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ORIGINAL ARTICLE

A novel size-dependent nonlocal strain gradient isogeometric model for functionally graded carbon nanotube-reinforced composite nanoplates

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Abstract
The paper presents a novel nonlocal strain gradient isogeometric model for functionally graded carbon nanotube-reinforced composite (FG-CNTRC) nanoplates. To observe the length scale and size-dependency effects of nanostructures, the nonlocal strain gradient theory (NSGT) is considered. The present model is efficient to capture both nonlocal effects and strain gradient effects in nanoplate structures. In addition, the material properties of the FG-CNTRC are assumed to be graded in the plate thickness direction. Based on the higher order shear deformation theory (HSDT), the weak form of the governing equations of motion of the nanoplates is presented using the principle of virtual work. Afterward, the natural frequency and deflection of the nanoplates are made out of isogeometric analysis (IGA). Thanks to higher order derivatives and continuity of NURBS basic function, IGA is suitable for the weak form of NSGT which requires at least the third-order derivatives in approximate formulations. Effects of nonlocal parameter, strain gradient parameter, carbon nanotube (CNT) volume fraction, distributions of CNTs and length-to-thickness ratios on deflection and natural frequency of the nanoplates are examined and discussed in detail. Numerical results are developed to show the phenomenon of stiffness-softening and stiffness-hardening mechanisms of the present model.

Keywords Nonlocal strain gradient theory (NSGT) · Size-dependent effects · Length scale effects · Isogeometric analysis (IGA) · Functionally graded carbon nanotube-reinforced composites (FG-CNTRC)

1 Introduction

Due to superior features, carbon nanotubes (CNTs) are considered as "new materials for the twenty-first century" [1] and a potential constituent of reinforcement for nanocomposites, and the strongest and the most prevalent and resilient material. Therefore, CNTs have many attached attentions from scientists and open a new research direction in nanotechnology.

With the advantageous of functionally graded/composites materials, a lot of numerical methods has been devised to study these materials such as smoothed finite element methods [2–4], isogeometric analysis [5, 6]. To study mechanical behaviors of CNTs, two popular models including continuum mechanics (CM) and molecular dynamics (MD) are often used. The MD can predict micro/nanostructures with the high precision, but there are some limitations to computationally expensive costs and calculating on too short time scale. Therefore, it must currently be left to CM for simulating of longer times and large systems. To overcome these shortcomings, several CM theories have been developed and used such as the first- and second-order strain gradient theory of Mindlin [7, 8], the nonlocal elasticity theory [9], the first-order strain gradient theory [10, 11], the second-order strain gradient theory [12–14], couple stress theory [15], modified couple stress theory [16] and modified strain gradient elasticity theory [17], etc. In those models, effect

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of nonlocal elasticity or strain gradient is only considered. Hence, Lim et al. [18] combined nonlocal elasticity and strain gradient effects to propose the nonlocal strain gradient theory (NSGT). Xu et al. [19] studied analytical solutions of bending and buckling analyses of Euler–Bernoulli beams using NSGT. Size-dependent analysis based NSGT and finite element method for bending, buckling, and free vibration responses of nanobeams was developed by Rajasekaran and Khaniki [20]. Li and Hu [21] used NSGT to investigate analytical solutions of nonlinear bending deflections and free vibration frequency of functionally graded nanobeams. Bending and buckling analyses of nanobeams using analytical methods based on NSGT [22] were reported. In their model, both Euler–Bernoulli and Timoshenko nanobeams were considered. Free vibration analysis of isotropic nanoplates [23] and buckling analysis of orthotropic nanoplates in thermal environment [24] using NSGT and analytical solutions was also developed. Arefi et al. [25] studied bending analysis of a sandwich porous nanoplate integrated with piezoelectric. Transient responses of porous functionally graded nanoplates under various pulse loads were reported in Ref. [26]. Geometrically nonlinear vibration analysis of sandwich nanoplates was performed by Nematollahi and Mohammadi [27]. Jalaei and Thai [28] studied the dynamic stability of viscoelastic porous FG nanoplate under longitudinal magnetic field. For nanoshell structures, buckling and post-buckling behaviors [29], wave dispersion analysis [30], and free vibration analysis [31] have also examined.

In sum, available studies have only focused on using analytical methods based on NSGT to calculate and simulate nanostructures. As we have known, analytical solutions are suitable for the problems with simple geometries and boundary conditions. It is very difficult to apply real structures in practices. So, numerical methods have been developed and considered as a potential candidate to perform the real structures. To the best of author's knowledge, a combination of NSGT and numerical methods has not been studied and published in the literature so far. Hence, this topic is very potential and interesting for researchers to explore. Due to that lack of the researches, the authors intend to fill this gap in the literature by developing a size-dependent numerical model using NSGT theory for nanoplate structures and plate structures. In nanoplates and plate structures, it always requires higher order continuity of basic functions for higher order derivative of approximate variables. IGA with NURBS basis functions proposed by Hughes and coworkers [32] can be considered as a very strong candidate to study plate structures. Nguyen et al. [33] reported an overview and computer implementation aspects of IGA. To express local refinement, an approach to generalize the geometry independent field approximation approach [34–37] was also proposed. This provides greater flexibility in the choice of the discretisation for the geometry and the field variables, hence allow accounting for localisation and local refinement without refining the patch in all coordinate directions and shows good convergence of IGA compared to alternatives [5, 6, 38]. The IGA has been successfully developed and applied to investigate size-dependent plate problems such as nonlocal theory [39–43], modified strain gradient elasticity theory [44, 45], modified couple stress theory [46, 47], structures with cutout [38, 48–50], locking free in plate/shell structures [38, 51–54] etc. Therefore, in this study, a novel size-dependent nonlocal strain gradient isogeometric model for static and free vibration analyses of functionally graded carbon nanotube-reinforced composites is developed.

2 Theoretical formulation

2.1 Functionally graded carbon nanotube-reinforced composite materials

Four configurations of CNTs including UD, FG-V, FG-O, and FG-X can be formulated as

$$V_{\mathrm{CNT}}=\begin{cases} V_{\mathrm{CNT}}^{*} & \text{(UD)} \\ \left(1+\frac{2z}{h}\right)V_{\mathrm{CNT}}^{*} & \text{(FG-V)} \\ 2\left(1-\frac{2|z|}{h}\right)V_{\mathrm{CNT}}^{*} & \text{(FG-O)} \\ 2\left(\frac{2|z|}{h}\right)V_{\mathrm{CNT}}^{*} & \text{(FG-X)} \end{cases}, \tag{1}$$

where

$$V_{\mathrm{CNT}}^{*}=\frac{w_{\mathrm{CNT}}}{w_{\mathrm{CNT}}+(\rho_{\mathrm{CNT}}/\rho_{\mathrm{m}})-(\rho_{\mathrm{CNT}}/\rho_{\mathrm{m}})w_{\mathrm{CNT}}}, \tag{2}$$

in which w_{CNT} is the mass fraction of CNTs.

Based on the rule of mixtures, the effective material properties are expressed as

$$E_{11}=\eta_1 V_{\mathrm{CNT}}E_{11}^{\mathrm{CNT}}+V_{\mathrm{m}}E^{\mathrm{m}}, \tag{3}$$

$$\frac{\eta_2}{E_{22}}=\frac{V_{\mathrm{CNT}}}{E_{22}^{\mathrm{CNT}}}+\frac{V_{\mathrm{m}}}{E^{\mathrm{m}}}, \tag{4}$$

$$\frac{\eta_3}{G_{12}}=\frac{V_{\mathrm{CNT}}}{G_{12}^{\mathrm{CNT}}}+\frac{V_{\mathrm{m}}}{G^{\mathrm{m}}}, \tag{5}$$

$$\nu_{12}=V_{\mathrm{CNT}}^{*}\nu_{12}^{\mathrm{CNT}}+V_{\mathrm{m}}\nu^{\mathrm{m}}, \tag{6}$$

$$\rho=V_{\mathrm{CNT}}\rho^{\mathrm{CNT}}+V_{\mathrm{m}}\rho^{\mathrm{m}}, \tag{7}$$

where ν_{12}^{CNT}, ρ^{CNT} and ν^{m}, ρ^{m} are Poisson's ratio and density of CNTs and matrix, respectively; E^{m} and G^{m} are the

Young's modulus and shear modulus of the isotropic matrix; E_{11}^{CNT}, E_{22}^{CNT} and G_{12}^{CNT} are the Young's and shear modulus of CNTs; η_1, η_2 and η_3 are CNT efficiency parameters listed in Table 1; V_{CNT} and V_{m} are the CNTs and matrix volume fractions, in which $V_{\mathrm{CNT}} + V_{\mathrm{m}} = 1$.

2.2 Nonlocal strain gradient theory (NSGT)

Equations of motion of nonlocal elastic solids are given as follows

$$t_{ij,j} + f_i = \rho \ddot{u}_i \quad \text{in } V, \tag{8}$$

$$t_{ij} n_i = g_i \text{ on } S, \tag{9}$$

where t_{ij} is the total nonlocal stress components; f_i and g_i are the body force and traction components, respectively; ρ is density mass; $\ddot{u}_i$ is the acceleration field; V is the volume and S is the Neumann boundary.

Based on the NSGT [18], the nonlocal stress tensor is expressed by

$$t_{ij} = t_{ij}^0 - \nabla t_{ij}^1, \tag{10}$$

where

$$t_{ij}^0(\mathbf{x}) = \int_V \alpha\left(\left|\mathbf{x}' - \mathbf{x}\right|\right)\sigma_{ij}\left(\mathbf{x}'\right)\mathrm{d}V'\left(\mathbf{x}'\right), \ldots t_{ij}^1(\mathbf{x}) = l^2 \int_V \alpha\left(\left|\mathbf{x}' - \mathbf{x}\right|\right)\nabla\sigma_{ij}\left(\mathbf{x}'\right)\mathrm{d}V'\left(\mathbf{x}'\right), \tag{11}$$

in which t_{ij}^0 and t_{ij}^1 are the nonlocal stress and higher order stress tensors; $\alpha\left(\left|\mathbf{x}' - \mathbf{x}\right|\right)$ is the nonlocal kernel function; $\mathbf{x}$ is a reference point in the body; l is the material length scale parameter; $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ is the differential operator; σ_{ij} is the local Cauchy stress tensors of classical elasticity theory and satisfies

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \quad \varepsilon_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right), \tag{12}$$

where ε_{kl} is strain components; C_{ijkl} is elastic modulus coefficients and u_k is displacements.

Using the special Helmholtz averaging kernel, the nonlocal constitutive in Eq. (11) can be rewritten as [55]

$$Lt_{ij}^0 = \sigma_{ij}, \; Lt_{ij}^1 = \lambda\nabla\sigma_{ij}, \; Lt_{ij,j}^0 = \sigma_{ij,j}, \; Lt_{ij,j}^1 = \lambda\nabla\sigma_{ij,j}, \tag{13}$$

where $L = \left(1 - \mu\nabla^2\right)$ defines the linear differential operator, in which $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\lambda = l^2$.

Similarly, Eq. (8) can be represented by

$$Lt_{ij} = Lt_{ij}^0 - L\nabla t_{ij}^1, \; Lt_{ij,j} = Lt_{ij,j}^0 - L\nabla t_{ij,j}^1, \tag{14}$$

Substituting Eq. (13) into Eq. (14), the equilibrium equation is performed as

$$\sigma_{ij,j} - \lambda\nabla^2\sigma_{ij,j} + \left(1 - \mu\nabla^2\right)f_i = \left(1 - \mu\nabla^2\right)\rho\ddot{u}_i, \tag{15}$$

Applying the principle of virtual displacement, Eq. (15) can be presented

$$\int_V \sigma_{ij,j}\delta u_i \mathrm{d}V - \lambda\int_V \nabla^2\sigma_{ij,j}\delta u_i \mathrm{d}V + \int_V \left(1 - \mu\nabla^2\right)f_i\delta u_i \mathrm{d}V = \int_V \left(1 - \mu\nabla^2\right)\rho\ddot{u}_i\delta u_i \mathrm{d}V, \tag{16}$$

where δu_i is the virtual displacement.

The first term in Eq. (16) can be given by applying the integration by parts and divergence theorem

$$\int_V \sigma_{ij,j}\delta u_i \mathrm{d}V = -\int_V \sigma_{ij}\delta u_{i,j}\mathrm{d}V + \int_S \sigma_{ij}n_i\delta u_i \mathrm{d}S. \tag{17}$$

Similarly, the second term in Eq. (16) can be rewritten as

$$\int_V \nabla^2\sigma_{ij,j}\delta u_i \mathrm{d}V = -\int_V \nabla^2\sigma_{ij}\delta u_{i,j}\mathrm{d}V + \int_S \nabla^2\sigma_{ij}n_i\delta u_i \mathrm{d}S. \tag{18}$$

Substituting Eqs. (17) and (18) into Eq. (16), the final equation is expressed as

$$-\int_V \sigma_{ij}\delta u_{i,j}\mathrm{d}V + \int_S \sigma_{ij}n_i\delta u_i \mathrm{d}S + \lambda\int_V \nabla^2\sigma_{ij}\delta u_{i,j}\mathrm{d}V - \lambda\int_S \nabla^2\sigma_{ij}n_i\delta u_i \mathrm{d}S + \int_V \left(1 - \mu\nabla^2\right)f_i\delta u_i \mathrm{d}V = \int_V \left(1 - \mu\nabla^2\right)\rho\ddot{u}_i\delta u_i \mathrm{d}V. \tag{19}$$

Table 1 The CNTs' efficiency parameters

V_{CNT}^*	η_1	η_2	η_3
0.11	0.149	0.934	0.934
0.14	0.150	0.941	0.941
0.17	0.140	1.381	1.381

The traction on the Neumann boundary in this study is ignored, Eq. (19) is rewritten as

$$\int_V \sigma_{ij}\delta u_{i,j}\mathrm{d}V - \lambda \int_V \nabla^2\sigma_{ij}\delta u_{i,j}\mathrm{d}V + \int_V \left(1-\mu\nabla^2\right)\rho\ddot{u}_i\delta u_i\mathrm{d}V = \int_V \left(1-\mu\nabla^2\right)f_i\delta u_i\mathrm{d}V. \quad (20)$$

Using the symmetric condition, the virtual displacement vector can be defined as

$$\delta u_{i,j} = \frac{1}{2}\left(\delta u_{i,j} + \delta u_{j,i}\right) = \delta\varepsilon_{ij}. \quad (21)$$

Substituting Eq. (21) into Eq. (20), the final form is described as

$$\int_V \sigma_{ij}\delta\varepsilon_{ij}\mathrm{d}V - \lambda \int_V \nabla^2\sigma_{ij}\delta\varepsilon_{ij}\mathrm{d}V + \int_V \left(1-\mu\nabla^2\right)\rho\ddot{u}_i\delta u_i\mathrm{d}V = \int_V \left(1-\mu\nabla^2\right)f_i\delta u_i\mathrm{d}V. \quad (22)$$

3 Displacement field

Based on HSDT [56], the displacement components can be formulated as

$$\bar{\mathbf{u}}(x,y,z) = \mathbf{u}^1(x,y) + z\mathbf{u}^2(x,y) + f(z)\mathbf{u}^3(x,y), \quad (23)$$

where

$$\bar{\mathbf{u}} = \begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{Bmatrix},\ \mathbf{u}^1 = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix},\ \mathbf{u}^2 = -\begin{Bmatrix} \beta_x \\ \beta_y \\ 0 \end{Bmatrix},\ \mathbf{u}^3 = \begin{Bmatrix} \theta_x \\ \theta_y \\ 0 \end{Bmatrix}, \quad (24)$$

in which u, v and w are the in-plane and transverse displacement components, respectively; θ_x and θ_y are two rotations; $\beta_x = w_{,x}$ and $\beta_y = w_{,y}$; $f(z) = z - 4z^3/3h^2$ [57].

The bending and shear strain components are defined as

$$\boldsymbol{\varepsilon} = \left\{ \varepsilon_{xx}\ \varepsilon_{yy}\ \gamma_{xy} \right\}^{\mathrm{T}} = \boldsymbol{\varepsilon}^1 + z\boldsymbol{\varepsilon}^2 + f(z)\boldsymbol{\varepsilon}^3,\ \boldsymbol{\gamma} = \left\{ \gamma_{xz}\ \gamma_{yz} \right\}^{\mathrm{T}} = \boldsymbol{\varepsilon}^{s1} + f'(z)\boldsymbol{\varepsilon}^{s2}, \quad (25)$$

where

$$\boldsymbol{\varepsilon}^1 = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{Bmatrix},\ \boldsymbol{\varepsilon}^2 = -\begin{Bmatrix} \beta_{x,x} \\ \beta_{y,y} \\ \beta_{x,y} + \beta_{y,x} \end{Bmatrix},\ \boldsymbol{\varepsilon}^3 = \begin{Bmatrix} \theta_{x,x} \\ \theta_{y,y} \\ \theta_{x,y} + \theta_{y,x} \end{Bmatrix},$$
$$\boldsymbol{\varepsilon}^{s1} = \begin{Bmatrix} w_{,x} - \beta_x \\ w_{,y} - \beta_y \end{Bmatrix},\ \boldsymbol{\varepsilon}^{s2} = \begin{Bmatrix} \theta_x \\ \theta_y \end{Bmatrix} \quad (26)$$

in which $f'(z) = 1 - 4z^2/h^2$.

The constitutive relations can be defined as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}, \quad (27)$$

where

$$C_{11} = \frac{E_{11}}{1-\nu_{12}^2},\ C_{22} = \frac{E_{22}}{1-\nu_{12}^2},\ C_{12} = C_{21} = \frac{\nu_{12}E_{11}}{1-\nu_{12}^2},$$
$$C_{66} = C_{55} = C_{44} = G_{12}. \quad (28)$$

Substituting Eqs. (25) and (27) into Eq. (22), the discrete equations for bending analysis of the nanoplates subjected to a transverse load f_0 can be described as [58]:

$$\int_\Omega \delta\bar{\boldsymbol{\varepsilon}}^T\mathbf{C}^b\bar{\boldsymbol{\varepsilon}}\mathrm{d}\Omega - \lambda\int_\Omega \delta\left(\nabla^2\bar{\boldsymbol{\varepsilon}}^T\right)\mathbf{C}^b\bar{\boldsymbol{\varepsilon}}\mathrm{d}\Omega$$
$$+ \int_\Omega \delta\bar{\boldsymbol{\gamma}}^T\mathbf{C}^s\bar{\boldsymbol{\gamma}}\mathrm{d}\Omega - \lambda\int_\Omega \delta\left(\nabla^2\bar{\boldsymbol{\gamma}}^T\right)\mathbf{C}^s\bar{\boldsymbol{\gamma}}^T\mathrm{d}\Omega$$
$$= \int_\Omega \left(1-\mu\nabla^2\right)f_0\delta w\mathrm{d}\Omega, \quad (29)$$

where

Table 2 Non-dimensional natural frequency ($\varpi = \omega a^2\sqrt{\rho h/D}$, where $D = \frac{Eh^3}{12(1-\nu^2)}$) of the simply supported isotropic square plate ($a/h = 100$)

Method	$l = \sqrt{\lambda}$		
	0	0.5	1
FEM-C [59]	19.7205	20.1971	21.5632
FEM-NC [59]	19.7205	20.2010	21.5785
Analytical solution [59]	19.7205	20.2012	21.5792
Present (11×11)	19.7323	20.6810	21.5711
Present (9×9)	19.7330	20.6860	21.5696
Present (7×7)	19.7366	20.7069	21.5775
Present (5×5)	19.7653	20.7879	21.5991

$$\bar{\boldsymbol{\varepsilon}} = \begin{Bmatrix} \boldsymbol{\varepsilon}^1 \\ \boldsymbol{\varepsilon}^2 \\ \boldsymbol{\varepsilon}^3 \end{Bmatrix}, \ \bar{\boldsymbol{\gamma}} = \begin{Bmatrix} \boldsymbol{\varepsilon}^{s1} \\ \boldsymbol{\varepsilon}^{s2} \end{Bmatrix}, \ \mathbf{C}^b = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{E} \\ \mathbf{B} & \mathbf{D} & \mathbf{F} \\ \mathbf{E} & \mathbf{F} & \mathbf{H} \end{bmatrix}, \ \mathbf{C}^s = \begin{bmatrix} \mathbf{A}^s & \mathbf{B}^s \\ \mathbf{B}^s & \mathbf{D}^s \end{bmatrix}$$

$$\left(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}\right) = \int_{-h/2}^{h/2} \left(1, z, z^2, f(z), zf(z), f^2(z)\right) C_{ij} \mathrm{d}z \text{ where } (i,j = 1, 2, 6)$$

$$\left(A^s_{ij}, B^s_{ij}, D^s_{ij}\right) = \int_{-h/2}^{h/2} \left(1, f'(z), f'^2(z)\right) C_{ij} \mathrm{d}z \text{ where } (i,j = 4, 5). \quad (30)$$

Similarly, the discrete equations for free vibration analysis are expressed as

$$\int_\Omega \delta\bar{\boldsymbol{\varepsilon}}^T \mathbf{C}^b \bar{\boldsymbol{\varepsilon}} \mathrm{d}\Omega - \lambda \int_\Omega \delta\left(\nabla^2 \bar{\boldsymbol{\varepsilon}}^T\right) \mathbf{C}^b \bar{\boldsymbol{\varepsilon}} \mathrm{d}\Omega + \int_\Omega \delta\bar{\boldsymbol{\gamma}}^T \mathbf{C}^s \bar{\boldsymbol{\gamma}} \mathrm{d}\Omega - \lambda \int_\Omega \delta\left(\nabla^2 \bar{\boldsymbol{\gamma}}^T\right) \mathbf{C}^s \bar{\boldsymbol{\gamma}}^T \mathrm{d}\Omega + \int_\Omega \left(1 - \mu\nabla^2\right) \delta\bar{\mathbf{u}} \mathbf{I}_m \ddot{\bar{\mathbf{u}}} \mathrm{d}\Omega = \mathbf{0}, \quad (31)$$

where

$$\bar{\mathbf{u}} = \begin{Bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \\ \mathbf{u}^3 \end{Bmatrix}, \mathbf{I}_m = \begin{bmatrix} \mathbf{I}_1 & \mathbf{I}_2 & \mathbf{I}_4 \\ \mathbf{I}_2 & \mathbf{I}_3 & \mathbf{I}_5 \\ \mathbf{I}_4 & \mathbf{I}_5 & \mathbf{I}_6 \end{bmatrix},$$

$$\left(\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6\right) = \int_{-h/2}^{h/2} \rho_e \left(1, z, z^2, f(z), zf(z), f^2(z)\right) \mathbf{I}_{3x3} \mathrm{d}z, \quad (32)$$

where $\mathbf{I}_{3x3}$ is the identity matrix of size 3×3.

$$\mathbf{u}^h(x,y) = \sum_{I=1}^{mxn} \begin{bmatrix} N_I(x,y) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & N_I(x,y) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_I(x,y) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_I(x,y) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_I(x,y) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & N_I(x,y) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_I(x,y) \end{bmatrix} \begin{Bmatrix} u_I \\ v_I \\ w_I \\ \theta_{xI} \\ \theta_{yI} \\ \beta_{xI} \\ \beta_{yI} \end{Bmatrix}, \quad (33)$$

$$= \sum_{I=1}^{mxn} \mathbf{N}_I(x,y) \mathbf{q}_I,$$

where $\mathbf{q}_I$ is a vector that contains degrees of freedom combined with the control point I; N_I is the shape function defined as in Ref. [32].

Substituting Eq. (33) into Eq. (26), Eq. (26) can be rewritten as:

3.1 The NSGT formulation using NURBS basis function

The displacements using NURBS basis functions can be expressed as

Table 3 The central non-dimensional deflection $\left(\overline{w} = \frac{1000wD}{q_0 a^4}\right)$ of the simply supported isotropic square plate ($a/h = 100$)

Method	$l = \sqrt{\lambda}$		
	0	0.5	1
FEM-C [59]	4.0624	3.8856	3.4271
FEM-NC [59]	4.0624	3.8845	3.4233
Analytical solution [59]	4.0624	3.8844	3.4231
Present (15 × 15)	4.0645	3.7172	3.4133
Present (11 × 11)	4.0645	3.7131	3.3956
Present (9 × 9)	4.0649	3.6966	3.3598
Present (7 × 7)	4.0696	3.6559	3.3039
Present (5 × 5)	4.0754	3.6020	3.2799

Table 4 The three lowest non-dimensional frequencies $\overline{\omega} = (\omega a^2/h)\sqrt{\rho^m/E^m}$ of the SSSS FG-CNTRC nanoplates

V^*_{CNT}	a/h	Mode	$\mu=1.5$							
			UD		FG-V		FG-O		FG-X	
			Ref. [43]	Present	Ref. [43]	Present	Ref. [32]	Present	Ref. [32]	Present
0.11	10	1	2.044	2.0385	1.969	1.9636	1.865	1.8559	2.152	2.1481
		2	2.201	2.2243	2.183	2.2053	2.062	2.0457	2.266	2.2958
		3	2.448	2.4491	2.262	2.2512	2.121	2.1366	2.489	2.6091
	20	1	2.477	2.471	2.292	2.2862	2.123	2.1164	2.738	2.7319
		2	2.615	2.6078	2.546	2.5394	2.428	2.4209	2.774	2.7673
		3	3.046	3.0424	2.729	2.7238	2.433	2.4252	3.167	3.1646
	50	1	2.688	2.6864	2.429	2.4273	2.225	2.224	3.026	3.0248
		2	2.790	2.7892	2.683	2.6824	2.543	2.5427	3.074	3.0727
		3	3.271	3.2763	2.929	2.9281	2.576	2.5749	3.446	3.4525
0.14	10	1	2.123	2.1183	2.049	2.0437	1.934	1.9248	2.233	2.2302
		2	2.256	2.2802	2.238	2.2622	2.164	2.1792	2.331	2.3642
		3	2.498	2.5945	2.412	2.4001	2.205	2.1889	2.549	2.6769
	20	1	2.637	2.6309	2.431	2.4255	2.238	2.2306	2.899	2.8926
		2	2.713	2.706	2.635	2.6277	2.496	2.4879	2.920	2.9131
		3	3.124	3.1201	2.986	2.9805	2.658	2.6499	3.275	3.2738
	50	1	2.909	2.9078	2.605	2.6037	2.366	2.3641	3.211	3.2096
		2	2.923	2.9226	2.793	2.7919	2.619	2.6183	3.355	3.353
		3	3.369	3.3757	3.249	3.2477	2.851	2.8494	3.595	3.6014
0.17	10	1	2.530	2.5403	2.442	2.4513	2.299	2.3023	2.680	2.6923
		2	2.744	2.7787	2.730	2.764	2.523	2.5444	2.854	2.9020
		3	3.016	3.0422	2.765	2.792	2.628	2.6491	3.137	3.2855
	20	1	3.033	3.067	2.814	2.8413	2.588	2.6096	3.373	3.4107
		2	3.241	3.2509	3.171	3.1764	2.925	2.9843	3.484	3.4970
		3	3.789	3.8005	3.290	3.3585	2.990	2.9918	4.012	4.0208
	50	1	3.268	3.3259	2.966	3.0095	2.698	2.7337	3.756	3.8187
		2	3.445	3.4717	3.332	3.3511	3.082	3.129	3.783	3.8286
		3	4.067	4.0896	3.508	3.6000	3.114	3.1608	4.366	4.3920

Table 5 Non-dimensional deflection $\overline{w} = w/h$ of the SSSS FG-CNTRC square nanoplates

V^*_{CNT}	Types	$\mu=1$		$\mu=2$		$\mu=4$	
		Ref. [43]	Present	Ref. [43]	Present	Ref. [43]	Present
0.11	UD	0.158	0.1574	0.304	0.3033	0.596	0.5950
	FG-V	0.251	0.2511	0.486	0.4857	0.956	0.9548
	FG-O	0.344	0.3443	0.667	0.6670	1.313	1.3123
	FG-X	0.097	0.0962	0.185	0.1846	0.362	0.3612
0.14	UD	0.117	0.1167	0.225	0.2242	0.440	0.4393
	FG-V	0.191	0.1911	0.369	0.3689	0.726	0.7246
	FG-O	0.264	0.2641	0.511	0.5108	1.005	1.0042
	FG-X	0.071	0.0711	0.136	0.1360	0.267	0.2657
0.17	UD	0.112	0.1035	0.216	0.1995	0.424	0.3915
	FG-V	0.178	0.1662	0.345	0.3216	0.679	0.6325
	FG-O	0.242	0.2260	0.468	0.4379	0.921	0.8616
	FG-X	0.070	0.0645	0.135	0.1238	0.264	0.2425

Table 6 Effect of length scale and nonlocal parameters on the first five non-dimensional frequencies $\overline{\omega} = (\omega a^2/h)\sqrt{\rho^m/E^m}$ of the CCCC FG-CNTRC square nanoplate with $V^*_{CNT} = 0.11$

λ	h/a	Mode	$\mu=1$				$\mu=3$			
			UD	FG-V	FG-O	FG-X	UD	FG-V	FG-O	FG-X
1	10	1	13.4558	12.4306	11.3002	14.4937	7.8921	7.2918	6.6289	8.4992
		2	15.4771	14.4449	13.2996	16.5866	8.9585	8.3640	7.7028	9.5981
		3	16.9283	16.3762	15.2006	17.5035	9.8254	9.5051	8.8228	10.1588
		4	18.2131	17.7490	16.6003	18.7723	10.5421	10.2732	9.6082	10.8656
		5	19.9251	19.3961	18.1947	20.5574	11.5232	11.2169	10.5234	11.8903
	20	1	17.4000	15.1584	13.4351	19.9642	10.2095	8.8944	7.8832	11.7135
		2	19.4765	17.1891	15.4101	22.1732	11.2876	9.9655	8.9355	12.8455
		3	27.1626	24.8578	22.4150	29.4246	15.7737	14.4357	13.0172	17.0860
		4	28.6424	26.5141	24.1226	30.7974	16.5817	15.3493	13.9647	17.8293
		5	31.4788	29.1868	26.6278	33.7978	18.2067	16.8837	15.4055	19.5458
2	10	1	18.3969	16.9611	15.4169	19.8642	10.7868	9.9465	9.0415	11.6447
		2	21.2987	19.8642	18.2968	22.8515	12.3284	11.5022	10.5973	13.2235
		3	23.8017	23.0189	21.3605	24.614	13.8151	13.3610	12.3985	14.286
		4	25.6668	25.0059	23.3778	26.4595	14.8562	14.4734	13.5308	15.3149
		5	27.9848	27.2674	25.5882	28.8427	16.1899	15.7738	14.8038	16.6884
	20	1	23.5291	20.4873	18.1671	27.0354	13.8021	12.0184	10.6574	15.8575
		2	26.5638	23.447	21.0372	30.2671	15.3953	13.5938	12.1986	17.5348
		3	38.1951	34.942	31.5014	41.387	22.1813	20.2926	18.2945	24.0333
		4	40.3883	37.3774	33.9974	43.4336	23.3816	21.6381	19.6813	25.1446
		5	44.3157	41.0993	37.4979	47.5618	25.6385	23.7806	21.6992	27.5146
3	10	1	22.2711	20.5192	18.6506	24.0672	13.0571	12.0320	10.9371	14.1071
		2	25.8429	24.0970	22.1991	27.7375	14.9588	13.9532	12.8575	16.0509
		3	29.0945	28.1349	26.1056	30.0889	16.8873	16.3306	15.1528	17.4638
		4	31.3983	30.5870	28.5916	32.3698	18.1735	17.7037	16.5484	18.7358
		5	34.1956	33.3293	31.2807	35.2318	19.7850	19.2824	18.0988	20.3874
	20	1	28.3778	24.7055	21.9115	32.6222	16.6450	14.4919	12.8531	19.1326
		2	32.1328	28.3648	25.4571	36.6221	18.623	16.4450	14.7616	21.2166
		3	46.6894	42.7079	38.4999	50.5958	27.1146	24.8030	22.3591	29.3811
		4	49.4172	45.7291	41.5904	53.1459	28.6086	26.4730	24.0769	30.7673
		5	54.1915	50.2626	45.8592	58.1534	31.3548	29.0848	26.5396	33.6451

$$\overline{\boldsymbol{\varepsilon}} = \left\{ \boldsymbol{\varepsilon}^1 \ \boldsymbol{\varepsilon}^2 \ \boldsymbol{\varepsilon}^3 \right\}^T = \sum_{I=1}^{mxn} \left\{ \mathbf{B}_I^1 \ \mathbf{B}_I^2 \ \mathbf{B}_I^3 \right\}^T \mathbf{q}_I = \sum_{I=1}^{mxn} \overline{\mathbf{B}}_I^b \mathbf{q}_I, \overline{\boldsymbol{\gamma}} = \left\{ \boldsymbol{\varepsilon}^{s1} \ \boldsymbol{\varepsilon}^{s2} \right\}^T = \sum_{I=1}^{mxn} \left\{ \mathbf{B}_I^{s1} \ \mathbf{B}_I^{s2} \right\}^T \mathbf{q}_I = \sum_{I=1}^{mxn} \overline{\mathbf{B}}_I^s \mathbf{q}_I, \quad (34)$$

where

$$\mathbf{B}_I^1 = \begin{bmatrix} N_{I,x} & 0 & 0\,0\,0\,0\,0 \\ 0 & N_{I,y} & 0\,0\,0\,0\,0 \\ N_{I,y} & N_{I,x} & 0\,0\,0\,0\,0 \end{bmatrix}, \mathbf{B}_I^2 = -\begin{bmatrix} 0\,0\,0\,0\,0 & N_{I,x} & 0 \\ 0\,0\,0\,0\,0\,0 & & N_{I,y} \\ 0\,0\,0\,0\,0 & N_{I,y} & N_{I,x} \end{bmatrix},$$
$$\mathbf{B}_I^3 = \begin{bmatrix} 0\,0\,0 & N_{I,x} & 0 & 0\,0 \\ 0\,0\,0\,0 & & N_{I,y} & 0\,0 \\ 0\,0\,0 & N_{I,y} & N_{I,x} & 0\,0 \end{bmatrix}, \mathbf{B}_I^{s1} = \begin{bmatrix} 0\,0 & N_{I,x} & 0\,0 & -N_I & 0 \\ 0\,0 & N_{I,y} & 0\,0\,0 & & -N_I \end{bmatrix},$$
$$\mathbf{B}_I^{s2} = \begin{bmatrix} 0\,0\,0 & N_I & 0 & 0\,0 \\ 0\,0\,0\,0 & & N_I & 0\,0 \end{bmatrix}. \quad (35)$$

Inserting Eq. (33) into Eq. (24), Eq. (24) can be performed

$$\overline{\mathbf{u}} = \left\{ \mathbf{u}^1 \ \mathbf{u}^2 \ \mathbf{u}^3 \right\}^T = \sum_{I=1}^{mxn} \left\{ \mathbf{N}_I^1 \ \mathbf{N}_I^2 \ \mathbf{N}_I^3 \right\}^T \mathbf{q}_I = \sum_{I=1}^{mxn} \overline{\mathbf{N}}_I \mathbf{q}_I, \quad (36)$$

where

$$\mathbf{N}_I^0 = \begin{bmatrix} N_I & 0 & 0 & 0\,0\,0\,0 \\ 0 & N_I & 0 & 0\,0\,0\,0 \\ 0 & 0 & N_I & 0\,0\,0\,0 \end{bmatrix}, \mathbf{N}_I^1 = -\begin{bmatrix} 0\,0\,0\,0\,0 & N_I & 0 \\ 0\,0\,0\,0\,0\,0 & & N_I \\ 0\,0\,0\,0\,0\,0 & & 0 \end{bmatrix},$$
$$\mathbf{N}_I^2 = \begin{bmatrix} 0\,0\,0 & N_I & 0 & 0\,0 \\ 0\,0\,0\,0 & & N_I & 0\,0 \\ 0\,0\,0\,0 & & 0 & 0\,0 \end{bmatrix}. \quad (37)$$

Table 7 Effect of length scale and nonlocal parameters on the first normalized natural frequency $\overline{\omega} = (\omega a^2/h)\sqrt{\rho^m/E^m}$ of the SSSS FG-CNTRC square nanoplate with $a/h = 50$

λ	V^*_{CNT}	Type	μ			
			1	2	3	4
1	0.11	UD	18.7298	13.4056	10.9907	9.5379
		FG-V	15.8296	11.3299	9.2889	8.0611
		FG-O	13.9114	9.9569	8.1633	7.0842
		FG-X	22.3967	16.0300	13.1423	11.4051
	0.14	UD	20.8485	14.9220	12.2339	10.6168
		FG-V	17.5643	12.5715	10.3068	8.9444
		FG-O	15.4018	11.0237	9.0379	7.8432
		FG-X	24.9352	17.8468	14.6318	12.6977
	0.17	UD	23.0822	16.5208	13.5447	11.7543
		FG-V	19.4596	13.928	11.4190	9.9096
		FG-O	17.0755	12.2216	10.0200	8.6956
		FG-X	27.6421	19.7843	16.2203	14.0763
2	0.11	UD	26.1265	18.6996	15.3310	13.3045
		FG-V	22.0802	15.8036	12.9567	11.2440
		FG-O	19.4038	13.8880	11.3862	9.8812
		FG-X	30.8353	21.8625	17.8666	15.4799
	0.14	UD	29.0820	20.8149	17.0652	14.8095
		FG-V	24.5003	17.5357	14.3768	12.4764
		FG-O	21.4833	15.3764	12.6064	10.9401
		FG-X	31.4461	22.2955	18.2205	15.7865
	0.17	UD	32.1975	23.0449	18.8935	16.3961
		FG-V	27.1433	19.4275	15.9278	13.8224
		FG-O	23.8171	17.0468	13.9760	12.1286
		FG-X	38.5582	27.4798	22.4572	19.4573
3	0.11	UD	30.7544	21.8050	17.8196	15.4392
		FG-V	26.9158	19.2646	15.7942	13.7065
		FG-O	23.6531	16.9294	13.8797	12.0450
		FG-X	30.8355	21.8625	17.8666	15.4799
	0.14	UD	31.3034	22.1943	18.1378	15.7148
		FG-V	29.8663	21.3763	17.5255	15.2089
		FG-O	26.1882	18.7438	15.3672	13.3360
		FG-X	31.4463	22.2955	18.2205	15.7865
	0.17	UD	38.4796	27.2822	22.2958	19.3174
		FG-V	33.0878	23.6821	19.4160	16.8495
		FG-O	29.0329	20.7799	17.0365	14.7846
		FG-X	38.7583	27.4798	22.4573	19.4573

Inserting Eqs. (34) and (36) into Eqs. (29) and (31), respectively, the final compact forms are expressed as follows:

$$\mathbf{K}\mathbf{q} = \mathbf{f}, \tag{38}$$

$$(\mathbf{K} - \omega^2\mathbf{M})\overline{\mathbf{q}} = \mathbf{0}, \tag{39}$$

where **K**, **M** and **f** are the global stiffness matrix, mass matrix and force vector defined as

$$\begin{aligned} \mathbf{K} &= \int_\Omega \left(\overline{\mathbf{B}}^b\right)^T \mathbf{C}^b \overline{\mathbf{B}}^b \mathrm{d}\Omega + \int_\Omega \left(\overline{\mathbf{B}}^s\right)^T \mathbf{C}^s \overline{\mathbf{B}}^s \mathrm{d}\Omega - \lambda \int_\Omega \left(\nabla^2 \overline{\mathbf{B}}^b\right)^T \mathbf{C}^b \overline{\mathbf{B}}^b \mathrm{d}\Omega - \lambda \int_\Omega \left(\nabla^2 \overline{\mathbf{B}}^s\right)^T \mathbf{C}^s \overline{\mathbf{B}}^s \mathrm{d}\Omega, \\ \mathbf{M} &= \int_\Omega (1 - \mu\nabla^2) \overline{\mathbf{N}}^T \mathbf{I}_m \overline{\mathbf{N}} \mathrm{d}\Omega,\ \mathbf{f} = \int_\Omega f_0 (1 - \mu\nabla^2) \left\{ 0\ 0\ w_I\ 0\ 0\ 0\ 0 \right\}^T \mathrm{d}\Omega, \mathbf{q} = \overline{\mathbf{q}} e^{i\omega t}, \end{aligned} \tag{40}$$

Table 8 Effect of length scale and nonlocal parameters on deflection $\overline{w} = w/h$ of the SSSS FG-CNTRC square nanoplate

V^*_{CNT}	λ	Types	μ			
			1	2	3	4
0.11	1	UD	0.0112	0.0218	0.0324	0.0430
		FG-V	0.0160	0.0311	0.0462	0.0613
		FG-O	0.0209	0.0406	0.0604	0.0801
		FG-X	0.0077	0.0150	0.0223	0.0296
	2	UD	0.0058	0.0112	0.0167	0.0221
		FG-V	0.0082	0.0160	0.0238	0.0315
		FG-O	0.0107	0.0209	0.0310	0.0412
		FG-X	0.0040	0.0077	0.0115	0.0152
	3	UD	0.0039	0.0075	0.0112	0.0149
		FG-V	0.0055	0.0108	0.0160	0.0212
		FG-O	0.0072	0.0140	0.0209	0.0277
		FG-X	0.0027	0.0052	0.0077	0.0103
0.14	1	UD	0.0089	0.0173	0.0257	0.0341
		FG-V	0.0128	0.0249	0.0370	0.0490
		FG-O	0.0167	0.0326	0.0484	0.0643
		FG-X	0.0061	0.0119	0.0177	0.0235
	2	UD	0.0046	0.0089	0.0132	0.0176
		FG-V	0.0066	0.0128	0.0190	0.0252
		FG-O	0.0086	0.0168	0.0249	0.0331
		FG-X	0.0032	0.0062	0.0091	0.0121
	3	UD	0.0031	0.0060	0.0089	0.0118
		FG-V	0.0044	0.0086	0.0128	0.0170
		FG-O	0.0058	0.0113	0.0168	0.0222
		FG-X	0.0021	0.0041	0.0062	0.0082
0.17	1	UD	0.0073	0.0142	0.0211	0.028
		FG-V	0.0105	0.0204	0.0303	0.0402
		FG-O	0.0137	0.0266	0.0395	0.0525
		FG-X	0.0050	0.0098	0.0145	0.0193
	2	UD	0.0037	0.0073	0.0108	0.0144
		FG-V	0.0054	0.0105	0.0156	0.0206
		FG-O	0.0070	0.0137	0.0203	0.0270
		FG-X	0.0026	0.0050	0.0075	0.0099
	3	UD	0.0025	0.0049	0.0073	0.0097
		FG-V	0.0036	0.0070	0.0105	0.0139
		FG-O	0.0047	0.0092	0.0137	0.0181
		FG-X	0.0017	0.0034	0.0050	0.0067

in which ω and $\overline{\mathbf{q}}$ are the natural frequency and modes shape, respectively.

4 Numerical examples

4.1 Verification

4.1.1 Strain gradient effect

We now consider an isotropic square plate (length $a = 10$, thickness h). Material properties can be given as Young's modulus $E = 30 \times 10^6$, Poisson's ratio $v = 0.3$ and density mass $\rho = 2300\ \text{kg/m}^3$. Without nonlocal effects, the present model is Aifantis's pure strain gradient theory [10]. Analytical and finite element solutions based on Kirchhoff theory were investigated by Babu and Patel [59]. Effects of length scale parameter on convergence studies of the normalized first natural frequency and central deflection of the isotropic plate are listed in Tables 2 and 3, respectively. According to Ref. [59], FEM-C and FEM-NC were conforming and nonconforming finite elements. We see that the convergence of the proposed method is really good and the present

Table 9 Effect of length scale and nonlocal parameters on the first natural frequency ($\overline{\omega} = \omega R^2 \sqrt{\rho h / D}$, where $D = \frac{Eh^3}{12(1-\nu^2)}$) of the fully clamped circular nanoplate with $R/h = 10$

λ	V^*_{CNT}	Type	μ			
			1	2	3	4
1	0.11	UD	11.0212	8.0063	6.5980	5.7409
		FG-V	9.6572	7.0221	5.7890	5.0379
		FG-O	8.4895	6.1765	5.0929	4.4326
		FG-X	12.4736	9.0531	7.4581	6.4881
	0.14	UD	11.9464	8.6739	7.1468	6.2178
		FG-V	10.5506	7.6682	6.3206	5.5000
		FG-O	9.3191	6.7770	5.5871	4.8623
		FG-X	13.3616	9.6928	7.9837	6.9447
	0.17	UD	13.6393	9.9091	8.1663	7.1056
		FG-V	11.9110	8.6620	7.1412	6.2148
		FG-O	10.4696	7.6174	6.2811	5.4668
		FG-X	15.4296	11.2003	9.2276	8.0278
2	0.11	UD	12.8049	9.3022	7.6661	6.6704
		FG-V	11.1623	8.1162	6.6909	5.8228
		FG-O	9.7735	7.1101	5.8626	5.1025
		FG-X	14.5876	10.5881	8.7232	7.5889
	0.14	UD	13.9194	10.1069	8.3279	7.2455
		FG-V	12.2268	8.8863	7.3247	6.3738
		FG-O	10.7536	7.8196	6.4467	5.6104
		FG-X	15.6792	11.3752	9.3701	8.1509
	0.17	UD	15.8394	11.5075	9.4838	8.2521
		FG-V	13.7605	10.0064	8.2495	7.1793
		FG-O	12.0322	8.7535	7.2178	6.2820
		FG-X	18.0524	13.105	10.7973	9.3936
3	0.11	UD	14.5055	10.5378	8.6846	7.5566
		FG-V	12.6200	9.1761	7.5647	6.5833
		FG-O	11.0296	8.0238	6.6160	5.7582
		FG-X	16.5662	12.0247	9.9069	8.6188
	0.14	UD	15.7846	11.4616	9.4443	8.2169
		FG-V	13.8375	10.0571	8.2898	7.2137
		FG-O	12.1471	8.8329	7.2821	6.3374
		FG-X	17.8278	12.9346	10.6548	9.2687
	0.17	UD	17.9400	13.0339	10.7419	9.3469
		FG-V	15.5545	11.3110	9.3251	8.1154
		FG-O	13.5683	9.8709	8.1392	7.0839
		FG-X	20.5055	14.8863	12.2651	10.6707

results match very well with reference solutions. Besides, an increase of length scale parameter leads to a rise of the natural frequency and a decrease of the central deflection of the plate. Hence, it can be found that using strain gradient theory makes the stiffness of the plate rise.

4.1.2 Nonlocal effect

In this section, nonlocal effect on FG-CNTRC nanoplates is studied. The matrix is made of PmPV with $E^m = 2.1$ GPa, $\nu^m = 0.34$, $\rho^m = 1.16$ g/cm^2, and the reinforcements are (10,10) SWCNTs with $E_{11}^{CNT} = 5.6466$ (TPa), $E_{22}^{CNT} = 7.08$ (TPa), $G_{12}^{CNT} = 1.9445$ (TPa), $\alpha_{11}^{CNT} = 3.4584$, $\alpha_{22}^{CNT} = 5.1682$, and $G_{23} = G_{13} = G_{12}$.

First, a simply supported (SSSS) square nanoplate (length $a = 1$, $\mu = 1.5$) is considered. Without strain gradient effects, the proposed model becomes Erigen's nonlocal elasticity theory model. Table 4 shows the first three non-dimensional frequencies of the FG-CNTRC nanoplate. We observe that the present results match very well with reference solutions [43] for all cases of a/h and V^*_{CNT}.

Next, the central deflection of SSSS FG-CNTRC square nanoplate (length $a = 1$, $a/h = 50$) under a uniform load $f_0 = 0.1 \times 10^4$ is investigated. Effect of nonlocal parameter

Table 10 Effect of length scale and nonlocal parameters on the first natural frequency ($\overline{\omega} = \omega R^2 \sqrt{\rho h / D}$, where $D = \frac{Eh^3}{12(1-\nu^2)}$) of the fully clamped circular nanoplate with $R/h = 20$

λ	V^*_{CNT}	Type	μ			
			1	2	3	4
1	0.11	UD	12.3162	8.9555	7.3829	6.4250
		FG-V	10.2291	7.4429	6.1374	5.3418
		FG-O	8.7096	6.3394	5.2281	4.5507
		FG-X	14.8578	10.7965	8.8983	7.7428
	0.14	UD	13.7922	10.0251	8.2634	7.1908
		FG-V	11.4643	8.3391	6.8756	5.9839
		FG-O	9.8001	7.1307	5.8799	5.1177
		FG-X	16.5310	12.0075	9.8949	8.6093
	0.17	UD	15.1660	11.0284	9.0919	7.9124
		FG-V	12.5456	9.1292	7.5282	6.5524
		FG-O	10.6678	7.7648	6.4036	5.5739
		FG-X	18.3066	13.3043	10.9658	9.5420
2	0.11	UD	13.9002	10.1069	8.3321	7.2511
		FG-V	11.5051	8.3708	6.9025	6.0077
		FG-O	9.7282	7.0803	5.8391	5.0825
		FG-X	16.8583	12.2503	10.0968	8.7858
	0.14	UD	15.6001	11.3390	9.3467	8.1335
		FG-V	12.9187	9.3965	7.7474	6.7427
		FG-O	10.9715	7.9825	6.5823	5.7290
		FG-X	18.8087	13.6624	11.2592	9.7966
	0.17	UD	17.1112	12.4423	10.2576	8.9269
		FG-V	14.1060	10.2640	8.4638	7.3668
		FG-O	11.8990	8.6604	7.1422	6.2167
		FG-X	20.7845	15.1051	12.4503	10.834
3	0.11	UD	15.5529	11.3087	9.3230	8.1135
		FG-V	12.8494	9.3489	7.7090	6.7097
		FG-O	10.8229	7.8770	6.4961	5.6544
		FG-X	18.9128	13.7435	11.3277	9.8570
	0.14	UD	17.4747	12.7018	10.4701	9.1112
		FG-V	14.4428	10.5050	8.6614	7.5382
		FG-O	12.2222	8.8925	7.3327	6.3821
		FG-X	21.1282	15.3478	12.6483	11.0054
	0.17	UD	19.1426	13.9196	11.4756	9.9869
		FG-V	15.7515	11.4612	9.4511	8.2261
		FG-O	13.2287	9.6282	7.9403	6.9115
		FG-X	23.3232	16.9504	13.9715	12.1578

on defection of the nanoplate is indicated in Table 5. Again, we can see that an excellent agreement between the present results and reference ones [43] is obtained. Besides, the deflection of the nanoplate increases with a rise of the nonlocal parameter. So, we also find that the stiffness of the plate decreases once using the Erigen's nonlocal elasticity theory.

4.2 Square plate

To consider both strain gradient and nonlocal effects in nanoplates, a FG-CNTRC square nanoplate (length $a = 1$) is exampled.

First, effect of length scale and nonlocal parameters on free vibration analysis of the nanoplate is investigated. Table 6 shows the first five frequencies of the fully clamped (CCCC) nanoplate. It can be seen that the natural frequencies decrease with a rise of the nonlocal parameter and increase with a rise of both the length scale parameter and length-to-thickness ratio. Table 7 lists the first natural frequency of the SSSS FG-CNTRC square nanoplate with $a/h = 50$. Again, we can see that when the length scale parameter increases the natural frequency increases as well, while the natural frequency reduces when the nonlocal parameter rises.

Next, effect of length scale and nonlocal parameters on deflection of the SSSS square nanoplate (length $a = 1$, $a/h = 50$) subjected to a uniform load $f_0 = 0.1 \times 10^4$ is examined. Deflections of the SSSS FG-CNTRC square nanoplate are indicated in Table 8. We can see that the deflection increases with a rise of the nonlocal parameter and decreases with an increase of the length scale parameter. Besides, we can see the smallest deflection in FG-X case and the largest deflection in FG-O case. So, the FG-X pattern can be considered the best reinforcement.

As obtained in results of static and free vibration analyses of the plates, it is observed that mechanical characteristics of the nanoplates including stiffness-softening and stiffness-hardening mechanisms are significantly influenced using the NSGT.

4.3 Circular plate

A fully clamped FG-CNTRC circular nanoplate with radius R is now considered. Effects of length scale and nonlocal parameters on the first natural frequency of the nanoplate with $R/h = 10$ and 20 are listed in Tables 9 and 10, respectively. Again, the non-dimensional natural frequency of the nanoplate decreases with an increase of the nonlocal parameter and increases with a rise of the length scale parameter. Therefore, it can be observed that the phenomenon of stiffness-softening and stiffness-hardening mechanisms is found using the NSGT.

5 Conclusions

In this paper, a novel nonlocal strain gradient isogeometric model for FG-CNTRC nanoplates was proposed. The material properties of the FG-CNTRC are assumed to be graded in the thickness direction. To consider the length scale and size-dependency effects of nanostructures, the nonlocal strain gradient theory (NSGT) was considered. Through the numerical results, it can be withdrawn some interesting points:

1. The proposed model is capable of capturing both nonlocal effects and strain gradient effects in nanoplate structures.
2. The governing equation is approximated using isogeometric analysis (IGA) which easily satisfy at least the third-order derivatives in weak form of nanoplates.
3. Phenomenon of stiffness-softening and stiffness-hardening mechanisms of the proposed model was fully observed.
4. The distribution of CNTs has a lot of influence the stiffness of the plate.

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