Visual Comput (2008) 24: 213–225
DOI 10.1007/s00371-007-0190-z

ORIGINAL ARTICLE

Gabriela González Castro
Hassan Ugail
Philip Willis
Ian Palmer

A survey of partial differential equations in geometric design

Published online: 5 December 2007

G. González Castro (✉) · I. Palmer ·
H. Ugail
University of Bradford, EIMC Department, School of Informatics, Richmond Road, Bradford, BD7 1DP, UK
G.Gonzalezcastro1@bradford.ac.uk

P. Willis
University of Bath, Media Technology Research Centre, Department of Computer Science, UK

Abstract Computer-aided geometric design is an area where the improvement of surface generation techniques is an everlasting demand, since faster and more accurate geometric models are required. Traditional methods for generating surfaces were initially mainly based upon interpolation algorithms. Recently, partial differential equations (PDE) were introduced as a valuable tool for geometric modelling, since they offer a number of features from which these areas can benefit. This work summarizes the uses given to PDE surfaces as a surface generation technique together with some other applications to computer graphics.

Keywords PDE surfaces · Geometric modelling · PDE method

1 Introduction

The systematization and characterization of certain surfaces dates as far back as the times of the Roman Empire. They were interested in creating a mould for ship hulls since their conquering aspirations demanded mass production of war vessels [23]. However, the introduction of drawings defining the shape of a hull became popular in England in the 17th century when a wooden beam, known as a spline, was used to draw smooth curves. Nowadays, geometric design is aided by computational tools where a large number of surface generation techniques are readily available.

The majority of methods used in computer-aided geometric design for generating surfaces are commonly based on a specific type of implicit surface, namely polynomial surfaces. This kind of surface is characterized by a number of control points and weights. However, the manipulation of such surfaces is not as straightforward as one would desire since the relationship between the changes in geometry and the manipulation of the control points is not intuitive.

Parametric surfaces are, in general, easier to manipulate than implicit ones, since it is only necessary to modify some of the parameters to obtain a different surface. Parametric surfaces are commonly represented by splines, which are a popular representation of curves in computer-aided geometric design due to the advantages they offer, namely: the simplicity of their construction and the accuracy with which they can be evaluated.

1.1 Common surface generation techniques for geometric design

Today, in the geometric design literature there exist numerous methods for surface generations. In particular, spline-based techniques have become increasingly popular over the years. A brief description of the most frequently occurring surface generation techniques is given below.

B-splines are curves that can be described by a given set of points. This technique was originally based on polynomial interpolation through the complete set of points. However, as high order polynomials are obtained from such a procedure, the resulting surfaces lacked smoothness. Afterwards, in the interest of providing smoothness to such surfaces, piecewise interpolation was carried out. De Casteljau and Bézier, both involved in automotive design, are the pioneers in the area [23]. The most common functions used to achieve piecewise interpolation are third-order polynomials and conics.

The following are the most common types of spline used in computer design:

Bézier surfaces. These surfaces are a special case of Hermite interpolation. They are constructed as a sequence of cubic segments rather than linear ones [32]. They are determined by

$$S(u, v) = \sum_{j=0}^{m} \sum_{k=0}^{n} P_{j,k} Bez_{j,m}(v) Bez_{k,n}(u), \tag{1}$$

where $P_{j,k}$ denote the control points and

$$Bez_{k,n} = \frac{n!}{k!(n-k)!} u^k (1-u)^{n-k}.$$

B-splines. These are a generalization of Bézier curves where each of the control points is multiplied by its respective basis function. The basis functions are determined by a well established rule and depend on the number of knots (joining points) required. Thus, a B-spline surface is defined by

$$S(u, v) = \sum_{j=0}^{m} \sum_{k=0}^{n} N_{i,p}(u) N_{j,q}(v) P_{j,k}, \tag{2}$$

where $P_{j,k}$ denote the control points, $N_{i,p}$ and $N_{i,q}$ are the B-spline basis functions of degree p and q respectively. A B-spline basis function of degree r is given by

$$N_{i,r}(u) = \frac{u - u_i}{u_{i+r-1} - t_i} N_{i,r-1} + \frac{u_{i+k} - u}{u_{i+r} - u_{i+1}} N_{i+1,r-1},$$

where u_i denotes a component of a pre-defined knot vector.

NURBS. Non-uniform rational B-splines differ from B-splines and Bézier curves. The difference is that NURBS include the weighting of the non-equidistant control points, which is also the reason for which they are regarded as rational. These surfaces are mathematically described by

$$S(u, v) = \frac{\sum_{j=0}^{m} \sum_{k=0}^{n} N_{i,p}(u) N_{j,q}(v) w_{j,k} P_{j,k}}{\sum_{j=0}^{m} \sum_{k=0}^{n} N_{i,p}(u) N_{j,q}(v) w_{j,k}}, \tag{3}$$

where $w_{j,k}$ represents the weight associated with the control point $P_{j,k}$.

The most common types of parametric surfaces used in computer-aided geometric design vary from rectangular surfaces to Coons patches. A brief description of each of these types is given below. However, if the reader is interested in further details, the work outlined in [23] would be an excellent further reading since it is a self-contained reference with a thorough description of each of the surface types mentioned in this work.

Rectangular surfaces, commonly known as tensor product surfaces, were based on bicubic spline interpolation in their early developments [23]. This type of surfaces, as its name points out, maps a rectangular domain into a three-dimensional region.

Coons patches are regarded as surfaces fitted through a given set of four boundary curves. The only condition imposed on the boundary curves associated with a Coons patch surface is that these curves have to meet at the patch corners.

Triangular surfaces take their name from the geometric arrangement from where each of their points is computed. The domain is divided into triangular elements, and then each point of the surface is evaluated at the barycentric coordinates of its respective triangular element in the domain. This type of surface was first used in finite element theory, where their formulation was complicated and exhausting. However, the formulation of triangular patches in Bernstein form was more elegant [23].

Subdivision is a surface generation technique for finding a smooth surface from a rough one. This technique consists of an iterative process by which new points within the surface are found according to a given subdivision rule and, unlike parametric surfaces, they can represent surfaces with arbitrary topology [15]. However, subdivision surfaces present some problems concerning the absence of a mechanism by which inner collision can be detected. Solution to some of these problems are presented in De Rose et al. [15].

Nevertheless, traditional surface generation techniques are not capable of guaranteeing global smoothness. Recently, this problem has been overcome by the introduction of partial differential equations as a tool for surface manipulation. Therefore, an overview containing some mathematical details concerning partial differential equations, relevant to this work, is given below.

1.2 Partial differential equations

Partial differential equations (PDE) are equations in which the unknown function depends on a set of partial derivatives of this unknown function with respect to two or more independent variables. For instance, let $U(x, y)$ be the unknown function depending on two independent variables x and y; then, the general form of a second-order partial differential equation is given by

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU = G(x, y), \tag{4}$$

where A, B, C, D, E and F are all general functions of $U(x, y)$, x and y and subscripts denote derivatives. Note that Eq. 4 contains terms with different orders of derivatives.

The importance of such a mathematical tool is that almost every physical phenomenon is modelled by a PDE. For instance, the heat equation in either one or two spatial dimensions describes how heat is distributed in a given length or area, respectively. Other examples of PDEs describing physical phenomena are the wave equation and the Laplace equation. Their use has also been extended to areas such as finance where the Black–Scholes equation models the variation of stock prices with time.

PDEs can be classified according to different features such as:

– *Order*. This is determined by the order of the highest partial derivative present in the equation.
– *Homogeneity*. This feature classifies PDEs as homogeneous and nonhomogeneous according to $G(x, y)$. If this term is identically equal to zero the PDE is said to be homogenous and otherwise is non-homogeneous.
– *Linearity*. A PDE is said to be linear when the coefficients do not depend on $U(x, y)$ and no derivative term is multiplied by any other including itself. It is nonlinear otherwise.

Additionally, linear PDEs can be also classified according to the type of coefficients. Such a classification is divided into three categories: parabolic, hyperbolic and elliptic. For instance, Eq. 4 can fall into any of these categories as follows:

– *Parabolic*. The PDE must satisfy $B^2 - 4AC = 0$.
– *Hyperbolic*. The equation falls into this category if $B^2 - 4AC > 0$.
– *Elliptic*. The partial differential equation is regarded as such when $B^2 - 4AC < 0$.

This classification extends to PDEs of higher order. However, the classification criterion varies depending on the order of the PDE. Additionally this classification has been useful in characterizing the type of phenomenon described by each class of equation.

The task of solving PDEs in general is not easy. However, several methods have been developed for finding their solution. These methods vary from purely analytical schemes to full numerical techniques. The methods available for solving PDEs are beyond the scope of this work and therefore, the reader is referred to [24] for further details.

Now, PDEs have been introduced to areas such as computer graphics and animation where they have been capable of solving a variety of problems [45] very efficiently. Furthermore, this mathematical tool has met the ever increasing demand of realism in the mentioned areas.

This work summarizes all the aspects concerning the use of PDEs as a surface generation technique and it is divided as follows: Sect. 2 provides a brief description of PDE surfaces. Section 3 includes general information about implicit PDE surfaces and lists the most common velocity fields used in computer-aided geometric design. Section 4 describes parametric PDE surfaces and some of the methods available for producing them. Section 5 describes some of the applications of PDEs in geometric design, in particular, in areas associated with computer-aided geometric design whereas some other applications given to PDEs in computer graphics are named in Sect. 6. Finally, Sect. 7 outlines the conclusions of this work.

2 Geometric PDE surfaces

The term PDE surfaces refers to surfaces that have been generated or modified by the solution of a given partial differential equation. These surfaces are the graphical representation of the solution to a given PDE subject to a set of boundary conditions. The advantages offered by the use of PDEs to generate surfaces over other surface generation techniques such as splines or NURBS are numerous, namely:

– Surface generation techniques based on PDEs require a smaller number of parameters than spline-based techniques to represent a given surface since PDE surfaces are characterized by a set of boundary curves, whereas spline-based techniques are defined by a set of control points. Thus, PDE surfaces are more likely to be easier to manipulate than others.
– PDE surfaces automatically guarantee some degree of intrinsic smoothness during blending processes, whereas such smoothness is not necessarily guaranteed when blending surfaces obtained when using spline-based techniques. The smoothness obtained by blending PDE surfaces increases with the order of the PDE giving rise to such a surface.
– PDE-based surface generation techniques potentially unify geometric and physical aspects of surface modelling. This result is particularly useful for engineering design.

The type and order of the PDE to use is generally not restricted. For instance, parabolic nonlinear PDEs of second, fourth and higher order have been used [45] with the aim of describing the evolution of an initial surface subject to the chosen PDE. Notice that the degree of smoothness is determined by the order of the equation. PDE surfaces comprise both implicit and parametric surfaces. Moreover, given the diversity of areas where PDE surfaces have been used, PDE surfaces can also be classified according to the problem they are aiming to solve within the scope of computer-aided geometric design. However, a simple and well defined classification taking into account both criteria simultaneously is very difficult to portray. Thus, a brief description of both implicit and parametric PDE surfaces is outlined in the next two sections, followed by a section entirely dedicated to describe the uses given to PDE surfaces in computer design.

3 Implicit PDE surfaces

Implicit PDE surfaces are those that result from an evolution equation for which its original domain is a pre-established surface; that is, these surfaces are generally calculated as the collection of points p satisfying a given geometric flow [45]. The general representation of the geometric flow is determined by

$$\frac{\partial p}{\partial t} = V(p, t), \tag{5}$$

where $V(p, t)$ denotes an arbitrary velocity field. It is important to stress that the initial surface for which the geometric flow is applied must be a closed and orientable one. Thus, Eq. 5 gives rise to a family of closed and orientable surfaces $S(t)$ determined by

$$\frac{\partial p}{\partial t} = N(p(t))V_n(k_1, k_2, p), \tag{6}$$

where $p(t)$ is a point in $S(t)$, $V_n(k_1, k_2, p)$ and $N(p)$ represent the normal velocity and vector of the surface at p respectively, whilst k_1 and k_2 denote the principal curvatures of $S(t)$.

Several velocity fields have been implemented for studying different problems present in computer-aided geometric design, such as surface blending, N-sided hole filling, free-form surface construction, noise reduction and image inpainting [3, 4, 45]. Examples of the most common velocity fields are listed below and, for the sake of brevity, only the PDE associated with them is given. The uses of these velocity fields will be discussed in a further section.

– *Mean curvature flow.* This velocity flow is described by

$$V_n = -\frac{1}{2}(k_1 + k_2).$$

– *Averaged mean curvature flow.* This flow is given by

$$V_n = \frac{1}{2}(k_1 + k_2) + h(t),$$

where

$$h(t) = \frac{\int_{S(t)} \frac{1}{2}(k_1 + k_2)\, \mathrm{d}\sigma}{\int_{S(t)} \mathrm{d}\sigma}.$$

– *Surface diffusion flow.* A fourth-order velocity field is determined by

$$V_n = \nabla^2 \frac{1}{2}(k_1 + k_2),$$

where ∇^2 represents the Laplacian.

– *Higher-order geometric flows.* The general PDE determining such flows is,

$$V_n = (-1)^{k+1}\nabla^{2k}\frac{1}{2}(k_1 + k_2),$$

where $k \geq 2$.

– *Heat flow.* This flow is described by

$$V_n = (-1)^{k+1}\nabla^{2k} p(t),$$

where $k > 0$ and $p(t)$ represents a point in $S(t)$.

– *Willmore flow.* This is a fourth-order flow given by

$$V_n = \nabla^2(k_1 + k_2) + 2(k_1 + k_2)\big((k_1 + k_2)^2 - K\big),$$

where K represents the Gaussian curvature.

The usual approach for solving PDEs related to problems in computer-aided geometric design consists of using finite differences. Details concerning the spatial and temporal discretizations together with the criteria applied to some other aspects such as the evaluation of boundary conditions, mesh regularization, stopping criteria and the generation of the initial mesh are exhaustively explained in [45].

Generally, the velocity fields are geometry intrinsic; i.e., they are applicable to surfaces with arbitrary topology. Additionally, these velocity fields are volume preserving and in the vast majority of cases, they are area reducing. However, they are volume preserving if and only if the surface to which they are applied is closed. Therefore, in the event of applying them to an open surface with a fixed boundary the area and volume preserving assertions stated before are not necessarily valid.

So far, implicit PDE surfaces result from the evolution of PDEs through time; that is, parabolic PDEs have been discussed. However, implicit PDE surfaces can also be obtained from elliptic PDEs [20]. A brief description on how elliptic PDEs give rise to implicit PDE surfaces is given below, where the most relevant mathematical details are outlined.

3.1 Accounting for the use of elliptic PDEs to obtain implicit PDE surfaces

The use of elliptic PDEs for generating implicit surfaces was introduced with the aim of taking advantage of the powerful parametric PDE formulation avoiding topological restrictions [19]. This type of implicit surface is regarded as the solution to

$$\left(a^2\frac{\partial^2}{\partial x^2}+b^2\frac{\partial^2}{\partial y^2}+c^2\frac{\partial^2}{\partial z^2}\right)^2 \boldsymbol{X}(x, y, z)=0, \tag{7}$$

where x, y and z denote the coordinate system and vary from 0 to 1 whereas a, b and c represent shape control parameters inherent to the PDE.

In principle, four boundary conditions are required to solve Eq. 7; however, analytic solutions, in general, does not exist for solving it and numerical methods are used to obtain an approximation of the surface. For example, standard finite differences schemes are implemented to find the surface satisfying Eq. 7. Therefore, the incorporation of additional constraints, either hard or flexible ones, is very straightforward since such constraints can be represented by additional algebraic equations that can be added to the original system inherent to the finite difference representation of Eq. 7.

The work presented in [19] provides further mathematical details of such a formulation together with some of its uses in computer-aided design, such as shape design, blending, reconstruction of surfaces from either curves or scattered points, sculpting, and the implementation of interactive tools.

4 Parametric PDE surfaces

Parametric PDE surfaces are regarded as the solution to an elliptic partial differential equation in the parametric domain. This is an excellent surface generation technique since the discretization of the operator associated with elliptic PDE is an averaging process of the solution neighbourhood of the PDE guaranteeing that the surface obtained will possess a certain degree of smoothness depending upon the order of the PDE.

Parametric PDEs have proved to be extremely useful for the implementation of surface generation methods and to address problems, such as shape blending [5], optimization [38], interactive design [41] and interactive sculpting [16]. Furthermore, the work presented in [6] shows that parametric PDE surfaces obtained from closed analytic solutions to the generating PDE can be represented in terms of B-splines [6].

For the purposes of illustrating the most relevant mathematical details concerning the formulation of a parametric PDE surface, the Bloor–Wilson PDE method, which is a standard method for surface generation, is discussed below.

4.1 The Bloor–Wilson PDE method

The Bloor–Wilson PDE method has been developed in [10] and was originally introduced as a blending tool [5] from where its use has been extended to several other areas. This method is a surface generation technique overcoming a number of problems inherent to polynomial surfaces. Additionally, it is an excellent choice for free-form surface generation, since it only requires boundary curves as input, which can be determined in a very intuitive manner [7]. A summarized description of the mathematical foundations of the Bloor–Wilson PDE method is outlined below.

In principle, there is no restriction upon the type and order of the PDE to be solved. However, elliptic PDEs have been chosen to develop this technique since this kind of PDE is regarded as an averaging process throughout the entire surface. The order of the PDE determines the smoothness of the surface since the boundary conditions required to solve the PDE are usually given in terms of positional and derivative requirements.

The original formulation of Bloor–Wilson PDE method consists of producing a parametric surface $\boldsymbol{X}(u, v)$ by finding the solution to a PDE of the form

$$\left(\frac{\partial^2}{\partial u^2}+a^2\frac{\partial^2}{\partial v^2}\right)^r \boldsymbol{X}(u, v)=0, \tag{8}$$

where u and v represent the parametric surface coordinates, which are then mapped into the physical space; i.e., $(x(u, v), y(u, v), z(u, v))$, a is a parameter inherent to the PDE mostly restricted to $a \geq 1$ and r determines the order of the PDE.

Equation 8 is a PDE of order $2r$. However, most of the work related to this method is based on fourth-order PDEs; i.e., $r = 2$; therefore four boundary conditions are required. These are generally given by a set of two positional boundary conditions and the value of the first derivative at the same positions. Notice that when $a = 1$ and $r = 2$, Eq. 8 is known as the biharmonic equation, which models some phenomena occurring within areas such as fluid and solid mechanics and therefore, many alternatives for solving it have been developed.

The solution to Eq. 8 can be found using different approaches, varying from analytical to fully numerical ones. However, the selection of full analytical methods gives rise to some topological restrictions on the objects represented by this type of solution. A typical example of a PDE surface obtained by using the Bloor–Wilson PDE method is presented in Fig. 1. The generating boundary curves are shown in Fig. 1a, the top and bottom circles represent the positional boundary conditions whereas the inner ones are used to calculate the value of the derivative boundary conditions. The resulting PDE surface is outlined in Fig. 1b.

The former example shows, in a schematic manner, the foundations of this method, enhancing the fact that only some boundary curves are required to obtain a smooth surface. However, the simplicity of the surface in this example does not give an idea of the surface that can be generated using this method. For this purpose, additional

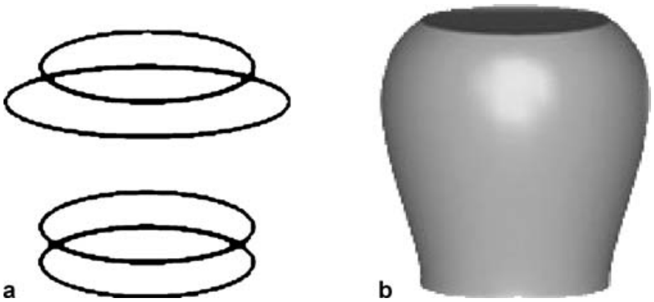

Fig. 1a,b. Example of a PDE surface generated using the Bloor–Wilson PDE method. The boundary curves are shown in **a** and the corresponding PDE surface is shown in **b**

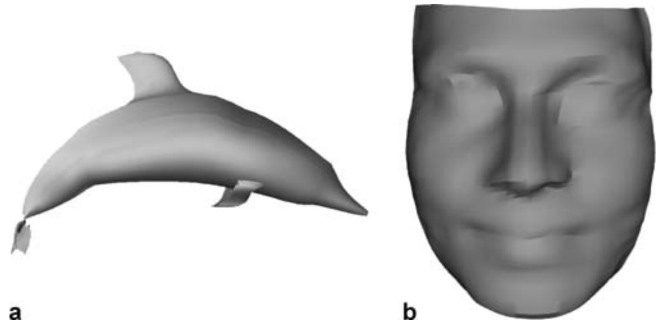

Fig. 3a,b. Examples of a PDE surfaces of complex geometries. These surfaces were obtained by using the Bloor–Wilson PDE method. A representation of a dolphin is shown in **a** and a PDE surface representation of a face is shown in **b**

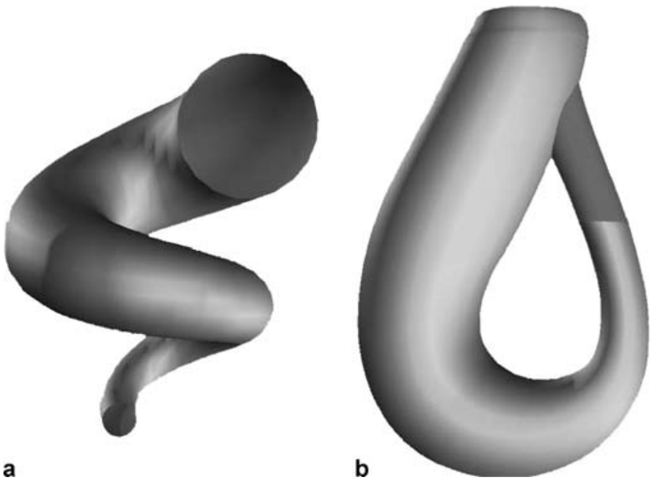

Fig. 2a,b. Examples of a PDE surfaces of geometries possessing mathematical expressions. A PDE surface corresponding to a sea shell is shown in **a** and a Klein bottle is represented in **b**

examples of complex geometries are presented in Figs. 2 and 3. The PDE representation of two objects that can be mathematically characterized are shown in Fig. 2, where a PDE surface corresponding to a sea shell is schematized in Fig. 2a and a surface representation of a Klein bottle is outlined in Fig. 2b.

PDE surface representations of complex geometries are shown in Fig. 3. A PDE surface representation of a dolphin is shown in Fig. 3a, and a PDE surface associated with a human face is shown in Fig. 3b. These PDE surfaces have been obtained by blending several surface patches.

It is noteworthy to mention that when the domain is restricted to $0 \geq u \geq 1$ and $0 \geq v \geq 2\pi$, the solution is also restricted to the use of periodic boundary conditions. The solution to Eq. 8 can then be expressed in terms of a Fourier series. The full details of such a solution are presented in [10]. The Fourier series associated with the solution of Eq. 8 is in general an infinite series and therefore, the solution is approximate with the property of exactly satisfying the boundary conditions through the addition of a remainder term. However, the solution is exact if all the boundary conditions can be expressed in terms of a finite Fourier series.

As for the case when the domain consists of the rectangular region restricted by $0 \geq u \geq 1$ and $0 \geq v \geq 1$, an eigenvalue solution has been developed. The mathematical details of this solution are presented in [11].

Notice that the method itself is not restricted to these cases since full numerical techniques such as finite differences or finite element can be employed for finding the solution to Eq. 8.

Furthermore, the compatibility between surfaces generated by the PDE method and the ones generated by more traditional techniques such as B-splines and Bézier surfaces has been thoroughly studied. Works such as [32] and [33] present Bézier solutions to elliptic PDEs. PDE surfaces have also been shown to be compatible with B-spline representations [6].

4.2 Alternatives to the Bloor–Wilson PDE method

Variations of the PDE method formulated by Bloor and Wilson have been accomplished. For instance, the work presented in [47] uses the following elliptic PDE,:

$$\left(a\frac{\partial^6}{\partial u^6} + b\frac{\partial^6}{\partial u^4 \partial v^2} + c\frac{\partial^6}{\partial u^2 \partial v^4} + d\frac{\partial^6}{\partial v^6} \right) \boldsymbol{X}(u,v) = 0, \quad (9)$$

where a, b, c and d are shape control parameters.

The solution to Eq. 9 is similar to that presented in [10] with the difference that the series is given in terms of a set of basis functions (this set is found according to the set of boundary conditions for each problem in particular). This formulation offers three additional shape control parameters, represented by a, b and c, which may be thought an improvement; however, the added advantage of having such parameters is difficult to assess since no physical meaning can be associated with them. Eq. 9 has also been used for blending surfaces where curvature continuity is guaranteed [46].

4.3 Parametric PDE surfaces obtained from physics-based models

The imposition of energy constraints to parametric surfaces has proved useful in application where local and global deformations are required. Such constraints have been applied under a number of different circumstances. Some examples of this are,

- B-spline surfaces have been sculpted interactively by applying linear constraints such as local pressures and sectional forces [12, 13].
- Elasticity theory has been employed to manipulate surfaces representing flexible materials, providing the model with more realism especially when such a model is used for animation purposes. The work presented in [37] shows different examples, whereby different energy constraints have been imposed according to the elastic properties required by each example.
- Geometric constraints have also been employed. Such constraints are expressed as energy functions which are then responsible for parametrically deforming a given geometric model. Such a deformation may take place either locally or globally, depending on the type of constraint employed [2, 44].

Elliptic PDEs have also been used for generating parametric PDE surfaces satisfying physics-based models where the acceleration and velocity of the surface can be included so that the surface is allowed to deform according to external forces [17]. This approach introduces new general and flexible constraints; and, the system can be solved using finite differences. This technique has successfully achieved local and global deformations of PDE-based surface models such as sculpting and blending [21]. This approach expands the topological limitations of the original formulation of the Bloor–Wilson PDE method.

A nonlinear second-order system of differential equations describes the behaviour of a physics-based model, which in general form is given by

$$M\frac{\partial^2 \boldsymbol{X}}{\partial t^2} + D\frac{\partial \boldsymbol{X}}{\partial t} + K\boldsymbol{X} = \boldsymbol{f}, \tag{10}$$

where M, D and K account for the mass, damping and stiffness matrices associated with the object respectively and f represent the total external force acting on the surface.

Equation 8 can be rewritten as

$$A\boldsymbol{X} = \boldsymbol{g}, \tag{11}$$

where A is the matrix associated with the finite difference expansion and $\boldsymbol{g}$ is vector. Thus, the dynamic version of the surface is given by

$$M\frac{\partial^2 \boldsymbol{X}}{\partial t^2} + D\frac{\partial \boldsymbol{X}}{\partial t} + (K + A)\boldsymbol{X} = \boldsymbol{f} + \boldsymbol{g}. \tag{12}$$

Again, semi-implicit finite difference methods can be employed to find the solution to Eq. 12. Further details concerning the physics-based formulation are given in [18].

Another type of physics-based parametric PDE surface is dynamic NURBS, also known as D-NURBS. They are a generalization of NURBS in which physical properties such as mass distribution, internal deformation energies and external forces are incorporated to the model. This incorporation leads to solving a set of nonlinear differential equations by integrating them numerically [35]. This type of PDE surfaces avoids the complications inherent in the manipulation of standard NURBS, since the designer is no longer required to manipulate control points directly.

Variational geometry formulations have also been used to produce parametric PDE surfaces [43] where different geometric constraints on the resulting surface can be imposed. These constraints are generally related to the smoothness of the surface and are particularly useful in surface fairing [29].

5 Applications of PDE surfaces

PDE surfaces have been successfully employed for developing techniques relevant to computer-aided geometric design. Given the versatility with which PDE surfaces can be formulated to address such techniques, there are occasions where more than one formulation is available for solving a specific problem and in some cases, both implicit and parametric PDE surfaces can be employed in the same given problem. Thus, the classification of PDE surfaces according to their use in computer-aided geometric design is given by providing a brief description of the problem followed by the different PDE formulations developed so far addressing such a problem.

5.1 Surface generation

The increasing demand of realism and real-time applications in computer-aided geometric design has resulted in the constant development of efficient surface generation techniques. PDE surfaces have proven to be powerful in such tasks and given the versatility of this kind of surface, several techniques have been developed. Two are the main areas where PDEs have been introduced as a surface generation tool are interactive design (also known as free-form design) and blending.

5.1.1 Interactive design

Computer-aided design (CAD) systems can benefit from interactive design tools based on PDE formulations where the main control of the surface is gained through the manipulation of the boundary conditions. An important remark must be made stressing that the user requires no familiarity with the mathematical details concerning PDEs since the user is only required to define the boundary conditions in terms of curves. Both implicit and parametric PDE surfaces have been used as the foundations of tools promoting interactive design.

Implicit PDE surfaces arising from elliptic PDEs have been used in [19] where either scattered points, cross-sectional or sketch curves outlining the rough shape of the object are chosen as boundary conditions. Additionally, [19] presents graphical examples obtained for these three types of boundary conditions.

Parametric PDE surfaces have also been used for interactive design purposes. The PDE method formulated by Bloor and Wilson has served as the foundation for the development of surface generation interactive tools. The work presented in [28] uses a sixth-order elliptical PDE for this purpose and gives a detailed explanation regarding the mathematics inherent in the method. For instance, the procedure by which the required derivative conditions are obtained is outlined. Given that this interactive tool has been designed assuming that the user possesses no mathematical background, the user is required to generate six boundary curves from which the interactive tool calculates the required derivatives. Free-form generation examples of objects such as a ship hull, a phone handset and a marine propeller are discussed in [7] where explicit mathematical functions determine the boundary conditions.

5.1.2 Shape blending

Shape blending refers to the process by which two or more surface patches are joined. This process must be carried out in a way such that a certain degree of smoothness is achieved at the regions where these patches join. PDEs offer a natural approach for addressing this problem. The degree of smoothness is determined by the order of the PDE in use.

Parabolic PDEs have been used for surface blending, obtaining excellent results. For instance, the work outlined in [45] presents the results of blending three cylinders at different angles. Three different velocity fields have been used to carry out the blending process namely: mean curvature flow, fourth- and sixth-order flows. These results are clearly illustrated by the graphical results. However a number of parameters are necessary to achieve these results.

By contrast, the blending properties of the Bloor–Wilson PDE method are such that the only requirement to achieve a fast and smooth blend between two surface patches is either a common boundary condition at the joining region or a boundary condition lying on one of the surface patches itself. Works such as [5, 30] describe the use of this PDE method for achieving smooth blends. The reader is referred to [5], where several examples of second-, fourth- and mixed-order blends are detailed that are easy to reproduce.

The alternative formulation of the Bloor–Wilson PDE method has also been used for surface blending [46] where a sixth-order elliptic PDE has been chosen for accomplishing such a purpose. Comparisons between closed-form solutions and the one proposed in this work are shown together with some interesting examples, one of which involves the blending of a wrinkled surface with a conic section.

5.2 Surface processing

Another major category in which the use of PDE surfaces in computer-aided geometric design is surface processing. The phenomena falling into this category are those concerning the total or partial enhancing of an already existing surface. Image inpainting, noise reduction, N-sided hole filling and surface fairing are some of the most relevant processes.

5.2.1 Image inpainting

Image inpainting is defined as the technique of modifying an image in an unnoticeable manner. Implicit PDE-based methods have been adopted for such purpose. In particular, Willmore-type flows have been used in [4]. This work presents some examples illustrating how this technique has been successfully employed to remove unwanted objects from pictures. However, there are some restrictions on the size of the region where this technique is applied when the texture of such a region needs to be preserved.

5.2.2 Noise reduction

Noise reduction is another area where PDE surfaces have been used. In particular, parabolic PDEs giving rise to implicit PDE surfaces have proved to be a useful tool for reducing noise whilst preserving the image details. Examples of some noise-reducing surface diffusion flows are presented in [3] where two approaches for the problem are outlined. The first of these approaches consists of using an isotropic diffusion flow; i.e., the flow acts uniformly along every direction. The graphical example associated with this flow illustrates how the noise within affected areas is gradually removed. The second approach exploits the advantages of anisotropic diffusion flows, which act differently along different directions. Again, it can be appreciated how the noise is removed leaving the details of the image unaffected. Diffusion flows are also used in [31]. However, these flows are used for mapping noisy images into three-dimensional implicit surfaces. The work presented in [1] is another example on how anisotropic diffusion is used as a noise reduction tool. The latter is capable of smoothing surface successfully, whilst noise is also reduced, which is particularly useful in cases where the surface is generated from unfiltered or corrupted data.

5.2.3 N-sided hole filling

The problems of N-sided hole filling consists of constructing a surface to fill such a hole with specific continuity requirements at the boundary. Velocity fields, such as mean curvature flow, fourth- and sixth-order flows, have been employed to construct implicit PDE surfaces, aiming to fill such holes successfully. An example to this effect is presented in [45], where the nose of a human face is reconstructed. As expected, the results obtained by the sixt-order flow are the smoothest.

5.2.4 Surface fairing

Surface fairing accounts for the process of generating free-form surfaces satisfying aesthetics requirements. Implicit PDE surfaces resulting from second- and fourth-order flows have been employed in [36] providing an excellent example of an algorithm for smoothing arbitrary triangular meshes. This algorithm uses a fourth-order velocity field. Each of the steps considered in the algorithm is outlined and the examples presented show the potential of the technique. For the interested readers in the pragmatic mathematical details of the implicit PDE formulation, this work is highly recommended as a further reading.

5.3 Design analysis and optimization

Geometric PDE surfaces offer a natural environment for developing tools capable of carrying out design analysis and optimization based on the physical properties inherent to a specific problem. The process of optimizing the shape of an object involves a target function establishing the requirements to be fulfilled. This target function is given in terms of a set of design parameters that are iteratively changed until the target function is satisfied.

Furthermore, PDE-based optimization formulations present a significant time reduction when compared to other optimization techniques available. Examples of the progress achieved by using PDE surfaces in these areas are presented in [9, 14, 27, 42] and [38], all of which are based on the Bloor–Wilson PDE method.

5.3.1 Design analysis

Such design analysis is carried out in [14, 27] and [9], where parameters characterizing a given specific object are found. For instance, [27] and [9] focus on the extraction of parameters characterizing aircraft geometry. The former aims to extract the design geometry from a given geometry, whereas the latter illustrates the changes of geometry by changing some of the design parameters. The work presented in [14] portrays the characterization of a marine propeller using a small set of parameters and a sixth-order version of the Bloor–Wilson PDE formulation. Emphasis must be made that these works manipulate the set of parameters defining each object for purposes of illustration.

The work presented in [8] portrays another application related to design analysis. The Bloor–Wilson PDE method is employed here to visualize the physical properties or functional performance of a given object. Temperature and stress distributions are some of these properties. Here, the target function mathematically describes the physical property of interest and no change is made to the set of design parameters.

5.3.2 Optimization

By contrast, optimization processes are required to satisfy a target function; thus, the manipulation of the design parameters is oriented towards this purpose. Again, the method formulated by Bloor and Wilson has proved to be useful for such purposes as shown in [42] and [38]. In particular, [42] presents an example with biological applications. This example predicts stable structures of vesicles by using the surface energy of the membrane as the target function. Industrial applications have also been found. For example, the work presented in [38] describes the optimal design of yoghurt containers where their thickness is minimized subject to constraints on the stress distribution around the walls of the container.

5.4 Other applications

There are a number of applications that uses PDE surfaces. However, given the nature of such applications, their classification is not a trivial exercise. Among such applications are subdivision, geometric manipulations and animation.

5.4.1 Subdivision

Alternative surface generation techniques such as subdivision can also benefit from the versatility offered by PDE surfaces. Subdivision is a surface generation technique by which an original set of points defining a rough-shaped object is modified by adding additional surface points according to a given subdivision rule. In spite of presenting a certain degree of smoothness, subdivision surfaces lack collision detection mechanisms, and in some cases this leads to unwanted overlapping from which topological restrictions are imposed [15]. Implicit PDE surfaces have been employed to address this problem; thus, several velocity flows have been tried. Gaussian curvature, surface diffusion, and Willmore flows have been tried in [45] for addressing this problem. This work outlines the necessary mathematical background together with some graphical examples where the different flows are employed to evolve the surfaces after subdividing the surface one.

Parametric surfaces have also been employed in subdivision processes. For instance, the work presented in [15] presents an alternative for modelling the dynamics of clothes in animation where an energy functional is proposed and finite element techniques are used to find the solution.

5.4.2 Animation

Animation is an area where PDE surfaces are potentially useful. Processes such as morphing, facial expression and movement are some of the processes that can directly benefit from their use. One of the reasons why PDE surfaces represent a powerful mechanism for overcoming the limitations inherent to these problems is the number of parameters describing a given PDE surface is relatively small. A mathematical characterization of such a process can be obtained through the manipulation of the PDE or its boundary conditions.

As mentioned before, cloth dynamics in animation has been modelled in [15]; however, this approach does not take full advantage of the PDE formulation.

Morphing refers to the process by which an object is transformed into another and, in the vast majority of cases, this process is required to take place as smoothly and aesthetically as possible. Parametric PDE surfaces are especially useful for such purposes since objects are basically determined by a set of boundary conditions. Thus, a suitable parameterized combination of the boundary conditions of the two objects to be morphed will lead to a smooth and fast transition between the objects. The Bloor–Wilson PDE method in particular offers some other alternatives when restricted to the periodic case.

The mathematical characterization of the movement of an object in animation will significantly reduce time and work involved in such a process. Again, the Bloor–Wilson PDE method is regarded as an excellent choice to accomplish this purpose. Both morphing and characterization of movement can take advantage of mathematical properties inherent to the solution of Eq. 8 restricted to the exclusive use of periodic boundary conditions where the resulting PDE surfaces are characterized by the spine. Mathematically speaking, the spine of a surface is a curve described by a polynomial function, which can be thought as the skeleton of the PDE surface [39]. The spine of a PDE surface has been proved to be a powerful tool for geometric manipulations of the entire surface [40].

6 Other aspects of computer graphics related to PDEs

Computer graphics industry has recently been highly interested in the simulation of natural phenomena since the increasing demand for special effects concerning the movement of smoke, water and fire among others have posed some interesting problems. The physics associated with these phenomena are described by very complicated PDEs for which only elaborate and time-consuming numerical solutions are available. Thus, the task of simulating such phenomena in computer graphics is not a simple one since the PDE associated with a particular phenomenon has to be simplified so that a certain amount of realism is kept, solved in virtually real time and properly applied to the graphics environment. Level set methods have proved useful in addressing these problems [25].

6.1 Water

One of the greatest challenges in computer graphics is the animation of water or other liquids. The specific scenarios where such animation may take place vary from cascades to pouring liquids into transparent containers. The key feature to achieve this kind of modelling consists of accurately separating the liquid from the air. A solution to this problem is presented in [22]. An approach called the *particle level set method* has been employed where the front of the water surface is thickened. For the sake of brevity, the reader is referred to this work if he or she is interested in details of the mathematical formulation. The results obtained by this technique are illustrated by examples in which a glass is filled with water, achieving an acceptable degree of realism.

6.2 Smoke

Realistic modelling of smoke is still regarded as a complex and challenging problem due to the complex nature inherent in the motion of gases. Computer-based fluid dynamics algorithms which enable one to model such a phenomenon that can be modified and adapted to computer graphics and animation are already available in the literature. The work described in [26] outlines an algorithm that uses Euler equations for inviscid incompressible fluids and solves them using a technique called *vorticity confinement*, which is extensively found in computer fluid dynamics literature. This technique is capable of retaining the short-scale features inherent in smoke, such as rolling features.

6.3 Fire

The intrinsically dangerous nature of fire is one of the reasons for which its modelling is constantly being improved. Combustion processes are characterized by the expansion of fuel, which is responsible for the turbulence associated with this process. The method proposed in [34] uses a dynamic implicit surface to represent the reaction zone, where the thin film approximation produces acceptable visual results. The reader is referred to [34] for further details where the modelling of this process involves several stages which require careful consideration.

7 Conclusions

This work presents a synopsis of the uses given to partial differential equations in areas related to computer-aided geometric design. PDEs are a very powerful tool in geometric design since some degree of smoothness is guaranteed, depending on the order of the PDE generating or modifying a surface. PDE surfaces are mainly classified as implicit or parametric PDE surfaces. Implicit PDE surfaces are generally obtained from parabolic PDEs whereas parametric PDE surfaces are associated with elliptic ones.

A brief description of some of the flows employed to find implicit PDE surfaces together with the problems each of these flows address has been given. The latter vary from surface generation to noise reduction and fairing. As far as parametric PDE surfaces are concerned, the Bloor–Wilson PDE method, which is based on the solution to an elliptic PDE, has proved to be a powerful tool for interactive surface generation since a very small number of parameters is required to characterize the surface. Alternative formulations to the Bloor–Wilson PDE method have been presented where physics-based models have been incorporated.

Elliptic PDEs have also been used to generate implicit PDE surfaces with the aim of taking advantage of the parametric PDE formulation without any restrictions on the topology of the object to be created. However, numerical techniques are likely to be used to find their solution.

PDEs have also been used for addressing other problems related to surfaces in computer graphics. The graphical modelling of natural phenomena such as water, smoke and fire is not a simple task since the PDE equations modelling these phenomena in the real world are very complicated. Thus, the aim is to simplify these equations so that they can be solved rapidly and accurately enough to preserve the realism with which these phenomena behave.

PDEs are powerful tools for applications related to geometric modelling. These limitations are directly associated with the mathematical challenges related to the stability and accuracy of numerical PDEs together with the speed with which such solutions can be obtained. Additionally, like polynomial surfaces, most PDE methods generate parametric surfaces and therefore problems inherent to parametric representation of surfaces can be considered as limitations. For example it is difficult to represent objects with arbitrary topology using parametric PDEs.

The introduction of PDEs to computer-aided geometric design has occurred fairly recently and therefore their full potential remains to be fully exploited, offering further lines of investigation where the mathematical properties of PDEs can be fully used.

Acknowledgement The authors wish to acknowledge the support received from the UK Engineering and Physical Sciences Research Council through the grants EP/C015118/1 and EP/D000017/1 by which this work has been endeavoured.

References

1. Bajaj, C., Xu, G.: Anisotropic diffusion on surfaces and functions on surfaces. ACM Trans. Graph. **22**, 4–32 (2003)
2. Barr, A.: Global and local deformations of solid primitives. Comput. Graph. **18**, 21–30 (1984)
3. Bertalmio, M., Cheng, L., Osher, S., Sapiro, S.: Variational problems and partial differential equations on implicit surfaces. J. Comput. Phys. **174**(2), 759–780 (2001)
4. Bertalmio, M., Sapiro, G., Caselles, V., Ballester, C.: Image inpainting, pp. 417–424. SIGGRAPH 2000, New Orleans (2000)
5. Bloor, M.I.G., Wilson, M.J.: Generating blend surfaces using partial differential equations. Comput. Aided Des. **21**(3), 165–171 (1989)
6. Bloor, M.I.G., Wilson, M.J.: Representing PDE surfac es in terms of B-splines. Comput. Aided Des. **22**(6), 324–331 (1990)
7. Bloor, M.I.G., Wilson, M.J.: Using partial differential equations to generate free-form surfaces. Comput. Aided Des. **22**(4), 202–212 (1990)
8. Bloor, M.I.G., Wilson, M.J.: Functionality in solids obtained from partial differential equations. Computing **8**, 21–42 (1993)
9. Bloor, M.I.G., Wilson, M.J.: The efficient parametrization of generic aircraft geometry. J. Aircraft **32**(6), 1269–1275 (1995)
10. Bloor, M.I.G., Wilson, M.J.: Spectral approximations to PDE surfaces. Comput. Aided Des. **28**(2), 145–152 (1996)
11. Bloor, M.I.G., Wilson, M.J.: An analytic pseudo-spectral method to generate a regular 4-sided PDE surface patch. Comput. Aided Geom. Des. **22**(3), 203–219 (2005)
12. Celniker, G., Gossard, D.C.: Deformable curve and surface finite-elements for free-form shape design. Comput. Graph. **25**, 257–266 (1991)
13. Celniker, G., Welch, W.: Linear constraints for deformable non-uniform B-spline surfaces. In: SI3D '92: Proceedings of the 1992 symposium on Interactive 3D graphics, pp. 165–170. ACM Press, New York, NY (1992)
14. Dekanski, C.W., Bloor, M.I.G., Wilson, M.J.: The representation of marine propeller blades using the PDE method. J. Ship Res. **38**(2), 108–116 (1995)
15. DeRose, T., Kass, M., Truong, T.: Subdivision surfaces in character animation, pp. 85–94. SIGGRAPH 1999, New York, NY (1999)
16. Du, H., Qin, H.: Direct manipulation and interactive sculpting of PDE surfaces.

Comput. Graph. Forum **19**(3), C261–C270 (2000a)
17. Du, H., Qin, H.: Dynamic PDE Surfaces with Flexible and General Geometric Constraints, pp. 213–222. Pacific Graphics 2000, Hong Kong (2000b)
18. Du, H., Qin, H.: Integrating Physics-Based Modeling with PDE Solids for Geometric Design, pp. 198–207. Pacific Graphics 2001, Tokyo (2001)
19. Du, H., Qin, H.: Interactive shape design using volumetric implicit PDEs. In: SMI '03: Proceedings of the Eighth ACM Symposium on Solid Modeling and Applications, pp. 235–246. ACM Press, New York, NY (2003)
20. Du, H., Qin, H.: A shape design system using volumetric implicit PDEs. Comput. Aided Des. **36**(11), 1101–1116 (2004)
21. Du, H., Qin, H.: Dynamic PDE-based surface design using geometric and physical constraints. Graph. Models **67**, 43–71 (2005)
22. Enright, D., Marschner, S., Fedkiw, R.: Animation and rendering of complex water surfaces. In: SIGGRAPH '02: Proceedings of the 29th Annual Conference on Computer Graphics and Interactive Techniques, pp. 736–744. ACM Press, New York, NY (2002)
23. Farin, G.: Curves and Surfaces for Computer Aided Geometric Design, a Practical Guide, 5th edn. Morgan Kaufmann, San Diego, CA (2001)
24. Farlow, S.J.: Partial Differential Equations for Scientists and Enginneers, 1st edn. Dover Publications Inc., Mineaol, NY (1993)
25. Fedkiw, R.: Simulating Natural Phenomena for Computer Graphics, Geometric Level Set Methods in Imaging, Vision and Graphics, 5th edn. Springer, New York, NY (2003)
26. Fedkiw, R., Aslam, T., Jensen, H.: Visual simulation of smoke. In: Siggraph 2001 Annual Conference, pp. 23–30. ACM, New York, NY (2001)
27. Huband, J., Li, W.: Extracting design parameters from airplane wing data by using Bloor–Wilson PDE surface model. Math. Eng. Ind. **8**, 239–252 (2001)
28. Kubiesa, S., Ugail, H., Wilson, M.J.: Interactive design using higher order PDEs. Visual Comput. **20**, 682–693 (2004)
29. Lin, V.C., Gossard, D.C., Light, R.A.: Variational geometry in computer-aided design. Comput. Graph. **15**, 171–177 (1981)
30. Lowe, T.W., Bloor, M.I.G., Wilson, M.J.: Functionality in blend design. In: Computer Aided and Computational Design ASME, vol. 1, pp. 43–50. Butterworth, New York (1990)
31. Mémoli, F., Sapiro, G., Osher, S.: Solving variational problems and partial differential equations mapping into general target manifolds. J. Comput. Phys. **195**(1), 263–292 (2004)
32. Monterde, J., Ugail, H.: On harmonic and biharmonic Bézier surfaces. Comput. Aided Geom. Des. **21**(7), 607–715 (2004)
33. Monterde, J., Ugail, H.: A general fourth order PDE method to generate Bézier surfaces from the boundary. Comput. Aided Geom. Des. **23**, 208–225 (2006)
34. Nguyen, D., Fedkiw, R., Jensen, H.: Physically based modeling and animation of fire. In: Siggraph 2002 Annual Conference. ACM, New York, NY (2002)
35. Qin, H., Terzopoulos, D.: D-NURBS: A physics-based framework for geometric design. IEEE Trans. Vis. Comput. Graph. **2**(1), 85–96 (1996)
36. Schneider, R., Kobbelt, L.: Geometric fairing of irregular meshes for free-form surface design. Comput. Aided Geom. Des. **18**(4), 359–379 (2001)
37. Terzopoulos, D., Platt, J., Barr, A., Fleischer, K.: Elastically deformable models. Comput. Graph. **21**, 205–214 (1987)
38. Ugail, H.: Parametric design and optimisation of thin-walled structure for food packaging. J. Optimisation Eng. **4**(4), 291–307 (2003a)
39. Ugail, H.: On the Spine of a PDE Surface, pp. 366–376. Springer, New York, NY (2003b)
40. Ugail, H.: Spine based shape parametrisation for PDE surfaces. Computing **72**, 195–204 (2004)
41. Ugail, H., Bloor, M.I.G., Wilson, M.J.: Techniques for interactive design using the PDE method. ACM Trans. Graph. **18**(2), 195–212 (1999)
42. Ugail, H., Wilson, M.J.: Efficient shape parametrisation for automatic design optimisation using a partial differential equation formulation. Comput. Struct. **81**(29), 2601–2609 (2003)
43. Westgaard, G., Nowacki, H.: Construction of fair surfaces over irregular meshes. In: SMA '01: Proceedings of the Sixth ACM Symposium on Solid Modeling and Applications, pp. 88–98. ACM Press, New York, NY (2001)
44. Witkin, A., Fleischer, K., Barr, A.: Energy constraints on parameterized models. Comput. Graph. **21**, 225–232 (1987)
45. Xu, G., Pan, Q., Bajaj, C.L.: Discrete surface modelling using partial differential equations. Comput. Aided Geom. Des. **23**, 125–145 (2006)
46. You, L., Comninos, P., Zhang, J.J.: PDE blending surfaces with C2 continuity. Comput. Graph. **28**(6), 895–906 (2004)
47. Zhang, J.J., You, L.: Fast surface modelling using a 6th order PDE. Comput. Graph. Forum **23**(3), 311–320 (2004)




GABRIELA GONZÁLEZ CASTRO obtained her bachelor's degree in physics from UNAM in Mexico and her PhD in industrial applied mathematics from the University of Southampton in the United Kingdom. Currently, she is a postdoctoral research assistant at the School of Informatics of the University of Bradford in the United Kingdom.

HASSAN UGAIL received his first degree in mathematics and a postgraduate degree in education from King's College London in 1995 and 1996, respectively. Between 1996 and 1999 he was a research student at the School of Mathematics at University of Leeds and received his PhD in 2000. He then worked at the School of Mathematics at University of Leeds as a postdoctoral research fellow until 2002. He was appointed to a lecturer at the School of Informatics at University of Bradford in 2002. He is currently a senior lecturer and head of distributed virtual environments research at Bradford.

PHILIP WILLIS has been active in computer graphics for 30 years. He was until 1997 one of the two chief editors of Computer Graphics Forum, the academic journal of the Eurographics Association. He then became the professional board chair and a member of the executive board, moving to be its chair. His research interests include human animation, 2++D animation, digital image compositing, and visualisation. His recent human animation work is based on an emotional model, used to influence an animated face. Professor Willis leads the Media Technology Research Centre at Bath, which includes prize-winning work in vision as well as in graphics.

IAN PALMER is a senior lecturer in digital media and Head of Department of Electronic Imaging and Media Communications at the School of Informatics of the University of Bradford in the United Kingdom. His research interests centre on computer animation and virtual environments, including the use of artificial intelligence to improve the usability and realism of interactive 3D environments. Previous work has included behavioural animation, the use of sphere-trees for collision detection and multiprocessing architectures for computer animation. Recent work has been in the areas of modified A* algorithms for dynamic, multi-agent environments, action recognition for response and performance monitoring in intelligent virtual environments and the use of multi-modal interfaces for 3D computer games.