathcal{E}_2^l$ by $\mathcal{E}_2^l = \{\pi_2(e) \mid e \in \mathcal{B}\}$. Otherwise, if $\mathcal{B} = \mathcal{C} = \emptyset$ then $\mathcal{E}_2^l = \emptyset$ and if $\mathcal{B} = \emptyset \neq \mathcal{C}$, then for every type (c) hyperedge $e \in \mathcal{C}$ there is exactly one $i^e \in V_1$ such that $\pi_2^{i^e}(e) = \{j \mid (i^e, j) \in e\} \subset V_2$. (Note that $e = N_\cdot^-((i^e, j'))$ with a certain $j' \in V_2 \backslash \pi_2^{i^e}(e)$.) Using this notation, we obtain $\mathcal{E}_2^l = \{\pi_2^{i^e}(e) \mid e \in \mathcal{C}\}$. □

Proof (Theorem 3). Hyperedges $e \in \mathcal{E}_\cdot^l$ of cardinality 1 can be omitted because they lead only to hyperedges of cardinality 1 of $\mathcal{CH}^l(D_1)$ and $\mathcal{CH}^l(D_2)$ which do not occur in $\mathcal{CH}(D_1)$ and $\mathcal{CH}(D_2)$, respectively. □

Corollary 3. (1) *If $|V_2| \geq 2$ and $\mathcal{B} = \emptyset$ (this is equivalent to $\forall\, i \in V_1 : N_1^-(i) \neq \emptyset$), then the l-competition hypergraphs $\mathcal{CH}^l(D_1)$ and $\mathcal{CH}^l(D_2)$ can be obtained from $\mathcal{CH}(D_1 \cdot D_2)$.*

(2) *If there are $\widetilde{e} \in \mathcal{E}_{\cdot}$ and $k \in V_1$ with $\emptyset \neq \pi_2^k(\widetilde{e}) \subset V_2$, then $D_1 = (V_1, A_1)$ can be obtained from $\mathcal{CH}(D_1 \cdot D_2)$.*

Proof. (1) In $\mathcal{CH}^l(D_1 \cdot D_2)$ there are no hyperedges of cardinality 1: this follows from $|V_2| \geq 2$ for hyperedges of type (a); since $\mathcal{B} = \emptyset$, no hyperedges of type (b) exist and, evidently, hyperedges of type (c) have at least cardinality $|V_2|+1 \geq 3$. Hence $\mathcal{CH}(D_1 \cdot D_2) = \mathcal{CH}^l(D_1 \cdot D_2)$ and (1) follows from Proposition 2.

(2) Let $i \in V_1$. If $\exists\, e' \in \mathcal{E}_{\cdot} : \pi_1(e') = \{i\} \wedge \pi_2(e') \subset V_2$, then this hyperedge e' is of type (b): $e' \in \mathcal{B}$. Consequently, $N_1^-(i) = \emptyset$.
On the other hand, let $\forall\, e' \in \mathcal{E}_{\cdot} : \pi_1(e') \neq \{i\} \vee \pi_2(e') = V_2$. The existence of $\widetilde{e} \in \mathcal{E}_{\cdot}$ and $k \in V_1$ with $\emptyset \neq \pi_2^k(\widetilde{e}) \subset V_2$ provides $e_2 := \pi_2^k(\widetilde{e}) \in \mathcal{E}_2^l$. This implies $\emptyset \neq N_1^-(i) = e_1 \in \mathcal{E}_1^l$ and therefore there exists a hyperedge e of type (c) of the form $e = (e_1 \times V_2) \cup (\{i\} \times e_2)$ in $\mathcal{CH}(D_1 \cdot D_2)$.
e is uniquely determined by $\pi_2^i(e) = e_2$, because e_2 is known. Finally, $N_1^-(i) = \pi_1(e) \setminus \{i\}$.

Consequently, the arc set A_1 and the digraph $D_1 = (V_1, A_1)$ can be obtained. □

3.5. The Disjunction $D_1 \vee D_2$

Again we find three types of hyperedges in $\mathcal{CH}^l(D_1 \vee D_2)$:

(a) $\mathbf{e} = \mathbf{e_1} \times \mathbf{V_2}$ with $e_1 \in \mathcal{E}_1^l$.
The comments to type (a) hyperedges of the lexicographic product can be taken over word-for-word.

(b) $\mathbf{e} = \mathbf{V_1} \times \mathbf{e_2}$ with $e_2 \in \mathcal{E}_2^l$.
In $\mathcal{CH}^l(D_1 \vee D_2)$ there are such hyperedges if and only if an $i \in V_1$ exists with $N_1^-(i) = \emptyset$. Obviously, if $e = V_1 \times e_2$ then there is a $j \in V_2$ with $\pi_2(e) = N_2^-(j) = e_2 \subset V_2$ and we have $e = N_\vee^-((i, j))$. Therefore, each $i \in V_1$ appears exactly $d_2^-(j) = |e_2|$-times as first component of a vertex of e. (Note that in general neither the vertex $i \in V_1$ nor $j \in V_2 \setminus e_2$ is uniquely determined by e or e_2.)

(c) $\mathbf{e} = (\mathbf{e_1} \times \mathbf{V_2}) \cup (\mathbf{V_1} \times \mathbf{e_2})$ with $e_1 \in \mathcal{E}_1^l$, $e_2 \in \mathcal{E}_2^l$.
Obviously, $N_1^-(i) = e_1 \subset V_1$ and $e_2 = N_2^-(j) \subset V_2$ for certain (in general unknown) vertices $i \in V_1 \setminus e_1$ and $j \in V_2 \setminus e_2$, respectively; i.e. $e = N_\vee^-((i, j))$. In contrast to (a) and (b) now we have $e_1 = N_1^-(i) \subset \pi_1(e) = V_1$ and $e_2 = N_2^-(j) \subset \pi_2(e) = V_2$.

Proof (Proposition 2). In case of $\mathcal{E}_\vee^l = \emptyset$ both $\mathcal{E}_1^l$ and $\mathcal{E}_2^l$ are empty, too.

So let $\mathcal{E}_\vee^l \neq \emptyset$. Additionally to the notation $\pi_2^i(e)$ (cf. 3.4, for the disjunction instead of the lexicographic product) for an arbitrary hyperedge $e \in \mathcal{E}_\vee^l$ and $j \in \pi_2(e)$ we define $\pi_1^j(e) := \{i \mid (i, j) \in e\}$. Let

$$\mathcal{A} := \{e \in \mathcal{E}_\vee^l \mid \pi_1(e) \subset V_1\},\ \mathcal{B} := \{e \in \mathcal{E}_\vee^l \mid \pi_2(e) \subset V_2\} \quad \text{and}$$
$$\mathcal{C} := \{e \in \mathcal{E}_\vee^l \mid \pi_1(e) = V_1 \wedge \pi_2(e) = V_2\}.$$

$\mathcal{CH}^l(D_1 \vee D_2) = \mathcal{CH}^l(D_1 \vee D_2'):$

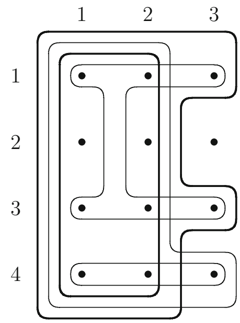

Fig. 6.

Then $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ contain exactly the type (a), (b) and (c) hyperedges, respectively, and

$$\mathcal{A} = \mathcal{C} = \emptyset \text{ if and only if } A_1 = \emptyset = \mathcal{E}_1^l \text{ and } \mathcal{E}_2^l = \{\pi_2(e) \mid e \in \mathcal{E}_\vee^l\};$$
$$\mathcal{B} = \mathcal{C} = \emptyset \text{ if and only if } A_2 = \emptyset = \mathcal{E}_2^l \text{ and } \mathcal{E}_1^l = \{\pi_1(e) \mid e \in \mathcal{E}_\vee^l\};$$
$$\mathcal{C} \neq \emptyset \text{ if and only if } A_1 \neq \emptyset \neq A_2.$$

It remains to investigate the case $\mathcal{C} \neq \emptyset$. Obviously, to determine $\mathcal{E}_1^l$ and $\mathcal{E}_2^l$ it suffices to make use of the type (c) hyperedges (in $\mathcal{C}$):

$$\mathcal{E}_1^l = \{\{i \in V_1 \mid \pi_2^i(e) = V_2\} \mid e \in \mathcal{C}\} \text{ and } \mathcal{E}_2^l = \{\{j \in V_2 \mid \pi_1^j(e) = V_1\} \mid e \in \mathcal{C}\}.$$

(Note that in case $\mathcal{A} \neq \emptyset$ we have $\mathcal{E}_1^l = \{\pi_1(e) \mid e \in \mathcal{A}\}$ and, analogously, if $\mathcal{B} \neq \emptyset$ it follows $\mathcal{E}_2^l = \{\pi_2(e) \mid e \in \mathcal{B}\}$.) □

Proof (Theorem 3). Replacing $\mathcal{E}_1^l$, $\mathcal{E}_2^l$ and $\mathcal{E}_\vee^l$ by $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_\vee$, respectively, the above proof can be taken over word-for-word. □

If $|V_1|, |V_2| \geq 2$ then there are no hyperedges of cardinality 1 in $\mathcal{E}_\vee^l$, i.e. $\mathcal{CH}(D_1 \vee D_2) = \mathcal{CH}^l(D_1 \vee D_2)$. Hence Proposition 2 and Theorem 3 can be strengthened to

Corollary 4. *If $|V_1|, |V_2| \geq 2$ then the l-competition hypergraphs $\mathcal{CH}^l(D_1)$ and $\mathcal{CH}^l(D_2)$ can be obtained from $\mathcal{CH}(D_1 \vee D_2)$.*

Proposition 4. *In general, it is impossible to obtain the digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ from $\mathcal{CH}^l(D_1 \vee D_2)$.*

To prove Proposition 4, we consider the same digraphs used for the verification of Proposition 1, i.e. $D_1 = (V_1, A_1)$, $D_1' = (V_1, A_1')$ and $D_2 = (V_2, A_2)$ with $V_1 = \{1, 2, 3, 4\}$, $V_2 = \{1, 2, 3\}$, $A_1 = \{(1, 2), (3, 2), (4, 3)\}$, $A_1' = \{(1, 4), (3, 4), (4, 2)\}$ and $A_2 = \{(1, 3), (2, 3)\}$, respectively (cf. Fig. 1).

Obviously, $D_1 \not\cong D_1'$ but $\mathcal{CH}^l(D_1 \vee D_2) = \mathcal{CH}^l(D_1' \vee D_2)$ (cf. Fig. 6), since $\mathcal{E}(\mathcal{CH}^l(D_1 \vee D_2)) = \mathcal{E}(\mathcal{CH}^l(D_1' \vee D_2))$ consists of the same hyperedges, namely:

$$
\begin{aligned}
N_\vee^-((1,3)) &= N_\vee^-((4,3)) = N_{\vee'}^-((1,3)) = N_{\vee'}^-((3,3))\\
&= \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2),(4,1),(4,2)\},\\
N_\vee^-((2,1)) &= N_\vee^-((2,2)) = N_{\vee'}^-((4,1)) = N_{\vee'}^-((4,2))\\
&= \{(1,1),(1,2),(1,3),(3,1),(3,2),(3,3)\},\\
N_\vee^-((2,3)) &= N_{\vee'}^-((4,3))\\
&= \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1),(3,2),(3,3),(4,1),(4,2)\},\\
N_\vee^-((3,1)) &= N_\vee^-((3,2)) = N_{\vee'}^-((2,1)) = N_{\vee'}^-((2,2))\\
&= \{(4,1),(4,2),(4,3)\},\\
N_\vee^-((3,3)) &= N_{\vee'}^-((2,3)) = \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2),(4,1),(4,2),(4,3)\},
\end{aligned}
$$

where $N_{\vee'}^-((i,j)) := N_{D_1' \vee D_2}^-((i,j))$, for $(i,j) \in V_1 \times V_2$.

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