Engineering with Computers (2022) 38 (Suppl 3):S2271–S2287
https://doi.org/10.1007/s00366-020-01279-2

ORIGINAL ARTICLE

Jacobi wavelet collocation method for the modified Camassa–Holm and Degasperis–Procesi equations

İbrahim Çelik[1]

Received: 9 July 2020 / Accepted: 30 December 2020 / Published online: 25 March 2021

Abstract

Matrices representations of integrations of wavelets have a major role to obtain approximate solutions of integral, differential and integro-differential equations. In the present work, operational matrix representation of rth integration of Jacobi wavelets is introduced and to find these operational matrices, all details of the processes are demonstrated for the first time. Error analysis of offered method is also investigated in present study. In the planned method, approximate solutions are constructed with the truncated Jacobi wavelets series. Approximate solutions of the modified Camassa–Holm equation and Degasperis–Procesi equation linearized using quasilinearization technique are obtained by presented method. Applicability and accuracy of presented method is demonstrated by examples. The proposed method is also convergent even when a minor number of grid points. The numerical results obtained by offered technique are compatible with those in the literature.

Keywords Jacobi wavelets · Nonlinear modified Camassa–Holm and Degasperis–Procesi equations · Convergence · Quasilinearization technique · Collocation method · Approximate solution

1 Introduction

Wavelets, recognized as good-localized functions, are an influential instrument used in signal and image processing, computer science, quantum mechanics, communications and various further areas of science. The wavelets methods allow the improvement of very quick algorithms and give accurate solutions when compared to the normally used algorithms. Some wavelet methods such as Haar wavelets [1–10], Legendre wavelets [11–16], Chebyshev wavelets [17–28] and Gegenbauer wavelets [29–39] are given special attention in the literature.

Many real-life problems are related to nonlinear models occurring in various fields of science and engineering, especially in plasma physics, plasma wave, chemical physics, fluid mechanics, and solid-state physics. They can be expressed in terms of nonlinear partial differential equations. Nonlinear equations also include surface waves in compressible liquids, acoustic waves in a harmonic crystal, and hydromagnetic waves in cold plasma [47]. Nonlinear partial differential equation of the important physical model called the modified w-equation is expressed as follows:

$$u_t - u_{xxt} + (w+1)u^2u_x - wu_xu_{xx} - uu_{xxx} = 0, \quad a < x < b, \tag{1}$$

have been solved by the suggested method for $w = 2$ and $w = 3$. When $w = 2$, Eq. (1) is transformed to

$$u_t - u_{xxt} + 3u^2u_x - 2u_xu_{xx} - uu_{xxx} = 0, \tag{2}$$

and called as modified Camassa–Holm (mCH) equation. If initial condition is taken as $u(x,0) = -2\text{sech}^2\left(\frac{x}{2}\right)$, the exact solution of Eq. (2) is [40]

$u(x,t) = -2\text{sech}^2\left(\frac{x}{2} - t\right)$.

When $w = 3$, Eq. (1) is transformed to

$$u_t - u_{xxt} + 4u^2u_x - 3u_xu_{xx} - uu_{xxx} = 0, \tag{3}$$

and called as modified Degasperis–Procesi (mDP) equation. If initial condition is taken as $u(x,\ 0) = -\frac{15}{8}\sec h^2\left(\frac{x}{2}\right)$, the exact solution of Eq. (3) is [40]

$u(x,0) = -\frac{15}{8}\text{sech}^2\left(\frac{x}{2} - \frac{5t}{4}\right)$.

For nonlinear partial differential equation in

✉ İbrahim Çelik
i.celik@pau.edu.tr

[1] Department of Mathematics, Faculty of Arts and Sciences, Pamukkale University, 20100 Denizli, Turkey

$\dot{u}(x,t) = F(u, u', u'', \ldots, u^{(r)})$,the quasilinearization method gives a sequence of repetition for linear partial differential equations:

$$\dot{u}_{s+1}(x,t) = F(u_s, u_s', u_s'', \ldots, u_s^{(r)}) + \sum_{i=0}^{r} \left(u_{s+1}^{(i)} - u_s^{(i)}\right) F_{u_s^{(i)}}(u_s, u_s', u_s'', \ldots, u_s^{(r)}), \quad (4)$$

where $F_{u_s^{(i)}}(u_s, u_s', u_s'', \ldots, u_s^{(r)}) = \frac{\partial}{\partial u_s^{(i)}}\left(F(u_s, u'_s, u''_s, \ldots, u_s^{(r)})\right)$, $\dot{u}(x,t) = \frac{\partial u(x,t)}{\partial t}$, $u'(x,t) = \frac{\partial u(x,t)}{\partial x}$ and $u_0(x,t)$ is selected as any function that provides boundary and initial conditions [41].

In this study, integration of the Jacobi polynomial $P_m^{(\alpha,\beta)}(x)$ from -1 to x has been found, the general procedures for obtaining operational matrices of integration of Jacobi wavelets have been introduced and operational matrix of rth integration of Jacobi wavelets and two theorems about error analysis of presented method have been given for the first time in this study. Presented technique is built on the approach to the solution of problem by the truncated Jacobi wavelet series. System of algebraic equations is attained by handling the Chebyshev collocation points. If system of algebraic equations is solved, unknown coefficients of the Jacobi wavelet series may be obtained. Therefore, implicit shape of the approximate solutions of nonlinear partial differential equations can be found using Jacobi wavelet series with the obtained coefficients. This process can be performed to the modified Camassa–Holm and Degasperis–Procesi equations by utilization of quasilinearization technique. Approximate results indicated that the Jacobi wavelet collocation method has a quite superior accuracy even at a minor number of grid points.

2 Jacobi polynomials

For $m \in Z^+$, Jacobi polynomials of degree m are defined as $P_m^{(\alpha,\beta)}(x)$ where $\alpha > -1$ and $\beta > -1$ on the range $[-1, 1]$. Recurrence formulae of their may be given as:

$$P_0^{(\alpha,\beta)}(x) = 1, \quad P_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta),$$

$$P_{m+1}^{(\alpha,\beta)}(x) = (a_m^{(\alpha,\beta)}\, x + b_m^{(\alpha,\beta)})P_m^{(\alpha,\beta)}(x) - c_m^{(\alpha,\beta)} P_{m-1}^{(\alpha,\beta)}(x), \quad m \geqslant 1 \quad (5)$$

where

$$a_m^{(\alpha,\beta)} = \frac{(2m+\alpha+\beta+1)(2m+\alpha+\beta+2)}{2(m+1)(m+\alpha+\beta+1)},$$

$$b_m^{(\alpha,\beta)} = \frac{(2m+\alpha+\beta+1)(\alpha^2-\beta^2)}{2(m+1)(m+\alpha+\beta+1)(2m+\alpha+\beta)},$$

$$c_m^{(\alpha,\beta)} = \frac{(2m+\alpha+\beta+2)(m+\alpha)(m+\beta)}{(m+1)(m+\alpha+\beta+1)(2m+\alpha+\beta)}.$$

The generating function for Jacobi polynomials are given as:

$$2^{\alpha+\beta}R^{-1}(1-t+R)^{-\alpha}(1+t+R)^{-\beta} = \sum_{m=0}^{\infty} P_m^{(\alpha,\beta)}(x)\ t^m,$$

where $R = (1-2xt+t^2)^{1/2}$. Some relations of Jacobi polynomials can be given as [42]:

$$\frac{\mathrm{d}}{\mathrm{d}x}(P_m^{(\alpha,\beta)}(x)) = \frac{1}{2}(m+\alpha+\beta+1)P_{m-1}^{(\alpha+1,\beta+1)}(x), \quad (6)$$

$$(m+\alpha+\beta+1)P_{m-1}^{(\alpha+1,\beta+1)}(x) = (m+\alpha)P_{m-1}^{(\alpha,\beta+1)}(x) + (m+\beta)P_{m-1}^{(\alpha+1,\beta)}(x),$$

$$(2m+\alpha+\beta)P_m^{(\alpha-1,\beta)}(x) = (m+\alpha+\beta)P_m^{(\alpha,\beta)}(x) - (m+\beta)P_{m-1}^{(\alpha,\beta)}(x), \quad (7)$$

$$(2m+\alpha+\beta)P_m^{(\alpha,\beta-1)}(x) = (m+\alpha+\beta)P_m^{(\alpha,\beta)}(x) + (m+\alpha)P_{m-1}^{(\alpha,\beta)}(x), \quad (8)$$

$$P_m^{(\alpha,\beta)}(-x) = (-1)^m P_m^{(\beta,\alpha)}(x),$$

$$P_m^{(\alpha,\beta)}(1) = \frac{\Gamma(m+\alpha+1)}{m!\,\Gamma(\alpha+1)},$$

$$P_m^{(\alpha,\beta)}(-1) = \frac{(-1)^m\Gamma(m+\beta+1)}{m!\,\Gamma(\beta+1)},$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}P_{m+1}^{(\alpha+1,\beta+1)}(x)\right] = 2m(1-x)^{\alpha}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x),$$

$$2m\int (1-x)^{\alpha}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x)\ \mathrm{d}x = (1-x)^{\alpha+1}(1+x)^{\beta+1}P_{m+1}^{(\alpha+1,\beta+1)}(x).$$

The following relation can be obtained by integration of the Jacobi polynomial $P_m^{(\alpha,\beta)}(x)$ from -1 to x:

$$\int_{-1}^{x} P_m^{(\alpha,\beta)}(t)\,dt = \frac{2}{(m+\lambda-1)}\left[P_{m+1}^{(\alpha-1,\beta-1)}(t)\right]\Big|_{-1}^{x}$$
$$= \frac{2}{(m+\lambda-1)}\left[\frac{(m+\lambda-1)}{(2m+\lambda)}P_{m+1}^{(\alpha,\beta-1)}(t) - \frac{(m+\beta)}{(2m+\lambda)}P_m^{(\alpha,\beta-1)}(t)\right]\Big|_{-1}^{x}$$
$$= \frac{2}{(m+\lambda-1)}\left[\frac{(m+\lambda-1)}{(2m+\lambda)}\left(\frac{(m+\lambda)}{(2m+\lambda+1)}P_{m+1}^{(\alpha,\beta)}(t) + \frac{(m+\alpha+1)}{(2m+\lambda+1)}P_m^{(\alpha,\beta)}(t)\right)\right.$$
$$\left.- \frac{(m+\beta)}{(2m+\lambda)}\left(\frac{(m+\lambda-1)}{(2m+\lambda-1)}P_m^{(\alpha,\beta)}(t) + \frac{(m+\alpha)}{(2m+\lambda-1)}P_{m-1}^{(\alpha,\beta)}(t)\right)\right]\Big|_{-1}^{x}$$
$$= 2\left\{\frac{(m+\lambda)}{(2m+\lambda+1)(2m+\lambda)}P_{m+1}^{(\alpha,\beta)}(x) + \frac{(\alpha-\beta)}{(2m+\lambda+1)(2m+\lambda-1)}P_m^{(\alpha,\beta)}(x)\right.$$
$$\left.-\frac{(m+\alpha)(m+\beta)}{(2m+\lambda)(2m+\lambda-1)(m+\lambda-1)}P_{m-1}^{(\alpha,\beta)}(x)\right\} - 2\left\{\frac{(m+\lambda)}{(2m+\lambda+1)(2m+\lambda)}P_{m+1}^{(\alpha,\beta)}(-1)\right.$$
$$\left.+\frac{(\alpha-\beta)}{(2m+\lambda+1)(2m+\lambda-1)}P_m^{(\alpha,\beta)}(-1) - \frac{(m+\alpha)(m+\beta)}{(2m+\lambda)(2m+\lambda-1)(m+\lambda-1)}P_{m-1}^{(\alpha,\beta)}(-1)\right\}, \quad m \geq 1$$

where $\lambda = \alpha + \beta + 1$.

Jacobi polynomials are orthogonal polynomials according to the weight function $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ on the range $[-1, 1]$ as [42]:

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\,dx = \begin{cases} L_m^{(\alpha,\beta)}, & m = n \\ 0, & m \neq n \end{cases}$$

where

$$L_m^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{(2m+\alpha+\beta+1)\,m!\,\Gamma(m+\alpha+\beta+1)}$$

is the normalizing factor.

3 Jacobi wavelet method

Wavelets are composed of functions family generated by dilation (or contraction) and translation of a single function named the mother wavelet. a and b are named the as dilation parameter and translation parameter. If translation and dilation parameters change continuously, continuous wavelets family may be obtained as follows [43]:

$$\psi_{a,b}(x) = |a|^{1/2}\psi\left(\frac{x-b}{a}\right), \quad a, b \in R, \quad a \neq 0. \tag{9}$$

Jacobi wavelets are written as

$$\psi_{nm}(x) = \psi(k, n, m, x).$$

where $k = 0, 1, 2, \ldots, n = 1, 2, \ldots, 2^k$, degree of the Jacobi polynomial is shown as $m, \alpha, \beta > -1$ are parameters and $x \in$ [0,1). Jacobi wavelets can be defined as:

$$\psi_{nm}(x) = \begin{cases} \dfrac{2^{\frac{k+1}{2}}}{\sqrt{L_m^{(\alpha,\beta)}}} P_m^{(\alpha,\beta)}(2^{k+1}x - 2n + 1), & \dfrac{n-1}{2^k} \leqslant x < \dfrac{n}{2^k}, \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

where $P_m^{(\alpha,\beta)}(2^{k+1}x - 2n + 1)$ is the Jacobi polynomial whose degree is m and it is orthogonal polynomial according to the weight function

$$w_n(x) = w(2^{k+1}x - 2n + 1)$$
$$= (1 - (2^{k+1}x - 2n + 1))^{\alpha}(1 + (2^{k+1}x - 2n + 1))^{\beta}$$

on interval $0 \leqslant x \leqslant 1$. Any finite interval $a \leqslant y \leqslant b$ can be converted into the simple range $0 \leqslant x \leqslant 1$ by transformation of variable given as $y = (b-a)x + a$.

Any function $f(x) \in L_w^2[0,1][0,1]$ can be extended as:

$$f(x) = \sum_{n=1}^{2^k}\sum_{m=0}^{\infty} f_{nm}\psi_{nm}(x), \tag{11}$$

where

$$f_{nm} = \langle f(x), \psi_{nm}(x)\rangle. \tag{12}$$

$\langle\, . \,, . \,\rangle$ indicates the dot product according to weight function $w_n(x)$ in Eq. (12).

Truncated series of Eq. (11) may be given as:

$$f(x) \cong \sum_{n=1}^{2^k}\sum_{m=0}^{M-1} f_{nm}\psi_{nm}(x) = \mathbf{C}^{\mathbf{T}}\,\mathbf{\Psi}(x), \tag{13}$$

where $\mathbf{\Psi}(x)$ and $\mathbf{C}$ are $2^k M \times 1$ dimensional columns vectors assumed as:

$$\mathbf{C}^{\mathbf{T}} = [f_{10}, f_{11}, \ldots, f_{1M-1}, f_{20}, \ldots, f_{2M-1}, \ldots, f_{2^k0}, \ldots, f_{2^kM-1}],$$

$$\mathbf{\Psi}(x) = [\psi_{10}, \psi_{11}, \ldots, \psi_{1M-1}, \psi_{20}, \ldots, \psi_{2M-1}, \ldots, \psi_{2^k0}, \ldots, \psi_{2^kM-1}]^T. \tag{14}$$

If the function $\psi_{nm}(x)$ in Eq. (10) is integrated, it may be shown as follows:

$$p_{nm}(x) = \int_0^x \psi_{nm}(s)\mathrm{d}s. \tag{15}$$

The following equations may be obtained by calculating $p_{nm}(x)$ for $m = 0$, $m = 1$ and $m > 1$:

$$p_{n0}(x) = \begin{cases} 0, & 0 \leqslant x < \frac{n-1}{2^k} \\ \frac{2(\beta+1)}{\lambda+1}\psi_{n0}(u) + \frac{2}{\lambda+1}\sqrt{\frac{L_1^{(\alpha,\beta)}}{L_0^{(\alpha,\beta)}}}\psi_{n1}(u), & \frac{n-1}{2^k} \leqslant x < \frac{n}{2^k} \\ 2, & \frac{n}{2^k} \leqslant x < 1 \end{cases}$$

$$p_{n1}(x) = \begin{cases} 0, & 0 \le x < \frac{n-1}{2^k} \\ -2\left\{ \frac{(\lambda+1)}{(\lambda+3)(\lambda+2)}P_2^{(\alpha,\beta)}(-1) + \frac{(\alpha-\beta)}{(\lambda+3)(\lambda+1)}P_1^{(\alpha,\beta)}(-1) \right\}\sqrt{\frac{L_0^{(\alpha,\beta)}}{L_1^{(\alpha,\beta)}}}\psi_{n0}(u) & \\ +\frac{2(\alpha-\beta)}{(\lambda+3)(\lambda+1)}\psi_{n1}(u) + \frac{2(\lambda+1)}{(\lambda+3)(\lambda+2)}\sqrt{\frac{L_2^{(\alpha,\beta)}}{L_1^{(\alpha,\beta)}}}\psi_{n2}(u), & \frac{n-1}{2^k} \le x < \frac{n}{2^k} \\ 2\left\{ \frac{(\lambda+1)}{(\lambda+3)(\lambda+2)}\left(P_2^{(\alpha,\beta)}(1) - P_2^{(\alpha,\beta)}(-1)\right) + \frac{(\alpha-\beta)}{(\lambda+3)(\lambda+1)}\left(P_1^{(\alpha,\beta)}(1) - P_1^{(\alpha,\beta)}(-1)\right) \right. & \\ \left. -\frac{(\alpha+1)(\beta+1)}{(\lambda+2)(\lambda+1)\lambda}\left(P_0^{(\alpha,\beta)}(1) - P_0^{(\alpha,\beta)}(-1)\right) \right\}\sqrt{\frac{L_0^{(\alpha,\beta)}}{L_1^{(\alpha,\beta)}}}\psi_{n0}(u), & \frac{n}{2^k} \le x < 1 \end{cases}$$

$$p_{nm}(x) = \begin{cases} 0, & 0 \le x < \frac{n-1}{2^k} \\ -2\left\{ \frac{(m+\lambda)}{(2m+\lambda+1)(2m+\lambda)}P_{m+1}^{(\alpha,\beta)}(-1) + \frac{(\alpha-\beta)}{(2m+\lambda+1)(2m+\lambda-1)}P_m^{(\alpha,\beta)}(-1) \right. & \\ \left. -\frac{(m+\alpha)(m+\beta)}{(2m+\lambda)(2m+\lambda-1)(m+\lambda-1)}P_{m-1}^{(\alpha,\beta)}(-1) \right\}\sqrt{\frac{L_0^{(\alpha,\beta)}}{L_m^{(\alpha,\beta)}}}\psi_{n0}(u) & \\ -\frac{2(m+\beta)(m+\alpha)}{(2m+\lambda)(2m+\lambda-1)(m+\lambda-1)}\sqrt{\frac{L_{m-1}^{(\alpha,\beta)}}{L_m^{(\alpha,\beta)}}}\psi_{nm-1}(u) + \frac{2(\alpha-\beta)}{(2m+\lambda+1)(2m+\lambda-1)}\psi_{nm}(u) & \\ +\frac{2(m+\lambda)}{(2m+\lambda+1)(2m+\lambda)}\sqrt{\frac{L_{m+1}^{(\alpha,\beta)}}{L_m^{(\alpha,\beta)}}}\psi_{nm+1}(u), & \frac{n-1}{2^k} \le x < \frac{n}{2^k} \\ 2\left\{ \frac{(m+\lambda)}{(2m+\lambda+1)(2m+\lambda)}\left(P_{m+1}^{(\alpha,\beta)}(1) - P_{m+1}^{(\alpha,\beta)}(-1)\right) + \frac{(\alpha-\beta)}{(2m+\lambda+1)(2m+\lambda-1)}\left(P_m^{(\alpha,\beta)}(1) - P_m^{(\alpha,\beta)}(-1)\right) \right. & \\ \left. -\frac{(m+\alpha)(m+\beta)}{(2m+\lambda)(2m+\lambda-1)(m+\lambda-1)}\left(P_{m-1}^{(\alpha,\beta)}(1) - P_{m-1}^{(\alpha,\beta)}(-1)\right) \right\}\sqrt{\frac{L_0^{(\alpha,\beta)}}{L_m^{(\alpha,\beta)}}}\psi_{n0}(u), & \frac{n}{2^k} \le x < 1 \end{cases}$$

where $\lambda = \alpha + \beta + 1$ and $u = 2^{k+1}x - 2n + 1$. If $\boldsymbol{\Psi}(x)$ column vector is integrated, the following matrix representation may be obtained:

$$\int_0^x \boldsymbol{\Psi}(s)\mathrm{d}s = [p_{10}, p_{11}, \ldots, p_{1M-1}, p_{20}, \ldots, p_{2M-1}, \ldots, p_{2^k0}, \ldots, p_{2^kM-1}]^T = \mathbf{P_1}\,\boldsymbol{\Psi_1}(x), \tag{16}$$

where

$$\boldsymbol{\Psi_1}(x) = [\psi_{10}, \psi_{11}, \ldots, \psi_{1M}, \psi_{20}, \ldots, \psi_{2M}, \ldots, \psi_{2^k0}, \ldots, \psi_{2^kM}]^T,$$

$$\mathbf{L_1}=\begin{bmatrix}
\frac{2(\beta-1)}{\lambda+1} & \frac{2}{\lambda+1}\sqrt{\frac{L_1^{(\alpha,\beta)}}{L_0^{(\alpha,\beta)}}} & 0 & 0 & \cdots & 0 & 0 & 0\\
-2\left\{\frac{(\lambda+1)}{(\lambda+3)(\lambda+2)}P_2^{(\alpha,\beta)}(-1)+\frac{(\alpha-\beta)}{(\lambda+3)(\lambda+1)}P_1^{(\alpha,\beta)}(-1)\right\}\sqrt{\frac{L_0^{(\alpha,\beta)}}{L_1^{(\alpha,\beta)}}} & \frac{2(\alpha-\beta)}{(\lambda+3)(\lambda+1)} & \frac{2(\lambda+1)}{(\lambda+3)(\lambda+2)}\sqrt{\frac{L_2^{(\alpha,\beta)}}{L_1^{(\alpha,\beta)}}} & 0 & \cdots & 0 & 0 & 0\\
\begin{array}{c}-2\left\{\frac{(\lambda+2)}{(\lambda+5)(\lambda+4)}P_3^{(\alpha,\beta)}(-1)+\frac{(\alpha-\beta)}{(\lambda+5)(\lambda+3)}P_2^{(\alpha,\beta)}(-1)\right.\\ \left.-\frac{(\alpha+2)(\beta+2)}{(\lambda+4)(\lambda+3)(\lambda+1)}P_1^{(\alpha,\beta)}(-1)\right\}\sqrt{\frac{L_0^{(\alpha,\beta)}}{L_2^{(\alpha,\beta)}}}\end{array} & \frac{-2(\beta+2)(\alpha+2)}{(\lambda+4)(\lambda+3)(\lambda+1)}\sqrt{\frac{L_1^{(\alpha,\beta)}}{L_2^{(\alpha,\beta)}}} & \frac{2(\alpha-\beta)}{(\lambda+5)(\lambda+3)} & \frac{2(\lambda+2)}{(\lambda+5)(\lambda+4)}\sqrt{\frac{L_3^{(\alpha,\beta)}}{L_2^{(\alpha,\beta)}}} & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\begin{array}{c}-2\left\{\frac{(M+\lambda-1)}{(2M+\lambda-1)(2M+\lambda-2)}P_M^{(\alpha,\beta)}(-1)+\frac{(\alpha-\beta)}{(2M+\lambda-1)(2M+\lambda-3)}P_{M-1}^{(\alpha,\beta)}(-1)\right.\\ \left.-\frac{(M+\alpha-1)(M+\beta-1)}{(2M+\lambda-2)(2M+\lambda-3)(M+\lambda-2)}P_{M-2}^{(\alpha,\beta)}(-1)\right\}\sqrt{\frac{L_0^{(\alpha,\beta)}}{L_{M-1}^{(\alpha,\beta)}}}\end{array} & 0 & 0 & 0 & \cdots & \frac{-2(M+\beta-1)(M+\alpha-1)}{(2M+\lambda-2)(2M+\lambda-3)(M+\lambda-2)}\sqrt{\frac{L_{M-2}^{(\alpha,\beta)}}{L_{M-1}^{(\alpha,\beta)}}} & \frac{2(\alpha-\beta)}{(2M+\lambda-1)(2M+\lambda-3)} & \frac{2(M+\lambda-1)}{(2M+\lambda-1)(2M+\lambda-2)}\sqrt{\frac{L_M^{(\alpha,\beta)}}{L_{M-1}^{(\alpha,\beta)}}}
\end{bmatrix}$$

$$\mathbf{F_1}=\begin{bmatrix} 2 & 0\ \cdots\ 0 \\ 2\left\{\frac{(\lambda+1)}{(\lambda+3)(\lambda+2)}\left(P_2^{(\alpha,\beta)}(1)-P_2^{(\alpha,\beta)}(-1)\right)+\frac{(\alpha-\beta)}{(\lambda+3)(\lambda+1)}\left(P_1^{(\alpha,\beta)}(1)-P_1^{(\alpha,\beta)}(-1)\right)\right. & \\ \left.-\frac{(\alpha+1)(\beta+1)}{(\lambda+2)(\lambda+1)\lambda}\left(P_0^{(\alpha,\beta)}(1)-P_0^{(\alpha,\beta)}(-1)\right)\right\}\sqrt{\frac{L_0^{(\alpha,\beta)}}{L_1^{(\alpha,\beta)}}}\psi_{n0}(u) & 0\ \cdots\ 0 \\ \vdots & \vdots\ \ddots\ 0 \\ 2\left\{\frac{(M+\lambda)}{(2M+\lambda+1)(2M+\lambda)}\left(P_{M+1}^{(\alpha,\beta)}(1)-P_{M+1}^{(\alpha,\beta)}(-1)\right)+\frac{(\alpha-\beta)}{(2M+\lambda+1)(2M+\lambda-1)}\left(P_M^{(\alpha,\beta)}(1)-P_M^{(\alpha,\beta)}(-1)\right)\right. & \\ \left.-\frac{(M+\alpha)(M+\beta)}{(2M+\lambda)(2M+\lambda-1)(M+\lambda-1)}\left(P_{M-1}^{(\alpha,\beta)}(1)-P_{M-1}^{(\alpha,\beta)}(-1)\right)\right\}\sqrt{\frac{L_0^{(\alpha,\beta)}}{L_{M-1}^{(\alpha,\beta)}}}\psi_{n0}(u) & 0\ \cdots\ 0 \end{bmatrix}$$

$$\mathbf{P_1}=\frac{1}{2^{k+1}}\begin{bmatrix} \mathbf{L_1} & \mathbf{F_1} & \mathbf{F_1} & \cdots & \mathbf{F_1} & \mathbf{F_1} \\ \mathbf{0} & \mathbf{L_1} & \mathbf{F_1} & \cdots & \mathbf{F_1} & \mathbf{F_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{L_1} & \mathbf{F_1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{L_1} \end{bmatrix}$$

If the vector $\mathbf{\Psi}(x)$ is integrated two times, matrix representation of integration may be denoted as:

$$\int_0^x\int_0^{x_1}\mathbf{\Psi}(s)\mathrm{d}s\mathrm{d}x_1=\int_0^x\mathbf{P_1}\,\mathbf{\Psi_1}(x_1)\mathrm{d}x_1=\mathbf{P_1}\int_0^x\mathbf{\Psi_1}(x_1)\mathrm{d}x_1=\mathbf{P_1P_2\Psi_2}(x)\neq\mathbf{P_1^2\Psi}(x).$$

The rth integration of the vector $\mathbf{\Psi}(x)$ may be denoted as:where

$$\int_0^x\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_{r-1}}\mathbf{\Psi}(s)\,\mathrm{d}s\mathrm{d}x_{r-1}\mathrm{d}x_{r-2}\cdots\mathrm{d}x_1=\mathbf{P_1P_2}\cdots\mathbf{P_r}\,\mathbf{\Psi_r}(x)\neq\mathbf{P_1^r\Psi}(x),$$

$$\mathbf{\Psi_r}(x)=[\psi_{10},\ \psi_{11},\ \ldots,\ \psi_{1M+r-1},\ \psi_{20},\ \ldots,\ \psi_{2M+r-1},\ \ldots,\ \psi_{2^k0},\ \ldots,\ \psi_{2^kM+r-1}\ ]^T. \tag{17}$$

$\mathbf{L_r}$ and $\mathbf{F_r}$ are $(M+r-1)\times(M+r)$ dimension matrices, $\mathbf{P_r}$ is a $2^k(M+r-1)\times 2^k(M+r)$ dimension matrix and the matrices $\mathbf{P_r}$, $\mathbf{L_r}$ and $\mathbf{F_r}$ have the following form:

$$\mathbf{P_r}=\frac{1}{2^{k+1}}\begin{bmatrix} \mathbf{L_r} & \mathbf{F_r} & \mathbf{F_r} & \cdots & \mathbf{F_r} & \mathbf{F_r} \\ \mathbf{0} & \mathbf{L_r} & \mathbf{F_r} & \cdots & \mathbf{F_r} & \mathbf{F_r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{L_r} & \mathbf{F_r} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{L_r} \end{bmatrix}$$

$$\mathbf{F_r}=\begin{bmatrix} 2 & 0\ \cdots\ 0 \\ 2\left\{\frac{(\lambda+1)}{(\lambda+3)(\lambda+2)}\left(P_2^{(\alpha,\beta)}(1)-P_2^{(\alpha,\beta)}(-1)\right)+\frac{(\alpha-\beta)}{(\lambda+3)(\lambda+1)}\left(P_1^{(\alpha,\beta)}(1)-P_1^{(\alpha,\beta)}(-1)\right)\right. & \\ \left.-\frac{(\alpha+1)(\beta+1)}{(\lambda+2)(\lambda+1)\lambda}\left(P_0^{(\alpha,\beta)}(1)-P_0^{(\alpha,\beta)}(-1)\right)\right\}\sqrt{\frac{L_0^{(\alpha,\beta)}}{L_1^{(\alpha,\beta)}}}\psi_{n0}(u) & 0\ \cdots\ 0 \\ \vdots & \vdots\ \ddots\ \vdots \\ 2\left\{\frac{(M+\lambda)}{(2M+\lambda+1)(2M+\lambda)}\left(P_{M+1}^{(\alpha,\beta)}(1)-P_{M+1}^{(\alpha,\beta)}(-1)\right)+\frac{(\alpha-\beta)}{(2M+\lambda+1)(2M+\lambda-1)}\left(P_M^{(\alpha,\beta)}(1)-P_M^{(\alpha,\beta)}(-1)\right)\right. & \\ \left.-\frac{(M+\alpha)(M+\beta)}{(2M+\lambda)(2M+\lambda-1)(M+\lambda-1)}\left(P_{M-1}^{(\alpha,\beta)}(1)-P_{M-1}^{(\alpha,\beta)}(-1)\right)\right\}\sqrt{\frac{L_0^{(\alpha,\beta)}}{L_{M-1}^{(\alpha,\beta)}}}\psi_{n0}(u) & 0\ \cdots\ 0 \\ \vdots & \vdots\ \ddots\ \vdots \\ 2\left\{\frac{(M+r+\lambda-1)}{(2M+2r+\lambda-1)(2M+2r+\lambda-2)}\left(P_{M+r}^{(\alpha,\beta)}(1)-P_{M+r}^{(\alpha,\beta)}(-1)\right)+\frac{(\alpha-\beta)}{(2M+2r+\lambda-1)(2M+2r+\lambda-3)}\left(P_{M+r-1}^{(\alpha,\beta)}(1)-P_{M+r-1}^{(\alpha,\beta)}(-1)\right)\right. & \\ \left.-\frac{(M+r+\alpha-1)(M+r+\beta-1)}{(2M+2r+\lambda-2)(2M+2r+\lambda-3)(M+r+\lambda-2)}\left(P_{M+r-2}^{(\alpha,\beta)}(1)-P_{M+r-2}^{(\alpha,\beta)}(-1)\right)\right\}\sqrt{\frac{L_0^{(\alpha,\beta)}}{L_{M+r-2}^{(\alpha,\beta)}}}\psi_{n0}(u) & 0\ \cdots\ 0 \end{bmatrix}$$

$$
\mathbf{L_r} = \begin{bmatrix}
\frac{2(\beta-1)}{\lambda+1} & \frac{2}{\lambda+1}\sqrt{\frac{L_1^{(\alpha,\beta)}}{L_0^{(\alpha,\beta)}}} & 0 & \cdots & 0 & 0 & 0 \\
-2\left\{\frac{(\lambda+1)}{(\lambda+3)(\lambda+2)}P_2^{(\alpha,\beta)}(-1) + \frac{(\alpha-\beta)}{(\lambda+3)(\lambda+1)}P_1^{(\alpha,\beta)}(-1)\right\}\sqrt{\frac{L_0^{(\alpha,\beta)}}{L_1^{(\alpha,\beta)}}} & \frac{2(\alpha-\beta)}{(\lambda+3)(\lambda+1)} & \frac{2(\lambda+1)}{(\lambda+3)(\lambda+2)}\sqrt{\frac{L_2^{(\alpha,\beta)}}{L_1^{(\alpha,\beta)}}} & \cdots & 0 & 0 & 0 \\
\begin{matrix} -2\left\{\frac{(\lambda+2)}{(\lambda+5)(\lambda+4)}P_3^{(\alpha,\beta)}(-1) + \frac{(\alpha-\beta)}{(\lambda+5)(\lambda+3)}P_2^{(\alpha,\beta)}(-1)\right. \\ \left. - \frac{(\alpha+2)(\beta+2)}{(\lambda+4)(\lambda+3)(\lambda+1)}P_1^{(\alpha,\beta)}(-1)\right\}\sqrt{\frac{L_0^{(\alpha,\beta)}}{L_2^{(\alpha,\beta)}}} \end{matrix} & \frac{-2(\beta+2)(\alpha+2)}{(\lambda+4)(\lambda+3)(\lambda+1)}\sqrt{\frac{L_1^{(\alpha,\beta)}}{L_2^{(\alpha,\beta)}}} & \frac{2(\alpha-\beta)}{(\lambda+5)(\lambda+3)} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\begin{matrix} -2\left\{\frac{(M+r+\lambda-1)}{(2M+2r+\lambda-1)(2M+2r+\lambda-2)}P_{M+r}^{(\alpha,\beta)}(-1) + \frac{(\alpha-\beta)}{(2M+2r+\lambda-1)(2m+2r+\lambda-3)}P_{M+r-1}^{(\alpha,\beta)}(-1)\right. \\ \left. - \frac{(M+r+\alpha-1)(M+r+\beta-1)}{(2M+2r+\lambda-2)(2M+2r+\lambda-3)(M+r+\lambda-2)}P_{m-1}^{(\alpha,\beta)}(-1)\right\}\sqrt{\frac{L_0^{(\alpha,\beta)}}{L_{M+r-2}^{(\alpha,\beta)}}} \end{matrix} & 0 & 0 & \cdots & \frac{-2(M+r+\beta-2)\,(M+r+\alpha-2)}{(2M+2r+\lambda-4)(2M+2r+\lambda-5)(M+r+\lambda-3)}\sqrt{\frac{L_{M+r-3}^{(\alpha,\beta)}}{L_{M+r-2}^{(\alpha,\beta)}}} & \frac{2\,(\alpha-\beta)}{(2M+2r+\lambda-3)\,(2M+2r+\lambda-5)} & \frac{2(M+r+\lambda-2)}{(2M+2r+\lambda-3)\,(2M+2r+\lambda-4)}\sqrt{\frac{L_{M+r-1}^{(\alpha,\beta)}}{L_{M+r-2}^{(\alpha,\beta)}}}
\end{bmatrix}
$$

4 Jacobi wavelet collocation method for the mCH and mDP equations

Consider nonlinear partial differential equations in Eq. (1) with boundary and initial conditions:

$$u(x,0)=f(x),\quad u(0,t)=g_0(t),\quad u(1,t)=g_1(t),\quad u_x(0,t)=g_2(t). \tag{18}$$

We presume that $\dot{u}^{(3)}(x,t)$ may be extended like truncated Jacobi wavelets series as:

$$\dot{u}^{(3)}(x,t)=\sum_{n=1}^{2^k}\sum_{m=0}^{M-1} f_{nm}\psi_{nm}(x)=\mathbf{C}^{\mathbf{T}}\mathbf{\Psi}(x), \tag{19}$$

where “ ” and “(3)” imply differentiation according to t and x. If Eq. (19) is integrated according to t from t_s to t and three times according to x from 0 to x, we can find the following equations:

$$u^{(3)}(x,t)=u^{(3)}(x,t_s)+(t-t_s)\mathbf{C}^{\mathbf{T}}\mathbf{\Psi}(x), \tag{20}$$

$$u^{(2)}(x,t)=u^{(2)}(x,t_s)+u^{(2)}(0,t)-u^{(2)}(0,t_s)+(t-t_s)\mathbf{C}^{\mathbf{T}}\mathbf{P_1}\mathbf{\Psi_1}(x), \tag{21}$$

$$\dot{u}^{(2)}(x,t)=\dot{u}^{(2)}(0,t)+\mathbf{C}^{\mathbf{T}}\mathbf{P_1}\mathbf{\Psi_1}(x), \tag{22}$$

$$u_x(x,t)=u_x(x,t_s)+u_x(0,t)-u_x(0,t_s)\ +x(u^{(2)}(0,t)-u^{(2)}(0,t_s))+(t-t_s)\mathbf{C}^{\mathbf{T}}\mathbf{P_1}\mathbf{P_2}\mathbf{\Psi_2}(x), \tag{23}$$

$$\begin{aligned}u(x,t)=u(x,t_s)+u(0,t)-u(0,t_s)\ +x(u_x(0,t)-u_x(0,t_s))+\frac{x^2}{2}(u^{(2)}(0,t)-u^{(2)}(0,t_s))\\ +(t-t_s)\mathbf{C}^{\mathbf{T}}\mathbf{P_1}\mathbf{P_2}\mathbf{P_3}\mathbf{\Psi_3}(x),\end{aligned} \tag{24}$$

$$u_t(x,t)=u_t(0,t)\ +xu_{xt}(0,t)+\frac{x^2}{2}\dot{u}^{(2)}(0,t)+\mathbf{C}^{\mathbf{T}}\mathbf{P_1}\mathbf{P_2}\mathbf{P_3}\mathbf{\Psi_3}(x). \tag{25}$$

By substituting boundary and initial conditions into Eq. (18), the following equation may be obtained:

$$\begin{aligned}\frac{1}{2}(u^{(2)}(0,t)-u^{(2)}(0,t_s))=g_1(t)-g_1(t_s)-(g_2(t)-g_2(t_s))-(g_0(t)-g_0(t_s))\\ -(t-t_s)\mathbf{C}^{\mathbf{T}}\mathbf{P_1}\mathbf{P_2}\mathbf{P_3}\mathbf{\Psi_3}(1).\end{aligned} \tag{26}$$

If Eqs. (18) and (26) are replaced into Eqs. (20–25), we can obtain the following equations:

$$u^{(3)}(x,t)=(t-t_s)\mathbf{C}^{\mathbf{T}}\mathbf{\Psi}(x)+u^{(3)}(x,t_s), \tag{27}$$

$$\begin{aligned}u^{(2)}(x,t)=u^{(2)}(x,t_s)+(t-t_s)\mathbf{C}^{\mathbf{T}}\left[\mathbf{P_1}\mathbf{\Psi_1}(x)-2\mathbf{P_1}\mathbf{P_2}\mathbf{P_3}\mathbf{\Psi_3}(1)\right]\\ -2(g_2(t)-g_2(t_s))\ -2(g_0(t)-g_0(t_s))+2(g_1(t)-g_1(t_s)),\end{aligned} \tag{28}$$

$$\dot{u}^{(2)}(x,t)=\mathbf{C}^{\mathbf{T}}\left[\mathbf{P_1}\mathbf{\Psi_1}(x)-2\mathbf{P_1}\mathbf{P_2}\mathbf{P_3}\mathbf{\Psi_3}(1)\right]\ -2g_2'(t)\ -2g_0'(t)+2g_1'(t), \tag{29}$$

$$\begin{aligned}u_x(x,t)=&u_x(x,t_s)+(t-t_s)\mathbf{C}^{\mathbf{T}}\\ &\left[\mathbf{P_1}\mathbf{P_2}\mathbf{\Psi_2}(x)-2x\mathbf{P_1}\mathbf{P_2}\mathbf{P_3}\mathbf{\Psi_3}(1)\right]\\ &-(1-2x)(g_2(t)-g_2(t_s))\ -2x(g_0(t)\\ &-g_0(t_s))+2x(g_1(t)-g_1(t_s)),\end{aligned} \tag{30}$$

$$\begin{aligned}u(x,t)=&u(x,t_s)+(t-t_s)\mathbf{C}^{\mathbf{T}}\\ &\left[\mathbf{P_1}\mathbf{P_2}\mathbf{P_3}\mathbf{\Psi_3}(x)-x^2\mathbf{P_1}\mathbf{P_2}\mathbf{P_3}\mathbf{\Psi_3}(1)\right]\\ &+(x-x^2)(g_2(t)-g_2(t_s))\ +(1-x^2)(g_0(t)\\ &-g_0(t_s))+x^2(g_1(t)-g_1(t_s)),\end{aligned} \tag{31}$$

$$\begin{aligned}u_t(x,t)=&\mathbf{C}^{\mathbf{T}}\left[\mathbf{P_1}\mathbf{P_2}\mathbf{P_3}\mathbf{\Psi_3}(x)-x^2\mathbf{P_1}\mathbf{P_2}\mathbf{P_3}\mathbf{\Psi_3}(1)\right]\\ &+(x-x^2)g_2'(t)+(1-x^2)g_0'(t)+x^2)g_1'(t).\end{aligned} \tag{32}$$

Using quasilinearization technique, Eq. (1) can be transformed into the sequence of linear differential equations. Nonlinear terms in Eq. (1) may be written as:

$$\begin{aligned}(u^2(x,t)u_x(x,t))^{l+1}\cong\ &2u^l(x,t)u_x^l(x,t)u^{l+1}(x,t)\\ &+u^{l^2}(x,t)u_x^{l+1}(x,t)-2u^{l^2}(x,t)u_x^l(x,t)\end{aligned} \tag{33}$$

$$\begin{aligned}(u_x(x,t)\,u_{xx}(x,t))^{l+1}\cong\ &u_x^{l+1}(x,t)\,u_{xx}^l(x,t)\\ &+u_x^l(x,t)\,u_{xx}^{l+1}(x,t)-u_x^l(x,t)\,u_{xx}^l(x,t)\end{aligned} \tag{34}$$

$$\begin{aligned}(u(x,t)\,u_{xxx}(x,t))^{l+1}\cong\ &u^{l+1}(x,t)\,u_{xxx}^l(x,t)\\ &+u^l(x,t)\,u_{xxx}^{l+1}(x,t)-u^l(x,t)\,u_{xxx}^l(x,t).\end{aligned} \tag{35}$$

By replacing Eqs. (33), (34) and (35) into Eq. (1), transformed equation is found as:

$$\begin{aligned}&u_t^{l+1}(x,t) - u_{xxt}^{l+1}(x,t) + \left(2(w+1)u^l(x,t)\,u_x^l(x,t) - u_{xxx}^l(x,t)\right) u^{l+1}(x,t)\\&+\left((w+1)\left(u^l(x,t)\right)^2 - wu_{xx}^l(x,t)\right) u_x^{l+1}(x,t) - wu_x^l(x,t)u_{xx}^{l+1}(x,t) - u^l(x,t)u_{xxx}^{l+1}(x,t)\\&= 2(w+1)\left(u^l(x,t)\right)^2 u_x^l(x,t) - wu_x^l(x,t)u_{xx}^l(x,t) - u^l(x,t)\,u_{xxx}^l(x,t)\end{aligned} \tag{36}$$

where $l = 0, 1, 2, \ldots$ and it is called index of quasilinearization technique. $u^0(x,t)$ which provides boundary and initial conditions is taken as

$$u^0(x,t) = u(x,t_s) + (x-x^2)(g_2(t) - g_2(t_s)) + (1-x^2)(g_0(t) - g_0(t_s)) + x^2(g_1(t) - g_1(t_s)), \tag{37}$$

which satisfies initial/boundary conditions. Substituting Eqs. (27–32) into Eq. (36), we obtain the following equation:

$$\begin{aligned}&\Delta t\,\mathbf{C^T}\begin{bmatrix}\left(\frac{1}{\Delta t} + 2(w+1)u^l(x,t)\,u_x^l(x,t) - u_{xxx}^l(x,t)\right)\mathbf{P_1P_2P_3\Psi_3}(x) + \left((w+1)\left(u^l(x,t)\right)^2 - wu_{xx}^l(x,t)\right)\mathbf{P_1P_2\Psi_2}(x)\\ -\left(1 + wu_x^l(x,t)\right)\mathbf{P_1\Psi_1}(x) - u^l(x,t)\,mathbf\Psi(x)\\ +\left(\frac{2-x^2}{\Delta t} + x^2\left(2(w+1)u^l(x,t)\,u_x^l(x,t) - u_{xxx}^l(x,t)\right) - 2x\left((w+1)\left(u^l(x,t)\right)^2 - wu_{xx}^l(x,t)\right) + 2wu_x^l(x,t)\right)\mathbf{P_1P_2P_3\Psi}_1(1)\end{bmatrix}\\&= 2(w+1)\left(u^l(x,t)\right)^2 u_x^l(x,t) - wu_x^l(x,t)\,u_{xx}^l(x,t) - u^l(x,t)\,u_{xxx}^l(x,t)\\&-\left(2(w+1)u^l(x,t)\,u_x^l(x,t) - u_{xxx}^l(x,t)\right)\ \left(u^l(x,t_s) + (x-x^2)\left(g_2(t) - g_2(t_s)\right) + (1-x^2)\left(g_0(t) - g_0(t_s)\right) + x^2\left(g_1(t) - g_1(t_s)\right)\right)\\&-\left((w+1)\left(u^l(x,t)\right)^2 - wu_{xx}^l(x,t)\right)\left(u_x^l(x,t_s) + (1-2x)\left(g_2(t) - g_2(t_s)\right) - 2x\right)\left(g_0(t) - g_0(t_s)\right) + 2x\left(g_1(t) - g_1(t_s)\right)\Big)\\&+wu_x^l(x,t)\left(u_{xx}^l(x,t_s) - 2\left(g_2(t) - g_2(t_s)\right) - 2\left(g_0(t) - g_0(t_s)\right) + 2\left(g_1(t) - g_1(t_s)\right)\right) + u^l(x,t)u_{xxx}^l(x,t_s)\\&+(x^2 - x - 2)g_2'(t) + (x^2-3)g_0'(t) + (2-x^2)g_1'(t)\end{aligned} \tag{38}$$

where $\Delta t = t - t_s$.

The collocation points may be selected as $2^{k+1}x_{ni} - 2n + 1 = \cos\frac{((M+1)-i)\pi}{(M+1)}$ or

$$x_{ni} = \frac{1}{2^{k+1}}\left(2n - 1 + \cos\frac{((M+1)-i)\pi}{(M+1)}\right), \quad i = 1,\ 2,\ \ldots,\ M \quad n = 1,\ 2,\ \ldots,\ 2^k. \tag{39}$$

Replacing the collocation points $x \to x_{ni}$ and time variable $t \to t_{s+1}$ into Eq. (38), vectors $\mathbf{\Psi}(x_{\mathbf{ni}})$,$\mathbf{\Psi_1}(x_{\mathbf{ni}})$ and $\mathbf{\Psi_r}(x_{\mathbf{ni}})$ may be achieved. We can find system of algebraic equations from Eq. (38) whose matrix notation may be written as:

$$\mathbf{C^TU = B}, \tag{40}$$

where $\mathbf{B}$ and $\mathbf{C}$ are $2^kM \times 1$ dimensional vectors, $\mathbf{U}$ is a $2^kM \times 2^kM$ dimensional matrix. If system of algebraic equations in Eq. (40) is solved, we may achieve the coefficients of Jacobi wavelet series in Eq. (31) which provides Eq. (1) and initial/ boundary conditions in Eq. (18).

5 Convergence analysis

In this part, convergence and error analysis of the Jacobi wavelet series expansion of a function $f(x)$ are studied.

Lemma 1 *If the Jacobi wavelet series expansion of a continuous function $f(x)$ converges uniformly, then Jacobi wavelet series expansion converges to the function $f(x)$.*

Proof For Jacobi wavelet, proof may be shown similar way in [20]

Theorem 1 *A function $f(x) \in C^r[0,1]$ with the $\left|f^{(r+1)}(x)\right| < U$ may be expanded as an infinite sum of Jacobi wavelets series which converges uniformly to $f(x)$, that is,*

$$f(x) = \sum_{n=1}^{2^k}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x).$$

Proof The following equation can be found from the inner product in Eq. (12):

$$\begin{aligned}c_{nm} &= \int_0^1 f(x)\phi_{nm}(x)w_k(x)\mathrm{d}x\\&= \frac{2^{\frac{k+1}{2}}}{\sqrt{L_m^{(\alpha,\beta)}}}\int_{\frac{n-1}{2^k}}^{\frac{n}{2^k}} f(x)P_m^{(\alpha,\beta)}(2^{k+1}x - 2n + 1)\,w(2^{k+1}x - 2n + 1)\ \mathrm{d}x.\end{aligned}$$

By substituting $2^{k+1}x - 2n + 1 = t$ and successive integration by parts in the above equation, it yields

$$c_{nm} = \frac{2^{-\frac{k+1}{2}}}{\sqrt{L_m^{(\alpha,\beta)}}} \int_{-1}^{1} f\left(\frac{t+2n-1}{2^{k+1}}\right) P_m^{(\alpha,\beta)}(t)\,(1-t)^{\alpha}(1+t)^{\beta}\mathrm{d}t$$
$$= \frac{2^{-\frac{3}{2}(k+1)}}{2m\sqrt{L_m^{(\alpha,\beta)}}} \int_{-1}^{1} f'\left(\frac{t+2n-1}{2^{k+1}}\right) P_{m-1}^{(\alpha+1,\beta+1)}(t)\,(1-t)^{\alpha+1}(1+t)^{\beta+1}\,\mathrm{d}t$$
$$= \cdots$$
$$= \frac{2^{-\frac{(2r+3)(k+1)}{2}}}{2^{r+1}m(m-1)\ldots(m-r)\sqrt{L_m^{(\alpha,\beta)}}} \int_{-1}^{1} f^{(r+1)}\left(\frac{t+2n-1}{2^{k+1}}\right) P_{m-r-1}^{(\alpha+r+1,\beta+r+1)}(t)\,(1-t)^{\alpha+r+1}(1+t)^{\beta+r+1}\,\mathrm{d}t.$$

By applying Cauchy–Schwarz inequality, above equation can be written as:

$$c_{nm} \leqslant \frac{2^{\frac{-(2r+3)(k+1)}{2}-r-1}}{m(m-1)\ldots(m-r)\sqrt{L_m^{(\alpha,\beta)}}} \left(\int_{-1}^{1}\left(f^{(r+1)}\left(\frac{t+2n-1}{2^{k+1}}\right)\right)^2(1-t)^{\alpha+r+1}(1+t)^{\beta+r+1}\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_{-1}^{1}\left(P_{m-r-1}^{(\alpha+r+1,\beta+r+1)}(t)\right)^2(1-t)^{\alpha+r+1}(1+t)^{\beta+r+1}\,\mathrm{d}t\right)^{\frac{1}{2}}.$$

Thus, we get

$$|c_{nm}| \leqslant \frac{2^{\frac{-(2r+3)(k+1)}{2}-r-1}\,U}{m(m-1)\ldots(m-r)\sqrt{L_m^{(\alpha,\beta)}}}\sqrt{L_0^{(\alpha+r+1,\beta+r+1)}}\sqrt{L_{m-r-1}^{(\alpha+r+1,\beta+r+1)}}$$
$$\leqslant \frac{2^{\frac{-(2r+3)(k+2)+\alpha+\beta+3}{2}}\,U}{m(m-1)\ldots(m-r)}\sqrt{\frac{\Gamma(r+\alpha+2)\Gamma(r+\beta+2)}{\Gamma(2r+\alpha+\beta+4)}}.$$

Since $n \leqslant 2^k$, for $m > r$, we obtain

$$|c_{nm}| \leqslant \frac{2^{\frac{-(4r+3)+\alpha+\beta}{2}}\,U}{n^{\frac{(2r+3)}{2}}m(m-1)\ldots(m-r)}\sqrt{\frac{\Gamma(r+\alpha+2)\Gamma(r+\beta+2)}{\Gamma(2r+\alpha+\beta+4)}}.$$

If $m = r$, it can be shown

$$c_{nm} \leqslant \frac{2^{\frac{-(2m+1)(k+1)}{2}}}{2^m m!\sqrt{L_m^{(\alpha,\beta)}}} \int_{-1}^{1} f^{(m)}\left(\frac{t+2n-1}{2^{k+1}}\right)(1-t)^{\alpha+m}(1+t)^{\beta+m}\mathrm{d}t.$$

Thus, we get

$$|c_{nm}| \leqslant \frac{2^{\frac{-(4m+1)}{2}}}{n^{\frac{2m+1}{2}}m!\sqrt{L_m^{(\alpha,\beta)}}} L_0^{(\alpha+m,\beta+m)} \max_{0\leqslant x\leqslant 1} f^{(m)}(x).$$

It is referred in [43] that for $m = 0$, $\{\phi_{n0}\}_{n=1}^{2^k}$ construct an orthogonal system build by Haar scaling function according to the weight function $\omega(x)$, so $\sum_{n=1}^{2^k} c_{n0}\Psi_{n0}(x)$ is convergent. Hence, we have

$$\left|\sum_{n=1}^{2^k}\sum_{m=0}^{\infty} c_{nm}\Psi_{nm}(x)\right| \leqslant \left|\sum_{n=1}^{2^k} c_{n0}\Psi_{n0}(x)\right| + \sum_{n=1}^{2^k}\sum_{m=1}^{\infty}|c_{nm}|\,|\Psi_{nm}(x)| \leqslant \left|\sum_{n=1}^{2^k} c_{n0}\Psi_{n0}(x)\right| + \sum_{n=1}^{2^k}\sum_{m=1}^{\infty}|c_{nm}| < \infty.$$

Thus, with the help of Lemma 1, the series $\sum_{n=1}^{2^k}\sum_{m=0}^{\infty} c_{nm}\Psi_{nm}(x)$ converges to $f(x)$ uniformly.

Theorem 2 *Let $f(x) \in \mathrm{C}^r[0,1]$ with the $|f^{(r+1)}(x)| < U$ be a continuous function, then accuracy estimation is obtained as:*

$$\left\|f(x) - \mathbf{C}^{\mathrm{T}}\mathbf{\Psi}\right\|_2^2 \leqslant \frac{U^2\Gamma(r+\alpha+2)\Gamma(r+\beta+2)}{2^{4r+4-\alpha-\beta}\Gamma(2r+\alpha+\beta+4)}\frac{1}{(r+1)(2r+1)}\frac{1}{(M-r)^{2r+1}}\left[1-\frac{1}{2^{2k(r+1)}}\right].$$

Proof We have

$$\left\|f(x)-\mathbf{C}^{\mathrm{T}}\mathbf{\Psi}\right\|_2^2=\int_0^1\left[f(x)-\sum_{n=1}^{2^k}\sum_{m=0}^{M-1}c_{nm}\phi_{nm}(x)\right]^2 w_n(x)\,dx$$
$$=\int_0^1\left[\sum_{n=1}^{2^k}\sum_{m=0}^{\infty}c_{nm}\phi_{nm}(x)-\sum_{n=1}^{2^k}\sum_{m=0}^{M-1}c_{nm}\phi_{nm}(x)\right]^2 w_n(x)\,dx$$
$$=\int_0^1\left[\sum_{n=1}^{2^k}\sum_{m=M}^{\infty}c_{nm}\phi_{nm}(x)\right]^2 w_n(x)\,dx=\sum_{n=1}^{2^k}\sum_{m=M}^{\infty}c_{nm}^2.$$

By substituting the relation c_{nm}^2 in the above equation, desired result is obtained as:

the interval [−15,15]. It can be seen from Table 1 that the Jacobi wavelet collocation-based algorithm produces stable and converging solution by increasing the level of resolution of the Jacobi wavelet k. Absolute errors of Example 1 have been given in Table 2 to compare presented results with earlier results in the interval [−15, 15]. Graphics of the approximate and exact solutions of Example 1 have been given Fig. 2 [45], Fig. 3 [46] and Fig. 4.2 [47]. As can be seen in Tables 1 and 2 and Figs. 1, 2, 3, 4, 5 and 6, it is clear that suggested method provides more accurate results if we compare the tables and graphics in [44–47].

$$\left\|f(x)-\mathbf{C}^{\mathrm{T}}\mathbf{\Psi}\right\|_2^2\leqslant\frac{U^2\Gamma(r+\alpha+2)\Gamma(r+\beta+2)}{2^{4r+3-\alpha-\beta}\Gamma(2r+\alpha+\beta+4)}\left[\sum_{n=1}^{2^k}\sum_{m=M}^{\infty}\frac{1}{n^{2r+3}(m(m-1)\dots(m-r))^2}\right]$$
$$\leqslant\frac{U^2\Gamma(r+\alpha+2)\Gamma(r+\beta+2)}{2^{4r+3-\alpha-\beta}\Gamma(2r+\alpha+\beta+4)}\left[\int_1^{2^k}\frac{dt}{t^{2r+3}}\int_M^{\infty}\frac{dt}{(t-r)^{2(r+1)}}\right]$$
$$\leqslant\frac{U^2\Gamma(r+\alpha+2)\Gamma(r+\beta+2)}{2^{4r+3-\alpha-\beta}\Gamma(2r+\alpha+\beta+4)}\frac{1}{2(r+2)(2r+1)}\frac{1}{(M-r)^{2r+1}}\left[1-\frac{1}{2^{2k(r+1)}}\right].$$

6 Numerical results

The modified Camassa–Holm and modified Degasperis–Procesi equations are solved to show the effectiveness and the applicability of Jacobi wavelet collocation method. Approximate results of proposed method are compared with analytical and numerical results in the literature.

Example 1 Consider Eq. (1) for $w=2$, modified Camassa–Holm (mCH) equation may be obtained as:

$$u_t-u_{xxt}+3u^2u_x-2u_xu_{xx}-uu_{xxx}=0,\qquad a<x<b,$$

with the initial condition $u(x,0)=-2\mathrm{sech}^2\left(\frac{x}{2}\right)$ and boundary conditions:
$u(a,t)=-2\mathrm{sech}^2\left(\frac{a}{2}-t\right),\quad u(b,t)=-2\mathrm{sech}^2\left(\frac{b}{2}-t\right),$
$u_x(a,t)=2\mathrm{sech}^2\left(\frac{a}{2}-t\right)\tanh\left(\frac{a}{2}-t\right).$

Substituting boundary and initial conditions into Eq. (38) and solving system of algebraic equations in Eq. (40), we may find vector **C**, whose components give us coefficients of Jacobi wavelet series. By substituting **C** in Eq. (31), we have obtained approximate solution of mCH equation that provides initial and boundary conditions. Graphics of the exact solution, approximate solutions and absolute errors of Example 1 are shown in Figs. 1, 2, 3, 4, 5 and 6 when $\alpha=0,\beta=0,M=71,\ k=0$ and $\Delta t=0.001$ for various values of t in the interval [−10, 10]. In Table 1, absolute errors can be shown for different values of M and k when $\alpha=0$, $\beta=0$, $\Delta t=0.001$, $t=0.1$, and various values of x in

Example 2 Consider Eq. (1) for $w=3$, modified Degasperis–Procesi (mDP) equation may be obtained as:

$$u_t-u_{xxt}+4u^2u_x-3u_xu_{xx}-uu_{xxx}=0,\qquad a<x<b,$$

with the initial condition $u(x,0)=-\frac{15}{8}\mathrm{sech}^2(\frac{x}{2})$ and boundary conditions

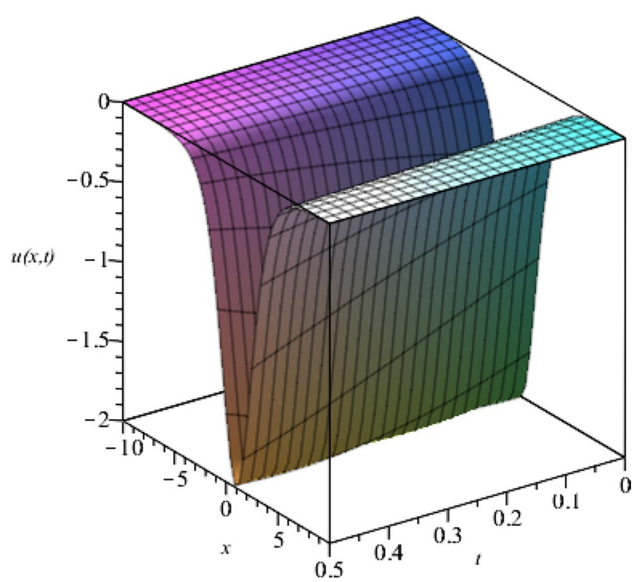

Fig. 1 Exact solution of Example 1

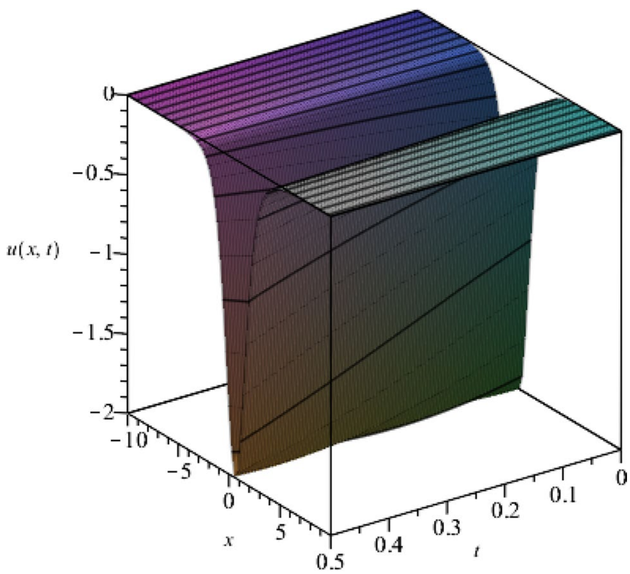

Fig. 2 Approximate solution of Example 1

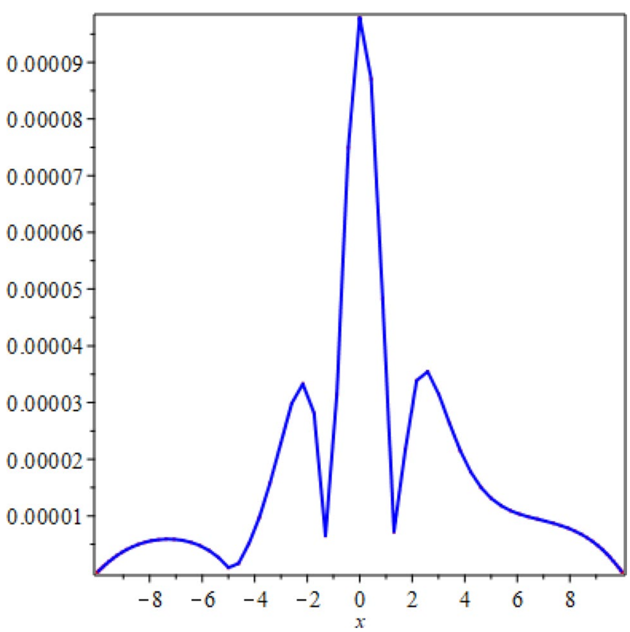

Fig. 3 Absolute errors of Example 1 at $t = 0.05$

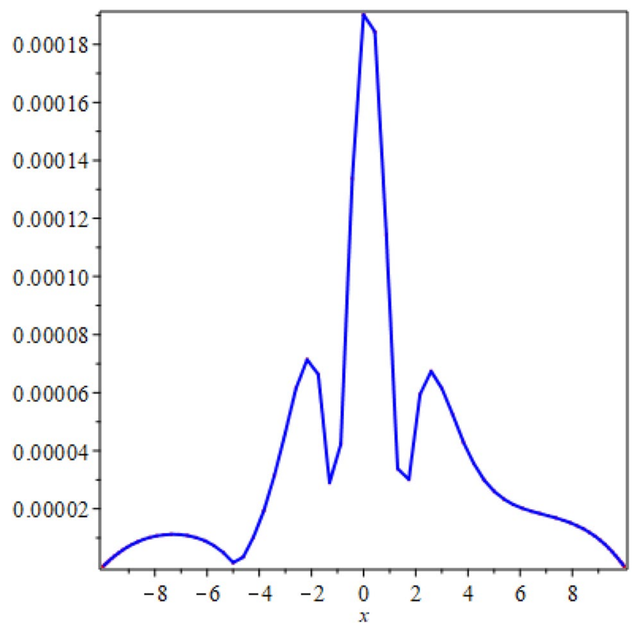

Fig. 4 Absolute errors of Example 1 at $t = 0.1$

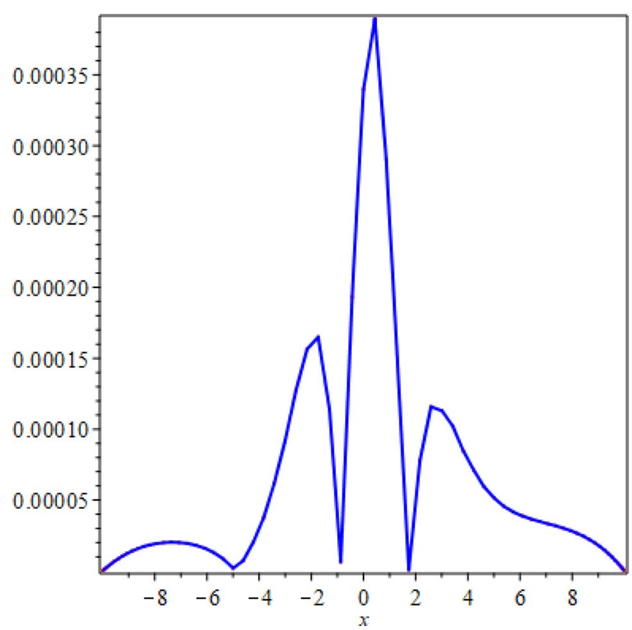

Fig. 5 Absolute errors of Example 1 at $t = 0.2$

$$u(a,t) = -\frac{15}{8}\mathrm{sech}^2(\frac{a}{2} - \frac{5t}{4}), \quad u(b,t) = -\frac{15}{8}\mathrm{sech}^2(\frac{b}{2} - \frac{5t}{4}), \quad u_x(a,t) = \frac{15}{8}\mathrm{sech}^2(\frac{a}{2} - \frac{5t}{4})\ \tanh(\frac{a}{2} - \frac{5t}{4}).$$

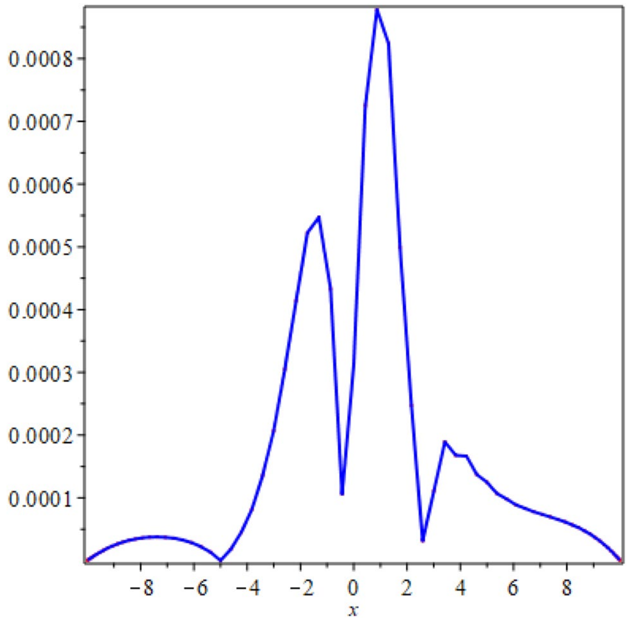

Fig. 6 Absolute errors of Example 1 at $t = 0.5$

Table 1 Absolute errors of Example 1 for various values of x

M	t	x	$k=0$	$k=1$	$k=2$
10	0.1	6	0.2200871e−1	0.8767495e−2	0.7759534e−5
		8	0.3877695e−1	0.1062448e−1	0.4621556e−3
		9	0.2688397e−1	0.7936734e−2	0.1643928e−4
		10	0.2005588e−2	0.1097291e−1	0.3012546e−3
		12	0.1716181e−1	0.8136334e−2	0.1753762e−3
16	0.1	6	0.3949731e−1	0.1211893e−2	0.3990758e−4
		8	0.3656204e−1	0.5439444e−3	0.1129990e−3
		9	0.4959233e−2	0.1201144e−2	0.3566888e−4
		10	0.2687046e−1	0.1034337e−2	0.4532343e−4
		12	0.2022488e−1	0.1007097e−3	0.3379564e−4
20	0.1	6	0.2023316e−1	0.3788986e−4	0.3655249e−5
		8	0.1602632e−1	0.1713941e−4	0.3073483e−6
		9	0.1389905e−1	0.4161723e−4	0.2468175e−6
		10	0.1668122e−1	0.5697625e−4	0.1268681e−6
		12	0.1160108e−1	0.4337532e−4	0.3455865e−6

Substituting boundary and initial conditions into Eq. (38) and solving system of algebraic equations in Eq. (40), we may find vector **C**, whose components give us coefficients of Jacobi wavelet series. By substituting **C** in Eq. (31), we have obtained approximate solution of mDP equation that provides boundary and initial conditions. Graphics of the absolute errors, approximate solution and exact solution of Example 2 are shown in Figs. 7, 8, 9, 10, 11 and 12 when $\alpha = 0, \beta = 0, M = 71,\ k = 0$ and $\Delta t = 0.001$ for various values of t in the interval $[-10,\ 10]$. Like in the Example 1, absolute errors can be shown in Table 3 for different values of M and k when $\alpha = 0, \beta = 0,\ \Delta t = 0.001,\ t = 0.1$ and various values of in the interval. It can be seen from Table 3 that the Jacobi wavelet collocation-based algorithm produces stable and converging solution by increasing the level of resolution of the Jacobi wavelet k. Absolute errors of Example 2 have been given in Table 4 to compare presented results with earlier results in the interval $[-15,\ 15]$. Graphics of the exact

Table 2 Comparison with absolute errors of offered method and existing methods in the literature at different time levels for mCH equation

x	t	Exact solution	VIM [44]	ADM [45]	HPM [46]	QuBSMs [47]	Present
6	0.05	− 0.02179598	2.005e−3	–	–	3.349e−4	4.268e−6
8		− 0.00296375	2.807e−4	3.332e−4	3.332e−4	4.359e−5	2.221e−7
9		− 0.00109081	–	1.229e−4	1.230e−4	1.596e−5	7.350e−7
10		− 0.00040136	3.817e−5	4.521e−5	4.530e−5	5.860e−6	6.831e−7
12		− 0.00005432	5.170e−6	–	–	7.900e−7	5.413e−7
6	0.10	− 0.02407444	4.226e−3	–	–	8.847e−4	1.097e−5
8		− 0.00327520	5.911e−4	7.108e−4	7.109e−4	1.159e−4	1.105e−6
9		− 0.00120550	–	2.623e−4	2.624e−4	4.248e−5	2.682e−6
10		− 0.00044356	8.035e−5	9.659e−5	9.664e−5	1.560e−5	2.509e−6
12		− 0.00006004	1.088e−5	–	–	2.100e−6	1.816e−6
8	0.15	− 0.00361934	–	1.139e−3	1.139e−3	2.238e−4	2.404e−6
9		− 0.00133224	–	4.203e−4	4.203e−4	8.208e−5	5.416e−6
10		− 0.00049021	–	1.547e−4	1.548e−4	3.014e−5	5.086e−6
8	0.20	− 0.00399961	–	1.624e−3	1.624e−3	3.765e−4	3.771e−6
9		− 0.00147230	–	5.992e−4	5.993e−4	1.381e−4	8.324e−6
10		− 0.00054176	–	2.207e−4	2.207e−4	5.073e−5	7.845e−6

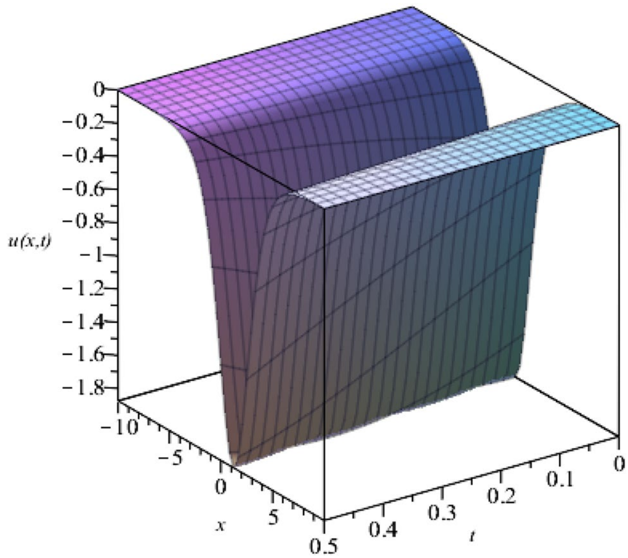

Fig. 7 Exact solution of Example 2

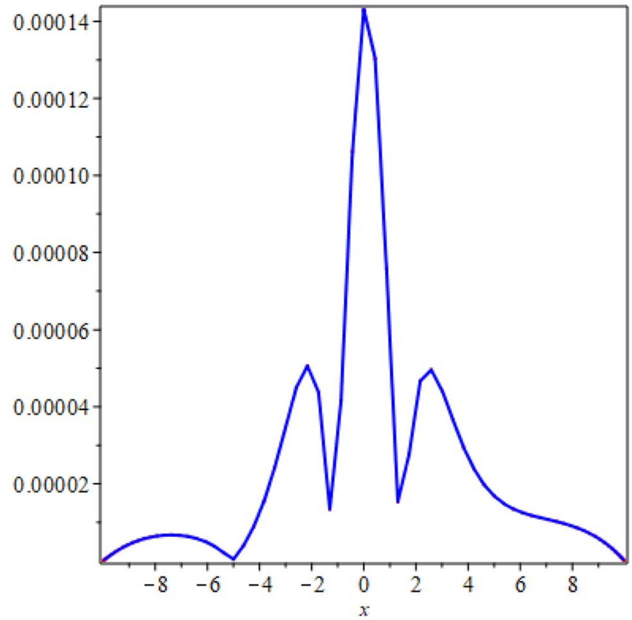

Fig. 9 Absolute errors of Example 2 at $t = 0.05$

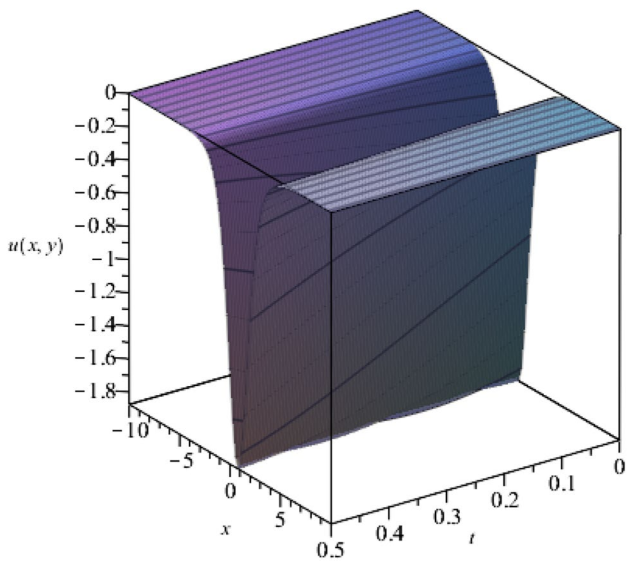

Fig. 8 Approximate solution of Example 2

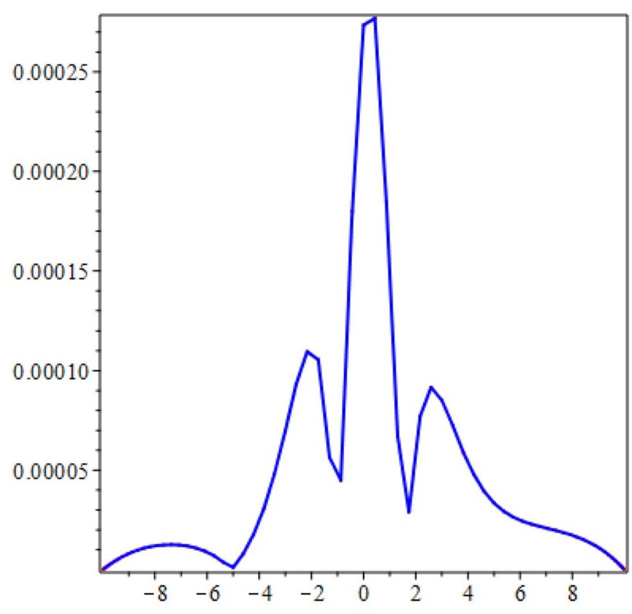

Fig. 10 Absolute errors of Example 2 at $t = 0.1$

solution and approximate solution of Example 2 have been given in Fig. 4 [45], Fig. 4 [46] and Fig. 4.4 [47]. As can be seen in Tables 3 and 4 and Figs. 7, 8, 9, 10, 11 and 12, it is clear that suggested method provides more accurate results if we compare the tables and graphics in [44–47].

7 Discussions and conclusion

Jacobi wavelets are general form of the Legendre, first-kind Chebyshev, second-kind Chebyshev and Gegenbauer wavelets. For Jacobi wavelet collocation method, matrix representation, called as operational matrices, of rth integration of

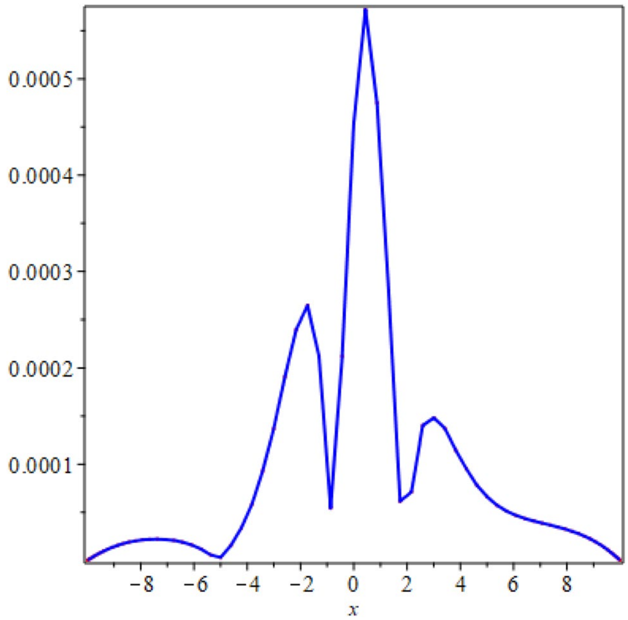

Fig. 11 Absolute errors of Example 2 at $t = 0.2$

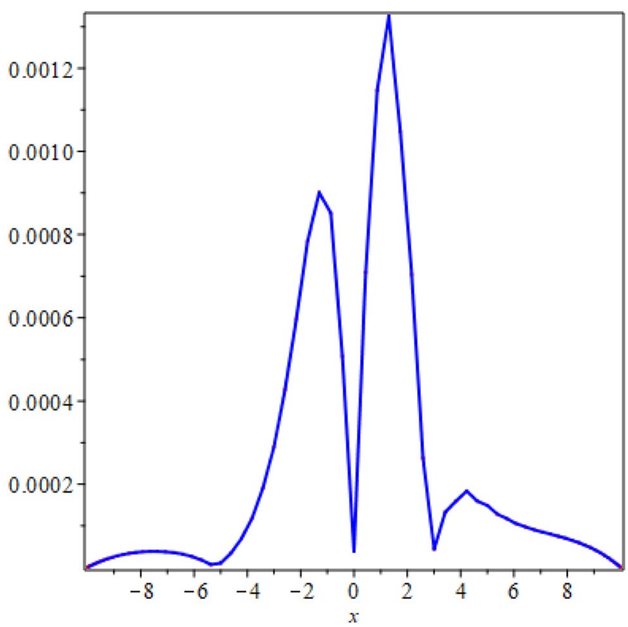

Fig. 12 Absolute errors of Example 2 at $t = 0.5$

Table 3 Absolute errors of Example 2 for various values of x

M	t	x	$k=0$	$k=1$	$k=2$
10	0.1	6	0.2587883e−1	0.9805439e−2	0.1110053e−4
		8	0.4547073e−1	0.1190818e−1	0.5053005e−3
		9	0.3152833e−1	0.8902112e−2	0.1792033e−4
		10	0.2355687e−2	0.1230602e−1	0.3294656e−3
		12	0.2013117e−1	0.9124526e−2	0.1917984e−3
16	0.1	6	0.4604518e−1	0.1262394e−2	0.4141376e−4
		8	0.4287237e−1	0.5665013e−3	0.1115217e−3
		9	0.5822258e−2	0.1254883e−2	0.3523935e−4
		10	0.3148907e−1	0.1081645e−2	0.4467676e−4
		12	0.2371258e−1	0.1052091e−3	0.3321313e−4
20	0.1	6	0.2365279e−1	0.3515931e−4	0.5747485e−5
		8	0.1866599e−1	0.1718983e−4	0.1041548e−6
		9	0.1618755e−1	0.4075308e−4	0.5361357e−7
		10	0.1945787e−1	0.5607931e−4	0.2011723e−7
		12	0.1353210e−1	0.4264685e−4	0.3628106e−6

Jacobi wavelets have been introduced and convergence analysis of offered method has been also investigated in present study. Suggested method has been applied to find approximate solutions of the nonlinear modified Camassa–Holm and Degasperis–Procesi equations linearized using quasilinearization technique. The advantage of this method is that it transforms problems into matrix products. Therefore, present method can be easily applied to obtain solutions of physical and engineering problems and its application is simple. When the Jacobi wavelets collocation method is applied to obtain approximate solutions of engineering problems using a small number of grids it has given convergent results. Algebraic equation systems depend on the number of grids so using a small number of grids reduces the size of equation systems. In Tables 1, 2, 3 and 4, convergence of the proposed method, even in the case of a small number of grid points, is observed. Comparison of the absolute errors given in Tables 1, 2, 3 and 4 and Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12 show that suggested method is superior to the results in [44–47] and offered method give highly accurate solution for the examples.

This study is applied to obtain approximate solutions of one-space dimensional problems. However, this method can also be applied for two- and three-dimensional problems as well.

Table 4 Comparison with absolute errors of offered method and existing methods in the literature at different time levels for mDP equation

x	*t*	Exact solution	VIM[44]	ADM[45]	HPM[46]	QuBSM[47]	Present
6	0.05	− 0.02094811	2.005e−3	–	–	4.490e−4	5.814e−6
8		− 0.00284880	2.807e−4	3.332e−4	3.332e−4	6.312e−5	2.392e−7
9		− 0.00104851	–	1.229e−4	1.230e−4	2.332e−5	9.460e−7
10		− 0.00038580	3.817e−5	4.521e−5	4.530e−5	8.590e−6	8.990e−7
12		− 0.00005222	5.169e−5	–	–	1.160e−6	7.063e−7
6	0.10	− 0.02371963	4.226e−3	–	–	9.037e−4	1.462e−5
8		− 0.00322779	5.911e−4	7.108e−4	7.109e−4	1.276e−4	1.273e−6
9		− 0.00118808	–	2.623e−4	2.624e−4	4.720e−5	3.378e−6
10		− 0.00043716	8.036e−5	9.659e−5	9.664e−5	1.740e−5	3.209e−6
12		− 0.00005917	1.088e−5	–	–	2.350e−6	2.325e−6
8	0.15	− 0.00365714	–	1.139e−3	1.139e−3	1.932e−4	2.608e−6
9		− 0.00134624	–	4.203e−4	4.203e−4	1.461e−5	6.438e−6
10		− 0.00049536	–	1.547e−4	1.548e−4	2.635e−5	6.138e−6
8	0.20	− 0.00414355	–	1.624e−3	1.624e−3	2.585e−4	3.632e−6
9		− 0.00152539	–	5.992e−4	5.993e−4	9.568e−5	9.038e−6
10		− 0.00056130	–	2.207e−4	2.207e−4	3.529e−5	8.672e−6

References

1. Gu JS, Jiang WS (1996) The Haar wavelets operational matrix of integration. Int J Syst Sci 27(7):623–628
2. Lepik U (2007) Numerical solution of evolution equations by the Haar wavelet method. Appl Math Comput 185:695–704
3. Hariharan G, Kannan K, Sharma KR (2009) Haar wavelet method for solving Fisher's equation. Appl Math Comput 211:284–292
4. Hariharan G, Kannan K (2010) Haar wavelet method for solving FitzHugh–Nagumo equation. Int J Math Stat Sci 2(2):59–63
5. Hariharan G, Kannan K (2010) A comparative study of a Haar wavelet method and a restrictive Taylor's series method for solving convection-diffusion equations. Int J Comput Methods Eng Sci Mech 11(4):173–184
6. Hariharan G, Kannan K (2010) Haar wavelet method for solving nonlinear parabolic equations. J Math Chem 48(4):1044–1061
7. Geng W, Chen Y, Li Y, Wang D (2011) Wavelet method for nonlinear partial differential equations of fractional order. Comput Inf Sci 4(5):28–35
8. Kaur H, Mittal RC, Mishra V (2011) Haar wavelet quasilinearization approach for solving nonlinear boundary value problems. Am J Comput Math 1:176–182
9. Çelik İ (2012) Haar wavelet method for solving generalized Burgers–Huxley equation. Arab J Math Sci 18:25–37
10. Çelik İ (2013) Haar wavelet approximation for magnetohydrodynamic flow equations. Appl Math Model 37:3894–3902
11. Razzaghi M, Yousefi S (2000) Legendre wavelets direct method for variational problems. Math Comput Simul 53:185–192
12. Razzaghi M, Yousefi S (2001) Legendre wavelets operational matrix of integration. Int J Syst Sci 32(4):495–502
13. Maleknejad K, Kajani MT, Mahmoudi Y (2003) Numerical solution of linear Fredholm and Volterra integral equation of the second kind by using Legendre wavelets. Kybernetes 32(9/10):1530–1539
14. Kajani MT, Vencheh AH (2004) Solving linear integro-differential equation with Legendre wavelet. Int J Comput Math 81(6):719–726
15. Heydari MH, Hooshmandasl MR, Ghaini FMM, Fereidouni F (2013) Two-dimensional Legendre wavelets for solving fractional Poisson equation with Dirichlet boundary conditions. Eng Anal Bound Elem 37:1331–1338
16. Hariharan G (2014) An efficient wavelet analysis method to film-pore diffusion model arising in mathematical chemistry. J Membr Biol 247(4):339–343
17. Babolian E, Fattahzadeh F (2007) Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration. Appl Math Comput 188:417–426
18. Babolian E, Fattahzadeh F (2007) Numerical computation method in solving integral equations by using Chebyshev wavelet operational matrix of integration. Appl Math Comput 188(1):1016–1022
19. Kajania MT, Vencheha AH, Ghasemib M (2009) The Chebyshev wavelets operational matrix of integration and product operation matrix. Int J Comput Math 86(7):1118–1125
20. Adibi H, Assari P (2010) Chebyshev wavelet method for numerical solution of Fredholm integral equations of the first kind. Math Probl Eng 2010:17 ((**Article ID 138408**))
21. Wang YX, Fan QB (2012) The second kind Chebyshev wavelet method for solving fractional differential equations. Appl Math Comput 218:8592–8601
22. Hooshmandasl MR, Heydari MH, Ghaini FMM (2012) Numerical solution of the one dimensional heat equation by using Chebyshev wavelets method. Appl Comput Math 1(6):1–7
23. Heydari MH, Hooshmandasl MR, Ghaini FMM, Li M (2013) Chebyshev wavelets method for solution of nonlinear fractional integrodifferential equations in a large interval. Adv Math Phys 2013:12 ((**Article ID 482083**))
24. Yang C, Hou J (2013) Chebyshev wavelets method for solving Bratu's problem. Bound Value Probl 2013:142
25. Hariharan G (2014) An efficient wavelet based approximation method to water quality assessment model in a uniform channel. Ain Shams Eng J 5(2):525–532
26. Çelik İ (2013) Numerical solution of differential equations by using Chebyshev wavelet collocation method. Cankaya Univ J Sci Eng 10(2):169–184
27. Çelik İ (2016) Chebyshev Wavelet collocation method for solving generalized Burgers–Huxley equation. Math Methods Appl Sci 39:366–377

28. Çelik İ (2018) Free vibration of non-uniform Euler–Bernoulli beam under various supporting conditions using Chebyshev wavelet collocation method. Appl Math Model 54:268–280
29. Pathak A, Singh RK, Mandal BN (2014) Solution of Abel's integral equation by using Gegenbauer wavelets. Investig Math Sci 4(1):43–52
30. Abd-Elhameed WM, Youssri YH (2014) New ultraspherical wavelets spectral solutions for fractional Riccati differential equations. In: Abstract and applied analysis, 2014; Hindawi
31. Abd-Elhameed WM, Youssri YH, Doha EH (2014) New solutions for singular Lane–Emden equations arising in astrophysics based on shifted ultraspherical operational matrices of derivatives. Comput Methods Differ Equ 2(3):171–185
32. Abd-Elhameed WM, Youssri YH (2015) New spectral solutions of multi-term fractional order initial value problems with error analysis. Comput Model Eng Sci 105(5):375–398
33. Rehman M, Saeed U (2015) Gegenbauer wavelets operational matrix method for fractional differential equations. J Korean Math Soc 52(5):1069–1096
34. Youssri YH, Abd-Elhameed WM, Doha EH (2015) Ultraspherical wavelets method for solving Lane–Emden type equations. Rom J Phys 60(9):1298–1314
35. Youssri YH, Abd-Elhameed WM, Doha EH (2015) Accurate spectral solutions of first-and second-order initial value problems by the ultraspherical wavelets-Gauss collocation method. Appl Appl Math Int J 10(2):835–851
36. Doha EH, Abd-Elhameed WM, Youssri YH (2016) New ultraspherical wavelets collocation method for solving 2nth-order initial and boundary value problems. J Egypt Math Soc 24(2):319–327
37. Çelik İ (2018) Generalization of Gegenbauer wavelet collocation method to the generalized Kuramoto–Sivashinsky equation. Int J Appl Comput Math 4(5):111
38. Çelik İ (2021) Squeezing flow of nanofluids of Cu–water and kerosene between two parallel plates by Gegenbauer wavelet collocation method. Eng Comput. 37:251–264
39. Çelik İ (2020) Gegenbauer wavelet collocation method for the extended Fisher-Kolmogorov equation in two dimensions. Math Method Appl Sci 43(8):5615–5628. https://doi.org/10.1002/mma.6300
40. Wazwaz AM (2006) Solitary wave solutions for modified forms of Degasperis–Procesi and Camassa–Holm Equations. Phys Lett A 352:500–504
41. Mandelzweig VB, Tabakin F (2001) Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs. Comput Phys Commun 141(2):268–281
42. Luke YL (1969) The special functions and their approximations, vol I. Academic Press, New York
43. Daubechies I (1992) Ten lectures on wavelets. SIAM, Philadelphia
44. Yildirim A (2010) Variational iteration method for modified Camassa–Holm and Degasperis-Procesi equations. Int J Numer Methods Biomed Eng 26:266–272
45. Ganji DD, Sadeghi EMM, Rahmat MG (2008) Modified forms of Degaperis–Procesi and Camassa–Holm equations solved by Adomian's decomposition method and comparison with HPM and exact solution. Acta Applicandae Mathematicae 104:303–311
46. Zhang B, Li S, Liu Z (2008) Homotopy perturbation method for modified Camassa–Holm and Degaperis–Procesi equations. Phys Lett A 372:1867–1872
47. Wasim I, Abbas M, Iqbal MK (2018) Numerical solution of modified forms of Camassa–Holm and Degasperis–Procesi equations via quartic B-spline collocation method. Commun Math Appl 9:393–409