Graphs and Combinatorics (2005) 21:1–26
Digital Object Identifier (DOI) 10.1007/s00373-004-0587-7

Graphs and Combinatorics

Connected Factors in Graphs – a Survey

Mekkia Kouider[1] and Preben Dahl Vestergaard[2,*]

[1] Laboratoire de Recherche en Informatique, UMR 8623, Bât. 490, Université Paris Sud, 91405 Orsay, France. e-mail: km@lri.fr
[2] Department of Mathematics, Aalborg University, Fredrik Bajers Vej 7G, DK–9220 Aalborg Ø, Denmark. e-mail: pdv@math.aau.dk

Key words. Connected factor, Spanning subgraph, k-walk, k-trail, k-tree

1. Introduction

A graph $G = (V, E)$ (or $(V(G), E(G))$ is here a finite, simple graph which has neither multiple edges nor loops. A subgraph F of G is called a *factor* of G if $V(F) = V$ and $E(F) \subseteq E$. In other words F is a spanning subgraph of G. Given an integer k, a k-factor of a graph G is a regular spanning subgraph of degree k. More generally, if f is a function from $V(G)$ into the nonnegative integers, then F is called an f-factor if $\deg_F(v) = f(v)$ for all v in $V(G)$. For vertex functions g, f from $V(G)$ into the non-negative integers satisfying $g(v) \leq f(v)$ for all v in $V(G)$ we call a factor F of G a (g, f)-factor if $g(v) \leq \deg_F(v) \leq f(v)$ for all v in $V(G)$. Tradition has evolved such that square brackets are used for constants bounding the degrees of the factor and round parentheses for functions. A family of edge disjoint factors (respectively edge disjoint (g, f)-factors) $F_1, F_2, \ldots, F_k$ with $\bigcup_{i=1}^{k} E(F_i) = E(G)$ is called a factorization of G (respectively a (g, f)-factorization).

As an introduction to our subject of connected factors we give a brief historical survey on factors. Plummer [120] (2004) has written an up to date survey on factors and factorizations with many references. An earlier survey is from 1985 by Akiyama and Kano [1], which appeared in an issue of Journal of Graph Theory devoted entirely to factors. Results on factors go back at least a century. In 1891 Petersen [119] proved that a graph is 2-factorable if and only if it is $2p$-regular, $p \geq 1$, and that a connected cubic graph with at most two

---

*Support from the Danish Natural Science Research Council, LRI University Paris XI and, Aalborg University is gratefully acknowledged
2000 Mathematics Subject Classification. 05C40, 05C45, 05C75

bridges has a 1-factor, later generalized by Bäbler [8] in 1938 and many others. In 1947 Tutte published his famous theorem [139] which states that a graph G has a 1-factor if and only if deletion of any vertex set S leaves $G - S$ with at most $|S|$ components of odd order. Independently, between 1931 and 1935, Hall [66] and König [84] gave some basic results on 1-factors in bipartite graphs. Rado [121] (1949) considered 1-factors in locally finite bipartite graphs and through the 1950's Belck [14] considered k-factors in regular graphs, Gallai [58] considered k-factors in graphs, and Tutte [138,137] considered f-factors. In 1970 Lovász [106] extended this work to (g, f)-factors. Later Tutte [132] (1981) proved that the (g, f)-factor theorem could be derived from his f-factor theorem, which in turn could be derived from his 1-factor theorem. Let us remark that M. Cai [27] (1991) and Holton and Sheehan [68] (1993) have illustrated uses of the (g, f)-factor theorem by giving applications. Anstee gives an algorithmic proof of Tutte's f-factor theorem [5] (1985) and variants of the (g, f)-factor theorem [4,3] (1990, 1998). If, in particular, g and f are constant, i.e., for two integers $0 \leq a \leq b$ we have $g(v) = a, f(v) = b$ for all v in $V(G)$, then the factor is said to be an $[a, b]$-factor, a special case of a (g, f)-factor. This was already considered by Tutte [133] and Thomassen [127] for $b = a + 1$; Kano and Saito [79] considered non-consecutive a and b. There is a vast literature on factors. The books by Bollobás [18] (1978), Lovász [105] (1979) and, more recently, the book by Volkmann [146] (1996) all have a chapter on factors.

Guiying [64], M. Cai [26], Kano [78], G.Y. Yan, J.F. Pan, C.K. Wong and Tokuda [152] have considered (g, f)-factorizations of graphs. Gutin [65] gave a condition for digraphs to have a connected (g, f)-factor.

As we have already seen above, factors may be selected by degree properties. One variant is that the parity of the degrees may be prescribed, see e.g. [131], [38], [2] for odd factors, [89] for even factors and [35] for both. Factors may also be selected by some structural property, and we shall, in this survey, do exactly that by putting our emphasis on connected factors. Some of the earliest results of graph theory on Hamiltonian cycles, spanning trees and walks fall naturally in that category. In the last decade the notion of connected factors has gained wider prominence, particularly after an international conference held in Beijing in 1993, see e.g. Kano [81].

1.1. Notation

Our preferred notation for the order of a graph G is n, i.e., $|G| = n$. We denote the minimum degree of G by $\delta(G)$ and its maximum degree by $\Delta(G)$. A set of vertices is said to be *stable* or *independent* if no two of them are joined by an edge. The *stability number* $\alpha(G)$ is the cardinality of a largest stable set of vertices in G. For a positive integer p we define the parameter $\sigma_p(G) = \min\{\deg(x_1) + \cdots + \deg(x_p)\}$, where the minimum is taken over all stable sets $x_1, \ldots, x_p$ of p vertices in G.

The graph G is called *k-connected* if at least k vertices must be removed to disconnect G or if G is a complete graph $K_n, n \geq k + 1$.

By $\omega(G)$ we indicate the number of components in G. A connected, non-complete graph G is said to be *t-tough* if $t \cdot \omega(G - S) \le |S|$ for every cutset S . The *toughness* of a connected, non-complete graph G is

$$tough(G) = \min\left\{\frac{|S|}{\omega(G-S)} \mid S \subset V \text{ and } \omega(G-S) \ge 2\right\}.$$

A complete graph K_n may be defined to have toughness ∞.

The *composition* of two graphs G and H is the graph $G[H]$ with vertex set $V(G) \times V(H)$ and edge set $\{(u_1v_1, u_2v_2)|u_1u_2 \in E(G) \textit{ or } u_1 = u_2 \textit{ and } v_1v_2 \in E(H)\}$.

A *walk* in the graph G is a sequence $v_1e_1v_2e_2v_3 \ldots v_ie_iv_{i+1} \ldots v_{p-1}e_{p-1}v_p$ of vertices and edges such that each edge e_i has ends v_i and v_{i+1}; repetition is allowed for edges as well as for vertices. The walk is closed if $v_1 = v_p$ and open if $v_1 \ne v_p$. A *trail* is a walk where repetition is permitted for vertices but not for edges. Let kG denote the multigraph obtained from G by replacing each edge by k parallel edges. A k-*walk*, respectively a k-*trail,* of a graph G is a connected spanning subgraph W of $(2k)G$, respectively of G, such that the degree of each vertex is even and at most $2k$. (This definition implies closedness, but some authors permit a k-walk/k-trail to be open or closed). So W projected down to G is a closed spanning walk passing each vertex at least once and at most k times. Note that throughout this paper k-walks and k-trails by definition are understood to be spanning subgraphs. A k-walk of G is in particular $[1, 2k]$-factor of G, and a k-trail is a $[2, 2k]$-factor.

2. Connected [1,k]-Factors

Every connected graph has a spanning tree. The number of spanning trees in the complete graph K_n is n^{n-2} (Cayley [33]). A collection of proofs of tree counting formulas for K_n are given by Moon [112], J. Matoušek and J. Nesětřil [107].

For any connected graph G, the matrix-tree theorem (implied by Kirchhoff [83]) gives a formula for the number of spanning trees, using the $n \times n$ adjacency matrix A of G, as follows. Let $D = \{d_{ij}\}_{1 \le i,j \le n}$ be the diagonal matrix with $d_{ii} = deg_G(v_i)$ and $d_{ij} = 0$ for $i \ne j$. For any integers s and t, $1 \le s, t \le n$, the number of spanning trees in G equals $(-1)^{s+t}$ times the determinant of the matrix obtained by deleting the s'th row and the t'th column from $D - A$. As a consequence, for a d-regular graph on n vertices this number is $\frac{1}{n}\prod_{j=1}^{n-1}(d - \lambda_j)$ where $\lambda_0 = d, \lambda_1, ..., \lambda_{n-1}$ are the eigenvalues of the adjacency matrix A of the graph G. See for example Cvetković, Doob and Sachs [41, p. 39]. Y. Jin and C. Liu [76] have counted spanning trees in $K_{m,n}$. Factorizations of K_n and $K_{m,n}$ into spanning trees have been given in [67,69,140,51].

Many authors have been interested in the existence of edge disjoint spanning trees, mainly in relation to the existence of Eulerian subgraphs. Tutte [135] and independently Nash-Williams [114] gave a condition for a graph to have a factorization into connected factors.

Theorem 1 ([135]). *Let k be a positive integer. A connected graph G can be decomposed into k edge disjoint connected factors if and only if*

$$k(\omega(G-L)-1) \leq |L| \text{ for every } L \subseteq E(G).$$

It follows that if G is m-edge connected then G is decomposable into $\lfloor m/2 \rfloor$ connected factors and consequently G has $\lfloor m/2 \rfloor$ edge disjoint spanning trees. In fact, we can see that an m-edge connected graph has a factorization with $\lfloor m/2 \rfloor$ factors by proving that all subsets $L \subseteq E(G)$ satisfy $\lfloor m/2 \rfloor(\omega(G-L)-1) \leq |L|$: if $G-L$ is connected we have $\lfloor m/2 \rfloor(\omega(G-L)-1) = 0 \leq |L|$, and if $G-L$ is not connected then for each component C of $G-L$ there are at least m edges of G connecting C to other components of $G-L$, and therefore we have $m\omega(G-L) \leq 2|L|$ implying $\lfloor m/2 \rfloor(\omega(G-L)-1) \leq |L|$.

In 1965 J. Edmonds proved the existence, in any 3-edge connected graph, of three spanning trees with no edge common to all three trees (referred to in [71]). This result implies the existence of three even subgraphs whose union covers all edges of G.

Theorem 1 implies that each 4-edge connected graph has two disjoint spanning trees. This was used by Itai, Lipton, Papadimitrou and Rodeh [71] to obtain Jaeger's result [73,74] on the existence of a cycle double cover of the edges of a 4-edge connected graph G as follows: for each spanning tree T_i, they construct an even subgraph H_i containing $E(G) - E(T_i)$; so with two disjoint trees T_1 and T_2 they get a cover of all the edges by $H_1 \cup H_2$, as an edge of G not covered should be in both $E(T_1)$ and $E(T_2)$. It follows that $\{H_1, H_2, H_1 \triangle H_2\}$, where $H_1 \triangle H_2$ denotes the symmetric difference of H_1 and H_2, gives a cycle double cover of $E(G)$.

Consideration of network reliability has lead several authors to investigate graphs with many edge disjoint spanning trees, see for example Lonc [104], Rescigno [122].

For a fixed integer k we shall in the following consider *k-trees*, i.e. spanning trees with maximum degree at most k. A k-tree of a graph G is in particular a connected $[1,k]$-factor of G. Note, that the existence of a k-tree is equivalent to the existence of a connected $[1,k]$-factor.

Caro, Krasikov and Roditty [28] prove that the square of a connected graph contains a 3-tree. They also show how close a graph comes to having a k-tree, they prove that a connected graph either has a k-tree or it contains a tree of maximum degree at most k and order at least $k\delta(G)+1$.

Ellingham, Nam and Voss consider graphs of high connectivity. They show that

Theorem 2 ([49]). *Let $m \geq 1$ be an integer. Then every m-edge connected graph G has a spanning tree T such that*

$$deg_T(v) \leq 2 + \lceil \deg_G(v)/m \rceil$$

for every vertex v of G.

This result implies that every m-edge connected m-regular graph has a 3-tree.

2.1. k-Trees and Degrees

Dirac's condition $\delta(G) \geq n/2$ ensures the existence of a Hamiltonian cycle in G. Later, the same conclusion was drawn from a weaker hypothesis, namely in Ore's Theorem, which states that if $\sigma_2(G) \geq n$ holds for a graph with at least three vertices, then G has a Hamiltonian cycle. We observe that a graph G of order n has a Hamiltonian path if and only if the graph $G' = G + K_1$, obtained from G by adding one new vertex joined to every vertex of G, has a Hamiltonian cycle. Consequently the condition $\sigma_2(G) \geq n-1$ implies $\sigma_2(G') \geq n+1$ and hence, by Ore's theorem applied to G', we see that G has a Hamiltonian path, i.e. a 2-tree.

Extending this result, Win proved

Theorem 3 ([144]). *Let G be any graph of order n. If $\sigma_k(G) \geq n-1$ then G has a k-tree.*

The result is sharp: it is sufficient to consider the complete bipartite graph $K_{r+1,rk+1}$ where k is any integer at least 2.

Recently, Czygrinov, G. Fan, Hurlbert, Kierstead and Trotter [43] studied the family of graphs satisfying the hypothesis of Theorem 3 and having only k-trees which actually have their maximum valency equal to k. Such a graph G either has a k-tree which is a *caterpillar*, i.e. a tree containing a path such that all other vertices have degree one, or G is constructed by joining one vertex to every vertex in the disjoint union of k complete graphs.

Theorem 3 has been generalized by Aung and Kyaw [6] and by Kyaw [91]. To shorten the statement of Theorem 4 below we define: a $(k+1)$-*frame* in G is a set S of $k+1$ independent vertices such that $G-S$ is connected. By $\deg_G(S)$ we understand the number $\deg_G(S) = \sum_{x \in S} \deg_G(x)$. Finally, let $N_i(S) = \{x \in V(G) \mid |N(x) \cap S| = i\}$ denote the set of vertices having exactly i neighbours in S.

Theorem 4 ([91]). *Let G be a connected graph of order n. Let $k \geq 2$ be an integer. If*

$$\deg_G(S) + \sum_{i=2}^{k+1} (k-i)|N_i(S)| \geq n-1$$

for every $(k+1)$-frame S in G, then G has a k-tree.

R. Xu generalized an earlier result by H. Wang [141]:

Theorem 5 ([147]). *Let k and n be integers such that $0 \leq k \leq n-2$ and let G be a connected bipartite graph with partition classes V_1, V_2 of the same size $|V_1| = |V_2| = n$. If for every $u \in V_1, v \in V_2$ we have*

$$deg_G(u) + deg_G(v) \geq n-k,$$

then G has a connected $[1, k+2]$-factor.

2.2. k-Walks and k-Trees

There are relations between k-walks and k-trees. Jackson and Wormald proved

Lemma 1 ([72]). *Let k be a positive integer.*

(i) If G has a k-tree then G has a k-walk.
(ii) If G has a k-walk then G has a $(k+1)$-tree.
(iii) If G has a k-walk then G is $1/k$-tough.
(iv) G has a k-walk if and only if $G[K_k]$ has a Hamiltonian cycle.

To see (i) double each edge in the k-tree to obtain an Eulerian multigraph with degrees at most $2k$ producing a k-walk of G.

To see (ii), given a k-walk, we traverse an Euler tour starting from a vertex x and we delete each edge entering in a vertex previously visited, unless the edge has been used earlier. Each vertex will have one edge entering and at most k edges going out. In this way we get a spanning tree with maximum degree at most $k+1$.

(iii) is seen by observing that a k-walk of G meets a vertex of a cutset S on passing between two components of $G-S$, thus $\omega(G-S) \le k|S|$ which implies that $\frac{1}{k} \le \frac{|S|}{\omega(G-S)}$ for any cutset S and hence tough $(G) \le 1/k$.

Inspired by this last lemma and by Theorem 2 , Ellingham, Nam and Voss pose the following conjecture:

Conjecture. *Let $m \ge 1$ be an integer. If G is an m-edge connected m-regular graph then G has a 2-walk.*

They remark that the conjecture is true for $m \le 4$.

In topological graph theory there are several interesting results on k-trees and k-walks. Combined results by Barnette [9, 11], Z. Gao and Richter [60], Brunet, Ellingham, Gao, Metzlar and Richter [21] give the next theorem.

Theorem 6. *Every 3-connected graph which embeds in the plane, the projective plane, the torus or the Klein bottle has a 2-walk and consequently also a 3-tree.*

The *Euler characteristic* χ of a polyhedron is $n-e+f$, where n, e, f are respectively the number of vertices, edges and faces. For a connected, planar graph G we define $\chi(G) = n-e+f$, a number which always turns out to be 2.

A surface homeomorphic to a polyhedron has the same Euler characteristic as the polyhedron. The Euler characteristic of a surface is closely related to its *genus* g, the number of handles/crosscaps put on a sphere to obtain the orientable/nonorientable surface. A surface has Euler characteristic $\chi = 2-2g$ if it is orientable and $\chi = 2-g$ if it is nonorientable. The Euler characteristic is 2 for the plane, 1 for the sphere and the projective plane and 0 for the Möbius band and the Klein bottle. Sanders and Zhao [125] proved that a 3-connected graph embeddable on a surface of Euler characteristic $\chi \le -46$ has a $\lceil \frac{8-2\chi}{3} \rceil$-tree and a $\lceil \frac{6-2\chi}{3} \rceil$-walk.

2.3. Toughness and k-Trees

The notion of toughness is due to Chvátal. He conjectured [39] that there exists $t > 0$ such that every t-tough graph has a 1-walk, i.e., a Hamiltonian cycle. Necessarily $t > 9/4$, because Bauer, Broersma and Veldman [13] constructed, for every $\epsilon > 0$, an example of a $(9/4 - \epsilon)$-tough graph which is non Hamiltonian. This improves earlier results by Chvátal [39], Thomassen (see [17]), Enomoto, Jackson, Katerinis and Saito [52]. Chvátal's conjecture is still open.

For each $k \geq 2$ there exists a positive real number t such that every t-tough graph has a k-walk. From Win's Theorem 7 below it follows that for $k \geq 3$ a $1/(k-2)$-tough graph has a k-walk. More precisely, Win established the following.

Theorem 7 ([145]). *Let $k \geq 2$ be an integer. If G is a connected graph such that $\omega(G - S) \leq (k-2) \mid S \mid +2$ holds for each subset S of $V(G)$, then G has a k-tree.*

Theorem 7 is sharp, as mentionned by Win, it is sufficient to consider $k+1$ copies of a complete graph and an extra vertex x_0 joined to all the other vertices. In that graph $\omega(G - S) \leq (k-2) \mid S \mid +3$ holds for each subset S and G has no k-tree.

Ellingham and Zha [48] gave a new proof for this result. In G let H be an induced subgraph having a k-tree of maximal order.

To obtain a contradiction, assume $H \neq G$. Through a number of steps they construct a subset S of $V(H)$ and a k-tree T of H, such that

(i) $\deg_T(v) = k$ for every vertex of S, and
(ii) every edge between H and $G - H$ has an end in S,
(iii) each component of $T - S$ has the same vertex set as the corresponding component of $H - S$, i.e. there is no H-edge between distinct components of $T - S$; in particular, $\omega(H - S) = \omega(T - S)$.

Consequently $\omega(G - S) > \omega(H - S)$ because G is connected and $H \neq G$ implies existence of an edge uv, $u \in V(G - H), v \in V(H)$; by (ii) the vertex v is necessarily in S. Let $S = \{x_1, x_2, \ldots, x_s\}$ and $S_i = \{x_1, x_2, \ldots, x_i\}$, $1 \leq i \leq s$. Deleting first x_1 from T creates k components. Next, deleting in succession x_i from T creates $k - 1 - |N(x_i) \cap S_i|$ additional components. The total number of components in $T - S$ is $k + \sum_{i=2}^{s}((k-1) - |N(x_i) \cap S_{i-1}|)$. At worst S spans a tree in which all $|S| - 1$ edges are counted. Therefore $\omega(T - S) \geq k + (|S| - 1)(k-1) - (|S| - 1) = (k-2)|S| + 2$. As $\omega(G - S) > \omega(H - S) = \omega(T - S) \geq (k-2)|S| + 2$, this contradicts the inequality of Theorem 7. Therefore $G = H$ and G has a k-tree.

Every graph that has a k-tree must be $1/k$-tough. Jackson and Wormald [72] conjectured that for $k \geq 2$ every $1/(k-1)$-tough graph has a k-walk.

For $k = 2$ Ellingham and Zha [48] pointed out that there are graphs which are 2/3-tough and have no 2-walk. For $k \geq 3$, from examples of Jackson and Wormald [72], Ellingham and Zha [48] it follows that there are graphs with asymptotic toughness $\frac{8k+1}{4k(2k-1)}$, i.e. toughness about $1/(k - 5/8)$ and with no

k-walk. This brings the gap to the constant of Jackson and Wormald's conjecture down to the order of $1/k^2$.

For $k = 2$, the condition of Win's Theorem 7 demands more than necessary for G to have a Hamiltonian path. His condition is equivalent to $\alpha(G) \leq 2$, and certainly it would suffice to demand $\sigma_2(G) \geq n - 1$. Ellingham, Nam and Voss generalize Win's result as follows:

Theorem 8 ([49]). *Let G be a connected graph and let h be a positive integer-valued function on V(G). Suppose that each* $S \subset V(G)$ *satisfies* $\omega(G - S) \leq \sum_{v \in S} h(v) - 2|S| + 2$. *Then G has a spanning tree T with* $\deg_T(v) \leq h(v)$ *for every vertex v of G.*

Finally, using other generalisations of Win's theorem and results on 2-factors, Ellingham and X. Zha [48] also proved that every 4-tough graph has a 2-walk. Nevertheless, no result is known on toughness and 2-trees. This problem is closely related to the conjecture of Chvátal.

2.4. $K_{1,\mathrm{h}}$-*Free Graphs*

There are several results for classes of graphs with forbidden subgraphs. From a result by Tokuda [129], we know that every connected $K_{1,h}$-free graph has a $[1, h]$-factor, not necessarily connected, for every $h \geq 3$; and by Caro, Krasikov and Roditty [29], that such a graph has in fact a h-tree. Jackson and Wormald generalized these results as follows

Theorem 9 ([72]). *Let* $h \geq 3$ *be an integer. Every connected,* $K_{1,h}$*-free graph has a* $(h - 1)$*-walk and hence a h-tree.*

By Lemma 1(ii) it suffices to prove that G has a $(h - 1)$-walk. The authors in [72] note that G has a $\Delta(G)$-walk, because $2G$ is Eulerian. Let W be a $\Delta(G)$-walk for which $|E(W)|$ is minimum. The walk W is, in fact, a $(h - 1)$-walk, for assume there exists v such that $\deg_G(v) = 2r \geq 2h$. Then W contains edges vu_1, $vu_2, \ldots, vu_r$, where $u_1, \ldots, u_r$ can be chosen to belong to distinct subcycles of W, so that $W \setminus \{vu_1, vu_2, \ldots, vu_r\}$ is connected. If $u_i = u_j$ for some $i \neq j$, we form $W' = W \setminus \{vu_i, vu_j\}$, otherwise $\alpha(N(v)) \leq h - 1 < r$ implies existence of i, j such that $u_iu_j \in E(G)$ and we form $W' = (W \setminus \{vu_i, vu_j\}) \cup u_iu_j$. In both cases we have obtained a contradiction to the minimality of $|E(W)|$.

Let $\langle N(v) \rangle_G$ denote the graph spanned in G by the neighbours of v and let us say that G is *locally connected* if $\langle N(v) \rangle_G$ is connected for each vertex v of G. A strengthening of the hypothesis in Theorem 9 now gives:

Theorem 10 ([72]). *Let* $h \geq 3$ *be an integer. If a graph is connected,* $K_{1,h}$*-free and also is locally connected, then it has a* $(h - 2)$-walk and hence also an $(h - 1)$-tree.

Theorem 11 ([72]). *Let* $j \geq 1$ *and* $h \geq 3$ *be integers such that j divides* $h - 1$. Then every j-connected and $K_{1,h}$*-free graph has a* $(2 + \frac{h-1}{j})$*-walk and hence also a* $(3 + \frac{h-1}{j})$*-tree.*

A. Kyaw presented in [92] a counterexample to the following conjecture of B. Jackson and N.C. Wormald [72, Conjecture 4.3], "Let $j \geq 1$ and $h \geq 3$ be integers such that $j+1$ divides $h-2$. If G is $K_{1,h}$-free, connected and locally j-connected then G has a $(\frac{h-2}{j+1}+1)$-tree". It remains to find the sharp value of k such that the precedent hypotheses imply the existence of a k-tree.

We conclude this section with an open problem.

Question of B. Jackson. Given integers s, κ, determine a sharp value of k such that any graph which is $K_{1,s}$-free, connected and locally κ-connected has a k-walk.

2.5. *Stability and k-Trees*

Win [143] states that

Every κ-connected graph with stability number at most $\kappa + c$ contains a spanning tree with no more than $c+1$ terminal vertices.

In this way he obtains the existence of an $\alpha(G) - \kappa + 1$-tree in every κ-connected graph and resolved a conjecture of Las Vergnas.

A classical result is the Chvátal-Erdös theorem,

Theorem 12 ([40]). *If the stability number α of a κ-connected graph is at most κ, then the graph has a Hamiltonian cycle.*

The theorem below follows from Lemma 1, stating that G has an h-walk if and only if $G[K_h]$ has a Hamiltonian cycle, and the Chvátal-Erdös theorem, applied to the graph $G[K_h]$ with $h = \lceil \alpha(G)/\kappa \rceil$.

Theorem 13 ([72]). *If G is κ-connected then G has an $\lceil \alpha(G)/\kappa \rceil$-walk.*

This, in turn, by Lemma 1(ii) implies that G has a $(\lceil \alpha(G)/\kappa \rceil + 1)$-tree.

A related result of Neumann-Lara and Rivera-Campo is slightly better for existence of k-trees since the maximum degree may be smaller for the tree in Theorem 14 than in Theorem 13.

Theorem 14 ([115]). *Let $\kappa \geq 1$ and $r \geq 2$ be two integers. If G is a κ-connected graph such that $\alpha(G) \leq 1 + (r-1)\kappa$ then G has an r-tree.*

We note that the complete graph $K_{k,1+k(s-1)}$ satisfies the hypothesis of Theorem 14 but has no $s-1$ tree.

2.6. *Matching and k-Trees*

Rivera-Campo considers extension of matchings into k-trees in graphs of low stability.

Theorem 15 ([123]). *Let κ be a positive integer and let G be a κ-connected graph having a perfect matching M. If $\alpha(G) \leq 1 + \frac{3\kappa}{2}$ then M can be extended to a 3-tree.*

He claims that, with the same proof, this last result can be extended as follows: *If G is κ-connected and has a perfect matching M such that $\alpha(G) \leq 1 + \frac{(2r-3)\kappa}{2}$ for some integer $r \geq 3$, then M can be extended to an r-tree.*

Recently Ellingham, Nam and Voss obtained extensions of matchings in tough graphs.

Theorem 16 ([49]). *Let G be a t-tough graph having a perfect matching M. Then there exists a $\left(2 + \left\lceil \frac{1}{t} \right\rceil\right)$-tree containing M.*

3. Connected $[2,k]$-Factors

An *even* factor is a factor in which all degrees are even, positive integers and an *odd* factor has all degrees odd.

3.1. Even Factors

By Fleischner [57], we know that

Theorem 17([57]). *If G is a 2-edge connected graph, without vertices of degree 2, then G has an even factor.*

Neither of the hypotheses in Theorem 17 can be weakened. Consider first, e.g., the graph composed by 3 copies of a complete graph and an extra vertex joined to exactly one vertex in each copy. A second example is the Petersen graph where each edge ab is replaced by three paths of length two joining a and b. None of these two graphs have an even factor.

Eulerian subgraphs are connected, even, spanning subgraphs, i.e., connected even $[2, n-1]$-factors, and Hamiltonian cycles are connected $[2,2]$-factors. On Hamiltonian cycles there have been several surveys, to mention but two, Bermond [17] (1978) and an up to date survey by Gould [63] (2003). Catlin [31] surveyed *supereulerian* graphs, i.e., graphs containing a connected even factor. Zelinka [153] showed that for every integer r there exists a 2-edge connected graph G, of order n, with no Eulerian subgraph covering more than n/r vertices of G. One important result about supereulerian graphs is the following.

Theorem 18 ([74]). *Every 4-edge connected graph is supereulerian.*

We have in Section 2 mentioned relations from spanning Eulerian subgraphs to spanning trees, and to the cycle double cover conjecture. The precedent Theorem 18 of Jaeger has been generalised by Catlin, Z.-Y. Han, H.-J. Lai [30]. They studied graphs which by addition of at most two edges would have two edge disjoint spanning trees and they gave a characterization for such graphs to be supereulerian. Then they proved that

Any 3-edge connected graph with at most 9 edge cuts of size 3 is supereulerian by using suitable contractions and thus reducing the graph to one which is at most 2 edges short of having two edge disjoint spanning trees.

A *wheel* is obtained from a cycle $v_1, ..., v_n$ and an extra vertex v joined to each of $v_1, ..., v_n$. A *rim-subdivision* of a wheel is obtained by replacing each edge v_iv_{i+1} of the cycle of the wheel by a path $v_iv_i'v_{i+1}$. We call a graph H a *minor* of G if H is isomorphic to the contraction image of a subgraph of G, and H is called an *induced minor* if it is isomorphic to the contraction image of an induced subgraph of G.

For graphs of lower edge connectivity, H.-J. Lai gives a sufficient condition for G to be supereulerian:

Theorem 19 ([94]). *Let G be a* 2*-edge connected graph. The following are equivalent:*

(i) *Every* 2*-edge connected induced subgraph of G is supereulerian.*
(ii) *G has no induced minor isomorphic to a rim-subdivision of a wheel.*

There are many other sufficient conditions for a graph to be supereulerian. Some conditions are in terms of forbidden subgraphs or require that each edge is in a short cycle or they are degree conditions. For example H.-J. Lai [93] established that if the minimum degree of 2-edge-connected triangle free graph is at least $n/10$ and $n \geq 30$ then the graph is supereulerian. Many authors have been interested in supereulerian graphs because the line graph of a supereulerian graph is Hamiltonian.

Spanning eulerian subgraphs with no upper bound on the number of times a vertex is used have been considered by several authors including Lesniak-Foster and Williamson [96], Benhocine, Clark, Köhler, and Veldman [15], Catlin [32], Z.H. Chen [37], Ellingham, X. Zha and Y. Zhang [50].

Theorem 20 ([15]). *Let G be a* 2*-edge connected graph of order* $n \geq 3$*. If* $\sigma_2(G) \geq (2n+3)/3$ *then G has a closed spanning trail.*

For Eulerian graphs see the work of Jaeger [73,74], the book of Fleischner [55,56], the book of C.Q. Zhang [154], the survey of Lesniak and Oellermann [95] or that of Catlin [31].

3.2. Even [a,b]-Factors

Kouider and Vestergaard obtained various sufficient conditions for a graph to have an even $[a,b]$-factor. One of them is

Theorem 21 ([90, 89]). *Let a and b be even integers such that* $2 \leq a \leq b$ *and let G be a 2-edge connected graph of order n.*

1) If $a \geq 4$ *and* $n \geq \dfrac{(a+b)^2}{b}$, *and* $\delta(G) \geq \dfrac{an}{a+b} + \frac{a}{2}$, *then G has an even* $[a,b]$*-factor.*
2) If $a=2$ *and* $n \geq 3$, $\delta(G) \geq max\{3, \dfrac{2n}{b+2}\}$, *then G has an even* $[2,b]$*-factor.*

Notice, that for $a = 2$ and $b = n - 2, or, n - 1$ we get as a corollary Fleishner's Theorem 17 cited above.

Several authors have considered *k-trails*. Broersma, Kriesell and Ryjácek proved [20] that every 4-connected claw-free graph has a 2-trail or in other words an even $[2, 4]$-factor. A *triangulated annulus* is a planar graph containing two circuits which each bound a face, while all other faces are triangles. Gao and Wormald [61] proved that a triangulated annulus has a 4-trail, i.e., it has a $[2, 8]$-factor. They derive from this that all triangulations in the projective plane, the torus and the Klein bottle have (closed) 4-trails. A graph with a (closed) k-trail contains an even factor and is thus by definition supereulerian.

X. Zha and Y. Zhang show the following.

Theorem 22 ([50]). *Let G be a connected graph of order n. If $\sigma_3(G) \geq n$, then G has either a closed 2-trail or a Hamiltonian path.*

We note that a closed 2-trail is a connected even $[2, 4]$-factor.

Theorem 22 is near to being sharp. Consider 3 copies of the same complete graph with exactly one vertex in common, then $\sigma_3(G) = n - 4$, and G has no closed 2-trail and no Hamiltonian path.

3.3. Degrees

It seems that Kano posed the following conjecture.

Conjecture: Let k be an integer and G a 2-edge connected graph of order n with $\delta(G) \geq 2$ and $n \geq k + 3$. If $\sigma_2(G) \geq \frac{4n}{k+2}$ then G has a 2-edge-connected $[2, k]$-factor.

As support he proved that such a graph has a connected $[2, k]$-factor. However, several authors (Y. Li, M. Cai [100] and R. Xu [148]) have given counterexamples to this conjecture but suggested that Kano's conjecture might hold if G is required to be 2-connected. Kouider and Maheo proved

Theorem 23 ([86]). *If G is a 2-edge connected graph of order n; $k \geq 2$; $n \geq k + 3$; $\sigma_2(G) \geq 4n/(k + 2)$, then if k is even, G has a 2-edge connected $[2, k]$-factor; if k is odd, G has a 2-edge connected $[2, k + 1]$-factor.*

Furthermore, if G is 2-connected, under the same hypothesis on the degrees they prove the existence of a 2-connected $[2, k]$-factor.

Theorem 23 is sharp. For $k = 2h - 1$, consider h copies of a complete graph and an extra vertex x joined to all the other vertices. In a 2-edge connected factor, the vertex x must have degree at least $2h = k + 1$. This graph satisfies the conditions of the theorem, but it has no 2-edge connected $[2, k]$-factor.

A related result is R. Xu's theorem [148]: *If G has connectivity $\kappa \geq 2$, order $n \geq 10\kappa$ and $\sigma_2(G) \geq 4n/5$, then G has a 2-connected $[2, 3]$-factor.*

3.4. Stability

Kouider [85] has shown that any κ-connected graph G has a covering of its vertices by at most $\lceil \alpha(G)/\kappa \rceil$ elementary cycles. Brandt has deduced

Theorem 24 ([19]). *Let $b \geq 2$ be an even integer, and $\kappa \geq 2$. If G is κ-connected and $\alpha(G) \leq \kappa b/2$, then G has a 2-connected $[2, b]$-factor.*

This proves, in other words, the existence of a 2-connected $[2, 2\lceil\alpha(G)/\kappa\rceil]$-factor, while Theorem 13 only gives a connected $[1, 2\lceil\alpha(G)/\kappa\rceil]$-factor. Brandt's result is sharp, it cannot be improved for the complete bipartite graphs.

3.5. Toughness

Chvátal conjectured that every k-tough graph on n vertices, with $n \geq k+1$ and kn even, has a k-factor. This was established in 1985 by Enomoto, Jackson, Katerinis and Saito [52]. They furthermore showed that the conjecture is best possible, because for any $k \geq 1$ and for any $\epsilon > 0$ there exists a (k-ϵ)-tough graph with n vertices, $n \geq k+1$ and kn even but with no k-factor.

There is a sufficient condition due to C. Chen, for existence of factors, which may not necessarily be connected,

Theorem 25 ([34]). *Let $b \geq 3$ be an integer, and let G be a graph of order $n \geq 3$.*

$$\textit{If } tough(G) \geq (1 + \frac{1}{b}) \textit{then } G \textit{ has } a[2, b] - \textit{factor}.$$

The author shows that the hypothesis on toughness may not be weakened. Ellingham, Voss and Nam extended this last result for connected factors as follows:

Theorem 26 ([49]). *If $b \geq 4, n \geq 3$ and tough(G) $\geq 1 + \frac{1}{b-2}$ then G has a connected $[2, b]$-factor that contains a $[2, b-2]$-factor.*

Using results from Enomoto, Jackson, Katerinis and Saito [52] on toughness and k-factors, they also proved that if $tough(G) \geq 4$ then G has a connected $[2, 3]$-factor. Generalizations from $[2, b]$-factors to $[a, b]$-factors are given in Theorems 48, 49 later.

3.6. Special Classes of Graphs

Thomassen conjectured [126] that every 4-connected line graph is Hamiltonian. That is equivalent to a seemingly stronger conjecture by Matthews and Sumner [111] that every 4-connected claw-free graph is Hamiltonian. Broersma, Kriesell and Ryjácek proved [20] that every 7-connected claw-free graph is Hamiltonian and that every 4-connected claw-free graph is Hamiltonian if it contains no induced hourglass, i.e., two triangles with exactly one common vertex. Furthermore, they proved that every 4-connected claw-free graph has a connected $[2, 4]$-factor in which each vertex has even degree, i.e., a 2-trail.

G. Li and Z. Liu [98] proved that if G is 2-connected and claw-free then G has a connected $[2, 3]$-factor.

Now, we recall some results on topological graphs and $[2, b]$-factors. Tutte proved [136,134] that every 4-connected planar graph is Hamiltonian, i.e., it has a 2-factor. Not all 3-connected planar graphs are Hamiltonian, so relaxations on Hamiltonicity have been considered. Barnette [12] proved that every 3-connected planar graph has a 3-tree. Enomoto, Iida and Ota proved [53] that every 3-connected, planar graph G with $\delta(G) \geq 4$ has a connected $[2,3]$-factor. Barnette proved [10] that every 3-connected, planar graph has a 2-connected $[2,15]$-factor. Later this was strengthened by Z. Gao who proved [59] that every 3-connected graph embeddable in the plane, the projective plane, the torus or the Klein bottle has a 2-connected $[2,6]$-factor. Also, we know by Sanders and Y. Zhao [124] that a 3-connected graph of Euler characteristic χ has a 2-connected $[2, 10-2\chi]$-factor.

4. (g,f)-Factors

Let us state three factor theorems which are fundamental. We shall consider ordered pairs of disjoint subsets X, Y of $V(G)$. We write $\deg_G(X) = \sum\{\deg_G(x) | x \in X\}$ and for a vertex function f we write $f(X) = \sum\{f(x) | x \in X\}$. By $e(X, Y)$ we denote the number of edges having one end in X and one end in Y. For a graph G and $S \subseteq V(G)$ we let $o(G-S)$ denote the number of components in $G-S$ with an odd number of vertices.

Theorem 27 (Tutte's 1-factor theorem [139]). *A graph G has a 1-factor if and only if $o(G-S) \leq |S|$ for all subsets S of V(G).*

Theorem 28 (Tutte's *f*-factor theorem [138,137]). *Let G be a graph and f be a nonnegative integer valued function defined on V(G). Let X,Y be disjoint subsets of V(G). A component C of $G-(X \cup Y)$ is called odd if $f(C) + e(C, Y) \equiv 1 (\mathrm{mod}\ 2)$. Let h(X,Y) be the number of odd components in $G-(X \cup Y)$. Then G has an f-factor if and only if $h(X, Y) \leq f(X) - f(Y) + \deg_{G-X}(Y)$ for all ordered pairs X,Y of disjoint subsets of V(G).*

Theorem 29 (Lovász's (g,f)-factor theorem [106]). *Let G be a graph and g,f nonnegative integer valued functions defined on V(G) satisfying $g(v) \leq f(v)$ for all v in V(G). Let X,Y be disjoint subsets of V(G). A component C of $G-(X \cup Y)$ is called odd if g(v)=f(v) for all v in V(C) and $e(C, Y) + f(C) \equiv 1 (\mathrm{mod}\ 2)$. Let $h(X, Y)$ be the number of odd components in $G-(X \cup Y)$. Then G has a (g,f)-factor if and only if $h(X, Y) \leq f(X) - g(Y) + deg_{G-X}(Y)$ for all ordered pairs X,Y.*

As a corollary, we have

Theorem 30. *For integers $1 \leq a < b$ the graph G has an $[a, b]$-factor if and only if*

$$b|X| - a|Y| + \sum_{v \in Y} deg_{G \setminus X}(v) \geq 0,$$

for all pairs of disjoint subsets X,Y of V(G).

By slightly changing the hypotheses, Lovász obtained

Theorem 31 (Lovász's parity factor theorem). *Let G be a graph and g,f nonnegative integer valued functions defined on V(G) satisfying* $g(v) \leq f(v)$ *and* $g(v) \equiv f(v)$ $(mod\ 2)$ *for all v in V(G). Let X,Y be disjoint subsets of V(G). A component C of* $G - (X \cup Y)$ *is called odd if* $e(C, Y) + f(C) \equiv 1$ $(mod\ 2)$*. Let h(X,Y) be the number of odd components in* $G - (X \cup Y)$*. Then G has a (g,f)-factor F with* $deg_F(v) \equiv g(v)$ *for all v in V(G) if and only if* $h(X, Y) \leq f(X) - g(Y) + deg_{G-X}(Y)$ *for all ordered pairs X,Y.*

Many authors have given sufficient conditions for a graph to have a (g, f)-factor and in several proofs Theorems 27–29 are used. Egawa and Kano [45] proved that $g(x) < f(x)$, $\dfrac{g(x)}{\deg(x)} \leq \dfrac{g(y)}{\deg(y)}$ for all adjacent vertices x, y in $V(G)$, is sufficient. Kano, Saito [79] proved that the existence of a real number θ, $0 < \theta \leq 1$, such that each vertex x of G satisfies $g(x) < f(x)$ and $g(x) \leq \theta\ \deg_G(x) \leq f(x)$, is sufficient.

Niessen [116] gave a sufficient condition for G to have an h-factor for any function h satisfying $g \leq h \leq f$.

The concept of *connected* (g, f)-factors is attributed to Kano [81]. This topic is closely related to the Hamiltonian cycle problem, as a connected 2-factor is obviously a Hamiltonian cycle. Existence of a connected $[a, b]$-factor, or of a connected (g, f)-factor, is an NP-complete problem, see for example the classic book of Garey and Johnson [62] and [7, 42] for an updated reference. Kano [81] proposed many conjectures and problems on the topic of connected factors.

4.1. Ore-type Conditions

Kouider and Maheo proved

Theorem 32 ([87]). *Let G be a connected graph of orden n and minimum degree* δ*. Let a and b be integers such that* $2a \leq b$*. Suppose that* $n \geq \dfrac{(a+b)(a+b-1)}{b}$ *and* $\delta \geq \frac{n}{1+\lfloor \frac{b}{a} \rfloor}$*. Then G has a connected* $[a, b]$*-factor.*

Note that the condition $\delta \geq \frac{n}{1+\frac{b}{a}}$ is necessary, because if the complete bipartite graph $K_{\delta, n-\delta}$ has an $[a, b]$ factor then $a(n - \delta) \leq b\delta$.

For 2-edge connected graphs having large order Matsuda has strengthened Theorems 23 and 32 as follows

Theorem 33 ([110]). *Let* $a \geq 2$ *and* $t \geq 2$ *be integers and G a 2-edge connected graph of order* $|G| \geq 2(t+1)((a-2)t + a) + t - 1$*. Suppose that* $\delta(G) \geq a$ *and* $\sigma_2(G) \geq 2|G|/(1+t)$*. Then G has an* $[a, at]$*-factor with the property that it contains a 2-edge connected* $[2, 2t]$-factor.

Matsuda's condition cannot be weakened to $\sigma_2(G) \geq 2|G|/(1+t) - 1$ as is seen from the following example.

Let G be a complete bipartite graph with partite sets A and B such that $|A| = m$ and $|B| = tm + 1$, where m is any positive integer. Then it follows that $|G| = |A| + |B| = (1+t)m + 1$, which for sufficiently large m satisfies the order condition, and

$$\frac{2|G|}{1+t} > \deg_G(x) + \deg_G(y) = 2m > \frac{2|G|}{1+t} - 1$$

for two nonadjacent vertices x and y in B. However, G has no $[a, at]$-factor, since $at|A| < a|B|$.

Nishimura proved

Theorem 34 ([118]). *Let $k \geq 2$ be an integer and let G be a connected graph of order n such that $n \geq 4k-3$, kn is even and $\delta(G) \geq k$. If $max\{deg(u), \deg(v)\} \geq n/2$ holds for every pair of independent vertices, then G has a k-factor.*

The theorem is sharp, the hypothesis on the degree cannot be weakened as can be seen by considering an unbalanced bipartite graph. The hypothesis on the order n cannot be weakened either, Nishimura considers the join of a complete graph K_{2k-4} with the disjoint union of a vertex and the cycle C_{2k-1}. This graph satisfies the hypothesis on the degrees but it is of order $4k-4$ and has no k-factor.

This factor is not necessarily connected, but Kano [81] observed that Ore's condition holds, hence G has a Hamiltonian cycle. Combined with the k-factor that gives a connected $[k, k+2]$-factor in G.

Kano has raised the following problem.

Problem: Find sufficient conditions for a graph to have a connected $[k, k+1]$-factor.

Answers to Kano's problem have been given in Theorems 35, 38, 36 and 50.

B. Wei and Y. Zhu proved

Theorem 35 ([142]). *Let $k \geq 2$ be an integer and let G be a graph of order n such that $n \geq 8k-4$ and kn is even. If $\delta(G) \geq n/2$, then G has a 2-connected k-factor containing a Hamiltonian cycle.*

The condition $\delta(G) \geq n/2$ ensures Hamiltonicity, but it does not ensure existence of a k-factor containing a given Hamiltonian cycle. For, let $n \geq 6$ be even, $k \geq 3$ and set $m = n/2$, form the cycle $C = v_1 v_2 \ldots v_m$ and the path $P = v_{m+1} v_{m+2} \ldots v_n$. Then the join $G = C + P$ obtained by adding all edges between C and P satisfies $\delta(G) \geq n/2$ but G has no k-factor containing the Hamiltonian cycle $v_1 v_2 \ldots v_n$.

Similarly, in Theorem 36 by M. Cai, Y. Li and Kano below, we observe that Ore's condition, $\sigma_2 \geq n$, implies Hamiltonicity.

Theorem 36 ([22,23]). *Let $k \geq 2$ be an integer and G a graph of order n with $\delta(G) \geq k$ and $\sigma_2(G) \geq n$. If $n \geq 8k-16$ for even n and $n \geq 6k-13$ for odd n, then for any given Hamiltonian cycle C, G has a $[k, k+1]$-factor containing C.*

Matsuda [108] obtained this last result with the condition $\sigma_2 \geq n$ replaced by the weaker condition $\max\{\deg_G(x), \deg_G(y)\} \geq n/2$ for nonadjacent x and y.

For $[a, b]$-factors Matsuda has proved an analogous result

Theorem 37 ([109]). *Let $2 \le a < b$ be integers and let G be a Hamiltonian graph of order $n \ge \frac{(a+b-4)(2a+b-6)}{b-2}$. Suppose that $\delta(G) \ge a$ and* $\max\{\deg_G(x), \deg_G(y)\} \ge \frac{(a-2)n}{a+b-4} + 2$ *for each pair of nonadjacent vertices x and y of V(G). Then G has an [a,b]-factor containing a given Hamiltonian cycle.*

Let us mention an unpublished result of Y. Li:

Theorem 38 ([97]). *Let $k \ge 2$ be an integer and G a graph of order n with $\delta(G) \ge n/2$ and $\delta(G-e) < n/2$ for all e in E(G). If $n \ge 4k+2$ then for any Hamiltonian cycle C of G there exists a $[k, k+1]$-factor containing C.*

Note that the graphs considered in that result are not necessarily regular. Let us consider for example the join of a stable set of p, $(p < n/4)$, vertices and an $(n/2 - p)$-regular Hamiltonian graph.

We state two conjectures about factors containing Hamiltonian cycles.

Conjecture (Y. Zhu, Z. Liu, M. Cai). *Let $k \ge 2$ be an integer, and G a 2-connected graph of order n with $n \ge 8k$, kn even and $\delta(G) \ge k$. If for any two nonadjacent vertices u and v of G* $\max\{\deg_G(u), \deg_G(v)\} \ge n/2$, *then G has a k-factor containing a Hamiltonian cycle.*

Conjecture (Y. Zhu, Z. Liu, M. Cai). *Let $k \ge 2$ be an integer, and G a 2-connected graph of order n with $n \ge 8k$, kn even and $\delta(G) \ge k$. If $|N_G(u) \cup N_G(v)| \ge \frac{2n-3}{3}$ holds for any two nonadjacent vertices u and v of G then G has a k-factor containing a Hamiltonian cycle.*

We get a $2k$-factor if we have for example a family of k edge-disjoint Hamiltonian cycles. Furthermore this factor is $2k$-edge connected. In 1971 Nash-Williams [113] established a sufficient condition, involving the minimum degree and the order of the graph, for the existence of such a factor. Several authors considered extensions of this result with conditions involving σ_2 instead of the minimum degree, but with different bounds on the order of the graph (Faudree, Rousseau and Schelp [54], H. Li [102], Egawa [46]).

Theorem 39 ([102]). *If G is a simple graph with $n \ge 20$ and minimum degree δ with $n \ge 2\delta^2$ and $\sigma_2 \ge n$, then G contains at least $\lfloor(\delta-1)/2\rfloor$ Hamiltonian cycles.*

Theorem 40 ([103]). *If G is a simple graph with $n \ge 20$ and minimum degree $\delta \ge 5$ and $\sigma_2 \ge n$, then G contains at least 2 edge disjoint Hamiltonian cycles.*

Theorem 41 ([46]). *Let k and n be integers such that $2 \le k \le \frac{n}{44} + 1$ and let G be a graph of order n with $\delta(G) \ge 4k-2$, $\sigma_2(G) \ge n$. Then G has k edge disjoint Hamiltonian cycles.*

Berman [16] has considered the parity of the number of connected f-factors avoiding or containing special edge sets.

4.2. $K_{1,h}$ -Free Graphs

G. Li, B. Zhu and C. Chen [99] proved that every 2-connected claw-free graph having a k-factor, $k \geq 2$, also has a connected $[k, k+1]$-factor.

Egawa and Ota [47] found that a connected claw-free graph with $\delta \geq (9k+12)/8$ and kn even has a k-factor.

B. Xu, Z. Liu and Tokuda prove

Theorem 42 ([149]). *Let $h \geq 3$ be an integer and let G be a connected $K_{1,h}$-free graph. Let g,f be maps from $V(G)$ into the nonnegative integers satisfying $g(v) \leq f(v)$, for each v in $V(G)$. If G has a (g,f)-factor then G contains a connected $(g, f+h-1)$-factor.*

For constant functions f and g, the result above is refined for $h = 3$ by B. Xu and Z. Liu and for $h \geq 4$ by Tokuda. The existence of an $[a, b]$-factor guarantees, also, the existence of a connected $[a, b]$-factor if b is big enough. More precisely,

Theorem 43 ([150,128]). *For integers h, a, b satisfying $h \geq 3, a \geq 1$ and $b \geq a(h-2)+2$, if G is connected, $K_{1,h}$-free and has an $[a, b]$-factor, then G has a connected* [a,b]-*factor.*

4.3. Stability

Egawa and Enomoto [44], and independently Nishimura [117] gave a sufficient condition for a graph to contain a k-factor, not necessarily connected, as a function of the stability number of the graph and the connectivity. This leads to the question of whether an integer $f(a, b, \kappa)$ exists such that $\alpha(G) \leq f(a, b, \kappa)$ guarantees existence of a *connected* $[a, b]$-factor in G.

Recently, Kouider and Lonc proved:

Theorem 44 ([88]). *Let G be a κ-connected graph, $a \geq 2$, $b \geq a+3$ and $(a, b) \neq (2,5), (2,7), (3,6), (4,7)$. If $\delta(G) \geq \dfrac{10(a+1)\kappa}{9(a-1)} + a$ and*

$$\alpha(G) \leq \begin{cases} \dfrac{4\kappa b}{(a+1)^2} & \text{for } a \text{ odd}, \\ \dfrac{4\kappa b}{a(a+2)} & \text{for } a \text{ even}, \end{cases}$$

then G has a connected $[a, b]$-factor.

Theorem 45 ([88]). *Let $b \geq a+1$ and let one of the following two conditions be satisfied*

(i) $a \geq 4$ and $(a, b) \neq (4, 7)$ or

(ii) $a = 3$ *and* b *is divisible by* 4.

If G *is a* κ*-connected graph such that* $\kappa \geq 2$, $\delta(G) \geq 2\kappa + a$, *and* $\alpha(G) \leq \frac{4b\kappa}{(a+1)^2}$, *then* G *has a* 2*-connected* $[a, b]$*-factor.*

4.4. Toughness

Win's theorem on trees and toughness, Theorem 7, combined with Theorem 53 below implies Theorem 42 of B. Xu, Z. Liu and Tokuda.

In fact Ellingham, Nam and Voss [Lemma 6,49] prove that every connected $K_{1,h}$-free graph satisfies the condition of Theorem 7 by Win, when k in Theorem 7 is taken to be equal to h. Thus G has an h-tree and a (g, f)-factor so that application of Theorem 53 with $f' = h$ produces a connected $(g, f + h - 1)$-factor in G as wanted in Theorem 42. Theorem 7 together with Theorem 46(ii) below also implies Theorem 42. B. Xu and Z. Liu observed [151] that any connected $1/(h-2)$-tough graph having a (g, f)-factor, $1 \leq g(x) \leq f(x)$ for all x in $V(G)$ has a connected $(g, f + h - 1)$-factor.

Let h be any integer positive valued function on the vertices of G. In [49] Ellingham, Nam and Voss consider the extension of a (g, f)-factor into a connected $(g, f + h)$-factor. Thus they generalize Win's theorem by giving four different sufficient conditions. For greater clarity we reformulate their result below for the case of a constant function h.

Theorem 46 ([49]). *Let* $h \geq 1$ *be an integer. Let* G *be a connected graph and* g, f *be positive integer-valued functions defined on* $V(G)$*. Suppose that* G *has a* (g, f)*-factor* F *in which each component has at least* c *vertices. Then* F *extends into a connected* $(g, f + h)$*-factor of* G *if for every nonempty subset* S *of* $V(G)$ *at least one of the following properties hold:*

(i) $\omega(G - S) < (h-2)|S| + 3$; *or,*
(ii) $c \geq 2$ *and* $\omega(G - S) < (h-1)|S| + 3$; *or,*
(iii) $\omega(G - S) < \lceil (\frac{h}{2} - \frac{1}{c})|S|) \rceil + 2$,
(iv) $c \geq 2$, *and* $\omega(G - S) < \lceil \frac{ch-2}{2(ch-1)} \cdot h|S| + \frac{2ch-1}{ch-1} \rceil$.

As a corollary they show that any $\frac{1}{h-2}$-tough graph has "the canonical" extension (see Tokuda's Theorem 43). We mention next the following result of Katerinis.

Theorem 47 ([82]). *Let* a *and* b *be two positive integers,* $1 \leq a \leq b$, *and let* G *be a graph of order* n *such that* $a \cdot n$ *is even if* $a = b$. *If* $tough(G) \geq a - 1 + \frac{a}{b}$, *then* G *has an* $[a, b]$*-factor.*

Ellingham, Nam and Voss use Theorem 47 above to derive the next two theorems.

Theorem 48 ([49]). *Let* a *and* b *be positive integers* $(4 \leq a + 2 \leq b)$. *If* G *is a graph with* $tough(G) \geq (a-1) + \frac{a}{b-2}$, *then* G *has a connected* $[a, b]$*-factor which contains an* $[a, b-2]$*-factor.*

Theorem 49 ([49]). *Let a and b be positive integers* $(3 \leq a+1 \leq b)$. *If G is a graph with* $tough(G) \geq \max\{a-1+\frac{a}{b-1}, \frac{2a}{a-1}\}$, *then G has a connected* $[a,b]$*-factor which contains an* $[a,b-1]$*-factor.*

4.5. Extensions of Factors

As an extension of Theorem 34 M. Cai proves

Theorem 50 ([25]). *Let k be an integer,* $k \geq 2$, *and G a connected graph of order n. If G has a k-factor F and, moreover, among any three independent vertices of G there is at least one pair of vertices with degree-sum at least* $n-k$, *then G has a matching M such that M and F are edge disjoint and* $M \cup F$ *is a connected* $[k,k+1]$*-factor.*

M. Cai and Y. Li further extend this in Theorems 51 and 52 below. They define an almost k-factor to be a factor F^- such that every vertex has degree k except at most one vertex with degree $k \pm 1$.

Theorem 51 ([24]). *Let k be an odd integer,* $k \geq 3$, *and G a connected graph of order n with* $n \geq 4k-3$ *and minimum degree at least k. If* $\max\{\deg_G(u), \deg_G(v)\} \geq n/2$ *for each pair of nonadjacent vertices u,v in G, then G has an almost k-factor* F^- *and a matching M such that* F^- *and M are edge disjoint and* $F^- \cup M$ *is a connected* $[k,k+1]$*-factor of G.*

Theorem 52 ([101]). *Let G be a connected graph of order n, let g and f be two positive integer functions defined on* $V(G)$ *which satisfy* $2 \leq g(v) \leq f(v)$ *for each vertex* $v \in V(G)$. *Let G have a* (g,f)*-factor F and put* $\mu = \min\{g(v) | v \in V(G)\}$. *Suppose that among any three independent vertices of G there is at least one pair of vertices with degree sum at least* $n-\mu$. *Then G has a matching M such that M and F are edge disjoint and* $M \cup F$ *is a connected* $(g,f+1)$*-factor of G.*

An extension by spanning trees is proved by Tokuda, B. Xu and J. Wang.

Theorem 53 ([130]). *Let G be a graph and* g, f, f' *positive integer-valued functions defined on* $V(G)$. *Assume G has a* (g,f)*-factor and an* $(1,f')$*-factor F such that F is a spanning tree for G. Then G contains a connected* $(g,f+f'-1)$*-factor.*

This implies that if G has a (g,f)-factor and a k-tree then G has a connected $(g,f+k-1)$-factor.

Note that there are not so many works on connected (g,f)-factors in which f and g are not constants. Factorization into even, connected $[a,b]$-factors has not yet been considered. It might in some years be the subject of another survey.

Acknowledgments. We would to thank M. Kano, B. Jackson and the referees for their helpfull remarks.

References

1. Akiyama, J., Kano, M.: Factors and factorizations of graphs – a survey. J. Graph Theory **9**, 1–42 (1985)
2. Amahashi, A.: On factors with all degrees odd. Graphs Comb. **1**, 11–114 (1985)
3. Anstee, R.P.: More sufficient conditions for a graph to have factors. Discrete Math. **184**, 15–24 (1998)
4. Anstee, R.P.: Simplified existence theorems for (g, f)-factors. Discrete Appl. Math. **27**, 29–38 (1990)
5. Anstee, R.P.: An algorithmic proof of Tutte's f-factor theorem. J. Algorithms **6**, 112–131 (1985)
6. Aung, M., Kyaw, A.: Maximal trees with bounded maximum degree in a graph. Graphs Comb. **14**, 209–221 (1998)
7. Ausiello, G., Crescenzi, P., Gambosi, G., Kann, V., Marchetti-Spaccamela, A., Protasi, M.: Complexity and Approximation, Combinatorial Optimization Problems and Their Approximability Properties". Springer Verlag 2002
8. Bäbler, F.: Über die Zerlegung regulärer Streckenkomplexe ungerader Ordninung. Comment. Math. Helv. **10**, 81–100 (1938)
9. Barnette, D.W.: Trees in polyhedral graphs. Can. J. Math. **18**, 731–736 (1996)
10. Barnette, D.W.: 2-connected spanning subgraphs of planar 3-connected graphs. J. Comb. Theory, Ser. B **61**, 210–216 (1994)
11. Barnette, D.W.: 3-trees in polyhedral maps. Isr. J. Math. **79**, 251–256 (1992)
12. Barnette, D.W.: Trees in polyhedral graphs. Can. J. Math. **18**, 731–736 (1966)
13. Bauer, D., Broersma, H.J., Veldman, H.J.: Not every 2-tough graph is Hamiltonian. Discrete Appl. Math. **99**, 317–321 (2000)
14. Belck, H.-B.: Reguläre Faktoren von Graphen. J. Reine Angew. Math. **188**, 228–252 (1950)
15. Benhocine, A., Clark, L., Köhler, N., Veldman, H.J.: On circuits and pancyclic line graphs. J. Graph Theory **10**, 411–425 (1986)
16. Berman, K.A.: Parity results on connected f-factors. Discrete Math. **59**, 1–8 (1986)
17. Bermond, J.-C.: Hamiltonian graphs. pp. 127–167 In: Beineke, W., Wilson, R.J. (Editors): Selected topics in graph theory. London: Academic Press 1978
18. Bollobás, B.: Extremal graph theory. London-New York: Academic Press 1978
19. Brandt, S.: Private communication
20. Broersma, H.J., Kriesell, M., Ryjacek, Z.: On factors of 4-connected claw-free graphs. J. Graph Theory **37**, 125–136 (2001)
21. Brunet, R., Ellingham, M.N., Gao, Z., Metzlar, A., Richter, R.B.: Spanning planar subgraphs of graphs in the torus and Klein bottle. J. Comb. Theory Ser. B **65**, 7–22 (1995)
22. Cai, M., Li, Y., Kano, M.: A $[k, k+1]$-factor containing a given Hamiltonian cycle. The Electron. J. Comb. **6**, 1–8 (1999)
23. Cai, M., Li, Y., Kano, M.: A $[k, k+1]$-factor containing given Hamiltonian cycle. Sci. China, Ser. A **41**, 933–938 (1998)
24. Cai, M.: Connected $[k, k+1]$-factors of graphs. Discrete Math. 169 (1997), 1–16.
25. Cai, M.: A degree condition for the existence of connected $[k, k+1]$-factors. Syst. Sci. Math. Sci. **8**, 364–368 (1995)
26. Cai, M.: $[a, b]$-factorizations of graphs. J. Graph Theory **15**, 283–302 (1991)
27. Cai, M.: On some factor theorems of graphs. Discrete Math. **98**, 223–229 (1991)
28. Caro, Y., Krasikov, I., Roditty, Y.: On the largest tree of given maximum degree in a connected graph. J. Graph Theory **15**, (1991), 7–13.
29. Caro, Y., Krasikov, I., Roditty, Y.: Spanning trees and some edge reconstructible graphs. Ars Comb. **20A**, 109–118 (1985)
30. Catlin, P.A., Han, Z.-Y., Lai, H.-J.: Graphs without spanning closed trails. Discrete Math. **160**, 81–91 (1996)

31. Catlin, P.A.: Supereulerian graphs: A survey. J. Graph. Theory **16**, 177–196 (1992)
32. Catlin, P.A.: Contractions of graphs with no spanning eulerian subgraphs. Combinatorica **8**, 313–321 (1988)
33. Cayley, A.: A theorem on trees. Q. J. Pure and Appl. Math, **23**, 376–378. (1889) Reprinted in Mathematical Papers, Vol. **13**. Cambridge University Press 26–28 (1889–1897)
34. Chen, C.: Toughness of graphs and $[2, b]$-factors. Graphs Comb. **10**, 97–100 (1994)
35. Chen, C., Kano, M.: Odd and even factors with given properties. Chin. Q. J. Math. **7**, 65–70 (1992)
36. Chen, Z.H.: Supereulerian graphs, independent sets, and degree-sum conditions. Discrete Math. **179**, 73–87 (1998)
37. Chen, Z.H.: A degree condition for spanning eulerian subgraphs. J. Graph Theory **17**, 5–21 (1993)
38. Cui, Y., Kano, M.: Some results on odd factors of a graph. J. Graph Theory **12**, 327–333 (1988)
39. Chvátal, V.: Tough graphs and Hamiltonian circuits. Discrete Math. **5**, 215–228 (1973)
40. Chvátal, V., Erdös, P.: A note on hamilton cycles. Discrete Math. **2**, 111–113 (1972)
41. Cvetković, D., Doob, M., Sachs, H.: Spectra of Graphs. New York: Academic Press 1979
42. Crescenzi, P., Kann, V.: A compendium of NP optimization problems (http://www.nada.kth.se/ viggo/)
43. Czygrinow, A., Fan, G., Hurlbert, G., Kierstead, H.A., Trotter, W.T.: Spanning trees of bounded degrees. Electron. J. Comb. **8**, #R33 (2001)
44. Egawa, Y., Enomoto, H.: Sufficient conditions for the existence of k-factors. In V.R. Kulli: Recent Studies in Graph Theory. Vishwa International 1989
45. Egawa, Y., Kano, M.: Sufficient conditions for graphs to have (g,f)-factors. Discrete Math. **151**, 87–90 (1996)
46. Egawa, Y.: Edge-disjoint Hamiltonian cycles in graphs of Ore type. SUT J. Math. **29**, 15–50 (1993)
47. Egawa, Y., Ota, K.: Regular factors in $K_{1,n}$-free graphs. J. Graph Theory **15**, 337–344 (1991)
48. Ellingham, M.N., Zha, Xiaoya.: Toughness, Trees and Walks. J. Graph Theory **33**, 125–137 (2000)
49. Ellingham, M.N., Nam, Y., Voss, H.-J.: Connected (g, f)-factors. J. Graph Theory **39**, 62–75 (2002)
50. Ellingham, M.N., Zha, X., Zhang, Y.: Spanning 2-trails from degree sum conditions. J. Graph Theory **45**, 298–319 (2004)
51. El-Zanati, S., Eynden, C.V.: Factorizations of $K_{m,n}$ into spanning trees. Graphs Comb. **15**, 287–293 (1999)
52. Enomoto, H., Jackson, B., Katerinis, P. Saito, A.: Toughness and the existence of k-factors, J. Graph Theory **9**, 87–95 (1985)
53. Enomoto, H., Iida, T., Ota, K.: Connected spanning subgraphs of 3-connected planar graphs. J. Comb. Theory, Ser. B **68**, 314–323 (1996)
54. Faudree, R.J., Rousseau, C.C, Schelp, R.H.: Edge disjoint hamiltonian cycles. in Graph Theory with Apllications to Algorithms and Computer Sci. (Kalamazoo, Mich. 1984), pp. 231–249 New York: Wiley-Intersci. 1985
55. Fleischner, H.: Eulerian Graphs and Related Topics, Part 1, Vol. 1, Ann. Discrete Math. **45**. New York: North-Holland 1990
56. Fleischner, H.: Eulerian Graphs and Related Topics, Part 1, Vol. 2, Ann. Discrete Math. **50**. New York: North-Holland 1991
57. Fleischner, H.: Spanning Eulerian subgraphs, the splitting lemma, and Petersen's theorem, Discrete. Math. **101**, 33–37 (1992)
58. Gallai, T.: On factorization of graphs. Acta. Math. Acad. Sci. Hungar. **1**, 133–153 (1950)

59. Gao, Z.: 2-connected coverings of bounded degree in 3-connected graphs. J. Graph Theory **20**, 327–338 (1995)
60. Gao, Z., Richter, R.B.: 2-walks in circuit graphs. J. Comb. Theory, Ser. B **62**, 259–267 (1994)
61. Gao, Z., Wormald, N.C.: Spanning Eulerian subgraphs of bounded degree in triangulations. Graphs Comb. **10**, 123–131 (1994)
62. Garey, M.R., Johnson, D.S.: Computers and Intractability: A guide to the theory of NP-completeness. San Francisco: W. H. Freeman and Company 1977
63. Gould, R.J.: Advances on the Hamiltonian problem - a survey. Graphs Comb. **19**, 7–52 (2003)
64. Guiying, Y.: Some theorems on (g, f)-factorizations of graphs. Proceedings of the Third China-USA International Conference on Graph Theory and its applications, River Edge N.J.: World Scientific Publishing 411–416 (1994)
65. Gutin, G.: Connected [g,f]-factors and supereulerian digraphs. Ars Comb. **54**, 311–317 (2000)
66. Hall, P.: On representative of subsets, J. Lond. Math. Soc. **10**, 26–30 (1935)
67. Haggard, J., McWha, P.: Decomposition of complete graphs into trees. Czech. Math. J. **25**, 31–36 (1975)
68. Holton, D.A., Sheehan, J.: The Petersen graph. Cambridge: Cambridge University Press 1993
69. Huang, C., Rosa, A.: Decomposition of complete graphs into trees. Ars Comb. **5**, 23–63 (1978)
70. Iida, H., Nishimura, T.: An Ore-type condition for the existence of k-factors in graphs. Graphs Comb. **7**, 353–361 (1991)
71. Itai, A., Lipton, R.J., Papadimitriou, C.H., Rodeh, M.: Covering graphs by simple circuits. SIAM J. Comput. **10**, 46–750 (1981)
72. Jackson, B., Wormald, N.C.: k-walks of graphs. Australas. J. Comb. **2**, 135–136 (1990)
73. Jaeger, F.: On nowhere-zero flows in multigraphs. Proceedings of the Fifth British Comb. Conference 1975, Congr. Numerantium **XV**, 373–378 (1975)
74. Jaeger, F.: Flows and generalized coloring theorems in graphs, J. Comb. Theory, Ser. B **26**, 205–216 (1979)
75. Jaeger, F.: A Note on Subeulerian Graphs, J. Graph Theory **3**, 91–93 (1979)
76. Jin, Y., Liu, C.: The enumeration of labelled spanning trees of $K_{m,n}$, Australas. J. Comb. **28**, 73–81 (2003)
77. Jin, Z., Li, X.: On a conjecture of k-walks of graphs. Australas. J. Comb. **28**, 135–143 (2003)
78. Kano, M.: $[a, b]$-factorization of a graph. J. Graph Theory **9**, 129–146 (1985)
79. Kano, M., Saito, A.: [a,b]-factors of graphs. Discrete Math. **47**(1983)(1983), 113–116 (1983)
80. Kano, M.: A sufficient condition for a graph to have $[a, b]$-factors. Graphs and Comb. **6**, 245–251 (1990)
81. Kano, M.: Some current results and problems on factors of graphs, Proceedings of the Third China-USA International Conference on Graph Theory and its applications, River Edge N.J.: World Scientific Publishing 93–98 (1994)
82. Katerinis, P.: Toughness of graphs and the existence of factors. Discrete Math. **80**, 81–92 (1990)
83. Kirchhoff, G.: Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird. Annalen der Physik und Chemie, **72**, 497–508 (1847) Reprinted in Gesammelte Abhandlungen, pp. 22–33 Leipzig 1882
84. König, D.: Graphs and matrices, Mat. Fiz. Lapok **38**, 166–119 (1931)
85. Kouider, M.: Cycles in graphs with prescribed stability number and connectivity, J. Comb. Theory, Ser. B **60**, 315–318 (1994)

86. Kouider, M., Maheo, M.: 2-edge-connected [2,k]-factors in graphs, *JCMCC*, **35**, 89–95 (2000)
87. Kouider, M., Maheo, M.: Connected $[a, b]$-factors in graphs. Combinatorica **22**, 71–82 (2002)
88. Kouider, M., Lonc, Z.: Stability number and $[a, b]$-Factors in Graphs. Accepted for publication in J. Graph Theory, 1–11
89. Kouider, M., Vestergaard, P.D.: Even [a,b]-factors in graphs. Accepted for publication in Discuss. Math. Graph Theory, 1–10
90. Kouider, M., Vestergaard, P.D.: On even [2,b]-factors in graphs. Australasian J. Comb., **27**, 139–147 (2003)
91. Kyaw, A.: A sufficient condition for a graph to have a k-tree. Graphs and Comb. **17**, 113–121 (2001)
92. Kyaw, A.: A counterexample to a conjecture of Jackson and Wormald. Australas. J. Comb. **19**, 259–260 (1999)
93. Lai, H.-J.: Contractions and hamiltonian line graphs. J. Graph Theory **12**, 11–15 (1988)
94. Lai, H.-J.: Supereulerian graphs and excluded induced minors. Discrete Math. **146**, 133–143 (1995)
95. Lesniak, L., Oellermann, O.R.: An Eulerian Exposition. J. Graph Theory **10**, 277–297 (1986)
96. Lesniak-Foster, L., Williamson, J.E.: On spanning and dominating circuits in graphs. Can. Math. Bull. **20**, 215–220 (1977)
97. Li, Y.: Personal communication 2004
98. Li, G., Liu, Z.: On Connected factors in $K_{1,3}$-free graphs. Acta Math. Appl. Sin. **14**, 43–47 (1998)
99. Li, G., Zhu, B., Chen, C.: On connected $[k, k+1]$-factors in claw-free graphs. Ars Comb. **62**, 207–219 (2002)
100. Li, Y., Mao-cheng, C.: A degree condition for a graph to have [a,b]-factors. J. Graph Theory **27**, 1–6 (1998)
101. Li, Y., Mao-cheng, C.: A degree condition for the existence of connected factors. Australa. J. Comb. **14**, 77–83 (1996)
102. Li, H.: Edge disjoint cycles in graphs. J. Graph Theory, **13**, 313–322 (1989)
103. Li, H., Zhu, Y.: Edge-disjoint hamiltonian cycles in graphs. Graph Theory and Its Applications : East West, Ann. N.Y. Acad. Sci. **576**, 311–322 (1990)
104. Lonc, Z., Parol, K., Wojciechowski, J.M.: "On the Asymptotic behavior of the Maximum Number of Spanning Trees in Circulant Graphs. Networks **30**, 47–56 (1997)
105. Lovász, L.: Comb. problems and exercises. North-Holland Publishing Co. Amsterdam-New York 1979
106. Lovász, L.: Subgraphs with prescribed valencies. J. Comb. Theory (B) **8**, 391–416 (1970)
107. Matoušek, J., Nesětřil, J.: Invitation to discrete Math. New York: Oxford University Press 1998
108. Matsuda, H.: Degree conditions for the existence of $[k, k+1]$-factors containing a given Hamiltonian cycle. Australa. J. Comb. **26**, 273–281 (2002)
109. Matsuda, H.: Degree conditions for Hamiltonian graphs to have $[a, b]$-factors containing a given Hamiltonian cycle. Discrete Math. 1–10 2004
110. Matsuda, H.: On 2-edge-connected $[a, b]$-factors of graphs with Ore-type conditions. manuscript 1–11 2004
111. Matthews, M.M., Summer, D.P.: Hamiltonian results in $K_{1,3}$-free graphs. J. Graph Theory **8**, 139–146 (1984)
112. Moon, J.W.: Various proofs of Cayley's formula for counting trees, in Harary, F., Editor: A seminar on graph theory. pp. 70–78 New York: Holt, Rinehart and Winston, 1967

113. Nash-Williams, C.St.J.A.: Edge-disjoint hamiltonian circuits in graphs with vertices of large valency, in Studies in Pure Mathematics, pp. 157–183 London: Academic Press, 1971
114. Nash-Williams, C.St.J.A.: Edge-disjoint spanning trees of finite graphs. J. Lond. Math. Soc. **36**, 451–460 (1961)
115. Neumann-Lara, V., Rivera-Campo, E.: Spanning trees with bounded degrees. Combinatorica **11**, 55–61 (1991)
116. Niessen, T.: A characterization of graphs having all (g,f)-factors. J. Comb. Theory, Ser. B **72**, 152–156 (1998)
117. Nishimura, T.: Independence number, connectivity and r-factors. J. Graph Theory **13**, 63–69 (1989)
118. Nishimura, T.: A degree condition for the existence of k-factor. J. Graph Theory **16**, 141–151 (1992)
119. Petersen, J.: Die Theorie der regulären graphs. Acta Math. **15**, 193–220 (1891)
120. Plummer, M.D.: Factors and factorization in graphs: an update. Discrete Math., 1–49 (to appear)
121. Rado, R.: Factorization of even graphs. Q. J. Math. Oxf. Ser. **20**, 94–104 (1949)
122. Rescigno, A.A.: Vertex-disjoint spanning trees of the star network with application to fault-tolerance and security. Inf. Sci. **137**, Issues **1–4** 259–274 (2001)
123. Rivera-Campo, E.: A note on matchings and spanning trees with bounded degrees. Graphs Comb. 159–165 (1997)
124. Sanders, D.P., Zhao, Y.: On 2-connected spanning subgraphs with low maximium degree. J. Comb. Theory, Ser. B **74**, 64–86 (1998)
125. Sanders, D.P. Zhao, Y.: On spanning trees and walks of low maximum degree. J. Graph Theory **36**, 67–74 (2001)
126. Thomassen, C.: Reflections on graph theory. J. Graph Theory **10**, 309–324 (1986)
127. Thomassen, C.: A remark on the factor theorems of Lovasz and Tutte. J. Graph Theory **5**, 441–442 (1981)
128. Tokuda, T.: Connected [a,b]-factors in $K_{1,n}$-free graphs containing an $[a, b]$-factor. Discrete Math. **207**, 293–298 (1999)
129. Tokuda, T.: A degree condition for the existence of $[a, b]$-factors in $K_{1,n}$-free graphs. Tokyo J. Math. **21**, 377–380 (1998)
130. Tokuda, T., Xu, B., Wang, J.: Connected factors and spanning trees in graphs. Graphs Comb. **19**, 259–262 (2003)
131. Topp, J., Vestergaard, P.D.: Odd factors of a graph. Graphs Comb. **9**, 371–381 (1993)
132. Tutte, W.T.: Graph factors. Combinatorica **1**, 79–97 (1981)
133. Tutte, W.T.: The subgraph problem. Ann. Discrete Math. **3**, 289–295 (1978)
134. Tutte, W.T.: Bridges and Hamiltonian circuits in planar graphs. Aequationes. Math. **15**, 1–33 (1977)
135. Tutte, W.T.: On the problem of decomposing a graph into n connected factors. J. Lond. Math. Soc. **36**, 221–230 (1961)
136. Tutte, W.T.: A theorem on planar graphs. Trans. Amer. Math. Soc. **82**, 99–116 (1956)
137. Tutte, W.T.: A short proof of the factor theorem for finite graphs. Can. J. Math. **6**, 347–352 (1954)
138. Tutte, W.T.: The factors of graphs. Can. J. Math. **4**, 314–328 (1952)
139. Tutte, W.T.: The factorization of linear graphs. J. Lond. Math. Soc. **22**, 107–111 (1947)
140. Wallis, W.D.: On decomposing complete graphs into trees. Ars Comb. **35**, 55–63 (1993)
141. Wang, H.: Partition of a bipartite graph into cycles. Discrete Math. **117**, 287–291 (1993)
142. Wei, B., Zhu, Y.: Hamiltonian k-factors in graphs. J. Graph Theory **25**, 217–227 (1997)
143. Win, S.: On a Conjecture of Las Vergnas Concerning Spanning Trees in Graphs. Result. Math. **2**, 215–224

144. Win, S.: Existenz von Gerusten mit vorgeschreibenem Maximalgrad in Graphen. Abh. Math. Semin. Univ. Hamb. **43**, 263–267 (1975)
145. Win, S.: On a connection between the existence of k trees and the toughness of a graph. Graphs Comb. **5**, 201–205 (1989)
146. Volkmann, L.: Fundamente der Graphentheorie. Wien/New York: Springer Verlag 1996
147. Xu, R.: Connected $[1,k]$-factors of balance bipartite graphs. Syst. Sci. Math. Sci. **12**, 149–153 (1999)
148. Xu, R.: Some Ore-type conditions for the existence of connected $[2,k]$-factors in graphs. Discrete Math. **194**, 295–298 (1999)
149. Xu, B., Liu, Z., Tokuda, T.: Connected factors in $K_{1,n}$-free graphs containing a (g,f)-factor. Graphs Comb. **14**, 393–395 (1998)
150. Xu, B., Liu, Z.: A note on connected factors in claw-free graphs. J. Syst. Sci. Complex. **14**, 91–92 (2001)
151. Xu, B., Liu, Z.: Connected factors and toughness. manuscript pp. 1–10
152. Yan, G.Y., Pan, J.F., Wong, C.K., Tokuda, T.: Decomposition of graphs into (g,f)-factors. J. Graphs Comb. **16**, 117–126 (2000)
153. Zelinka, B.: Some Remarks on Eulerian Graphs. Czech. Math. J. **29**, 564–567 (1979)
154. Zhang, C.Q.: Integer flows and cycle covers of cycles. New York, Basel, Hong Kong: Marcel Dekker, Inc., 1997

Received: May 6, 2004
Final version received: October 15, 2004