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Graphs and Combinatorics

On Ranks of Matrices Associated with Trees

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Abstract. A sign pattern matrix is a matrix whose entries are from the set $\{+, -, 0\}$. The purpose of this paper is to obtain bounds on the minimum rank of any symmetric sign pattern matrix A whose graph is a tree T (possibly with loops). In the special case when A is nonnegative with positive diagonal and the graph of A is "star-like", the exact value of the minimum rank of A is obtained. As a result, it is shown that the gap between the symmetric minimal and maximal ranks can be arbitrarily large for a symmetric tree sign pattern A.

Key words. Sign pattern matrix, Symmetric tree sign pattern, Minimal rank, Symmetric minimal rank

1. Introduction

In qualitative and combinatorial matrix theory, we study properties of a matrix based on combinatorial information, such as the signs of entries in the matrix. A matrix whose entries are from the set $\{+, -, 0\}$ is called a *sign pattern matrix* (or sign pattern, or pattern). We denote the set of all $n \times n$ sign pattern matrices by Q_n. For a real matrix B, $sgn(B)$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by $+$ (respectively, $-$, 0). If $A \in Q_n$, then the *sign pattern class of A* is defined by

$$Q(A) = \{B \ : sgn(B) = A\}.$$

For a symmetric sign pattern A, we define $smr(A)$, the *symmetric minimal rank* of A by

$$smr(A) = min\{rank\ B : \ B = B^T,\ B \in Q(A)\}.$$

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Similarly, the *symmetric maximal rank* of A, $SMR(A)$, is

$$SMR(A) = max\{rank\ B:\ B = B^T,\ B \in Q(A)\}.$$

For $A \in Q_n$, the *minimal rank* of A, denoted as $mr(A)$, is defined by

$$mr(A) = min\{rank\ B:\ B \in Q(A)\}.$$

The *maximal rank* of A, $MR(A)$, is given by $MR(A) = max\{rank\ B : B \in Q(A)\}$.

A key basic result of the paper [8] is that for a symmetric sign pattern A, we always have $SMR(A) = MR(A) = k$, where k is the maximum number of nonzero entries of A with no two of the nonzero entries in the same row or column (k is the *term rank* of A as defined in [5]). On the other hand, $mr(A) < smr(A)$ is possible. Recall that for a real symmetric matrix B, the *inertia* of B, written as $i(B)$, is the triple of integers $i(B) = (i_+(B), i_-(B), i_0(B))$, where $i_+(B)$ (respectively, $i_-(B), i_0(B)$) denotes the number of positive (respectively, negative, zero) eigenvalues of matrix B counted with their algebraic multiplicities. In [8] it was proved that a symmetric sign pattern A requires unique inertia (all the real symmetric matrices in $Q(A)$ have the same inertia) if and only if $smr(A) = SMR(A)$.

For a symmetric $n \times n$ sign pattern A, by $G(A)$ we mean the undirected graph of A, with vertex set $\{1, \ldots, n\}$ and (i, j) is an *edge* if and only if $a_{i,j} \neq 0$, where (i, j) and (j, i) are regarded as the same edge. A sign pattern A is a *symmetric tree sign pattern* if A is symmetric and $G(A)$ is a tree, possibly with loops. The symmetric tree sign patterns which require unique inertia were characterized in [8]. These are precisely the symmetric tree sign patterns A which require fixed rank, that is, all the matrices in $Q(A)$ have the same rank ($mr(A) = MR(A)$). In particular, when a symmetric tree sign pattern A has zero diagonal ($G(A)$ has no loops), it is always the case that

$$mr(A) = smr(A) = SMR(A) = MR(A) = 2t$$

where t is the maximum number of independent edges in $G(A)$. When A has some nonzero diagonal entries, $MR(A)$ is the maximum of the numbers of the form $q + 2t$, where t is a number of independent edges in $G(A)$ and q is the number of loops nonadjacent to those edges (each of t or q can be zero). Hence, when there is a loop at each vertex, $MR(A) = n$ when A is $n \times n$.

Several questions arise regarding symmetric tree sign patterns. Although $SMR(A)$ is known as mentioned above, how can we find $smr(A)$, or at least bounds on $smr(A)$, when A has some nonzero diagonal entries? In [7] examples of symmetric tree sign patterns A are given where $SMR(A) - smr(A) = 1$. How large can this difference be for an arbitrary symmetric tree sign pattern?

We note that for a symmetric tree sign pattern A, every matrix $B \in Q(A)$ is diagonally similar to a symmetric matrix in $Q(A)$ (that is to say, there exists a nonsingular diagonal matrix D such that $D^{-1}BD$ is a symmetric matrix in $Q(A)$). This can be proved inductively by using an end vertex. Hence,

$$mr(A) = smr(A)$$

and determining $mr(A)$ is a matter of determining $smr(A)$. Further, it can be shown (also by induction) that there exist suitable nonsingular diagonal matrices D_1 and D_2 with the same sign pattern such that D_1BD_2 has all the off-diagonal nonzero entries equal to 1. Thus $mr(A)$ can be achieved by a matrix (with the same sign pattern as D_1BD_2) whose off-diagonal nonzero entries are all equal to 1, and whose diagonal entries agree in sign with A. Finally, it should be pointed out that for a non-symmetric tree sign pattern $A = A_1S$, where A_1 is a symmetric tree sign pattern and S is a nonsingular diagonal sign pattern, we have

$$mr(A) = mr(A_1) = smr(A_1).$$

Hence, the study of the minimum rank of a non-symmetric tree sign pattern matrix reduces to the study of the minimum rank of a symmetric tree sign pattern matrix.

In this paper we obtain bounds on $mr(A)$ where A is any symmetric sign pattern matrix whose graph is a tree T (possibly with loops). We also obtain the exact value of $mr(A)$ in the special case when A is nonnegative with positive diagonal and $G(A)$ is "star-like". As a result, we show that the gap between $SMR(A)$ and $smr(A)$ can be arbitrarily large for a symmetric tree sign pattern A.

The work in this paper is related to that of [10, 13]. In [10, 13] the diagonal entries are allowed to be free, and in [13] a recursive algorithm is given for computing the free-diagonal minimum rank for a tree. In this paper, we work with a tree sign pattern A (hence each of whose diagonal entries is of fixed sign) and obtain bounds for $mr(A)$. The free-diagonal minimum rank obtained in [13] is a lower bound for the minimum rank in our sense. More generally, extensive work has been done on the ranks of matrices (especially the adjacency matrices) associated with graphs. For example, see [1], [2], [3], [4], [6], [9], [11], [12], [14], [15], and [16].

2. Some Bounds on the Minimum Rank of a Symmetric Tree Sign Pattern

We first give a path-loop bound on $mr(A)$. A collection of subsets of a graph is said to be *independent* if the subsets are pairwise disjoint; it is said to be *nonadjacent* if no two vertices from two different subsets in the collection are adjacent.

Theorem 2.1. *Let A be any symmetric sign pattern matrix, with graph G (possibly with loops). Let $\{P_1, \ldots, P_k, L_1, \ldots, L_m\}$ be any collection of independent and nonadjacent paths and loops in G. Let the lengths of the paths $P_1, \ldots, P_k$* be *$l_1, \ldots, l_k$, respectively. Then*

$$l_1 + \cdots + l_k + m \le mr(A).$$

Proof. Performing a suitable permutational similarity on A if necessary, we may assume that the vertices of $L_1, \ldots, L_m$ come first, followed by the vertices of $P_1, \ldots, P_k$. Thus A is permutationally similar to a sign pattern of the form

$$\left(\begin{array}{ccccccc|c} D_1 & & & & & & & \\ & \ddots & & & & & & \\ & & D_m & & & & & * \\ & & & A_1 & & & & \\ & & & & \ddots & & & \\ & & & & & A_k & & \\ \hline & & & * & & & & * \end{array}\right)$$

where each D_i is a 1×1 nonzero matrix and each A_j is an irreducible tri-diagonal pattern of order $l_j + 1$. Clearly, $mr(D_i) = 1$ for all i. Note that the last l_j rows of every matrix in $Q(A_j)$ are linearly independent. Hence, $mr(A_j) \geq l_j$ for each j. Then, since the upper left portion of the above partitioned matrix is block diagonal, it follows that

$$mr(A) \ \geq \ mr(D_1) + \cdots + mr(D_m) + mr(A_1) + \cdots + mr(A_k) \ \geq \ m + l_1 + \cdots + l_k.$$

□

The following example shows that the path-loop bound is tight.

Example 2.2. Let T_1 be a path of length 4 with a loop at each vertex. Let T_2 be formed from the star $K_{1,3}$ by subdividing each edge into a path of length 2, with a loop at each vertex.

Let A be the nonnegative (tri-diagonal) sign pattern whose graph is T_1. From Theorem 2.1, $mr(A) \geq 4$. It follows from Proposition 3.2 in [7] that $mr(A) = 4$ (and $MR(A) = 5$). In particular,

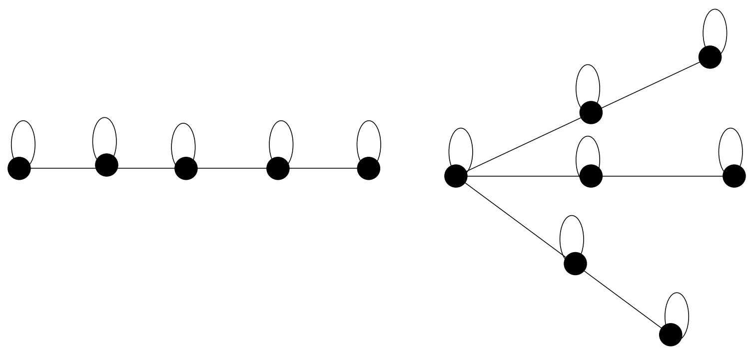

Fig. 1

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

is a matrix in $Q(A)$ that has rank 4. Similarly, for any path of length l with a loop at each vertex, $mr(A) = l$ for the corresponding nonnegative sign pattern A.

Next, let A be the nonnegative sign pattern whose graph is T_2, a "star-like" pattern. By considering a path of length 4 and a nonadjacent loop, it follows from Theorem 2.1 that $mr(A) \geq 5$. By a result in section 3,

$$mr(A) = 7 - 3 + 1 = 5,$$

which again shows the tightness of the path-loop bound. □

For a general path P, with only some (or none) of the vertices having loops, Proposition 3.3 of [7] could be used to find the minimum rank of any symmetric sign pattern whose graph is P. The conditions in this proposition are in terms of the locations of the nonzero diagonal entries of the sign pattern. From this, a sharper path-loop bound than the one given in Theorem 2.1 can be obtained.

For a tree T, there exists a unique path from any vertex to any other vertex. Hence, the diameter of T equals to the maximum path length in T, and we obtain the following weak bound.

Corollary 2.3. *Let A be any symmetric sign pattern matrix whose graph is a tree T (possibly with loops). Then diam $T \leq mr(A)$.*

We note that if A is any symmetric sign pattern, then it can be proved that *diam* $G(A) \leq mr(A)$.

Throughout the remainder of the paper, when we say an edge we mean an edge between two distinct vertices.

We shall next obtain an edge-loop bound on $mr(A)$. The shorter notation xy will be used for an edge (x, y) of a graph, and $d(x)$ denotes the degree of a vertex x. Recall that a forest is a disjoint union of trees.

Lemma 2.4. *If M is a (nonempty) maximum independent edge set of a forest F, then there exists $xy \in M$ with $d(x) = 1$, that is, xy is a leaf of F.*

Proof. Let x_1y_1 be a leaf of F, with $d(x_1) = 1$. If $x_1y_1 \in M$, we are done. Assume $x_1y_1 \notin M$. By the maximality of M, there exists $y_1x_2 \in M$ (otherwise, $M \cup \{x_1y_1\}$ is an independent set, contradicting the maximality of M). If $d(x_2) = 1$, we are done. Assume $d(x_2) \geq 2$, and let x_2y_2 be an edge where $y_2 \neq y_1$. Then $x_2y_2 \notin M$ since $y_1x_2 \in M$. If y_2 is not adjacent to any edge in M, then the independent edge set

$$M \cup \{x_1y_1, x_2y_2\} - \{y_1x_2\}$$

contradicts the maximality of M. So there exists x_3 such that $y_2x_3 \in M$. Since F is acyclic, $x_3 \notin \{x_1, y_1\}$. If $d(x_3) = 1$, we are done. Otherwise, we continue in the same way, and obtain

$$x_3y_3 \notin M, y_3x_4 \in M, x_4y_4 \notin M, \ldots.$$

Since F is finite, this process must terminate. Thus we obtain $y_{k-1}x_k \in M$, with $d(x_k) = 1$, for some positive integer k. □

The proof of the following result should be clear.

Lemma 2.5. *If M is a (nonempty) maximum independent edge set of a forest F, and $xy \in M$, then $M - \{xy\}$ is a maximum independent edge set in $F - \{x, y\}$.*

Theorem 2.6. *Let A be any symmetric sign pattern matrix whose graph is a forest F (possibly with loops). Let M be a maximum set of independent edges in F and l the number of loops in $F - V(M)$. Then*

$$l + |M| \le mr(A).$$

Proof. We proceed by complete induction on n. The result is trivial for $n \le 2$.

Since the result is clear if F does not contain any edge, we may assume $M \ne \phi$. By Lemma 2.4, there is an edge $xy \in M$ with $d(x) = 1$. Performing a permutation similarity on A if necessary, we assume that

$$A = \begin{pmatrix} * & a & 0 & \ldots & 0 \\ a & * & * & \ldots & * \\ 0 & * & & & \\ \vdots & \vdots & & A^* & \\ 0 & * & & & \end{pmatrix},$$

where $a \ne 0$.

Note that for any matrix in $Q(A)$, the second row cannot be written as a linear combination of the $n - 2$ rows below it. Thus,

$$mr(A) \ \ge \ 1 + mr(A^*).$$

Since $M^* = M - \{xy\}$ is a maximal independent edge set in $F^* = F - \{x, y\}$ by Lemma 2.5, the induction hypothesis implies that $mr(A^*) \ \ge l + |M^*|$. Hence, $mr(A) \ \ge \ 1 + mr(A^*) \ge 1 + l + |M^*| = 1 + l + |M| - 1 = l + |M|$. □

As with the path-loop bound, the edge-loop bound is also tight.

Example 2.7. Let F be a star $K_{1,s}$ (which has a vertex at the "center" and all the other s vertices adjacent to the center vertex), with a loop at each vertex.

Let A be any symmetric sign pattern matrix whose graph is this star. Using one edge and $s - 1$ loops, Theorem 2.6 says that $mr(A) \ \ge \ s$. The lower bound s is

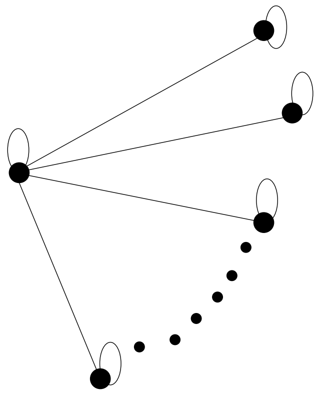

Fig. 2

achieved by the following matrix of order $s+1$ (where we have the center vertex listed last):

$$\begin{pmatrix} 1 & & & & 1 \\ & 1 & & & 1 \\ & & \ddots & & \vdots \\ & & & 1 & 1 \\ 1 & 1 & \dots & 1 & s \end{pmatrix}.$$

□

Depending on the tree, each of the path-loop or edge-loop bound can be a better bound.

Example 2.8. For the star in Example 2.7, but with the center vertex and one other vertex without loops, the edge-loop bound is again s, while the path-loop bound is $s-1$.

Next, consider the star-like graph

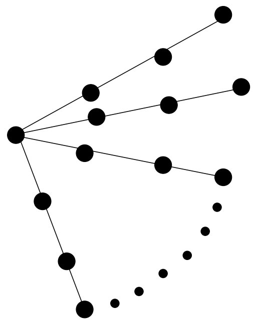

Fig. 3

obtained from $K_{1,s}$ by inserting two vertices on each edge. In this case, $mr(A) = 2s + 2$, the path loop bound is $2s$, and the edge-loop bound is $s + 1$. Note that the diameter bound is only 6. □

Using the edge-loop bound we arrive at a loop bound.

Corollary 2.9. *Let A be any symmetric sign pattern matrix whose graph is a tree T with k loops. Then*

$$\frac{k}{2} \leq mr(A).$$

Proof. We first consider the case where every vertex of T has a loop. Let A be $n \times n$, M be a maximum set of independent edges in T, and l be the number of loops in $T - V(M)$. Then $2|M| + l = n$ or $|M| = \frac{n-l}{2}$. From Theorem 2.6,

$$mr(A) \ \geq \ l + |M| \ = \ \frac{2l + (n-l)}{2} \ = \ \frac{n+l}{2} \ \geq \ \frac{n}{2},$$

that is, $mr(A) \ \geq \ \frac{n}{2}$.

Now, suppose k is the number of loops in T. Consider the principal submatrix of A associated with the subgraph of T induced by the k vertices with loops. This subgraph is a union of disjoint trees. Applying the above all loops result to the diagonal sub-blocks associated with these disjoint trees, we obtain

$$\frac{k}{2} \leq mr(A).$$ □

3. Star-Like Trees

A star-like graph is a tree that has exactly one vertex x (the center vertex) with $d(x) \ > \ 2$.

Theorem 3.1. *Let T be a star-like graph where every vertex has a loop, and suppose there are $k \geq 2$ paths of lengths ≥ 2 from the center vertex. Let A be an $n \times n$ nonnegative symmetric sign pattern matrix whose graph is T. Then*

$$mr(A) = n - k + 1.$$

Proof. First, we first construct a matrix $B \in Q(A)$ such that B has $k - 1$ rows that are linear combinations of the remaining rows of B. We then have $mr(A) \ \leq rank\ B \ \leq \ n - (k-1) = n - k + 1$.

Let v_0 be the center vertex of T and let $w_1, w_2, \ldots, w_t$ be the end-vertices adjacent to v_0. Label the vertices along the paths of lengths ≥ 2 as $v_{11}, v_{12}, \ldots, v_{1n_1}$; $v_{21}, v_{22}, \ldots, v_{2n_2}$; $\ldots$; $v_{k1}, v_{k2}, \ldots, v_{kn_k}$.

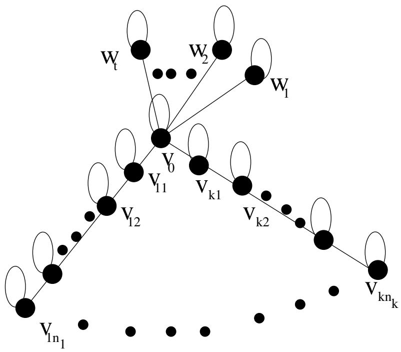

Fig. 4

We order the vertices as v_0, $v_{11}, v_{12}, \ldots, v_{1n_1}$, $v_{21}, v_{22}, \ldots, v_{2n_2}$, $\ldots$, v_{k1}, v_{k2}, $\ldots, v_{kn_k}, w_1, w_2, \ldots, w_t$. Let A be the nonnegative sign pattern matrix whose graph is T with the vertices ordered this way. Define a matrix $B = (b_{ij}) \in Q(A)$ as follows:

$$b_{ij} = \begin{cases} 0 & \text{if } \ a_{ij} = 0, \text{ (that is, the corresponding vertices are nonadjacent);} \\ 2 & \text{if } \ i = j \text{ and the corresponding vertex is of degree 2 in } \ T - v_0; \\ 1 & \text{otherwise.} \end{cases}$$

Thus the matrix B has the form

$$\begin{array}{c} \\ v_0 \\ v_{11} \\ v_{12} \\ v_{13} \\ \vdots \\ v_{1,n_1-1} \\ v_{1n_1} \\ v_{21} \\ \vdots \\ v_{2n_2} \\ \vdots \\ v_{k1} \\ \vdots \\ v_{kn_k} \\ w_1 \\ \vdots \\ w_t \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccccc} v_0 & v_{11} & v_{12} & v_{13} & \ldots & v_{1,n_1-1} & v_{1n_1} & v_{21} & \ldots & v_{2n_2} & \ldots & v_{k1} & \ldots & v_{kn_k} & w_1 & \ldots & w_t \end{array} \\
\begin{pmatrix}
1 & 1 & 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & \ldots & 1 & \ldots & 0 & 1 & \ldots & 1 \\
1 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 1 & 2 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 2 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & \ldots & 2 & 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 1 & 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 1 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 1 & 0 & \ldots & 0 \\
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 1
\end{pmatrix}
\end{array}.$$

For each vertex v_{ij}, let R_{ij} denote the corresponding row of B with label v_{ij}. Notice that after deleting the first row, each column of B labeled by some v_{ij} contains either two or three consecutive nonzero entries (of the forms $1,1$, or $1,2,1$). It can be seen that for each $i = 1, 2, \ldots, k$,

$$\sum_{j=1}^{n_i} (-1)^{j-1} R_{ij} = (1, 0, 0, \ldots, 0).$$

Thus, $R_{i,1}$ $(i \geq 2)$ can be written as a linear combination of $R_{11}, R_{12}, \ldots, R_{1n_1}$, $R_{i2}, \ldots, R_{in_i}$. Therefore, *rank* $B \leq n - (k-1) = n - k + 1$.

We now show that for every $B \in Q(A)$, *rank* $B \geq n - k + 1$. Recall that, without loss of generality, we may assume that all the nonzero off-diagonal entries of B are equal to 1. Label the rows and columns of B as above. Consider the submatrix of B obtained by deleting the rows with labels $v_{31}, v_{41}, \ldots, v_{k1}$ and the columns with labels $v_{3n_3}, n_{4n_4}, \ldots, n_{kn_k}$. The resulting submatrix has the following block form

$$B_1 = \begin{pmatrix} H & * & V \\ 0 & U & 0 \\ V^T & 0 & D \end{pmatrix}$$

where H is the submatrix whose graph is the path with vertices $v_0, v_{11}, \ldots, v_{1n_1}$, $v_{21}, \ldots, v_{2n_2}$, U is an upper triagular matrix with all diagonal entries equal to 1, D is a nonsingular diagonal matrix of order t, and V is a matrix whose first row consists of 1's and all other entries are 0. Using the last t nonzero diagonal entries of B_1 as pivot elements, we can perform elementary row/column operations on B_1 to eliminate the nonzero entries of V and V^T. The only other entry that may be affected is the (1, 1) entry. The modified upper left block of order $n_1 + n_2 + 1$ still has the path of length $n_1 + n_2$ (with at least $n_1 + n_2$ loops) as its graph, and hence (by Theorem 2.1 say), its rank is at least $n_1 + n_2$. Thus the resulting block upper triangular matrix has rank at least $n - 1 - (k-2) = n - k + 1$. It follows that $mr(A) \geq n - k + 1$. □

We can now show that the difference $SMR(A) - smr(A)$ may be arbitrarily large.

Example 3.2. This graph is star-like, with k paths (of length 2) adjacent to the center vertex, with a loop at each vertex. Let A be any nonnegative symmetric sign pattern matrix of order $2k+1$ whose graph is this tree. Since A has no zero diagonal entries, we have

$$SMR(A) = MR(A) = 2k + 1.$$

By Theorem 3.1,

$$smr(A) = mr(A) = (2k+1) - k + 1 = k + 2.$$

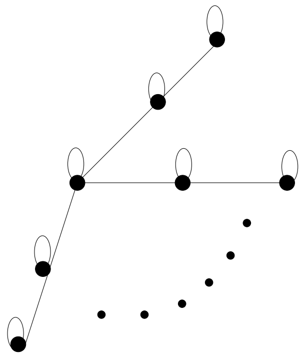

Fig. 5

Clearly, by taking k to be arbitrarily large, we can make $SMR(A) - smr(A)$ as large as desired. □

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