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ORIGINAL ARTICLE

A monotone iterative technique combined to finite element method for solving reaction-diffusion problems pertaining to non-integer derivative

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Abstract

This paper focuses on some mathematical and numerical aspects of reaction-diffusion problems pertaining to non-integer time derivatives using the well-known method of lower and upper solutions combined with the monotone iterative technique. First, we study the existence and uniqueness of weak solutions of the proposed models, then we prove some comparison results. Besides, linear finite element spaces on triangles are used to discretize the problem in space, whereas the generalized backward-Euler method is adopted to approximate the time non-integer derivative. Furthermore, the idea of this method is to construct two sequences of solutions of a linear initial value problem which are easier to compute and converge to the solution of the nonlinear problem. We show numerically through two examples that this convergence requires only few iterations. Some well-known examples with exact solutions and numerical results based on the finite element method in 2D are provided to validate the theoretical results. As a result, we confirm that the proposed method is efficient and easy to use to overcome the convergence and stability difficulties.

Keywords Reaction-diffusion problems · Non-integer derivative · Upper and lower solutions · Monotone iterative technique · Finite element method

Mathematics Subject Classification 35R11 · 35-XX · 34K37 · 47J35 · 58J35 · 35K55 · 35K91

1 Introduction

Partial differential equations (PDEs) with non-integer order derivatives have gained much more attention in the last years. Such derivatives have been applied recently to many problems in several fields, such as physics [1–6], engineering mechanics [7, 8], biology and ecology [9–13], epidemiology [14–19], stochastic process [20, 21] image processing [22–24], signal processing [25], medical imaging [26] chaos theory [27–29] and others.

In the present investigation, we consider the following initial boundary value problem with non-integer order time derivative

$$\begin{cases} \dfrac{\partial^\beta u}{\partial t^\beta}(t,x) + \mathcal{P}u(t,x) = f(t,x,u(t,x)) & \text{in } Q, \\ u(t,x) = g(t,x) & \text{on } \Gamma, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega, Q = (0,T)\times\Omega$ and $\Gamma = (0,T)\times\partial\Omega, T > 0$. $\mathcal{P}$ denotes the second-order partial differential operator in the divergence form

$$\mathcal{P}u(t,x) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial u}{\partial x_j}\right) + b(t,x)u,$$

with coefficients $a_{ij}, c \in L^\infty(Q)$ and

$$\sum_{i,j}^{N} a_{ij}(t,x)\xi_i\xi_j \geq \alpha|\xi|^2, \qquad \text{a.e. in } Q, \alpha > 0, \tag{1.2}$$

The main idea is to study the situation when the involved nonlinear function f admits a splitting of a difference of two monotone functions or equivalently, of a sum of two functions f_1 and f_2, i.e.

$$f(t,x,u) = f_1(t,x,u) + f_2(t,x,u),$$

which one of them is increasing and the other one is decreasing. The functions $f_1, f_2 : Q\times\mathbb{R} \to \mathbb{R}$ are of Caratheodory type, that is, f_1 and f_2 are measurable in $(t,x) \in Q$ for each $u \in R$ and continuous in u for a.e. $(t,x) \in Q$.

Generally, in applied mathematics it is difficult (if not impossible) to obtain exact solutions for most nonlinear partial differential equations (PDEs) with non-integer derivative. It is, therefore, an urgent necessity to look for efficient numerical methods to solve these types of PDEs. Indeed, several numerical methods have been proposed in the literature to solve such equations, the three best known are finite difference method (FDM), finite element method (FEM) and finite volume method (FVM). In [30], the authors have suggested a one and two dimensions finite difference method to numerically approximate the solutions of reaction-diffusion equations with Caputo time-fractional derivatives. Nonstandard finite difference method and Spectral collocation method were used in [31] to obtain numerical solutions of fractional-order advection-dispersion problem. Reference [32] used the Legendre spectral finite difference method to solve non-linear reaction-diffusion equation and non-linear Burger's-Huxleye quation with non-integer derivative in the ABC sense. In [33], the authors applied the finite difference method for solving parabolic equation with time Caputo derivative and fractional Laplacian. Error estimates for a semi-discrete finite element method for fractional order parabolic equations was obtained in [34]. The adaptive finite element method for fractional differential equations using hierarchical matrices was studied in [35]. In [36], the study used the finite element method to investigate the solutions of a space-time fractional Fokker-Planck equation. A Petrov–Galerkin finite element method was used by Ref. [37] to solve the convection-diffusion equations with non-integer derivatives. The authors of [38] presented the time discontinuous finite element method for numerical solution for the diffusion-wave equation with fractional derivative. In the Ref. [39], it was developed a finite Element Method for the numerical resolution of Space-Time Fractional Fokker-Planck Equation. Finally, [2] applied a finite element scheme with Newton's method for solving the time-fractional nonlinear diffusion equation using fractional Crank-Nicolson scheme. Some other outstanding studies PDEs with non-integer derivatives adopting finite element method have been made in [40–47, 47] and related references.

The best computational algorithms for a numerical solution are the numerically stable and convergent schemes, for problem (1.1) this is guaranteed easily if the second member f does not depend on the unknown function u. For nonlinear systems, most of works cited above and in related references in the literature use explicit approximation for the nonlinear term f in (1.1). Such formulation gives a linear finite difference approximation at each time step and is, therefore, more suitable for digital processing. Unfortunately, this approach can also lead to incorrect or misleading information about the solution, see for example [48, 49]. The problem of stability can be solved by using the Newton method for the implicit scheme, several works in this direction were carried out, see for example [50]. Furthermore, Newton's method has two drawbacks, the first is for solving some real multiphysical problems is the need for evaluating the entries of the Jacobian matrix. The second is that the Newton method is locally convergent.

We propose in this paper an iterative scheme to prove the existence and uniqueness of weak solutions of problem (1.1) as well as for the computation of numerical solutions. Furthermore, the iterative processes given in this article give two monotone sequences and are not more complicated than the processes used for the corresponding linear parabolic system when the spatial dimension is greater than one. This type of monotone iteration process was used firstly for elliptical equations in [51, 52], and for parabolic equations in [53]. Similar methods have been used in [54–58] for systems of parabolic and elliptical equations, but are mainly limited to problems with classical derivatives and by the finite difference method for numerical solutions. In this paper, we develop a monotone iterative method with upper and lower solutions for a parabolic problems with non-integer derivative, moreover we present a numerical scheme that combines this method to the finite element method. We prove firstly the existence and uniqueness of the weak solutions for the Nonlinear reaction-diffusion equation with non integer derivative. We show that the monotone sequences, which are solutions of the linear equation converge to the minimal and maximal solutions of the nonlinear equation, these comparison results are used to establish the last result. In addition, to compute the numerical solution, the linear P_1 finite element spaces on triangles are used for the space discretization, where the generalized backward-Euler method is used to approximate the time non-integer derivative.

The layout of this paper is as follow: Sect. 2 provides the definitions and preliminary results to be used in the article. In Sect. 3, main results are stated and proved: comparison results are obtained and the existence of a weak solution for the nonlinear equations (1.1) are proved. Some techniques to construct the lower and upper solutions are given in Sect. 4, where some application examples of the results are presented in Sect. 5. The linear finite element method on triangle is used in Sect. 6 to compute the numerical solution. In Sect. 7 some numerical simulations are discussed to confirm theoretical results. Finally, conclusion and future works are given in Sect. 8.

2 Definitions and preliminaries

In [59], the authors gave a new definition of no-integer derivative which is a natural extension to the usual first derivative as follows:

Definition 2.1 ([59]) Given a function $f : [0,\infty) \longrightarrow \mathbb{R}$. Then for all $t > 0, \quad \beta \in (0,1]$, let

$$\mathcal{D}_t^{\beta}(f)(t) = \lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\beta}\right)-f(t)}{\varepsilon},$$

$\mathcal{D}^{\beta}$ is called the *Conformable derivative* of f of order β. If f is β-differentiable in some $(0,b), b>0$, and $\lim_{t\to 0^+}\mathcal{D}^{\beta}(f)(t)$ exists, then let

$$\mathcal{D}^{\beta}(f)(0) = \lim_{t\to 0^+}\mathcal{D}^{\beta}(f)(t).$$

Now we shall present some properties of this new derivative, more properties of this type of derivation are given in [59–63].

Definition 2.2 (*Conformable Integral* [59]) Let $a \geq 0$ and $t \geq a$. Also, let f be a function defined on $(a, t]$ and $\beta \in \mathbb{R}$. Then, the β–Conformable integral of f is defined by,

$$\mathcal{I}_a^{\beta}(f)(t) := \int_a^t f(x)d_{\beta}x = \int_a^t x^{\beta-1}f(x)dx,$$

if the Riemann improper integral exists. When $a=0$ we write $\mathcal{I}_0^{\beta}(f)(t) = \mathcal{I}^{\beta}(f)(t)$

$f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is β-integrable on $[a, b]$ if and only if $t^{\beta-1}f$ is integrable on $[a, b]$.

Lemma 2.3 ([60]) *Assume that* $f : [a,\infty) \to \mathbb{R}$ *is continuous and* $0<\beta\leq 1$. *Then, for all* $t>a$ *we have*

$$\mathcal{D}^{\beta}\mathcal{I}_a^{\beta}f(t) = f(t).$$

Lemma 2.4 ([60]) *Let* $f : (a,\infty) \to \mathbb{R}$ *be differentiable and* $0<\beta\leq 1$. *Then, for all* $t>a$ *we have*

$$\mathcal{I}_a^{\beta}\mathcal{D}^{\beta}(f)(t) = f(t) - f(a).$$

In the rest of this section, we assume $a, b \in \mathbb{R}, 0 < a < b$.

Definition 2.5 ([61]) Let $p \in [1,+\infty]$ and let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a measurable function. Say that f belongs to $L^p_{\beta}([a,b])$ provided that either

$$\int_a^b |f(t)|^p d_{\beta}t < +\infty \quad \text{ if } 1 \leq p < +\infty,$$

or there exists a constant $C \in \mathbb{R}$ such that

$$|f| \leq C \quad \text{ a.e. on } [a,b] \text{ if } p = +\infty.$$

Theorem 2.6 ([61]) *Let* $p \in [1,+\infty]$. *Then the set* $L^p_{\beta}([a,b])$ *is a Banach space endowed with the norm defined for* $f \in L^p_{\beta}([a,b])$ *as*

$$\|f\|_{L^p_{\beta}([a,b])} = \begin{cases} \left(\int_a^b |f(t)|^p d_{\beta}t\right)^{1/p} & \text{if } 1 \leq p < +\infty, \\ \inf\{C \in \mathbb{R} : |f| \leq C \text{ a.e. on } [a,b]\} & \text{if } p = \infty. \end{cases}$$

Moreover, $L^2_{\beta}([a,b])$ *is a Hilbert space with the inner product given for every* $(f,g) \in L^2_{\beta}([a,b]) \times L^2_{\beta}([a,b])$ *by*

$$\langle f,g\rangle_{L^2_{\beta}([a,b])} = \int_{[a,b]} f(t)g(t)d_{\beta}t.$$

The integration by parts theorem and the Conformable version of granwal's inequality and other properties of this derivative can be found in [59, 60, 64]. Before the statement of the properties, we denote

$$C_0([a,b]) = \{f : [a,b] \to \mathbb{R}, f \text{ is continuous on } [a,b] \text{ with compact support in} [a,b]\}.$$

For $p \in [1,+\infty[$ we set the space

$$L^p_{\beta}([a,b],E) = \left\{u : [a,b] \to E : \int_{[a,b]} \|f(t)\|^p_E d_{\beta}t < +\infty\right\}.$$

Let $H^1(\Omega)$ denote the usual Sobolev space of square integrable functions and let $\left(H^1(\Omega)\right)'$ denote its dual space. Then by identifying $L^2(\Omega)$ with its dual space, $H^1(\Omega) \subset L^2(\Omega) \subset \left(H^1(\Omega)\right)'$ forms an evolution triple with all the embedding being continuous, dense and compact.

We set $V = L^2_\beta\big(0,T;H^1(\Omega)\big)$, denote its dual space by $V' = L^2_\beta(0,T;(H^1(\Omega))')$, and define a function space W by

$$W = \left\{ w \in V | \mathcal{D}^\beta_t u \in V' \right\},$$

where the conformable derivative $\mathcal{D}^\beta_t$ is understood in the sense of distributions,

Lemma 2.7 ([64]) *Let $V, H,$ and V' be Hilbert spaces such that $V \subset H \equiv H' \subset V'$, where V' is the dual of V and the injections are continuous. Suppose that $\mathbf{u} \in L^2_\beta([0,T];V)$ and $\mathcal{D}^\beta_t \mathbf{u} \in L^2_\beta\big([0,T];V'\big)$. Then u is equal a.e. to a continuous function from $[0,T]$ into H, and the following equality holds in the distribution sense on $(0,T)$*

$$\mathcal{D}^\beta_t(|\mathbf{u}|^2_H) = 2\langle \mathcal{D}^\beta_t \mathbf{u}, \mathbf{u}\rangle_{V',V}. \tag{2.1}$$

Moreover, the embedding $W \subset C([0,T];H)$ is continuous.

Remark 2.8 As a consequence of the previous identifications, the scalar product in H of $f \in H$ and $v \in V$ is the same as the scalar product of f and v in the duality between V' and V:

$$\langle f, v\rangle_{V',V} = (f,v)_H, \quad \forall f \in H, \forall v \in V. \tag{2.2}$$

3 Main results

Let $H^1_0(\Omega)$ be the subspace of $H^1(\Omega)$ whose elements have generalized homogeneous boundary values, and denote by $H^{-1}(\Omega)$ its dual space. One can consult [65, 66] for general properties of Sobolev spaces. Then obviously $H^1_0(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ forms also an evolution triple, and all statements made above remain true also in this situation when setting $V_0 = L^2_\beta\big(0,T;H^1_0(\Omega)\big) V'_0 = L^2_\beta\big(0,T;H^{-1}(\Omega)\big)$ and define

$$W_0 = \left\{ w \in V_0 | \quad \mathcal{D}^\beta_t w \in V'_0 \right\}.$$

where $\mathcal{D}^\beta_t$ is the conformable derivative, is understood in the sense of vector-valued distributions and characterized by

$$\int_0^T \mathcal{D}^\beta_t u(t)\phi(t) d_\beta t = -\int_0^T u(t)\mathcal{D}^\beta_t \phi(t) d_\beta t, \quad \text{for all } \phi \in C^\infty_0(0,T).$$

The space W endowed with the norm

$$\|u\|_{W_0} = \|u\|_{V_0} + \|\mathcal{D}^\beta_t u\|_{V'_0},$$

is a Banach space which is separable and reflexive [64]. Furthermore, by Lemma 2.7 the embedding $W \subset C([0,T];L^2(\Omega))$ is continuous. Finally, because $H^1(\Omega) \subset L^2(\Omega)$ is compactly embedded, by [64, Lemma 3.2] we have a compact embedding of $W \subset L^2_\beta\big(0,T;L^2(\Omega)\big)$

We denote the duality pairing between the elements of V'_0 and V_0 by $\langle\cdot,\cdot\rangle$, and define the bi-linear form $\mathcal{A}$ associated with the operator $\mathcal{P}$ by

$$\langle \mathcal{P}u, \varphi\rangle = \mathcal{A}(u,\varphi) \equiv \sum_{i,j=1}^N \int_{[0,T]}\int_\Omega [a_{ij}(t,x)\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j} + c(t,x)u\phi] \mathrm{d}x \mathrm{d}_\beta t, \quad \varphi \in V_0.$$

We set $L^2_\beta = L^2_\beta[0,T;L^2(\Omega)]$ the cone L^{2+}_β of all nonnegative elements of L^2_β. This induces a corresponding partial ordering also in the subspace W of $L^2(Q)$, and if $v, w \in W$ with $v \le w$ then $[v,w] := \{u \in W | v \le u \le w\}$ denotes the order interval formed by v and w. Let $(\cdot,\cdot)$ denote the inner product in L^2_β, define by

$$(u,v)_{L^2_\beta \times L^2_\beta} := \int_0^T \int_\Omega u(t,x)v(t,x) dx d_\beta t = \int_\Omega \int_0^T u(t,x)v(t,x) d_\beta t dx \quad \forall u,v \in L^2_\beta,$$

and denote by F_1 and F_2 the Nemytskij operators related with the functions f_1 and f_2, we have $F_1 u(t,x) = f_1(t,x,u(t,x))$ and $F_2 u(t,x) = f_2(t,x,u(t,x))$ for all $u \in V_0$.

Definition 3.1 A function $u, v \in W_0$ are said to be coupled weak solutions of (1.1) if

1. $F_1 u + F_2 v, F_1 v + F_2 u \in L^2_\beta$,
2. $u(0,x) = v(0,x) = u_0(x)$ in Ω,
3. $\langle \mathcal{D}^\beta_t(u), \varphi\rangle + \mathcal{A}(u,\varphi) = (F_1 u + F_2 v, \varphi)$ and $\langle \mathcal{D}^\beta_t(v), \varphi\rangle + \mathcal{A}(v,\varphi) = (F_1 v + F_2 u, \varphi)$ for all $\varphi \in V_0$.

Definition 3.2 A functions $v, w \in W_0$ are said to be a weak lower and upper solutions respectively of (1.1) if

1. $F_1 v + F_2 w, F_1 w + F_2 v \in L^2_\beta$,
2. $v(t,x) \le 0 \le w(t,x)$ on Γ and $v(0,x) \le u_0(x) \le w(0,x)$ in Ω,
3. $\langle \mathcal{D}^\beta_t(v), \varphi\rangle + \mathcal{A}(v,\varphi) \leqslant (F_1 v + F_2 w, \varphi)$ and $\langle \mathcal{D}^\beta_t(w), \varphi\rangle + \mathcal{A}(w,\varphi) \geqslant (F_1 w + F_2 v, \varphi)$ for all $\varphi \in V_0 \cap L^{2+}_\beta$.

Remark 3.3 In the definitions above, we have chosen $g \equiv 0$.

3.1 Comparison results

We can now prove the following comparison result.

Theorem 3.4 *Assume that $v, w \in W_0$ are coupled lower and upper solutions of (1.1) and f_1, f_2 satisfy the following inequalities*

$$\begin{cases} f_1(t,x,u_1) - f_1(t,x,u_2) & \leq c_1(t,x)(u_1 - u_2), \\ f_2(t,x,u_1) - f_2(t,x,u_2) & \geq -c_2(t,x)(u_1 - u_2), \end{cases} \tag{3.1}$$

whenever $u_1 \geq u_2$ for some $c_1, c_2 \in L_+^\infty(Q)$. Then we have $v \leq w$.

Remark 3.5 Clearly, the condition (3.1) is satisfied with $c_1 = c_2 = 0$ when f_1 is monotone nonincreasing in u and f_2 is is monotone non-decreasing in u.

Proof By the definition of coupled lower and upper solutions of (1.1) we get

$$v - w \leq 0 \text{ on } \Gamma, v(0,x) - w(0,x) \leq 0 \text{ in } \Omega,$$

and

$$\langle \mathcal{D}_t^\beta (v-w), \varphi \rangle + \mathcal{A}(v-w, \varphi) \leq (F_1 v - F_1 w + F_2 w - F_2 v, \varphi),$$

for all $\varphi \in V_0 \cap L_\beta^{2+}$. Choose $\varphi = (v-w)^+ = \max\{v-w, 0\} \in V_0 \cap L_\beta^{2+}$, so that $(v-w)^+(0,x) = 0$ and using (2.1) and the ellipticity condition (1.2), we get, for any $\xi \in [0,T]$, the following inequality

$$\frac{1}{2}\|(v-w)^+(\xi,\cdot)\|_{L^2(\Omega)}^2 + \alpha \|\nabla (v-w)^+\|_{L_\beta^2(Q_\xi)}^2 \\ \leq \|c_1 + c_2 - b\|_{L^\infty(Q)} \int_0^\xi \int_\Omega \left((v-w)^+\right)^2 dx d_\beta t,$$

where $Q_\xi = \Omega \times (0,\xi) \subset Q$. Now set $y(\xi) = \|(v-w)^+(\xi,\cdot)\|_{L^2(\Omega)}^2$ to obtain

$$y(\xi) \leq 2\|c_1 + c_2 - b\|_{L^\infty(\Omega)} \int_0^\xi y(t) d_\beta t, \quad \xi \in [0,T],$$

since $v - w \in C([0,T]; L^2(\Omega))$, $y(\xi) \geq 0$ and $y(0) = 0$, By applying the Conformable Gronwall inequality, we get $y(\xi) = 0$ for any $\xi \in [0,T]$ which means $(v-w)^+(t,x) = 0$ a.e. in Q, that is, $v \leq w$, proving the claim. □

The following corollary is also needed in our discussion.

Corollary 3.6 *For any $z \in W$ satisfying $z(t,x) \leq 0$ on Γ, $z(0,x) \leq 0$ in Ω and*

$$\langle \mathcal{D}_t^\beta z, \varphi \rangle + \mathcal{A}(z, \varphi) \leq 0 \text{ for all } \varphi \in V_0 \cap L_\beta^{2+},$$

we have $z \leq 0$.

Proof Let $\varphi(t,x) = z^+(t,x) = \sup\{z(t,x), 0\}$ then $\varphi \in V_0 \cap L_+^2(\Omega)$. Hence we have $z^+(0,x) = 0$ in Ω and

$$\left(\mathcal{D}_t^\beta(z), z^+\right) + \mathcal{A}(z, z^+) \leq 0,$$

we have

$$\mathcal{A}(z, z^+) \geq \alpha \|z\|^2 \geq 0,$$

we get for any $\xi \in (0,T]$

$$\langle \mathcal{D}_t^\beta(z), z^+ \rangle = \int_\Omega \int_0^\xi \mathcal{D}_t^\beta(z) z^+ d_\beta t \, dx \leq 0,$$

then

$$0 \leq \frac{1}{2}\|z^+(\xi,\cdot)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|z^+(0,\cdot)\|_{L^2(\Omega)}^2 = \frac{1}{2}\|z^+(\xi,\cdot)\|_{L^2(\Omega)}^2 \leq 0,$$

which implies $z^+(t,x) = 0$ a.e. $x \in Q$, for all $t \in [0,T]$. i.e. $z(t,.) \leq 0$ a.e. in Ω, for all $t \in [0,T]$. □

3.2 Existence and uniqueness

We consider the linear IBVP

$$\begin{aligned} &\mathcal{D}_t^\beta u(t,x) + \mathcal{P}u(t,x) = g(t,x) \text{ in } Q, \\ &u(t,x) = 0 \text{ on } \Gamma, \\ &u(0,x) = \phi(x) \text{ in } \Omega, \end{aligned} \tag{3.2}$$

where $\phi \in L^2(\Omega)$ and $g \in L_\beta^2 \subset V_0^*$.

Theorem 3.7 *There exists a unique weak solution $u \in W_0$ to the IBVP (3.2), and for some constant $C > 0$*

$$\|u\|_{W_0} \leq C\left(\|\phi\|_{L^2(\Omega)} + \|g\|_{L_\beta^2}\right).$$

Proof For a similar proof using Fedo-Galerkin technique see [64]. □

We shall prove the following result which includes several interesting special cases.

Theorem 3.8 *Assume that*

(A_1) *v_0, w_0 are coupled weak lower and upper solutions of (1.1) such that $v_0 \leq w_0$ a.e. in Q*

(A_2) *$f_1, f_2 : Q \times R \to R$ are Caratheodory functions such that $f_1(.,.,u)$ is nondecreasing in u and $g(.,.,u)$ is nonincreasing in u for $(t,x) \in Q$ a.e.*

(A_3) *for any* $\mu_1, \mu_2 \in [v_0, w_0], h \in L^2_\beta$ *where* $h(t,x) = f_1(t,x,\mu_1) + f_2(t,x,\mu_2)$

Then there exist monotone sequences $\{v_j(t,x)\}_j, \{w_j(t,x)\}_j \subset W_0$ *such that*

$$v_j \to v^*, w_j \to w^* \text{ in } W_0,$$

and v^*, w^* *are the coupled weak minimal and maximal solutions of* (1.1).

Proof For simplicity we set $g = u_0 \equiv 0$, without loss of generality we consider the following IBVPs for each $j = 1, 2, \dots$

$$\begin{cases} \mathcal{D}^\beta_t v_j + \mathcal{P} v_j = f_1(t,x,v_{j-1}) + f_2(t,x,w_{j-1}) & \text{in } Q, \\ v_j(t,x) = 0 \text{ on } \Gamma \text{ and } \quad v_j(0,x) = 0 \text{ in } \Omega, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \mathcal{D}^\beta_t w_j + \mathcal{P} w_j = f_1(t,x,w_{j-1}) + f_2(t,x,v_{j-1}) & \text{in } Q, \\ w_j(t,x) = 0 \text{ on } \Gamma \text{ and } w_j(0,x) = 0 \text{ in } \Omega, \end{cases} \tag{3.4}$$

whose variational forms associated with (3.3) and (3.4) are given by

$$\langle \mathcal{D}^\beta_t v_j, \varphi \rangle + \mathcal{A}(v_j, \varphi) = (F_1 v_{j-1} + F_2 w_{j-1}, \varphi), \tag{3.5}$$

and

$$\langle \mathcal{D}^\beta_t w_j, \varphi \rangle + \mathcal{A}[w_j, \varphi] = (F_1 w_{j-1} + F_2 v_{j-1}, \varphi), \tag{3.6}$$

for all $\varphi \in V_0$.

By (A_3), $h \in L^2_\beta$ where $h(t,x) = f_1(t,x,v_0) + f_2(t,x,w_0)$. Hence by Theorem 3.7 there is a unique weak solution $v_1 \in W_0$ of (3.3) for $j = 1$. In the same way, we get the existence of a unique weak solution $w_1 \in W_0$ of (3.4). We shall first show that

$$v_0 \le v_1 \le w_1 \le w_0. \tag{3.7}$$

To prove $v_0 \le v_1$, we find that v_1 satisfies the relation

$$\langle \mathcal{D}^\beta_t v_1, \varphi \rangle + \mathcal{A}(v_1, \varphi) = (F_1 v_0 + F_2 w_0, \varphi), \tag{3.8}$$

and note that v_0 satisfies

$$\langle \mathcal{D}^\beta_t v_0, \varphi \rangle + \mathcal{A}(v_0, \varphi) \le (F_1 v_0 + F_2 w_0, \varphi), \tag{3.9}$$

for each $\varphi \in V_0 \cap L^{2+}_\beta$ and $v_0(t,x) \le 0$ on Γ, $v_0(0,x) \le 0$ in Ω. Let $z = v_0 - v_1$. Then z satisfies $z(t,x) \le 0$ on Γ, $z(0,x) \le 0$ in Ω and

$$\langle \mathcal{D}^\beta_t z, \varphi \rangle + \mathcal{A}(z, \varphi) \le 0 \text{ for all } \varphi \in V_0 \cap L^{2+}_\beta,$$

Hence by Corollary 3.6, we get $z \le 0$, that is $v_0 \le v_1$. A similar reasoning proves that $w_1 \le w_0$. To show that $v_1 \le w_1$, we see from (3.6) that w_1 satisfies

$$\langle \mathcal{D}^\beta_t w_1, v \rangle + \mathcal{A}(w_1, \varphi) = (F_1 w_0 + F_2 v_0, v) \ge (F_1 v_0 + F_2 w_0, v),$$

for each $v \in V_0 \cap L^{2+}_\beta$, using the monotone character of F_1 and F_2. By Corollary 3.6 we readily obtain, in view of (3.8) and (3.9), $v_1 \le w_1$, proving (3.7)

Next by induction we shall prove that if for some $j > 1$

$$v_0 \le v_{j-1} \le v_j \le w_j \le w_{j-1} \le w_0, \tag{3.10}$$

then

$$v_j \le v_{j+1} \le w_{j+1} \le w_j, \tag{3.11}$$

According to (A_3) and the relation (3.10), Theorem 3.7 guarantees the existence of unique weak solutions $v_j, w_j, v_{j+1}, w_{j+1} \in W_0$ of (3.3) and (3.4). To show that $v_j \le v_{j+1}$, consider

$$\langle \mathcal{D}^\beta_t v_{j+1}, \varphi \rangle + \mathcal{A}(v_{j+1}, \varphi) = (F_1 v_j + F_2 w_j, \varphi), \tag{3.12}$$

and

$$\langle \mathcal{D}^\beta_t v_j, \varphi \rangle + \mathcal{A}(v_j, \varphi) = (F_1 v_{j-1} + F_2 w_{j-1}, \varphi) \le (F_1 v_j + F_2 w_j, \varphi), \tag{3.13}$$

for each $v \in V_0 \cap L^{2+}_\beta$. In the above inequality (3.13), we have used the monotone nature of F_1, F_2 and (3.10). Corollary 3.6 then yields that $v_j \le v_{j+1}$. Similarly, we can show that $w_{j+1} \le w_j$. To prove $v_{j+1} \le w_{j+1}$, we see, using the monotone character of F_1, F_2 and (3.10), that

$$\langle \mathcal{D}^\beta_t w_{j+1}, \varphi \rangle + \mathcal{A}(w_{j+1}, \varphi) = (F_1 w_j + F_2 v_j, \varphi) \ge (F_1 v_j + F_2 w_j, \varphi), \tag{3.14}$$

for each $\varphi \in V_0 \cap L^{2+}_\beta$. The relation (3.14), together with (3.12), prove by Corollary 3.6 that $v_{j+1} \le w_{j+1}$, showing that (3.11) is valid. Hence by induction, we arrive at

$$v_0 \le v_1 \le v_2 \le \dots \le v_j \le w_j \le \dots \le w_2 \le w_1 \le w_0 \quad a.e. \text{ in } Q, \tag{3.15}$$

for all j and for a.e. in Q. By the monotonicity of iterates $\{v_j\}, \{w_j\}$ and (3.15), there exist pointwise limits

$$v^*(t,x) = \lim_{j\to\infty} v_j(t,x), w^*(t,x) = \lim_{j\to\infty} w_j(t,x),$$

for a.e. in Q Furthermore, since $v_j, w_j \in [v_0, w_0]$, it follows by Lebesgue's dominated convergence theorem that

$$v_j \to v^* \text{ and } w_j \to w^* \text{ in } L^2_\beta. \tag{3.16}$$

By (A_3), we have for any $\mu_1, \mu_2 \in [v_0, w_0]$, $F_1\mu_1 + F_2\mu_2$ is continuous and bounded as a mapping from $[v_0, w_0] \subset L^2_\beta \to L^2_\beta$ and therefore in view of (3.16), it follows that

$$F_1v_j \to F_1v^*, F_2w_j \to F_2w^*, F_1w_j \to F_1w^*, F_2v_j \to F_2v^* \text{ in } L^2_\beta \text{ as } j \to \infty. \tag{3.17}$$

Since v_j, w_j are solutions of the linear IBVPs (3.3) and (3.4) with homogeneous initial boundary values, the estimate of Theorem 3.7 gives

$$\left\|v_j\right\|_W \leq C\left\|F_1v_{j-1} + F_2w_{j-1}\right\|_{L^2_\beta}, \tag{3.18}$$

and

$$\left\|w_j\right\|_W \leq C\left\|Fw_{j-1} + F_2v_{j-1}\right\|_{L^2_\beta}. \tag{3.19}$$

Due to (3.17), the foregoing estimates imply the strong convergence of the sequences $\{v_j\}, \{w_j\}$ in W whose limits must be $v^*$and w^* respectively, because of the compact embedding $W \subset L^2_\beta$ [64, Lemma 3.2.] . Since v_j, w_j for $j \geq 1$ belong to the subset $M = [u \in W : u(0,x) = 0]$ which is closed in W, it follows that their limits $v^*, w^* \in M$, which are the limits satisfying also homogeneous initial and boundary conditions. The convergence result (3.17) together with

$$v_j \to v^*, w_j \to w^* \text{ in } W_0 \quad \text{as } j \to \infty,$$

permit us to pass to the limit in the corresponding variational forms (3.5) and (3.6) as $j \to \infty$. This yields

$$\langle \mathcal{D}^\beta_t v^*, \varphi\rangle + \mathcal{A}(v^*, \varphi) = (F_1v^* + F_2w^*, \varphi),$$

and

$$\langle \mathcal{D}^\beta_t w^*, \varphi\rangle + \mathcal{A}(w^*, \varphi) = (F_1w^* + F_2v^*, \varphi),$$

for all $\varphi \in V_0$, showing (v^*, w^*) are coupled weak solutions of (1.1).

Let $u \in [v_0, w_0]$ be any weak solution of (1.1). Then we show that $v_1 \leq u \leq w_1$. Since v_1 satisfies (3.9), we get, using monotone nature of F_1 and F_2

$$\langle \mathcal{D}^\beta_t v_1, v\rangle + \mathcal{A}(v_1, \varphi) = \left(F_1v_0 + F_2w_0, \varphi\right) \leq (F_1u + F_2u, \varphi),$$

for each $v \in V_0 \cap L^{2+}_\beta$. Since u is a weak solution of (1.1) we have

$$\langle \mathcal{D}^\beta_t u, v\rangle + \mathcal{A}(u, \varphi) = (F_1u + F_2u, \varphi),$$

for each $\varphi \in V_0 \cap L^{2+}_\beta$. Corollary 3.6 therefore yields $v_1 \leq u$. A similar argument shows that $u \leq w_1$. Next we suppose that for some $j > 1$

$$v_0 \leq v_j \leq u \leq w_j \leq w_0.$$

Then utilizing the monotone character of F_1 and F_2, we obtain

$$\langle \mathcal{D}^\beta_t v_{j+1}, \varphi\rangle + \mathcal{A}(v_{j+1}, \varphi) = \left(F_1v_j + F_2w_j, \varphi\right) \leq (F_1u + F_2u, \varphi),$$

for each $\varphi \in V_0 \cap L^{2+}_\beta$. It then follows from Corollary 3.6 that $v_{j+1} \leq u$. In the same way, we can prove that $u \leq w_{j+1}$. Thus we arrive at

$$v_0 \leq v_j \leq v_{j+1} \leq u \leq w_{j+1} \leq w_j \leq w_0.$$

Hence by induction $v_j \leq u \leq w_j$ for all j and consequently, taking the limit as $j \to \infty$, we get $v^* \leq u \leq w^*$, proving that v^*, w^* are coupled weak minimal and maximal solutions of (1.1). The proof is now complete. □

Corollary 3.9 *In addition to the assumptions of Theorem 3.8, if we suppose that f_1,f_2 satisfy the conditions (3.1), then $u = v^* = w^*$ is the unique weak solution of (1.1).*

Proof Since $v^* \leq w^*$, it is sufficient to show that $w^* \leq v^*$. But this follows right away from Corollary 3.6 if we set $z = w^* - v^*$. Thus the claim of the Corollary is true. □

Several particular problems can be studied by the same steps and the conclusion of Theorem 3.8 is satisfied, for example if we consider the following problem

$$\begin{cases} \mathcal{D}^\beta_t u + \mathcal{P}u = h_1(t,x,u) + h_2(t,x,u) & \text{in } Q, \\ u(t,x) = 0 \text{ on } \Gamma, \\ u(0,x) = u_0(x) \text{ in } \Omega, \end{cases} \tag{3.20}$$

where the both h_1 and h_2 are not monotone. Then we find the following result

Corollary 3.10 *Suppose that $f_1(t,x,u) = h_1(t,x,u) + \eta_1(t,x)u$ is nondecreasing in u and $f_2(t,x,u) = h_2(t,x,u) - \eta_2(t,x)u$ is nonincreasing in u, where $\eta_1, \eta_2 \in L^\infty_+(Q)$. Then the same conclusion of Theorem 3.8 and Corollary 3.9 remains the same for problem (3.20).*

Proof We consider the problem (3.21) equivalent to (3.20)

$$\begin{cases} \mathcal{D}^\beta_t u + \mathcal{P}u + \eta_1 u - \eta_2 u = \tilde{f}_1(t,x,u) + \tilde{f}_2(t,x,u) & \text{in } Q, \\ u = 0 \quad \text{on } \Gamma, \\ u(0,x) = 0 \text{ on } \Omega. \end{cases} \tag{3.21}$$

Let v_0, w_0 are coupled weak lower and upper solutions of (3.20), then are also for the problem (3.21). Let $\tilde{\mathcal{P}}u = \mathcal{P}u + \eta_1 u - \eta_2 u$, all the conditions on $\mathcal{P}$ in Theorem 3.8 are satisfied for $\tilde{\mathcal{P}}$, so by application of the Theorem 3.8 and Corollary 3.1 we find the result of the Corollary 3.10. □

3.3 Nonuniqueness of solutions

We have shown in the Corollary 3.9 that the solution of problem (1.1) is unique if f_1 and f_2 satisfy the condition (3.1), this result is only valid inside sector $< v, w >$, so it does not give any information outside of sector $< v, w >$. In the following example, we show that if f_1 and f_2 do not satisfy (3.1) then the problem (1.1) can have several solutions. Let us consider the problem

$$\begin{array}{ll} \mathcal{D}_t^{\beta} u - u_{xx} = u + \left(\frac{3}{2}\sin^{2\beta/3}(x)\right)u^{(3-2\beta)/3} & (t>0, 0<x<\pi),\\ u(t,0)=u(t,\pi)=0 & (t>0),\\ u(0,x)=0 & (0<x<\pi). \end{array} \tag{3.22}$$

Clearly the functions

$$f_1(t,x,u) = u + \left(\frac{3}{2}\sin^{2\beta/3} x/2\right)u^{(3-2\beta)/3} \text{ and } f_2(t,x,u)=0,$$

satisfy (A_2) and (A_3). To apply Theorem 3.8 consider the functions v and w defined by

$$v=0, \quad w = Kt^{3/2}\sin x,$$

where $K \geq 1$ is a constant. We have

$$v(t,0)=v(t,\pi)=w(t,0)=w(t,\pi)=0,$$

it remains to verify that v and w satisfy the differential inequality given by definition of upper and lower solutions, we have

$$\mathcal{D}_t^{\beta} w - w_{xx} = K\frac{3}{2}t^{(3-2\beta)/2}\sin x + Kt^{3/2}\sin x = w + K\frac{3}{2}t^{(3-2\beta)/2}\sin x,$$

then the following inequality must be satisfied

$$w + K\frac{3}{2}t^{(3-2\beta)/2}\sin x \geq w + \left(\frac{3}{2}\sin^{2\beta/3}(x)\right)w^{(3-2\beta)/3}.$$

This implies

$$K \geq K^{(3-2\beta)/3},$$

which is clearly satisfied by any $K \geq 1$. The existence of solutions of (3.22) is guaranteed by the Theorem 3.8 but not the uniqueness. However $u_1 = 0$ and $u_2 = t^{3/2}\sin x$ are two different solutions to problem (3.22) in the sector $< v, w >$ and

$$0 \leq u_1(t,x), u_2(t,x) \leq Kt^{3/2}\sin x,$$

Additional solutions are given in the following form

$$u_\tau(t,x) = \begin{cases} 0 & \text{when } 0 \leq t \leq \tau,\\ (t-\tau)^{3/2}\sin x & \text{when } t \geq \tau. \end{cases}$$

4 Construction of lower and upper solutions

In the Conformable problem considered, to obtain the result of existence, we have to suppose that the problem has lower and upper solutions, therefore the results depend on the existence of the last ones. In many real problems the boundary and initial data are nonnegative and $f(.,.,0) = f_1(.,.,0) + f_2(.,.,0) \geq 0$. In this situation, it is only necessary to find a suitable non-negative upper solution.

In this section, let's use the same techniques in [56] to construct the lower and upper solutions, the construction usually depends on the nonlinear term $f(t, x, u)$, we usually manage to solve the problem linear.

To illustrate some methods for the construction of upper and lower solutions we consider a class of nonlinear functions which are sufficient to treat the reaction-diffusion models in (1.1). For this class of functions we require the following conditions:

- $f(t,x,0) \geq 0$ and there exist two functions K_1, K_2 with $K_2 \geq 0$ such that

  $$f(t,x,u) \leq K_1(t,x)u + K_2(t,x) \quad \text{when } u \geq 0, (t,x) \in \Omega.$$

Then, the construction of upper solution w and lower solutions v it is often practical to use the solution of the Conformable linear parabolic problems in the form

$$\begin{cases} \mathcal{D}_t^{\beta} w - \mathcal{P}w = K_1(t,x)w + K_2(t,x) \text{ in } Q_T = (0,T)\times\Omega\\ w(t,x) = h(t,x) \text{ on } \Gamma_T = [0,T]\times\partial\Omega\\ w(0,x) = u_0(x) \text{ in } \Omega \end{cases}, \tag{4.1}$$

and

$$\begin{cases} \mathcal{D}_t^{\beta} v - \mathcal{P}v = f(t,x,0) \text{ in } Q_T = (0,T)\times\Omega\\ v(t,x) = h(t,x) \text{ on } \Gamma_T = [0,T]\times\partial\Omega\\ v(0,x) = u_0(x) \text{ in } \Omega \end{cases}, \tag{4.2}$$

where u_0 and h are nonnegative functions. The choice of K_1 and K_2 depends on the reaction function f under consideration. Note that the unique weak solutions of (4.1) and (4.2) exists and is nonnegative in W.

Let us note that the construction of an efficient numerical scheme of problem (4.1) and (4.2) is not difficult because it is a linear problem, therefore it is enough to use in our case the implicit Conformable Euler scheme in time and the finite element method in space.

An other simple condition on f is the existence of constant functions $\mathbf{p}$ and $\mathbf{q}$ such that

$$\mathbf{p} \leq 0 \leq \mathbf{q} \quad \text{and} \quad f(.,.\mathbf{p}) \geq 0 \geq f(.,.,\mathbf{q}). \tag{4.3}$$

It is easy to verify that the pair $w = \mathbf{q}$ and $v = \mathbf{p}$ are ordered upper and lower solutions.

5 Applications

In this section, we give some example of application of the existence and construction results of problem 1.1.

5.1 Nonlinear Michaelis-Menten oxygen uptake kinetics model

We consider the time Nonlinear Michaelis-Menten oxygen uptake kinetics model without advection term

$$\begin{cases} \mathcal{D}_t^\beta C(t,x) - \dfrac{\partial^2 C}{\partial x^2} = \dfrac{-\sigma C}{C + K_m} \text{ in } Q_T = (0,T) \times (0,1) \\ C(t,0) = a, C(t,1) = b \text{ in } (0,T), \\ C(0,x) = C_0(x) \text{ in } (0,1), \end{cases} . \tag{5.1}$$

where σ, K_m are positive constants. This model is a special case of (1.1) with

$$f_1 \equiv 0 \text{ and } f_2(t,x,C) = \frac{-\sigma C}{C + K_m},$$

$\partial f_2/\partial C < 0$ then f_2 is monotone decreasing function, any constant $M \geq 0$ satisfying

$$M \geq \max\{a, b, u_0(x)/ \quad x \in (0,1)\}, \tag{5.2}$$

is an upper solution of (5.1), whose existence and uniqueness of solution u is obtained from the Theorem 3.8 and Corollary 3.9 with $0 \leq u \leq K$. By application of the technique given in section 4 we can obtain a non-constant upper solution w, since

$$f(0) = 0 \text{ and } f(u) \leq 0 \text{ for all } u \geq 0,$$

hence the solution $w(t,x)$ of problem (4.1) corresponding to $K_1 = K_2 = 0$ is an upper solution.

Another effect can make the reaction function in (5.1) in the form when the effect of inhibition in the enzyme-substrate reaction scheme is taken into consideration the reaction function becomes

$$f(t,x,C) = \frac{-\delta C}{K_m + C + \delta C^2}, \tag{5.3}$$

where δ is a positive constant. In this case the same function w as in the previous model is a nonnegative upper solution.

5.2 The time Conformable Fisher's model

We consider the time Conformable Fisher's model

$$\begin{cases} \mathcal{D}_t^\beta u - D\Delta u = u(u-\xi)(1-u) \quad \text{in } Q, \\ u(t,x) = g(t,x) \text{ in } \Gamma, \\ u(0,x) = u_0(x) \text{ in } \Omega. \end{cases} \tag{5.4}$$

where σ, θ are positive constants with $0 < \theta < 1$. According to the real signification of the density function u, it must be between 0 and 1, then $0 \leq u_0(x) \leq 1$ and $0 \leq h(t,x) \leq 1$. Under this requirement, it is easy to verify that 0 and 1 are respectively lower and upper solutions of (5.4). Hence by Theorem 3.8, and Corollary 3.9 time Conformable Fisher's model has a unique weak solution u such that $0 \leq u \leq 1$.

5.3 Conformable Models in reactor dynamics and heat conduction

The model described the neutron flux with an internal source with time Conformable derivative is given by

$$\begin{cases} \mathcal{D}_t^\beta u - D\Delta u = u(\alpha - \delta u) + \phi(x) \quad \text{in } Q, \\ u(t,x) = g(t,x) \text{ in } \Gamma, \\ u(0,x) = u_0(x) \text{ in } \Omega, \end{cases} \tag{5.5}$$

where α, δ are positive constants and $\phi \geq 0$. Since

$$f(0) = \phi(x) \geq 0 \text{ and } f(u) \leq f(\alpha/2\delta) \text{ for all } u \geq 0,$$

the solutions $v(t,x)$ and $w(t,x)$ of (4.1) and (4.2) respectively, with $K_1 = 0, K_2 = \alpha^2/(4\delta) + \phi(x)$ (or $K_1 = \alpha, K_2 = \phi(x)$) are lower and upper solutions. Since by Theorem 3.8 and Corollary 3.9 that the weak solution of (5.5) exists and $v \leq u \leq \bar{w}$.

Another Conformable heat-conduction problem is given by

$$\begin{cases} \mathcal{D}_t^\beta u - D\Delta u = K(\delta^4 - u^4) \text{ in } Q_T = (0,T) \times \Omega \\ u(t,x) = g(t,x) \text{ in } \Gamma, \\ u(0,x) = u_0(x) \text{ in } \Omega, \end{cases} \tag{5.6}$$

where δ, K are positive constants. Since $f(0) = K\delta^4$ and $f(\xi) \leq 0$ for $\xi \geq \delta$. We chose $w = \xi \geq \max\{g, u_0, \delta\}$, then Theorem 3.8 and Corollary 3.9 implies that the reaction model (5.6) has a unique weak solution u where $0 \leq u(t,x) \leq \xi$.

The nonisothennal chemical reaction model is in the form

$$\begin{cases} \mathcal{D}_t^\beta u - D\Delta u = \mu(\xi - u)\exp(\theta u/(1+u)) \\ u(t,x) = 0 \text{ in } \Gamma, \\ u(0,x) = u_0(x) \text{ in } \Omega, \end{cases} \tag{5.7}$$

where μ, ξ, and θ are positive constants. We have for any $K \geq \xi$, $f(0) = \mu\xi$ and $f(K) \leq 0$, Then by Theorem 3.8 and Corollary 3.9 the model (5.7) has a unique weak solution u where $0 \leq u(t,x) \leq \max\{u_0, \xi\}$.

6 Discrete Galerkin finite element method

To solve (3.2) by the finite element method, we construct a finite dimensional subspace V_h of $H_0^1(\Omega)$ using continuous piecewise basis functions, i.e.

$$V_j = \left\{ v_j \in C^0(\bar{\Omega}) : v_j\Big|_K \in P_k, \forall K \in T_h \right\},$$

where T_j is a regular family of triangulations of Ω, the elementary size of the elements K in T_j as defined by:

$$h = \max_{K \in T_j} \text{diam}(K).$$

Then we can define the semi-discrete finite element method for the weak problem of (1.1): Find $u_j : [0,T] \to V_j$ such that

$$\begin{aligned} \left(\mathcal{D}_t^\beta u_j, \varphi\right) + \mathcal{A}\left(u_j, \varphi\right) &= (f(u_j), \varphi) \quad \forall \varphi \in V_j \\ u_j(\cdot, 0) &= \Pi_j u_0, \end{aligned} \tag{6.1}$$

where Π_j is the elliptic projection operator [ref 45...] into V_j, i.e., for any $u \in H^1(\Omega), \Pi_j u \in V_j$ satisfies the equation

$$\mathcal{A}\left(\left(u - \Pi_j u\right), \varphi\right) = 0 \quad \forall \varphi \in V_j.$$

In semi-discrete approximation, the solution $u \in V_j$ is in the form

$$u(x,t) = \sum_{k=1}^{N} u_k(t)\phi_k(x), \tag{6.2}$$

where ϕ_k are the basis functions of V_j.

Now the semi-discrete finite element scheme of the linear Picard problems (3.3) and (3.4) is given for $m = 1, 2, \ldots$ by:

Find $v^m, w^m \in V_j$ such that

$$\begin{cases} \left(\mathcal{D}_t^\beta v^m, \varphi\right) + \mathcal{A}(v^m, \varphi) = (f_1(v^{m-1}) + f_2(w^{m-1}), \varphi) \quad \forall \varphi \in V_j, \\ v^m(0, \cdot) = \Pi_j v_0, \\ \left(\mathcal{D}_t^\beta w^m, \varphi\right) + \mathcal{A}(w^m, \varphi) = (f_1(w^{m-1}) + f_2(v^{m-1}), \varphi) \quad \forall \varphi \in V_j, \\ w^m(0, \cdot) = \Pi_j w_0. \end{cases} \tag{6.3}$$

In the above approximation, the solution v^m and w^m are in the form

$$v^m(x,t) = \sum_{k=1}^{N} u_k^m(t)\phi_k(x), \quad w^m(x,t) = \sum_{k=1}^{N} w_k^m(t)\phi_k(x), \tag{6.4}$$

where ϕ_k are the basis functions of V_j. First, we partition the time interval $[0, T]$ into N equal sub-intervals $\left[t_k, t_{k+1}\right]$ of length $\Delta t = \frac{T}{N}$. The time-discrete finite element method, such as the Conformable implicit Euler scheme [67] for (3.2) can be defined as: Find $(v, w) \in V_j \times V_j$ such that

$$\begin{cases} \left(\dfrac{((n+1)^\beta - n^\beta)(\Delta t)^\beta}{\beta}(v_{n+1}^m - v_n^m), \varphi\right) + \mathcal{A}\left(v_n^m, \varphi\right) = (f_1(v_{n+1}^{m-1}) + f_2(w_{n+1}^{m-1}), \varphi), \quad \forall \varphi \in V_j \\ \left(\dfrac{((n+1)^\beta - n^\beta)(\Delta t)^\beta}{\beta}(w_{n+1}^m - w_n^m), \varphi\right) + \mathcal{A}\left(w_n^m, \varphi\right) = (f_1(w_{n+1}^{m-1}) + f_2(v_{n+1}^{m-1}), \varphi), \quad \forall \varphi \in V_j \\ v^m(\cdot, 0) = \Pi_j v_0, \quad w^m(\cdot, 0) = \Pi_j w_0. \end{cases} \tag{6.5}$$

We find the flowing system for unknown vectors $U = \left(u_1, \cdots, u_N\right)^T$:

$$\begin{cases} L_n(\beta)k^\beta MV^{(n,m)} + AV^{(n,m)} = L_n(\beta)k^\beta MV^{(n-1,m)} \\ +F_1^n(V^{n,m-1}) + F_2^n(W^{n,m-1}), \\ L_n(\beta)k^\beta MW^{(n,m)} + AW^{(n,m)} = L_n(\beta)k^\beta MW^{(n-1,m)} \\ +F_1^n(W^{n,m-1}) + F_2^n(V^{n,m-1}), \\ V^{(0,m)} = V_0 = \left(\Pi_1 v_0, \Pi_2 v_0, \dots, \Pi_N v_0\right), \\ W^{(0,m)} = W_0 = \left(\Pi_1 w_0, \Pi_2 w_0, \dots, \Pi_N w_0\right), \end{cases} \tag{6.6}$$

where

$$M = \left(M_{ki}\right) = \left(\phi_k, \phi_i\right) = \int_K \varphi_k(x)\varphi_i(x)dx,$$

$$A = \left(A_{ki}\right) = \mathcal{A}\left(\phi_k, \phi_i\right) = \int_K a(x)\nabla\varphi_k(x)\nabla\varphi_i(x)dx,$$

$$F_s^n(U^{n,m}) = \int_K f_s^n(U^{(n,m)})\phi_i(x)dx, \quad s = 1, 2,$$

Table 1 $L_2^\beta(L_2)$-norm and $L_2^\beta(H_1)$-norm for $\Delta t = 0.1$

β	h	# Nodes	# Elements	$L_2(L2)$-error	$L_2(H_1)$-error
1	0.20	88	143	$2.06e-02$	$1.750e-01$
	0.10	362	661	$1.03e-02$	$1.047e-01$
	0.05	1452	2774	$1.02e-02$	$8.520e-02$
	0.02	9062	17,813	$1.01e-02$	$7.750e-02$
0.9	0.20	88	143	$2.077e-02$	$1.900e-01$
	0.10	362	661	$1.340e-02$	$1.235e-01$
	0.05	1452	2774	$1.290e-02$	$1.043e-01$
	0.02	9062	17,813	$1.250e-02$	$9.640e-02$
0.7	0.20	88	143	$3.230e-02$	$2.495e-01$
	0.10	362	661	$2.150e-02$	$1.783e-01$
	0.05	1452	2774	$1.990e-02$	$1.559e-01$
	0.02	9062	17,813	$1.890e-02$	$1.460e-01$

Table 2 Table of $L_2^\beta(L_2)$-norm and $L_2^\beta(H_1)$-norm for $\Delta t = 0.001$

β	h	# Nodes	# Elements	$L_2^\beta(L_2)$-error	$L_2^\beta(H_1)$-error
1	0.20	88	143	$7.900e-03$	$5.84e-02$
	0.10	362	661	$1.7e-03$	$2.420e-02$
	0.05	1452	2774	$3.874e-04$	$1.150e-02$
	0.02	9062	17,813	$5.432e-05$	$4.500e-03$
0.9	0.20	88	143	$1.114e-02$	$7.71e-02$
	0.10	362	661	$3.600e-03$	$3.530e-02$
	0.05	1452	2774	$2.000e-03$	$2.12e-02$
	0.02	9062	17,813	$1.700e-03$	$1.57e-02$
0.7	0.20	88	143	$2.020e-02$	$1.286e-01$
	0.10	362	661	$9.200e-03$	$7.43e-02$
	0.05	1452	2774	$7.000e-03$	$5.820e-02$
	0.02	9062	17,813	$6.300e-03$	$5.260e-02$

and

$$L_n(\beta) = \frac{n^\beta - (n-1)^\beta}{\beta}.$$

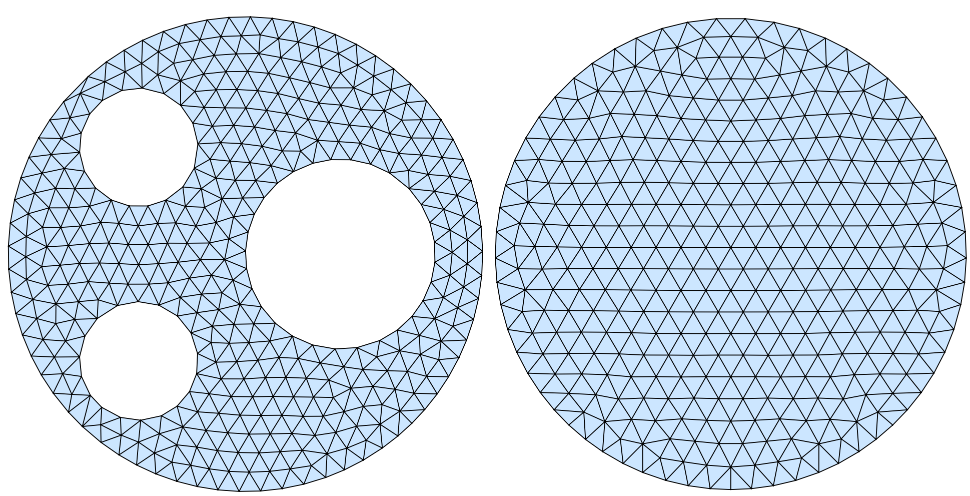
Note that if $\beta = 1$ we have $L_n(\beta) = 1$ then we find the classical Euler finite element scheme.

7 Examples and simulations results

7.1 Example with a known analytic solution on domain disk with three holes

We consider the flowing first test problem

Fig. 1 Two typs of Domain Ω



(a) Mesh with 670 elements and 399 nodes.

(b) mesh with 661 elements and 362 nodes.

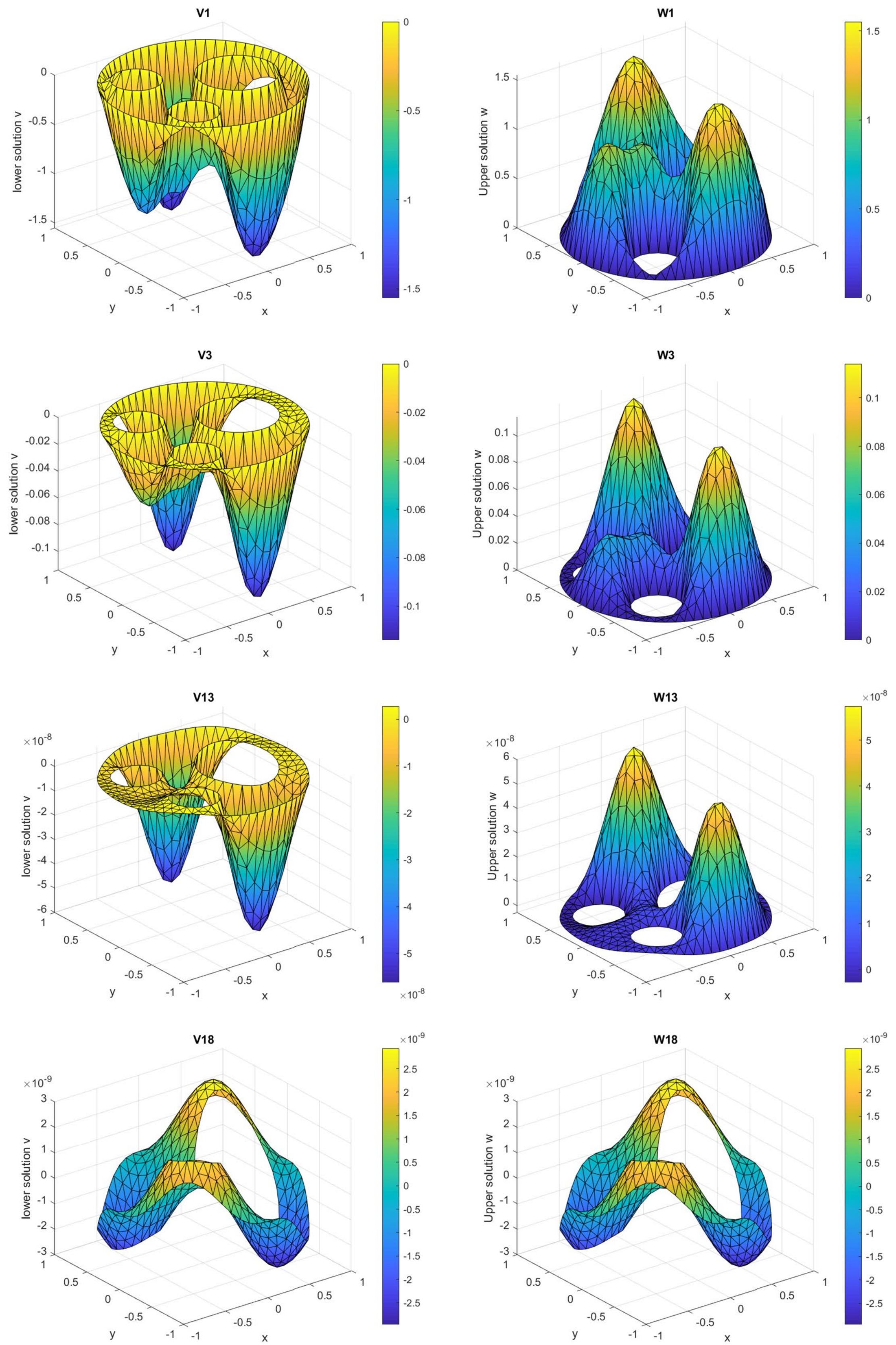


Fig. 2 Convergence of upper (W_n) and lower (V_n) sequences of Example 1 for $h = 0.08$, $\Delta t = 0.01$, $t_1 = 100\Delta t$ and $\beta = 0.5$

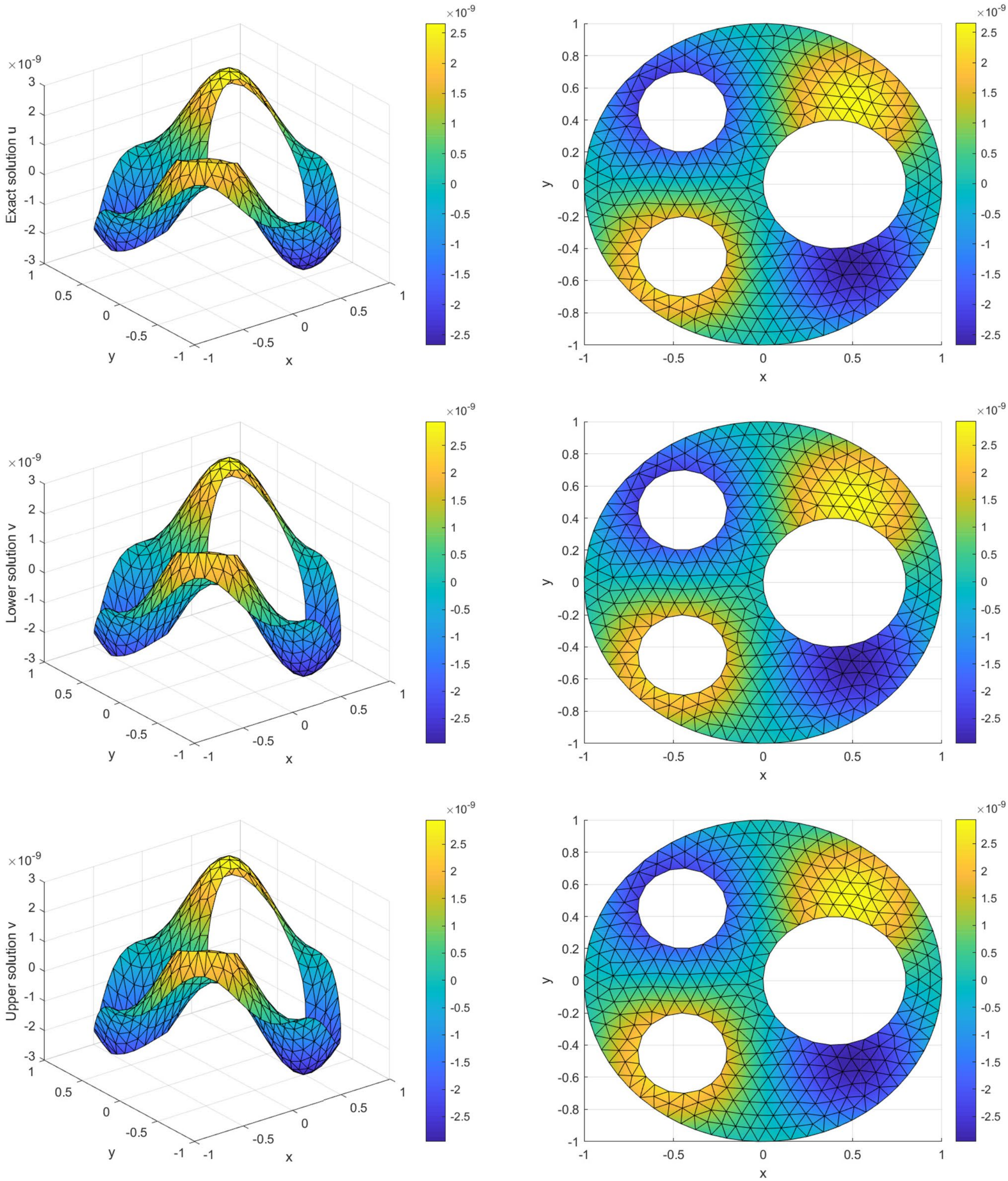


Fig. 3 Comparison of numerical and exact solution for test Example 1 for $\Delta t = 0.01$, $\beta = 0.5$ at time $t_1 = 100\Delta t$

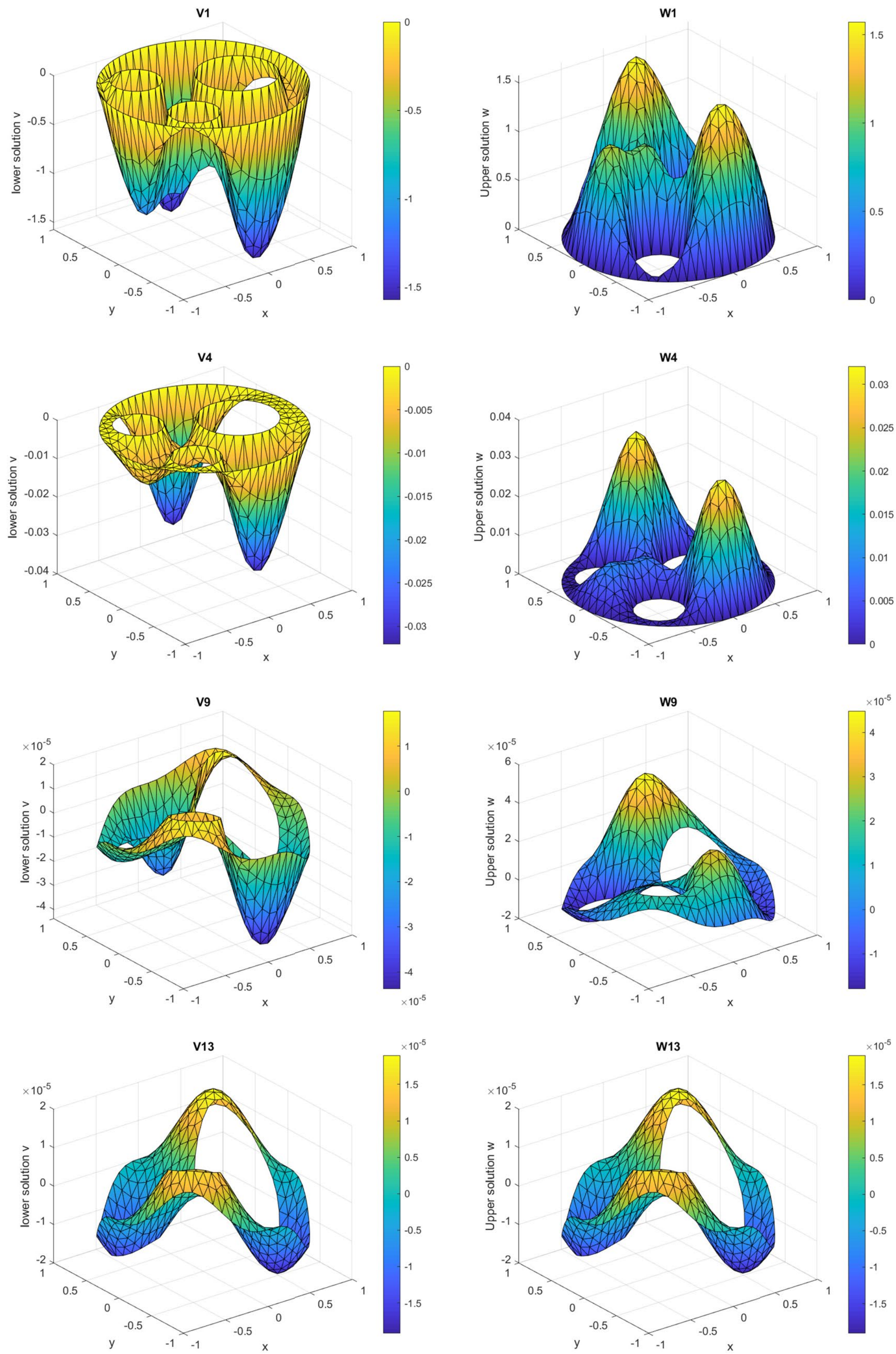


Fig. 4 Convergence of upper (W_n) and lower (V_n) sequences of Example 1 for $h = 0.08$, $\Delta t = 0.01$, $t_1 = 100\Delta t$ and $\beta = 0.9$

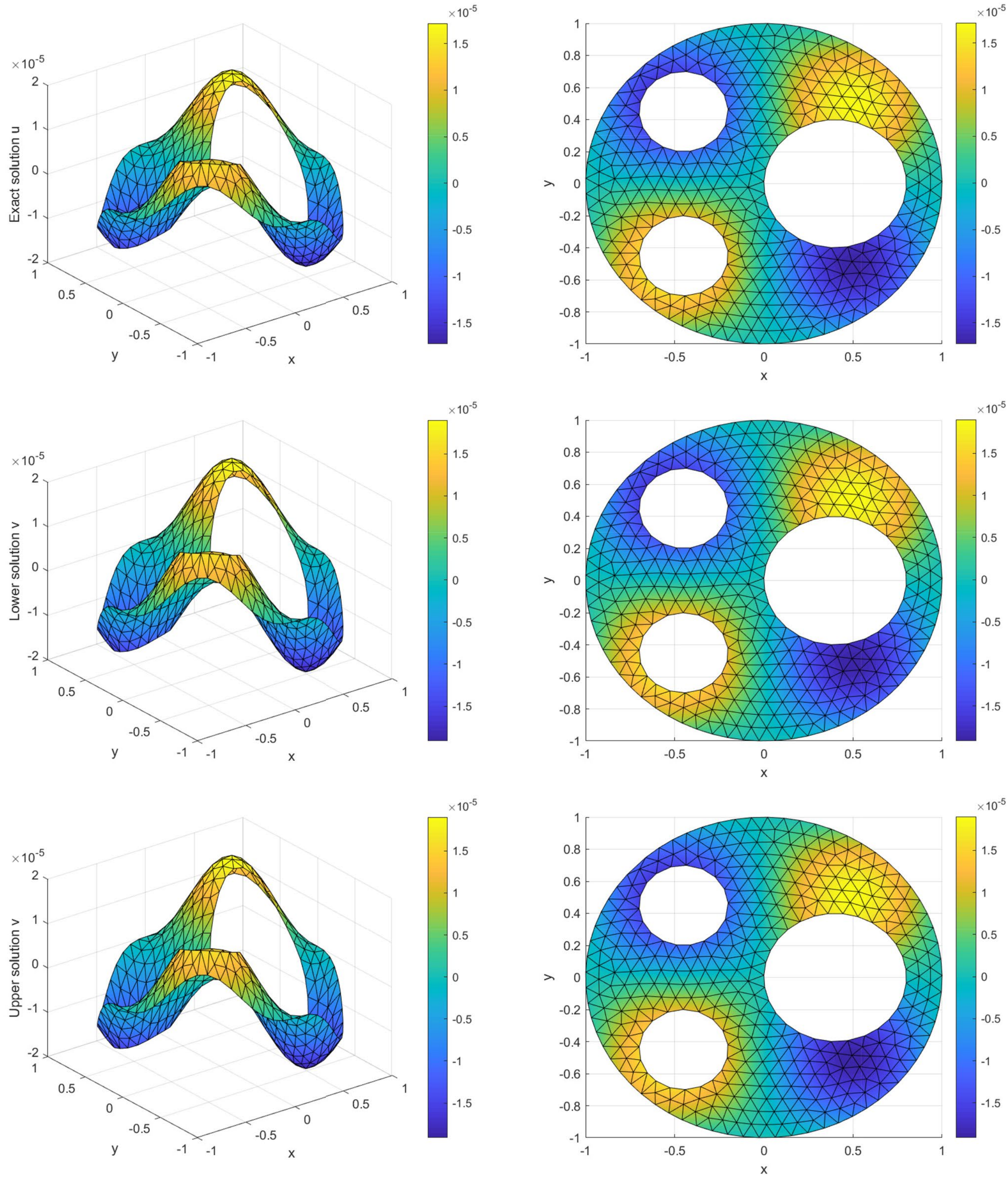


Fig. 5 Comparison of numerical and exact solution for test Example 1 for $\Delta t = 0.01$, $\beta = 0.5$ at time $t_1 = 100\Delta t$

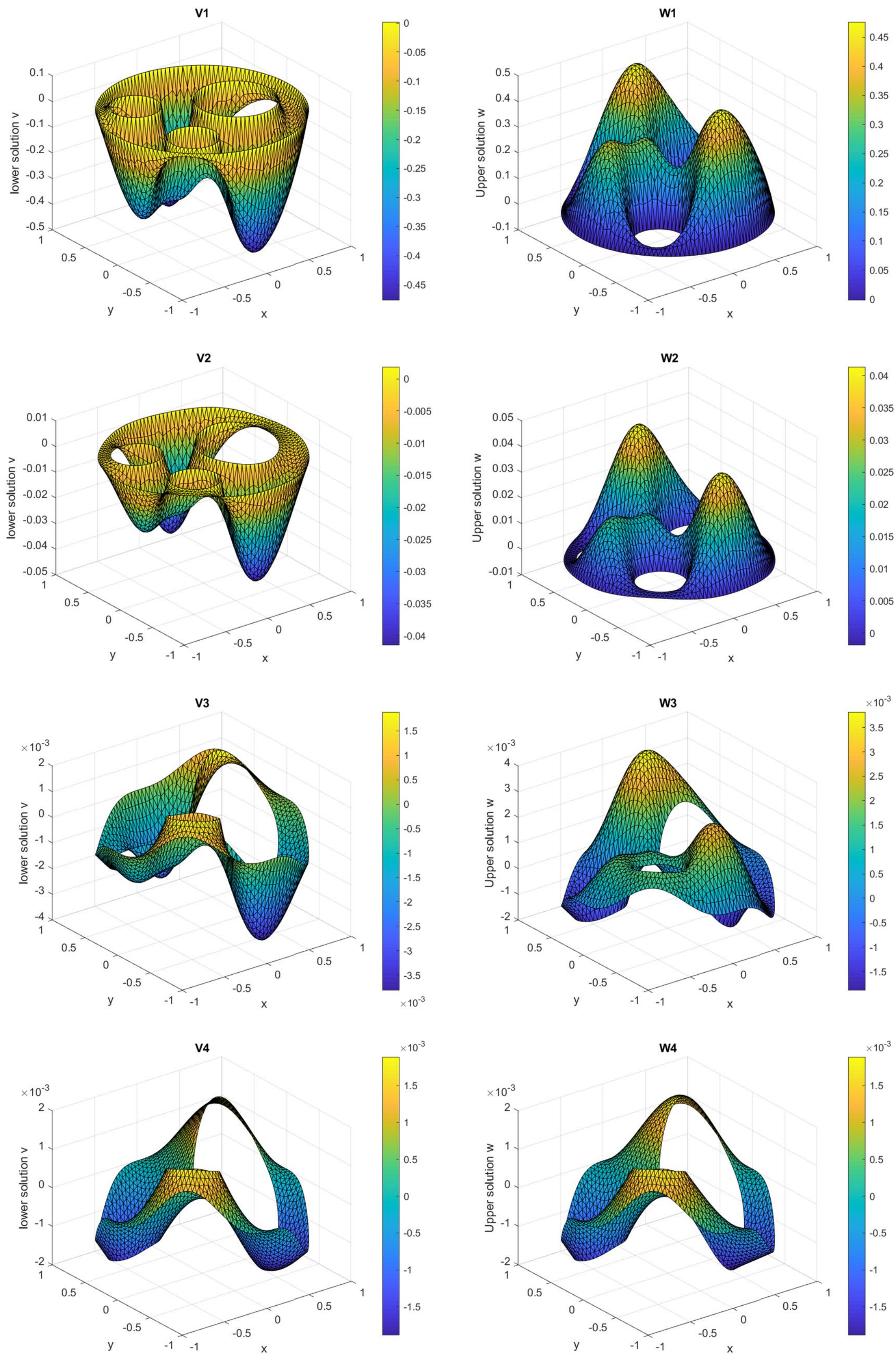


Fig. 6 Convergence of upper (W_n) and lower (V_n) sequences of Example 1 for $h = 0.04$, $\Delta t = 0.001$, $t_1 = 100\Delta t$ and $\beta = 0.5$

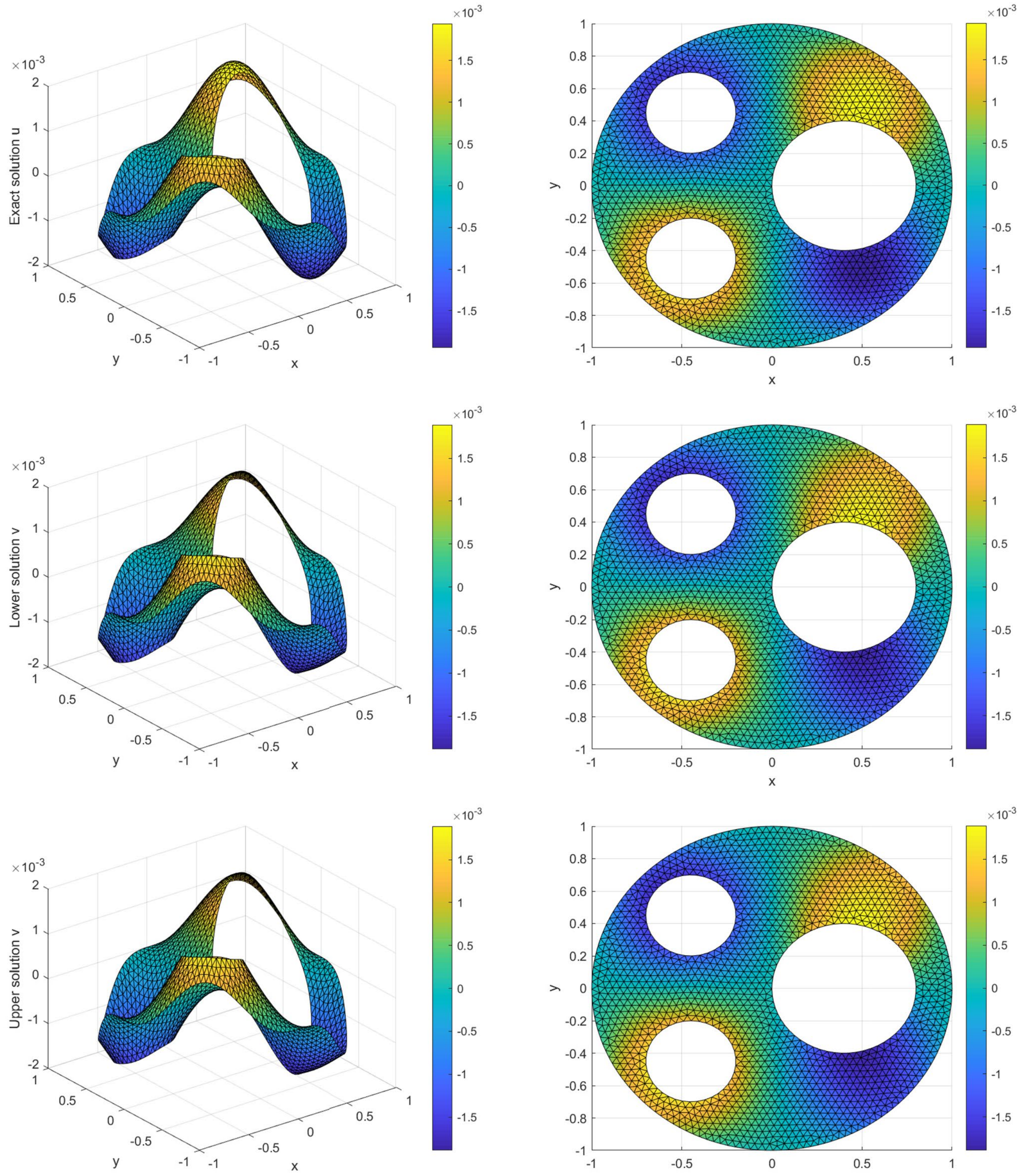


Fig. 7 Comparison of numerical and exact solution for test Example 1 for $h = 0.04$, $\Delta t = 0.001$, $\beta = 0.5$ at time $t_1 = 100\Delta t$

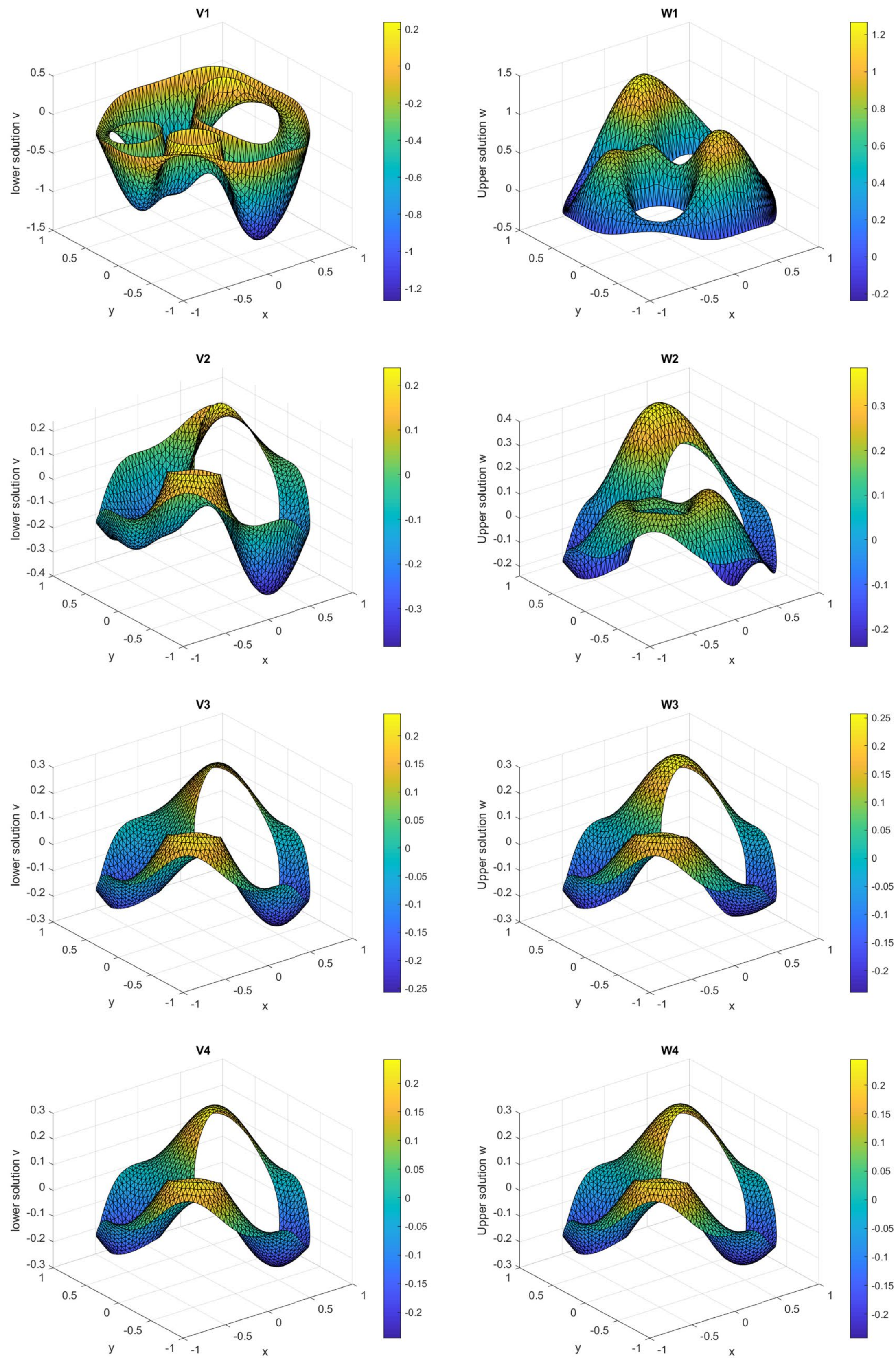


Fig. 8 Convergence of upper (W_n) and lower (V_n) sequences of Example 1 for $h = 0.08$, $\Delta t = 0.01$, $t_1 = 100\Delta t$ and $\beta = 0.9$

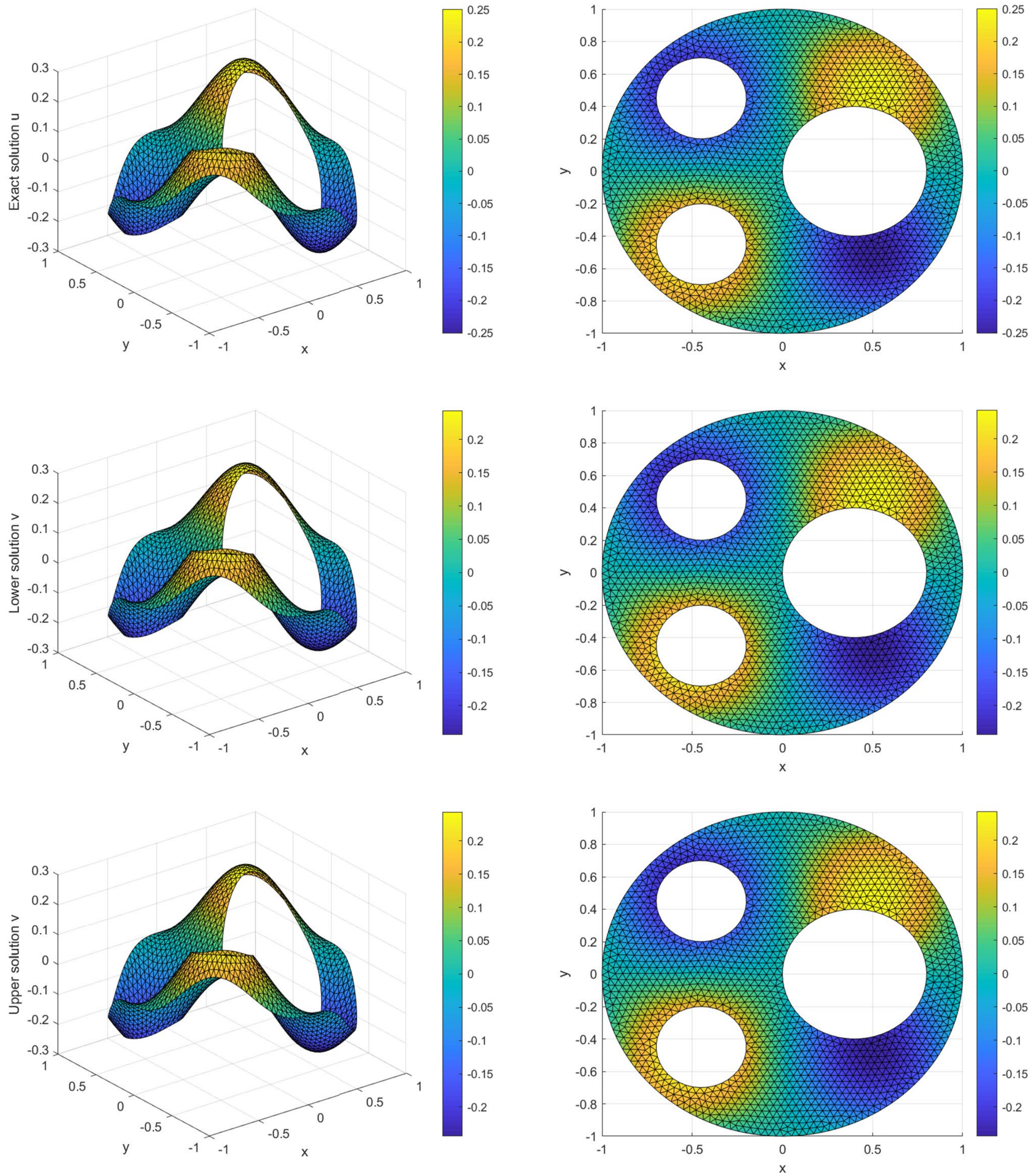
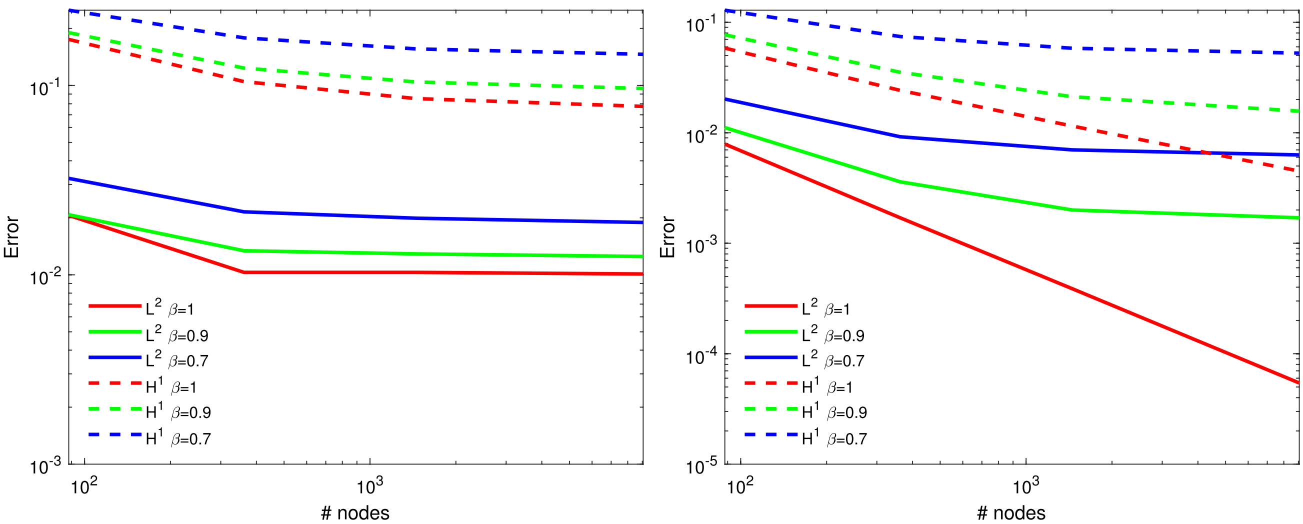


Fig. 9 Comparison of numerical and exact solution for test Example 1 for $h = 0.04$, $\Delta t = 0.01$, $\beta = 0.9$ at time $t_1 = 100\Delta t$



(a) $\Delta t = 0.1.$ **(b)** $\Delta t = 0.001.$

Fig. 10 Convergence of L_2 and H_1 errors with increasing number of nodes in the coarse discretization for Example 1

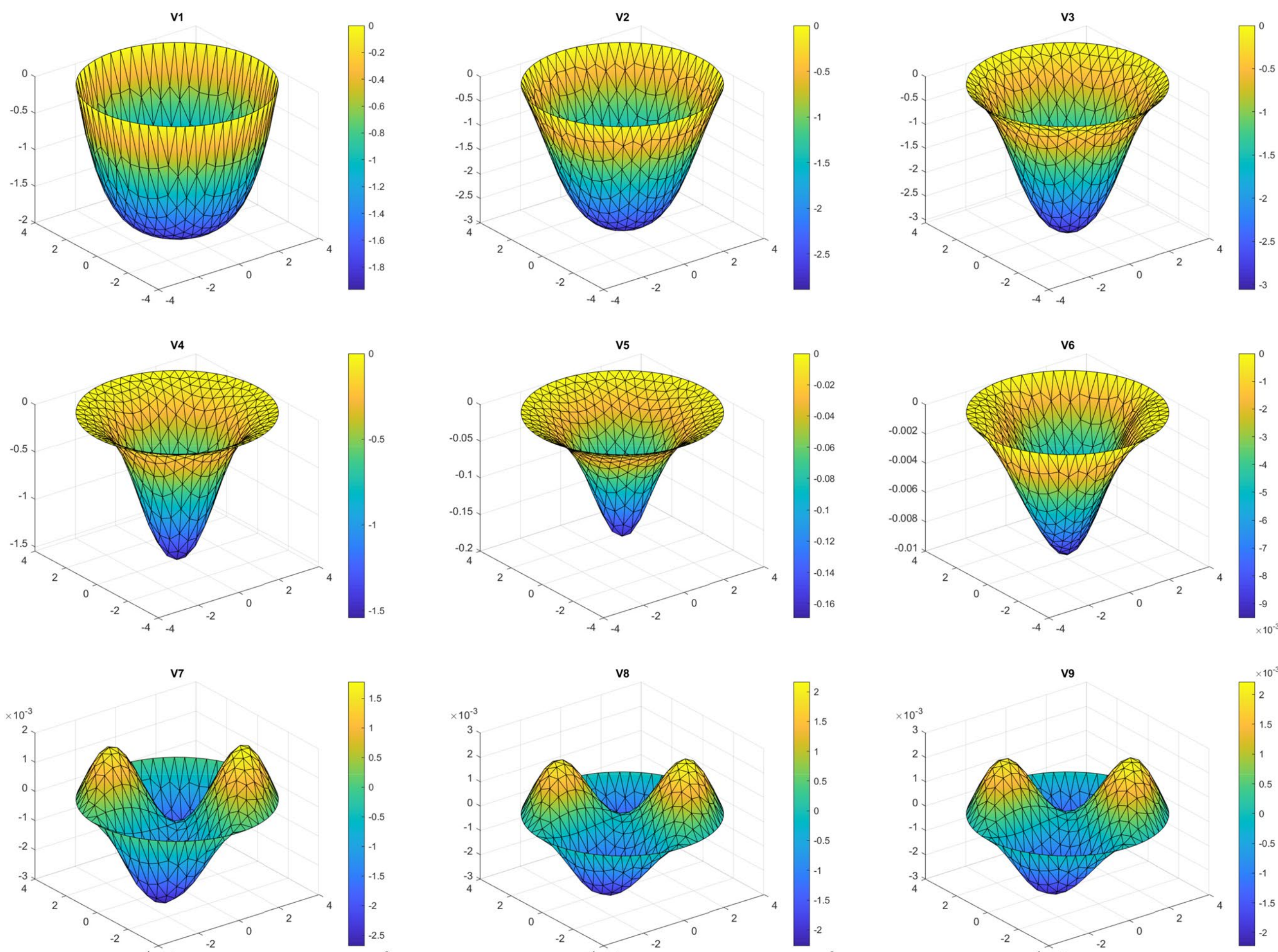


Fig. 11 Convergence of lower sequence (V_n) for test Example 2 with $\Delta t = 0.01$, $\beta = 0.95$ and at time $t = 100\Delta t$

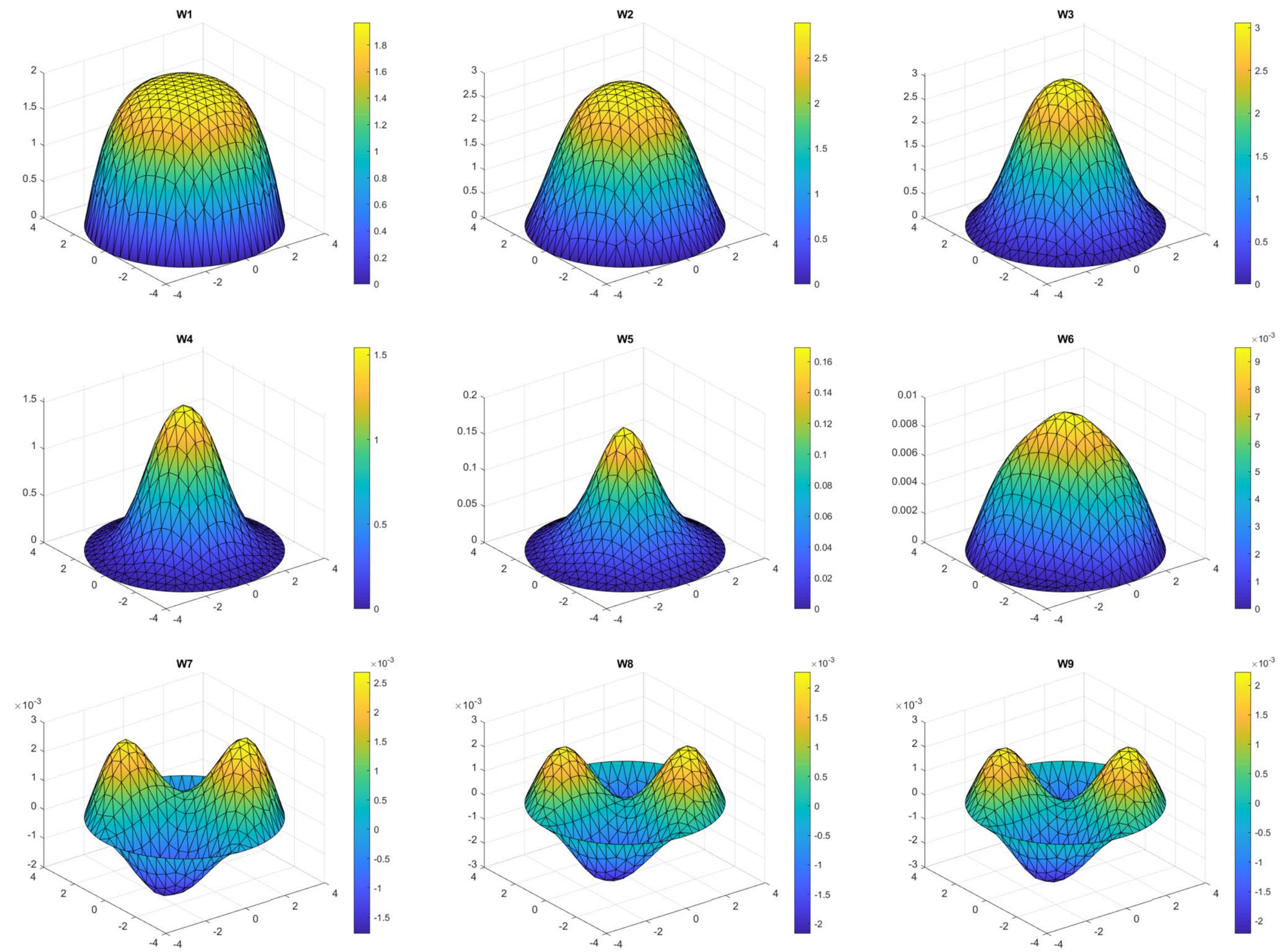


Fig. 12 Convergence of upper sequence (W_n) for test Example 2 with $\Delta t = 0.01$, $\beta = 0.95$ and at time $t = 1$

$$\begin{cases} \mathcal{D}_t^{\beta} u - D\Delta u = \pi^2 u \quad \text{in } Q_T = (0,T)\times\Omega, \\ u(t,x) = \exp(-\pi^2 t^{\beta}/\beta)\sin(\pi x)\sin(\pi y) \quad (t,x,y)\in(0,T)\times\Gamma, \\ u(0,x,y) = \sin(\pi x)\sin(\pi y) \quad (x,y)\in\Omega. \end{cases} \tag{7.1}$$

Here, we consider in the first case the domain Ω in the flowing form (see Fig. 1a):

$\Omega = D \cap \overline{D}_1 \cap \overline{D}_2 \cap \overline{D}_3$ where
$D = \{(x,y)\in\mathbb{R}^2 \quad / \quad x^2+y^2<1\}$
$\overline{D}_i = \{(x,y)\in\mathbb{R}^2 \quad / \quad (x-a_i)^2+(y-b_i)^2 > r_i^2\}, \quad i=1,2,3,$

with Lipschitz boundary given by

$\Gamma = C \cup C_1 \cup C_2 \cup C_3$,
where
$C = \{(x,y)\in\mathbb{R}^2 \quad / \quad x^2+y^2=1\}$.
$C_i = \{(x,y)\in\mathbb{R}^2 \quad / \quad (x-a_i)^2+(y-b_i)^2 = r_i^2\}, \quad i=1,2,3,$

The exact solution of (7.1) is $u(t,x,y) = \exp(-\pi^2 t^{\beta}/\beta)\sin(\pi x)\sin(\pi y)$.

Is not difficult to show that $v \equiv -1$ is a lower solution and $w \equiv 1$ is an upper solution of (7.1). By using the fully discrete finite element scheme given in (6.6) we solve the model (7.1) for two values of the non-integer order, 0.5 and 0.9. We chose firstly in our simulation $\Delta t = 0.01$ and $h = 0.08$. We show in Fig. 4 first 18 steps of V_n and W_n for $\beta = 0.5$ and in Fig. 2 first 13 steps of V_n and W_n for $\beta = 0.9$ at time step $t = 100\Delta t$. Observe that the upper and lower sequence converges more quickly to the numerical solution of the model and the two sequences coincide for $k \geq 18$ for $\beta = 0.5$ and $k \geq 13$ for $\beta = 0.9$. The comparison between the approximate and exact solution is presented in Fig. 3 for $\beta = 0.5$ and in Fig. 5 for $\beta = 0.9$.

When we chose now $\Delta t = 0.001$ and $h = 0.04$. We compute the numerical solution for $\beta = 0.5, 0.9$ at time step $t = 100\Delta t$. We show in Figs. 6 and 8 that the upper and lower sequence converges more quickly to the numerical

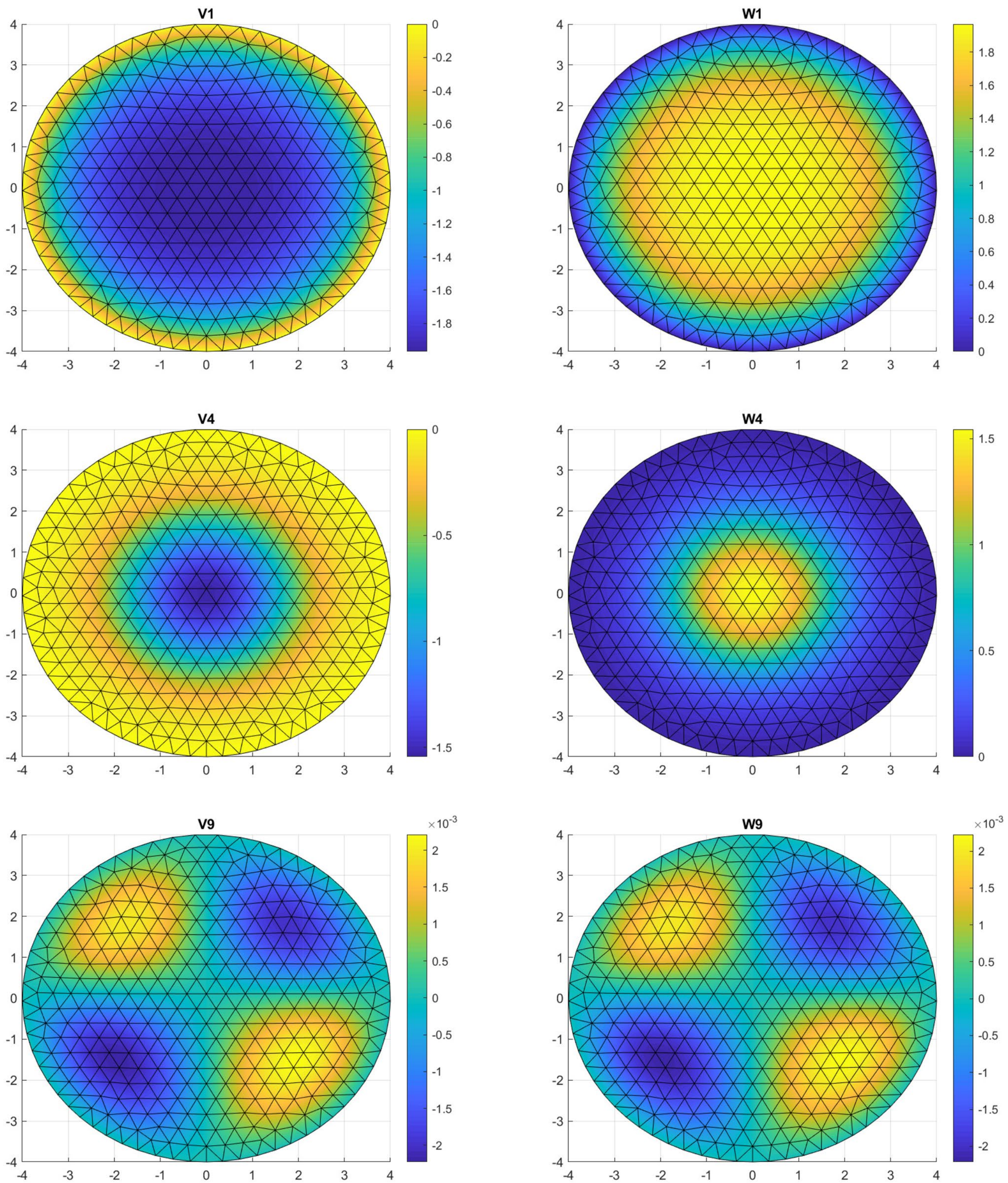


Fig. 13 Comparison between upper solutions W_1, W_4, W_9 and lower solutions V_1, V_4, V_9 for test Example 2 with $\Delta t = 0.01$, $\beta = 0.95$ and at time $t = 100\Delta t$

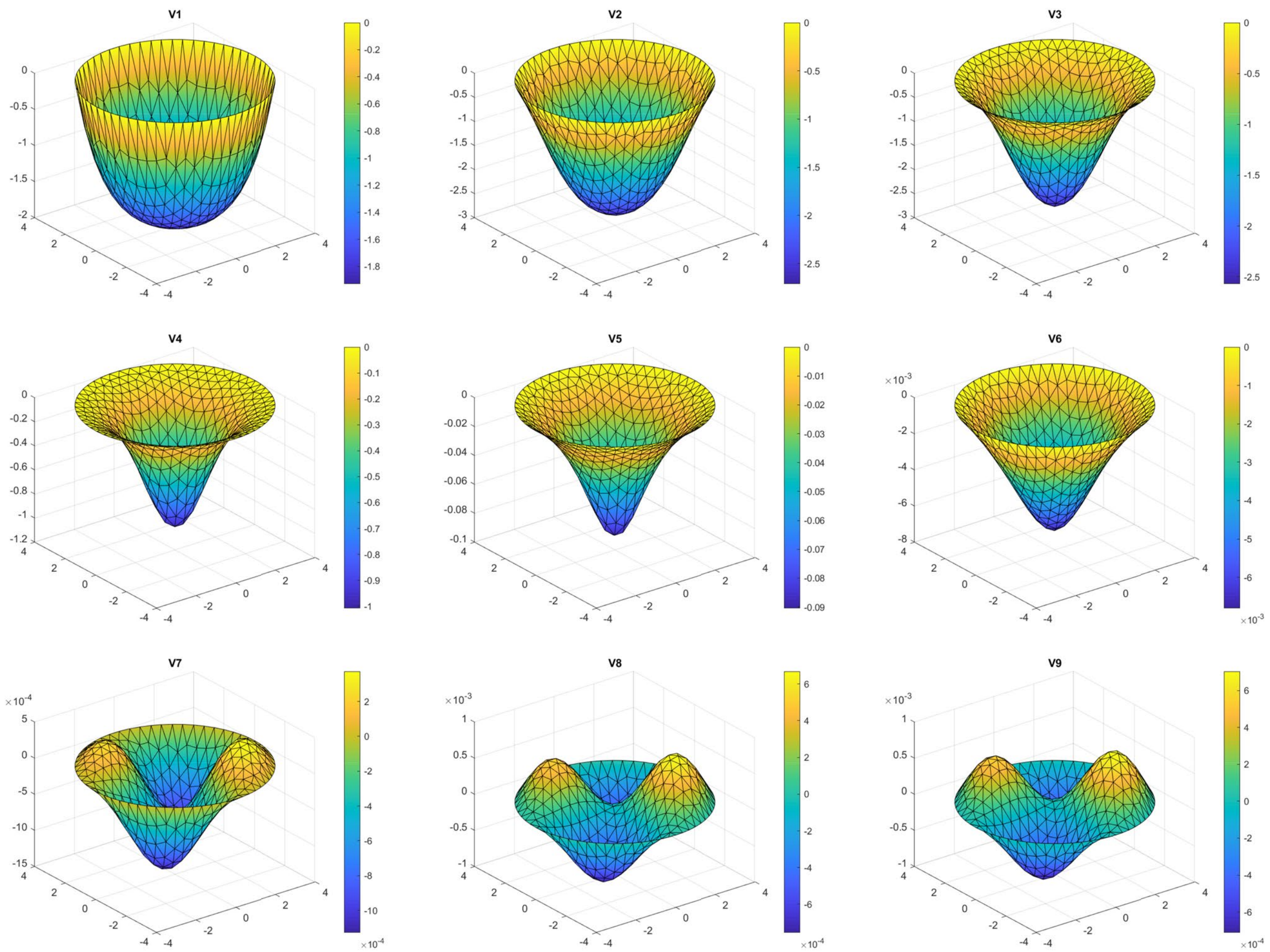


Fig. 14 Convergence of lower sequence (V_n) for test Example 2 with $\Delta t = 0.01$, $\beta = 0.5$ and at time $t = 100\Delta t$

solution of the model and the two sequences coincide for $k \geq 4$. The comparison between the approximate and exact solution is presented in Fig. 7 for $\beta = 0.5$ and in Fig. 9 for $\beta = 0.9$.

The corresponding $L^2_\beta(0, T, L^2(\Omega))$-norm errors and $L^2_\beta(0, T, H^1_0(\Omega))$-norm errors for test Example 1 are listed in Tables 1 and 2 and presented in Fig. 10 for various values of $\Delta t, h$ and the non-integer order β.

7.2 Example on circle domain with unknown solution

We consider the generalized time Conformable Fisher-Kolmogorov equation

$$\begin{cases} \frac{\partial^\beta u}{\partial t^\beta} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \frac{\alpha}{k} u(1 - u^q) \text{ in } Q_T = (0, T) \times \Omega, \\ u(t, x, y) = h(t, x) \text{ in } (0, T] \times \Gamma, \\ u(0, x, y) = u_0(x, y) \text{ in } \Omega, \end{cases} \tag{7.2}$$

where $h(t, x) = 0, u_0(x, y) = \sin(\pi x)\sin(\pi y)$ and k, α are positive constants and $q \in \mathbb{N}$. This model is a special case of problem (1.1) with $f_1(t, x, y) = \frac{\sigma}{k} u$ and $f_2(t, x, y) = -\frac{\sigma}{k} u^{q+1}$, the domain Ω is taken as a disk with radius equal to 4 since is in the form (see Fig. 1b)

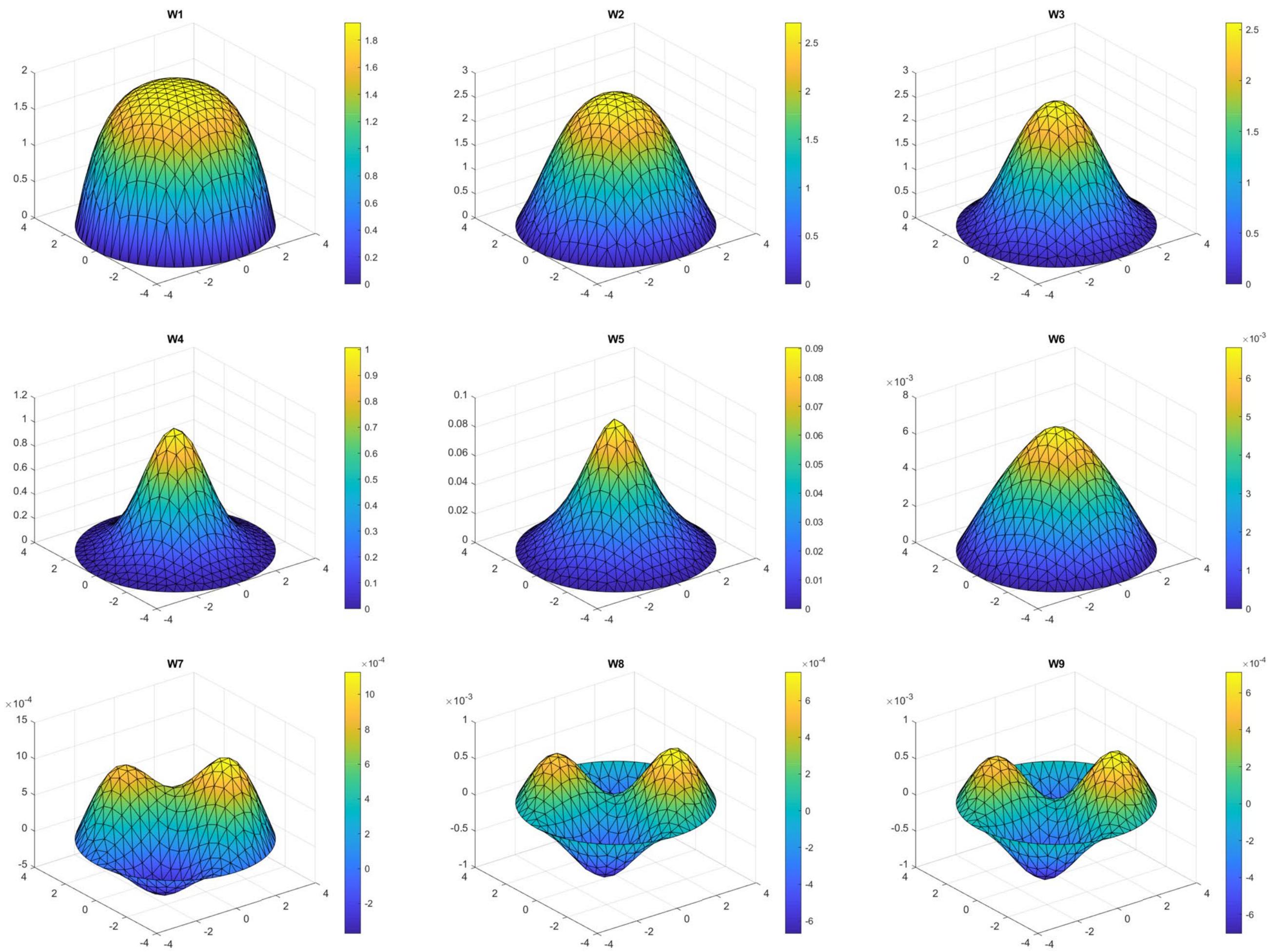


Fig. 15 Convergence of upper sequence (W_n) for test Example 2 with $\Delta t = 0.01$, $\beta = 0.5$ and at time $t = 100\Delta t$

$$\Omega = \{(x, y) \in \mathbf{R}^2 \quad / \quad x^2 + y^2 < 16\}$$
with boundary is circle with radius 4, $\partial\Omega$
$$= \{(x, y) \in \mathbf{R}^2 \quad / \quad x^2 + y^2 = 16\}.$$

We chose in our simulation $\alpha/k = 1$ and $q = 2$. Is not difficult to show that $v \equiv -1$ is a lower solution and $w \equiv 1$ is an upper solution of (7.2). By using the fully discrete finite element scheme given in (6.6) we solve the model (7.2) for two values of β, 0.5 and 0.95. We show in Figs. 11 and 12 first 9 steps of V_n and W_n for $\beta = 0.95$ at time step $t = 100\Delta t$ where $\Delta t = 0.01$. Observe that the upper and lower sequence converges more quickly to the numerical solution the model and the two sequences coincide for $m \geq 9$, see Fig. 13. the same conclusion shown in Figs. 14,15 and 13 for the solution $\beta = 0.5$ (Fig. 16).

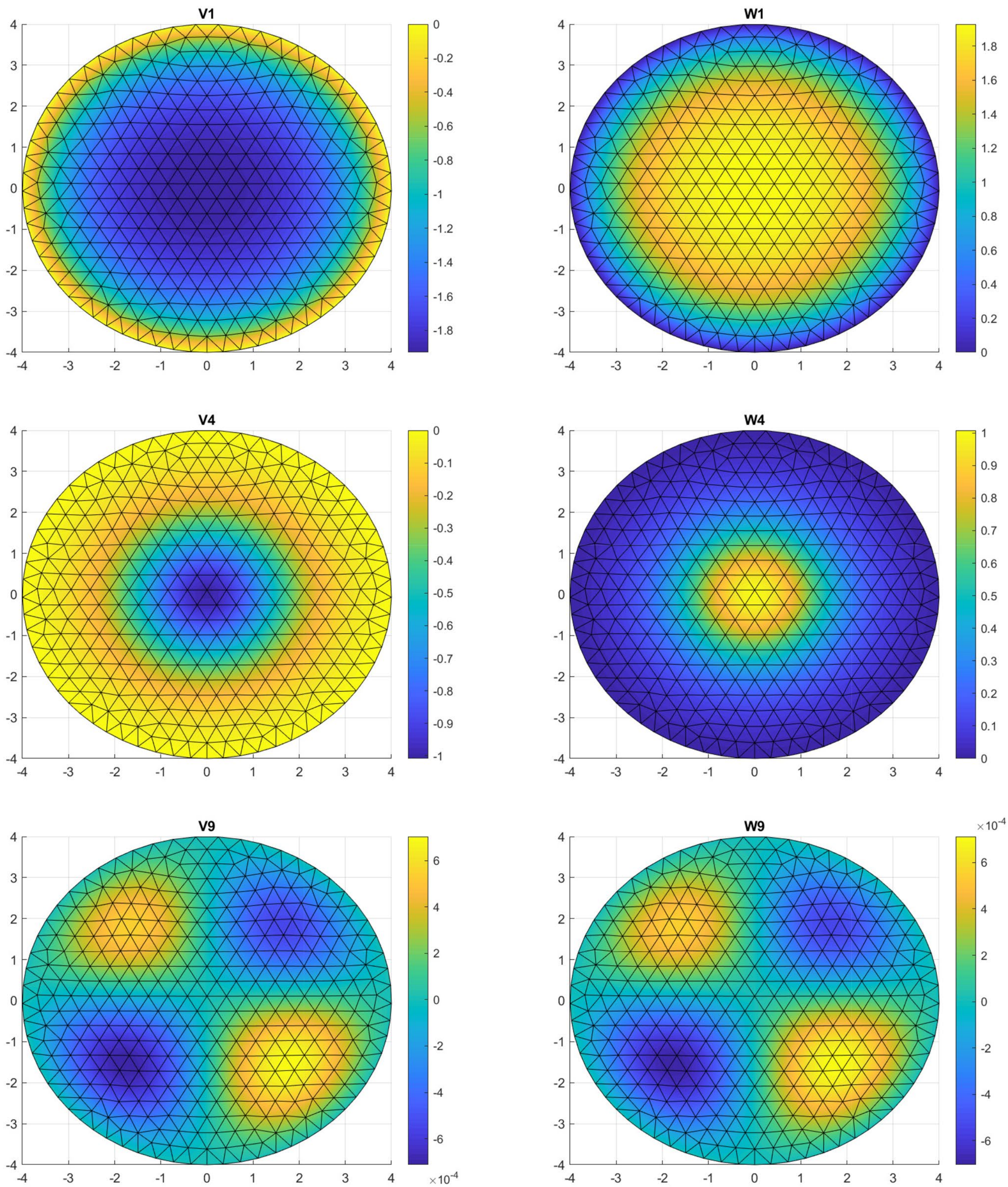


Fig. 16 Comparison between upper solutions W_1, W_4, W_9 and lower solutions V_1, V_4, V_9 for test Example 2 with $\Delta t = 0.01$, $\beta = 0.5$ and at time $t = 100\Delta t$

8 Conclusion and future works

We have shown in this current paper that the monotone iterative technique combined with the upper and lower solutions method are an efficient tools to study theoretically and numerically nonlinear PDEs with non-integer order time derivative. The numerical algorithm for the computation of the solution is obtained by using finite element method for space and the generalization to non-integer order of the implicit Euler scheme for time discetization and we presented real examples and obtained important results.

As perspective, the numerical method used can be improved to increase the speed of convergence by combining this method with Newton's method. The stability and the convergence, thus the error estimation of the scheme proposed in Sect. 4 are considered in our future works. This method can also be applied to other problems such as hyperbolic problems, systems with several equations, problems which contain nonlinear boundary conditions or problems that contain other types of non-integer derivations and this is the subject of our next articles.

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