Engineering with Computers (2022) 38:2051–2066
https://doi.org/10.1007/s00366-020-01194-6

ORIGINAL ARTICLE

Existence and uniqueness results and analytical solution of the multi-dimensional Riesz space distributed-order advection–diffusion equation via two-step Adomian decomposition method

Pratibha Verma[1] · Manoj Kumar[1]

Received: 28 July 2020 / Accepted: 6 October 2020 / Published online: 22 October 2020

Abstract
In this article, we introduced for the first time the two-step Adomian decomposition method (TSADM) for solving the multi-dimensional Riesz space distributed-order advection–diffusion (RSDOAD) equation. The TSADM was successfully applied to obtain the analytical solution of the multi-dimensional (RSDOAD) equation. The analytical solution has been obtained without approximation/discretization of the Riesz fractional operator. Furthermore, new results for the existence are obtained with the help of some fixed point theorems, while the uniqueness of the solution was investigated employing the Banach contraction principle. Finally, we included a generalized example to demonstrate the validity and application of the proposed method. The obtained results conclude that the proposed method is powerful and efficient for the considered problem compared to the other existing methods.

Keywords Fractional derivatives · Riesz space distributed-order advection–diffusion equation · Riesz derivative, Two-step Adomian decomposition method · Fixed point theorem

Mathematics Subject Classification 26A33 · 35R11 · 47H10

1 Introduction

The theory of fractional calculus has been more focused and getting attention of the researches in the different areas of science and engineering [1–3]. Fractional calculus is an updated version of the standard calculus, in which the the operators (derivative/integral) are of fractional order. Nowadays, studies of fractional calculus have received considerable attention [4–8]. The authors proposed a unified method to solve time fractional Burgers' equation with the Caputo derivative. Using numerical methods, the authors obtained approximate solution and compared the obtained results with the exact [9, 10].

Furthermore, the analytical solution of fractional differential equations are difficult to find and has not been the focus of much attention. Therefore, a numerical approach is needed for solving fractional differential equations (FDEs) [11, 12]. The most developed methods for the numerical approximation of FDEs are spectral methods, spectral collocation method, Adomian decompostion method (ADM), improved collection method, Sinc collocation methods, and so on. Many works have done on the existence and uniqueness of the FDEs. The most urgent problems listed are how to confirm the existence and uniqueness of the FDEs for the solution and find the existence of the method which provides the analytical solution of the FDEs [13–15].

For the first time, Caputo investigated the use of differential equations with distributed-order derivatives for generalizing stress–strain relations of unelastic media. Further, He discussed distributed-order time/space fractional differential equations and obtained the solutions with closed-form

✉ Manoj Kumar
manoj@mnnit.ac.in

Pratibha Verma
pratibhaverma@mnnit.ac.in

[1] Department of Mathematics, Motilal Nehru National Institute of Technology Allahabad, Prayagraj, Uttar Pradesh 211004, India

formulae of the standard problems. The difference between these two problems is also significant, according to the results funded by the Caputo. There are several numerical methods developed for obtaining the approximate solution of these equations, but such a method does not discuss which gives the accurate solution without the use of the discretization, linearization, and other approximations process for the operators. In [16], the author uses the second-order accurate implicit scheme for the one/two-dimension Riesz space distributed-order advection–dispersion equations in which the Riesz space distributed-order advection–dispersion equations are converted into multi-terms Riesz space distributed-order advection–dispersion equations with the help of discretization by using the midpoint quadrature rule of the operators. Additionally, we discuss the stability and convergency of the method and also analyze the obtained results. Hui Zhang et al. [17] investigated the approximate solution of the two-dimensional Riesz space distributed-order advection–diffusion equation by using a Crank–Nicolson alternating direction implicit (ADI) Galerkin–Legendre spectral scheme, in which they have used approximation for the distributed-order Riesz space derivative to convert the considered problem into a multi-term fractional equation. The obtained solution has higher computational accuracy compared to other numerical methods. Moreover, discussion on the solution of the Riesz space distributed-order advection–dispersion equations was done in literature for one/two dimensions, not more than two dimensions, and the approximate solution w as obtained by using numerical techniques with high computation.

The Adomian decomposition method (ADM) was used to obtain the approximate solution of FDEs and is very reliable for these problems. Further modification was done to improve the method. Namely, the modified ADM [18], the new improved ADM [19], the new modification ADM [20] and also many improved versions of the ADM have been presented in previous works. All improved versions of the ADM always gives the approximate solution for non-linear FDEs. In this article, we consider the method, which is also an improved version of standard ADM, but reduces the computation and provides an analytical solution for non-linear FDEs in just one iteration called the two-step Adomain decomposition method (TSADM) [21–24]. If the TSADM applies to the problem, then the method always provides the analytical solution; applicability means the zeroth term of the series has included the term, which satisfies the problem and associated initial/boundary conditions.

We consider the multi-dimensional Riesz space distributed-order advection–diffusion (RSDOAD) equation and target to find the analytical solution of the RSDOAD with the help of the TSADM. The target was achieved without converting the operators into multi-operators, which reduces the difficulties of the problem. If we choose any approximation technique, the problem becomes more and more complicated by increasing the dimensions of the problem. All numerical methods can be used to solve these types of equations with some drawbacks for one/two dimensions, but after increasing the dimension finding an approximate solution becomes highly complicated. The proposed method in this paper is highly efficient than other numerical methods for the solution of the Riesz space distributed-order advection–diffusion equation for multi-dimension and gives the accurate solution without the use of the approximation/discretization such as numerical methods. The applicability of the proposed method can be proved by taking arbitrary dimensions of the equation and the analytical solution is obtained in just one iteration.

The primary objective of the present work is to find the analytical solution for the multi-dimensional RSDOAD equation by using the TSADM. Moreover, we have obtained new results for the existence and uniqueness of a solution. To prove the efficiency of the TSADM, we have considered a generalized example with a generalized source term, compared them with other numerical methods and then concluded the analysis of the result. The TSADM gives the analytical solution in just one iteration, and this is one of the main advantages of the proposed method.

The plan of this paper is listed as follows. In Sect. 2, we present some preliminaries from fractional calculus and introduce definitions, properties, and lemma of fractional operators based on Caputo sense, and also some essential theorems and lemmas, which are needed in the proof of the main results. In Sect. 3, the main results based on the Banach contraction principle and some fixed point theorems have shown and proved. In Sect. 4, we describe the TSADM for the muti-dimension RSDOAD equation. In Sect. 5, the proposed method was applied to solve the generalized example. Finally, conclusions are drawn in Sect. 6.

2 Preliminaries

We describe some basic concepts of fractional integrals and derivatives, along with their properties. We adopt the Caputo's definition of fractional derivatives, which is very popular in the field of applied mathematics. Furthermore, we describe some theorems and lemmas, which are further used to prove the existence and uniqueness conditions of the considered problems.

Definition 1 ([22, 23]). A real function $g(y)$, $y > 0$, is said to be in the space C_θ, if $\theta \in \mathbb{R}$, there exists a real number $v(> \theta)$, such that $g(y) = y^v g_1(y)$, where $g_1(y) \in C[0, \infty)$ and it is said to be in the space C_θ^n if $g^n \in C_\theta$, $n \in \mathbb{N} \cup \{0\}$.

Definition 2 ([22, 23]). The Riemann–Liouville fractional integral operator of order $\sigma \geq 0$, of a function $g \in C_\theta$, $\theta \geq -1$, is defined as

$$J^{\sigma}{}_{y}g(y) = \frac{1}{\Gamma(\sigma)}\int_0^y (y-\xi)^{\sigma-1}g(\xi)d\xi, \quad \sigma > 0, \quad y > 0. \qquad (1)$$
$$J^{0}{}_{y}g(y) = g(y).$$

The following properties of the operator $J^{\sigma}{}_{\xi}$ can be found in [22, 23]. For $g \in C_\theta, \theta \geq -1, \sigma, \omega \geq 0$ and $\zeta > -1$,

$$(i)\ J_y^{\sigma}J_y^{\omega}g(y) = J_y^{\sigma+\omega}g(y). \qquad (2)$$

$$(ii)\ J_y^{\sigma}J_y^{\omega}g(y) = J_y^{\omega}J_y^{\sigma}g(y). \qquad (3)$$

$$(iii)\ J_y^{\sigma}y^{\zeta} = \frac{\Gamma(\zeta+1)}{\Gamma(\sigma+\zeta+1)}y^{\sigma+\zeta}. \qquad (4)$$

The Caputo derivative is a modified version of the Riemann–Liouville derivative, which removes some disadvantages of the Riemann–Liouville derivative for fractional derivatives.

Definition 3 ([22, 23]). The fractional derivative of $g(y)$ in the Caputo sense is defined as

$$D_y^{\sigma}g(y) = J_y^{n-\sigma}D_y^{n}g(y) = \frac{1}{\Gamma(n-\sigma)}\int_0^y (y-\xi)^{n-\sigma-1}g^{n}(\xi)d\xi, \qquad (5)$$

for $n-1 < \sigma \leq n, \quad n \in \mathbb{N}, \quad y > 0, \quad g \in C_{-1}^{n}$, and

$$D_y^{\sigma}A = 0. \qquad (6)$$

The Caputo's fractional derivatives are linear operators, as we have

$$D_y^{\sigma}(As(y) + Bh(y)) = AD_y^{\sigma}s(y) + BD_y^{\sigma}h(y), \qquad (7)$$

where A and B are constants.

Definition 4 ([22, 23]). For every smallest integer n, which exceeds σ, we can define the Caputo time-fractional derivative operator of order $\sigma > 0$ as

$$D_{\xi}{}^{\sigma}w(y,\xi) = \frac{\partial^{\sigma}w(y,\xi)}{\partial\xi^{\sigma}} = \begin{cases} \frac{1}{\Gamma(n-\sigma)}\int_0^{\xi}(\xi-\tau)^{n-\sigma-1}\frac{\partial^{n}w(y,\tau)}{\partial\tau^{n}}d\tau\ ; & n-1<\sigma<n, \\ \frac{\partial^{n}w(y,\xi)}{\partial\xi^{n}}; & \sigma = n \in \mathbb{N}. \end{cases} \qquad (8)$$

Definition 5 ([22]). Consider the function $F_r(\eta) = \eta^{r}(1-\eta)^{r}$, $\eta \in [0,1]$, where $r = 1, 2, 3, \cdots$

The analytical expression for the Riesz derivatives of the above function is as follows:

$$\frac{\partial^{\rho}F_r(\eta)}{\partial|\eta|^{\rho}} = -\psi_{\rho} \times \sum_{L=0}^{r}\frac{(-1)^{L}r!(r+L)!}{L!(r-L)!\Gamma(r+L+1-\rho)}[\eta^{r+L-\rho} + (1-\eta)^{r+L-\rho}], \qquad (9)$$

where $\psi_{\rho} = \frac{1}{2\cos(\frac{\pi\rho}{2})}$, $\rho \neq 1$.

The following basic lemma also helps us to obtain the solution of some considered problems.

Lemma 1 ([22–24]). *If $n-1 < \sigma \leq n$, $n \in \mathbb{N}$ and $g \in C_{\theta}^{n}$, $\theta \geq -1$, then*

$$D_y{}^{\sigma}J_y^{\sigma}g(y) = g(y), \qquad (10)$$

$$J_y^{\sigma}D_y{}^{\sigma}g(y) = g(y) - \sum_{q=0}^{n-1}g^{q}(0^{+})\frac{y^{q}}{q!}, \quad y > 0. \qquad (11)$$

Definition 6 An operator is called completely continuous, if it is continuous and maps bounded sets into precompact sets.

Definition 7 Let $(Y, \|\cdot\|)$ be a normed space. A contraction of Y is mapping $P : Y \to Y$ that satisfies, for every $y_1, y_2 \in Y$,
$\|P(y_1) - P(y_2)\| \leq \delta\|y_1 - y_2\|$,
for some real values $0 \leq \delta < 1$.

The following lemma was obtained by using the Caputo's derivative definition, which has played a significant role in our analysis.

Lemma 2 ([22–24]). *Let $\sigma > 0$, then a general solution to the homogeneous equation*

$$D_{0^+}^{\sigma}\psi(y) = 0 \qquad (12)$$

is given by

$$\psi(y) = a_0 + a_1y + a_2y^2 + a_3y^3 + \cdots + a_{m-1}y^{m-1}, \quad a_j \in \mathbb{R}, j = 1, 2, \cdots, m-1 (m = [\sigma]+1). \qquad (13)$$

Lemma 3 ([22–24]). *Let $\sigma > 0$, then we have*

$$J_{0^+}^{\sigma}D_{0^+}^{\sigma}\psi(y) = \psi(y) + a_0 + a_1y + a_2y^2 + a_3y^3 + \cdots + a_{m-1}y^{m-1}, \qquad (14)$$
$$a_j \in \mathbb{R}, j = 1, 2, \cdots, m-1 (m = [\sigma]+1).$$

Theorem 1 (**Banach fixed point theorem,** [22–24]). *Every contraction mapping on a complete metric space has a unique fixed point.*

Theorem 2 (**Schaefer's fixed point theorems,** [22–24]). *Let $P : Y \to Y$ be a completely continuous operator. If the* *set* $S(P)=\{y\in Y : y=c^*P(y)$ *for some* $c^*\in[0,1]\}$ *is bounded, then, P has fixed points in P.*

Theorem 3 (**Arzelà-Ascoli Theorem,** [22–24]). *Let Y be a compact metric space. Let* $C(Y,\mathbb{R})$ *be given the sup norm metric. Then a set* $P\subset C(Y)$ *is compact iff P is bounded, closed and equicontinuous.*

3 Main results for the existence and uniqueness of solution

Consider the multi-dimensional Riesz space distributed-order advection–diffusion (RSDOAD) equation, which is described as follows:

$$\begin{aligned}\frac{\partial V(X_i,X_{m+j},\theta)}{\partial\theta} &= \sum_{i=0}^{m}\omega_i\int_1^2 Q_i(\alpha_i)\frac{\partial^{\alpha_i}V}{\partial|x_i|^{\alpha_i}}d\alpha_i\\ &+\sum_{\substack{j=1\\ i=j}}^{n}\omega_{m+j}\int_1^2 Q_{m+j}(\alpha_{m+j})\frac{\partial^{\alpha_{m+j}}V}{\partial|x_{m+j}|^{\alpha_{m+j}}}d\alpha_{m+j}\\ &+\sum_{i=0}^{m}\omega'_i\int_0^1 Q'_i(\alpha'_i)\frac{\partial^{\alpha'_i}V}{\partial|x_i|^{\alpha'_i}}d\alpha'_i\\ &+\sum_{\substack{j=1\\ i=j}}^{n}\omega'_{m+j}\int_0^1 Q'_{m+j}(\alpha'_{m+j})\frac{\partial^{\alpha'_{m+j}}V}{\partial|x_{m+j}|^{\alpha'_{m+j}}}d\alpha'_{m+j}\\ &+F(X_i,X_{m+j},\theta),(X_i,X_{m+j},\theta)\in\Omega\times I,\end{aligned}\tag{15}$$

with the initial and boundary conditions

$$\begin{aligned}&V(X_i,X_{m+j},\theta)=0,(X_i,X_{m+j},\theta)\\ &\in\partial\Omega\times I, X_i=(x_i)_{i=0,1,2,\cdots m}, X_{m+j}=(x_{m+j})_{j=1,2,\cdots n},\\ &V(X_i,X_{m+j},0)=V_0(X_i,X_{m+j}),(X_i,X_{m+j})\in\Omega,\end{aligned}\tag{16}$$

where $I=(0,\Theta]$, $\Omega=\prod_{i=0}^{m}(0,k)\times\prod_{\substack{j=1\\ i=j}}^{n}(0,k)$, $F(X_i,X_{m+j},\theta)$ is the source and sink $w_i,w'_i>0$, $i=0,1,2,\cdots,m$ are the diffusion coefficients, $w_{m+j},w'_{m+j}>0$, $j=1,2,\cdots,m$ are the average velocities, and $Q_i(\alpha_i)$, $Q_{m+j}(\alpha_{m+j})$, $Q'_i(\alpha'_i)$, $Q'_{m+j}(\alpha'_{m+j})$ are non-negative and bounded weight functions that satisfy the following conditions:

$$\begin{aligned}&Q_i(\alpha_i)\geq 0,\quad Q_i(\alpha_i)\not\equiv\ 0,\\ &\alpha_i\in(1,2),\quad 0<\int_1^2 Q_i(\alpha_i)d\alpha_i<\infty,\end{aligned}\tag{17}$$

$$\begin{aligned}&Q'_i(\alpha'_i)\geq 0,\quad Q'_i(\alpha'_i)\not\equiv\ 0,\quad \alpha'_i\in(1,2),\\ &0<\int_0^1 Q'_i(\alpha'_i)d\alpha'_i<\infty,\end{aligned}\tag{18}$$

$$\begin{aligned}&Q_{m+j}(\alpha_{m+j})\geq 0,\quad Q_{m+j}(\alpha_{m+j})\not\equiv\ 0,\quad \alpha_{m+j}\in(1,2),\\ &0<\int_1^2 Q_{m+j}(\alpha_{m+j})d\alpha_{m+j}<\infty,\end{aligned}\tag{19}$$

$$\begin{aligned}&Q'_{m+j}(\alpha'_{m+j})\geq 0,\quad Q'_{m+j}(\alpha'_{m+j})\not\equiv\ 0,\\ &\alpha'_{m+j}\in(1,2),\\ &0<\int_0^1 Q'_{m+j}(\alpha'_{m+j})d\alpha'_{m+j}<\infty.\end{aligned}\tag{20}$$

The Riesz fractional derivatives [25, 26] on a finite domain $[0,k]$ are described as follows:

$$\begin{aligned}&\frac{\partial^{\alpha_i}V}{\partial|x_i|^{\alpha_i}}\\ &=-\frac{1}{2\cos(\frac{\alpha_i\pi}{2})}({}_0D_{x_i}^{\alpha_i}V+{}_{x_i}D_k^{\alpha_i}V),\\ &0<\alpha_i<2,\alpha_i\neq 1, i=0,1,2,\cdots m,\end{aligned}\tag{21}$$

$$\begin{aligned}&\frac{\partial^{\alpha_{m+j}}V}{\partial|x_{m+j}|^{\alpha_{m+j}}}\\ &=-\frac{1}{2\cos(\frac{\alpha_{m+j}\pi}{2})}({}_0D_{x_{m+j}}^{\alpha_{m+j}}\\ &V+{}_{x_{m+j}}D_k^{\alpha_{m+j}}V),0<\alpha_{m+j}<2,\alpha_{m+j}\neq 1,j=1,2,\cdots n.\end{aligned}\tag{22}$$

For $c-1<\gamma\leq c, c\in\mathbb{N}$, the operators ${}_0D_{x_i}^{\gamma}$, ${}_0D_{x_{m+j}}^{\gamma}$ and ${}_{x_i}D_k^{\gamma}$, ${}_{x_{m+j}}D_k^{\gamma}$ are defined as follows:

$$\begin{aligned}&{}_0D_{x_i}^{\gamma}V=\frac{1}{\Gamma(c-\gamma)}\\ &\frac{\partial^c}{\partial x_i^c}\int_0^{x_i}(x_i-a)^{c-\gamma-1}V(x_1,x_2,\cdots,x_{i-1},a,\theta)da,\end{aligned}\tag{23}$$

$$\begin{aligned}&{}_{x_i}D_k^{\gamma}V=\frac{(-1)^c}{\Gamma(c-\gamma)}\\ &\frac{\partial^c}{\partial x_i^c}\int_{x_i}^{k}(a-x_i)^{c-\gamma-1}V(x_1,x_2,\cdots,x_{i-1},a,\theta)da,\end{aligned}\tag{24}$$

$$\begin{aligned}&{}_0D_{x_{m+j}}^{\gamma}V=\frac{1}{\Gamma(c-\gamma)}\\ &\frac{\partial^c}{\partial x_{m+j}^c}\int_0^{x_{m+j}}\\ &(x_{m+j}-a)^{c-\gamma-1}V(x_1,x_2,\cdots,x_{m+j-1},a,\theta)da,\end{aligned}\tag{25}$$

$$ {}_{x_{m+j}}D_k^{\gamma}V = \frac{(-1)^c}{\Gamma(c-\gamma)} \frac{\partial^c}{\partial x_{m+j}^c}\int_{x_{m+j}}^{k} (a-x_{m+j})^{c-\gamma-1}V(x_1,x_2,\cdots,x_{m+j-1},a,\theta)da. \quad (26) $$

In this section, we demonstrate the existence and uniqueness of the solution for the problem given by the Eqs. (15, 16) by using a fixed point theorem.

We denote $C(\psi,\mathbb{R})$ as the Banach space of all continuous functions from $\Omega\times I=\psi$ into $\mathbb{R}$ with the norm $\|\cdot\|_\infty$ defined by $\|V\|_\infty := \sup\left\{|V(X_i,X_{m+j},\theta)|;(X_i,X_{m+j},\theta)\in\psi\right\}$ (see [22–24]).

Definition 8 A function $V(X_i,X_{m+j},\theta)\in C(\psi,\mathbb{R})$ is known as a solution of the problem (15, 16), if $V(X_i,X_{m+j},\theta)$ satisfies the multi-dimensional Riesz space distributed-order advection–diffusion (RSDOAD) equation and the associated initial/boundary conditions.

The following assumptions are use to prove the uniqueness and existence of solution for the problem (15, 16).

(S_1) There exist non-negative constants M_{α_i}, $M_{\alpha_i'}$, $M_{\alpha_{m+j}}$ and $M_{\alpha'_{m+j}}$, $i=0,1,2,\cdots m$, $j=1,2,\cdots n$ such that

$$ \left|\frac{\partial^{\alpha_i}V_1}{\partial|x_i|^{\alpha_i}}-\frac{\partial^{\alpha_i}V_2}{\partial|x_i|^{\alpha_i}}\right|\le M_{\alpha_i}|V_1-V_2|, $$
$$ \left|\frac{\partial^{\alpha_i'}V_1}{\partial|x_i|^{\alpha_i'}}-\frac{\partial^{\alpha_i'}V_2}{\partial|x_i|^{\alpha_i'}}\right|\le M_{\alpha_i'}|V_1-V_2|, $$
$$ \left|\frac{\partial^{\alpha_{m+j}}V_1}{\partial|x_{m+j}|^{\alpha_{m+j}}}-\frac{\partial^{\alpha_{m+j}}V_2}{\partial|x_{m+j}|^{\alpha_{m+j}}}\right| \le M_{\alpha_{m+j}}|V_1-V_2|, \left|\frac{\partial^{\alpha'_{m+j}}V_1}{\partial|x_{m+j}|^{\alpha'_{m+j}}}-\frac{\partial^{\alpha'_{m+j}}V_2}{\partial|x_{m+j}|^{\alpha'_{m+j}}}\right| \le M_{\alpha'_{m+j}}|V_1-V_2|, \forall(X_i,X_{m+j},\theta)\in\psi. $$

(S_2) There exist non-negative constants χ_{α_i}, $\chi_{\alpha_i'}$, $\chi_{\alpha_{m+j}}$ and $\chi_{\alpha'_{m+j}}$, $i=0,1,2,\cdots m$, $j=1,2,\cdots n$ such that

$$ \chi_{\alpha_i}=\int_1^2|Q_i(\alpha_i)|d\alpha_i<\infty, \chi_{\alpha_i'}=\int_0^1|Q_i'(\alpha_i')|d\alpha_i'<\infty, $$
$$ \chi_{\alpha_{m+j}}=\int_1^2|Q_{m+j}(\alpha_{m+j})|d\alpha_{m+j} <\infty, \chi_{\alpha'_{m+j}}=\int_0^1|Q'_{m+j}(\alpha'_{m+j})|d\alpha'_{m+j}<\infty. $$

(S_3) The function $F: C(\psi)\to C(\psi)$ is continuous and there exists $B_1>0$ such that

$$ |F(X_i,X_{m+j},\theta)|\le B_1, \forall(X_i,X_{m+j},\theta)\in\psi. $$

(S_4) The function $V_0: C(\Omega)\to C(\Omega)$ is continuous and there exists $B_2>0$ such that

$$ |V_0(X_i,X_{m+j})|\le B_2, \forall(X_i,X_{m+j})\in\Omega. $$

(S_5) There exist non-negative constants ρ_{α_i}, $\rho_{\alpha_i'}$, $\rho_{\alpha_{m+j}}$ and $\rho_{\alpha'_{m+j}}$, $i=0,1,2,\cdots m$, $j=1,2,\cdots n$ such that

$$ \left|\frac{\partial^{\alpha_i}V}{\partial|x_i|^{\alpha_i}}\right|\le\rho_{\alpha_i}|V|, \left|\frac{\partial^{\alpha_i'}V}{\partial|x_i|^{\alpha_i'}}\right|\le\rho_{\alpha_i'}|V|, \left|\frac{\partial^{\alpha_{m+j}}V}{\partial|x_{m+j}|^{\alpha_{m+j}}}\right| \le\rho_{\alpha_{m+j}}|V|, \left|\frac{\partial^{\alpha'_{m+j}}V}{\partial|x_{m+j}|^{\alpha'_{m+j}}}\right|\le\rho_{\alpha'_{m+j}}|V|, \forall(X_i,X_{m+j},\theta)\in\psi. $$

(S_6) There exist non-negative constants μ_{α_i}, $\mu_{\alpha_i'}$, $\mu_{\alpha_{m+j}}$ and $\mu_{\alpha'_{m+j}}$, $i=0,1,2,\cdots m$, $j=1,2,\cdots n$ such that

$$ \left|\frac{\partial^{\alpha_i}V}{\partial|x_i'|^{\alpha_i}}\right|\le\mu_{\alpha_i}|x_i'|, \left|\frac{\partial^{\alpha_i'}V}{\partial|x_i'|^{\alpha_i'}}\right| \le\mu_{\alpha_i'}|x_i'|, \left|\frac{\partial^{\alpha_i}V}{\partial|x_i''|^{\alpha_i}}\right| \le\mu_{\alpha_i}|x_i''|, \left|\frac{\partial^{\alpha_i'}V}{\partial|x_i''|^{\alpha_i'}}\right|\le\mu_{\alpha_i'}|x_i''|, \left|\frac{\partial^{\alpha_{m+j}}V}{\partial|x'_{m+j}|^{\alpha_{m+j}}}\right| \le\mu_{\alpha_{m+j}}|x'_{m+j}|, \left|\frac{\partial^{\alpha'_{m+j}}V}{\partial|x'_{m+j}|^{\alpha'_{m+j}}}\right|\le\mu_{\alpha'_{m+j}}|x'_{m+j}|, $$
$$ \left|\frac{\partial^{\alpha_{m+j}}V}{\partial|x''_{m+j}|^{\alpha_{m+j}}}\right|\le\mu_{\alpha_{m+j}}|x''_{m+j}|, \left|\frac{\partial^{\alpha'_{m+j}}V}{\partial|x''_{m+j}|^{\alpha'_{m+j}}}\right| \le\mu_{\alpha'_{m+j}}|x''_{m+j}|, \forall(X_i,X_{m+j},\theta)\in\psi. $$

Our first result is based on the Banach fixed point theorem.

Theorem 4 *Let the hypotheses $(S1)--(S2)$ hold. If*

$$ \Theta\left(\sum_{i=0}^{m}|\omega_i|\chi_{\alpha_i}M_{\alpha_i} +\sum_{i=0}^{m}|\omega_i'|\chi_{\alpha_i'}M_{\alpha_i'}+\sum_{\substack{j=1\\ i=j}}^{n}|\omega_{m+j}|\chi_{\alpha_{m+j}}M_{\alpha_{m+j}} +\sum_{\substack{j=1\\ i=j}}^{n}|\omega'_{m+j}|\chi_{\alpha'_{m+j}}M_{\alpha'_{m+j}}\right)<1, \quad (27) $$

then, the problem (15, 16) *has a unique solution on* $C(\psi,\mathbb{R})$.

Proof We will convert the problem (15, 16) into a fixed point problem. Consider the operator $\Sigma: C(\psi,\mathbb{R})\to C(\psi,\mathbb{R})$ given by

$$\Sigma(V(X_i,X_{m+j},\theta)) = V_0(X_i,X_{m+j}) + J_\theta\Bigg(\sum_{i=0}^{m}\omega_i\int_1^2 Q_i(\alpha_i)\frac{\partial^{\alpha_i}V}{\partial|x_i|^{\alpha_i}}d\alpha_i + \sum_{\substack{j=1\\ i=j}}^{n}\omega_{m+j}\int_1^2 Q_{m+j}(\alpha_{m+j})\frac{\partial^{\alpha_{m+j}}V}{\partial|x_{m+j}|^{\alpha_{m+j}}}d\alpha_{m+j} + \sum_{i=0}^{m}\omega'_i\int_0^1 Q'_i(\alpha'_i)\frac{\partial^{\alpha'_i}V}{\partial|x_i|^{\alpha'_i}}d\alpha'_i + \sum_{\substack{j=1\\ i=j}}^{n}\omega'_{m+j}\int_0^1 Q'_{m+j}(\alpha'_{m+j})\frac{\partial^{\alpha'_{m+j}}V}{\partial|x_{m+j}|^{\alpha'_{m+j}}}d\alpha'_{m+j} + F(X_i,X_{m+j},\theta)\Bigg). \tag{28}$$

We observe that the fixed points of the operator Σ are the solution of the the problem (15, 16), and with the help of the Banach contraction principle, we prove that Σ has a fixed point. We shall first prove that Σ is a contraction. Let $V_1, V_2 \in C(\psi,\mathbb{R})$. Then, for every $(X_i,X_{m+j},\theta)\in\psi$

$$\begin{aligned}
|\Sigma V_1-\Sigma V_2| = \Bigg|\Bigg(& V_0(X_i,X_{m+j}) + J_\theta\Bigg(\sum_{i=0}^{m}\omega_i\int_1^2 Q_i(\alpha_i)\frac{\partial^{\alpha_i}V_1}{\partial|x_i|^{\alpha_i}}d\alpha_i + \sum_{\substack{j=1\\ i=j}}^{n}\omega_{m+j}\int_1^2 Q_{m+j}(\alpha_{m+j})\frac{\partial^{\alpha_{m+j}}V_1}{\partial|x_{m+j}|^{\alpha_{m+j}}}d\alpha_{m+j}\\
&+\sum_{i=0}^{m}\omega'_i\int_0^1 Q'_i(\alpha'_i)\frac{\partial^{\alpha'_i}V_1}{\partial|x_i|^{\alpha'_i}}d\alpha'_i + \sum_{\substack{j=1\\ i=j}}^{n}\omega'_{m+j}\int_0^1 Q'_{m+j}(\alpha'_{m+j})\frac{\partial^{\alpha'_{m+j}}V_1}{\partial|x_{m+j}|^{\alpha'_{m+j}}}d\alpha'_{m+j} + F(X_i,X_{m+j},\theta)\Bigg)\Bigg)\\
&-\Bigg(V_0(X_i,X_{m+j}) + J_\theta\Bigg(\sum_{i=0}^{m}\omega_i\int_1^2 Q_i(\alpha_i)\frac{\partial^{\alpha_i}V_2}{\partial|x_i|^{\alpha_i}}d\alpha_i + \sum_{\substack{j=1\\ i=j}}^{n}\omega_{m+j}\int_1^2 Q_{m+j}(\alpha_{m+j})\frac{\partial^{\alpha_{m+j}}V_2}{\partial|x_{m+j}|^{\alpha_{m+j}}}d\alpha_{m+j}\\
&+\sum_{i=0}^{m}\omega'_i\int_0^1 Q'_i(\alpha'_i)\frac{\partial^{\alpha'_i}V_2}{\partial|x_i|^{\alpha'_i}}d\alpha'_i + \sum_{\substack{j=1\\ i=j}}^{n}\omega'_{m+j}\int_0^1 Q'_{m+j}(\alpha'_{m+j})\frac{\partial^{\alpha'_{m+j}}V_2}{\partial|x_{m+j}|^{\alpha'_{m+j}}}d\alpha'_{m+j} + F(X_i,X_{m+j},\theta)\Bigg)\Bigg)\Bigg|\\
\le J_\theta\Bigg(&\sum_{i=0}^{m}|\omega_i|\int_1^2 |Q_i(\alpha_i)|\left|\frac{\partial^{\alpha_i}V_1}{\partial|x_i|^{\alpha_i}}-\frac{\partial^{\alpha_i}V_2}{\partial|x_i|^{\alpha_i}}\right|d\alpha_i + \sum_{\substack{j=1\\ i=j}}^{n}|\omega_{m+j}|\int_1^2 |Q_{m+j}(\alpha_{m+j})|\left|\frac{\partial^{\alpha_{m+j}}V_1}{\partial|x_{m+j}|^{\alpha_{m+j}}}-\frac{\partial^{\alpha_{m+j}}V_2}{\partial|x_{m+j}|^{\alpha_{m+j}}}\right|d\alpha_{m+j}\\
&+\sum_{i=0}^{m}|\omega'_i|\int_0^1 |Q'_i(\alpha'_i)|\left|\frac{\partial^{\alpha'_i}V_1}{\partial|x_i|^{\alpha'_i}}-\frac{\partial^{\alpha'_i}V_2}{\partial|x_i|^{\alpha'_i}}\right|d\alpha'_i + \sum_{\substack{j=1\\ i=j}}^{n}|\omega'_{m+j}|\int_0^1 |Q'_{m+j}(\alpha'_{m+j})|\left|\frac{\partial^{\alpha'_{m+j}}V_1}{\partial|x_{m+j}|^{\alpha'_{m+j}}}-\frac{\partial^{\alpha'_{m+j}}V_2}{\partial|x_{m+j}|^{\alpha'_{m+j}}}\right|d\alpha'_{m+j}\Bigg)\\
\le J_\theta\Bigg(&\sum_{i=0}^{m}|\omega_i|\chi_{\alpha_i}M_{\alpha_i} + \sum_{\substack{j=1\\ i=j}}^{n}|\omega_{m+j}|\chi_{\alpha_{m+j}}M_{\alpha_{m+j}} + \sum_{i=0}^{m}|\omega'_i|\chi_{\alpha'_i}M_{\alpha'_i} + \sum_{\substack{j=1\\ i=j}}^{n}|\omega'_{m+j}|\chi_{\alpha'_{m+j}}M_{\alpha'_{m+j}}\Bigg)|V_1-V_2|\\
\le \Theta\Bigg(&\sum_{i=0}^{m}|\omega_i|\chi_{\alpha_i}M_{\alpha_i} + \sum_{\substack{j=1\\ i=j}}^{n}|\omega_{m+j}|\chi_{\alpha_{m+j}}M_{\alpha_{m+j}} + \sum_{i=0}^{m}|\omega'_i|\chi_{\alpha'_i}M_{\alpha'_i} + \sum_{\substack{j=1\\ i=j}}^{n}|\omega'_{m+j}|\chi_{\alpha'_{m+j}}M_{\alpha'_{m+j}}\Bigg)\|V_1-V_2\|_\infty.
\end{aligned} \tag{29}$$

Hence, Σ is the contraction due to the condition (27). By Banach contraction principle, we can deduce that Σ has an unique fixed point which is the unique solution of the problem (15, 16). ☐

Proof We subdivide the proof into several steps.

Step 1: The map Σ is continuous.

Let V_ζ be a sequence such that $V_\zeta \to V$ in $C(\psi, \mathbb{R})$. Then for each $(X_i, X_{m+j}, \theta) \in \psi$, we have

$$
\begin{aligned}
|\Sigma V_\zeta - \Sigma V| = \Bigg| \Bigg(& V_0(X_i, X_{m+j}) + J_\theta \Bigg(\sum_{i=0}^{m} \omega_i \int_1^2 Q_i(\alpha_i) \frac{\partial^{\alpha_i} V_\zeta}{\partial |x_i|^{\alpha_i}} d\alpha_i \\
& + \sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j} \int_1^2 Q_{m+j}(\alpha_{m+j}) \frac{\partial^{\alpha_{m+j}} V_\zeta}{\partial |x_{m+j}|^{\alpha_{m+j}}} d\alpha_{m+j} + \sum_{i=0}^{m} \omega'_i \int_0^1 Q'_i(\alpha'_i) \frac{\partial^{\alpha'_i} V_\zeta}{\partial |x_i|^{\alpha'_i}} d\alpha'_i \\
& + \sum_{\substack{j=1 \\ i=j}}^{n} \omega'_{m+j} \int_0^1 Q'_{m+j}(\alpha'_{m+j}) \frac{\partial^{\alpha'_{m+j}} V_\zeta}{\partial |x_{m+j}|^{\alpha'_{m+j}}} d\alpha'_{m+j} + F(X_i, X_{m+j}, \theta) \Bigg) \Bigg) \\
& - \Bigg(V_0(X_i, X_{m+j}) + J_\theta \Bigg(\sum_{i=0}^{m} \omega_i \int_1^2 Q_i(\alpha_i) \frac{\partial^{\alpha_i} V}{\partial |x_i|^{\alpha_i}} d\alpha_i \\
& + \sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j} \int_1^2 Q_{m+j}(\alpha_{m+j}) \frac{\partial^{\alpha_{m+j}} V}{\partial |x_{m+j}|^{\alpha_{m+j}}} d\alpha_{m+j} + \sum_{i=0}^{m} \omega'_i \int_0^1 Q'_i(\alpha'_i) \frac{\partial^{\alpha'_i} V}{\partial |x_i|^{\alpha'_i}} d\alpha'_i \\
& + \sum_{\substack{j=1 \\ i=j}}^{n} \omega'_{m+j} \int_0^1 Q'_{m+j}(\alpha'_{m+j}) \frac{\partial^{\alpha'_{m+j}} V}{\partial |x_{m+j}|^{\alpha'_{m+j}}} d\alpha'_{m+j} + F(X_i, X_{m+j}, \theta) \Bigg) \Bigg) \Bigg| \\
\leq \Theta \Bigg(& \sum_{i=0}^{m} |\omega_i| \chi_{\alpha_i} M_{\alpha_i} + \sum_{\substack{j=1 \\ i=j}}^{n} |\omega_{m+j}| \chi_{\alpha_{m+j}} M_{\alpha_{m+j}} + \sum_{i=0}^{m} \omega'_i \chi_{\alpha'_i} M_{\alpha'_i} + \sum_{\substack{j=1 \\ i=j}}^{n} |\omega'_{m+j} \chi_{\alpha'_{m+j}} M_{\alpha'_{m+j}} \Bigg) \| V_\zeta - V \|_\infty .
\end{aligned}
\tag{30}
$$

Our second result is based on the Schaefer's fixed point theorem.

Theorem 5 *Assume that* $(S1) - -(S6)$ *hold. Then, the problem* (15, 16) *has at least one solution on* $C(\psi, \mathbb{R})$.

Since Σ is continuous, we imply that $\|\Sigma V_\zeta - \Sigma V\|_\infty \to 0$ as $\zeta \to \infty$.

Step 2: The map Σ maps bounded sets into bounded sets in $C(\psi, \mathbb{R})$.

Let $\pi_\epsilon := \{ V \in C(\psi, \mathbb{R}) : \|V\|_\infty \leq \epsilon, \epsilon > 0 \}$, $\|V\|_\infty := \{ \sup |V|; (X_i, X_{m+j}, \theta) \in \psi \}$. We have, for all $V \in \pi_\epsilon$,

$$
\begin{aligned}
|\Sigma V(X_i, X_{m+j}, \theta)| &= \Bigg| V_0(X_i, X_{m+j}) + J_\theta\Bigg(\sum_{i=0}^{m} \omega_i \int_1^2 Q_i(\alpha_i)\frac{\partial^{\alpha_i} V}{\partial |x_i|^{\alpha_i}} d\alpha_i \\
&+ \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j} \int_1^2 Q_{m+j}(\alpha_{m+j})\frac{\partial^{\alpha_{m+j}} V}{\partial |x_{m+j}|^{\alpha_{m+j}}} d\alpha_{m+j} + \sum_{i=0}^{m} \omega'_i \int_0^1 Q'_i(\alpha'_i)\frac{\partial^{\alpha'_i} V}{\partial |x_i|^{\alpha'_i}} d\alpha'_i \\
&+ \sum_{\substack{j=1\\ i=j}}^{n} \omega'_{m+j} \int_0^1 Q'_{m+j}(\alpha'_{m+j})\frac{\partial^{\alpha'_{m+j}} V}{\partial |x_{m+j}|^{\alpha'_{m+j}}} d\alpha'_{m+j} + F(X_i, X_{m+j}, \theta)\Bigg)\Bigg| \\
&\le |V_0(X_i, X_{m+j})| + J_\theta\Bigg(\sum_{i=0}^{m} |\omega_i| \int_1^2 |Q_i(\alpha_i)|\left|\frac{\partial^{\alpha_i} V}{\partial |x_i|^{\alpha_i}}\right| d\alpha_i \\
&+ \sum_{\substack{j=1\\ i=j}}^{n} |\omega_{m+j}| \int_1^2 |Q_{m+j}(\alpha_{m+j})|\left|\frac{\partial^{\alpha_{m+j}} V}{\partial |x_{m+j}|^{\alpha_{m+j}}}\right| d\alpha_{m+j} + \sum_{i=0}^{m} |\omega'_i| \int_0^1 |Q'_i(\alpha'_i)|\left|\frac{\partial^{\alpha'_i} V}{\partial |x_i|^{\alpha'_i}}\right| d\alpha'_i \\
&+ \sum_{\substack{j=1\\ i=j}}^{n} |\omega'_{m+j}| \int_0^1 |Q'_{m+j}(\alpha'_{m+j})|\left|\frac{\partial^{\alpha'_{m+j}} V}{\partial |x_{m+j}|^{\alpha'_{m+j}}}\right| d\alpha'_{m+j} + |F(X_i, X_{m+j}, \theta)|\Bigg) \\
&\le B_2 + +J_\theta\Bigg(\sum_{i=0}^{m} |\omega_i|\chi_{\alpha_i}\rho_{\alpha_i}|V| + \sum_{\substack{j=1\\ i=j}}^{n} |\omega_{m+j}|\chi_{\alpha_{m+j}}\rho_{\alpha_{m+j}}|V| + \sum_{i=0}^{m} |\omega'_i|\chi_{\alpha'_i}\rho_{\alpha'_i}|V| + \sum_{\substack{j=1\\ i=j}}^{n} |\omega'_{m+j}|\chi_{\alpha'_{m+j}}\rho_{\alpha'_{m+j}}|V| + B_1\Bigg) \\
&\le B_2 + \theta\Bigg(\sum_{i=0}^{m} |\omega_i|\chi_{\alpha_i}\rho_{\alpha_i} + \sum_{\substack{j=1\\ i=j}}^{n} |\omega_{m+j}|\chi_{\alpha_{m+j}}\rho_{\alpha_{m+j}} + \sum_{i=0}^{m} |\omega'_i|\chi_{\alpha'_i}\rho_{\alpha'_i} + \sum_{\substack{j=1\\ i=j}}^{n} |\omega'_{m+j}|\chi_{\alpha'_{m+j}}\rho_{\alpha'_{m+j}}\Bigg)|V| + \theta B_1.
\end{aligned}
\tag{31}
$$

Hence,

$$
\begin{aligned}
|\Sigma V(X_i, X_{m+j}, \theta)| &\le B_2 + \Theta\Bigg(\sum_{i=0}^{m} |\omega_i|\chi_{\alpha_i}\rho_{\alpha_i} \\
&+ \sum_{\substack{j=1\\ i=j}}^{n} |\omega_{m+j}|\chi_{\alpha_{m+j}}\rho_{\alpha_{m+j}} + \sum_{i=0}^{m} |\omega'_i|\chi_{\alpha'_i}\rho_{\alpha'_i} \\
&+ \sum_{\substack{j=1\\ i=j}}^{n} |\omega'_{m+j}|\chi_{\alpha'_{m+j}}\rho_{\alpha'_{m+j}}\Bigg)\epsilon + \Theta B_1 \ := C.
\end{aligned}
\tag{32}
$$

Finally, we observe

$$
\left\| \Sigma V(X_i, X_{m+j}, \theta) \right\|_\infty < \infty, \forall (X_i, X_{m+j}, \theta) \in \psi. \tag{33}
$$

Therefore, we conclude that the operator Σ maps bounded sets into bounded sets in $C(\psi, \mathbb{R})$.

Step 3: The map Σ maps bounded sets into equicontinuous sets of $C(\psi, \mathbb{R})$.

Let us consider $V \in \pi_\epsilon, (X_i, X_{m+j}, \theta) \in \psi$, where $i = 0, 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$ and $X'_i < X''_i$, $X'_{m+j} < X''_{m+j}$, $\theta' < \theta''$. Then,

$$
\begin{aligned}
&|\Sigma V(X_i', X_{m+j}', \theta') - \Sigma V(X_i'', X_{m+j}'', \theta'')| = \Bigg|\Bigg(V_0(X_i', X_{m+j}') \\
&+ J_{\theta'}\Bigg(\sum_{i=0}^{m}\omega_i \int_1^2 Q_i(\alpha_i)\frac{\partial^{\alpha_i}V}{\partial|x_i'|^{\alpha_i}}d\alpha_i + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}\int_1^2 Q_{m+j}(\alpha_{m+j})\frac{\partial^{\alpha_{m+j}}V}{\partial|x_{m+j}'|^{\alpha_{m+j}}}d\alpha_{m+j} \\
&+ \sum_{i=0}^{m}\omega_i' \int_0^1 Q_i'(\alpha_i')\frac{\partial^{\alpha_i'}V}{\partial|x_i'|^{\alpha_i'}}d\alpha_i' + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}'\int_0^1 Q_{m+j}'(\alpha_{m+j}')\frac{\partial^{\alpha_{m+j}'}V}{\partial|x_{m+j}'|^{\alpha_{m+j}'}}d\alpha_{m+j}' + F(X_i', X_{m+j}', \theta')\Bigg)\Bigg) \\
&- \Bigg(V_0(X_i'', X_{m+j}'') + J_{\theta''}\Bigg(\sum_{i=0}^{m}\omega_i \int_1^2 Q_i(\alpha_i)\frac{\partial^{\alpha_i}V}{\partial|x_i''|^{\alpha_i}}d\alpha_i + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}\int_1^2 Q_{m+j}(\alpha_{m+j})\frac{\partial^{\alpha_{m+j}}V}{\partial|x_{m+j}''|^{\alpha_{m+j}}}d\alpha_{m+j} \\
&+ \sum_{i=0}^{m}\omega_i' \int_0^1 Q_i'(\alpha_i')\frac{\partial^{\alpha_i'}V}{\partial|x_i''|^{\alpha_i'}}d\alpha_i' + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}'\int_0^1 Q_{m+j}'(\alpha_{m+j}')\frac{\partial^{\alpha_{m+j}'}V}{\partial|x_{m+j}''|^{\alpha_{m+j}'}}d\alpha_{m+j}' + F(X_i'', X_{m+j}'', \theta'')\Bigg)\Bigg)\Bigg| \\
&\le |V_0(X_i', X_{m+j}') - V_0(X_i'', X_{m+j}'')| + \Bigg|J_{\theta'}\Bigg(\sum_{i=0}^{m}\omega_i \int_1^2 Q_i(\alpha_i)\frac{\partial^{\alpha_i}V}{\partial|x_i'|^{\alpha_i}}d\alpha_i \\
&+ \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}\int_1^2 Q_{m+j}(\alpha_{m+j})\frac{\partial^{\alpha_{m+j}}V}{\partial|x_{m+j}'|^{\alpha_{m+j}}}d\alpha_{m+j} + \sum_{i=0}^{m}\omega_i' \int_0^1 Q_i'(\alpha_i')\frac{\partial^{\alpha_i'}V}{\partial|x_i'|^{\alpha_i'}}d\alpha_i' \\
&+ \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}'\int_0^1 Q_{m+j}'(\alpha_{m+j}')\frac{\partial^{\alpha_{m+j}'}V}{\partial|x_{m+j}'|^{\alpha_{m+j}'}}d\alpha_{m+j}'\Bigg) - J_{\theta''}\Bigg(\sum_{i=0}^{m}\omega_i \int_1^2 Q_i(\alpha_i)\frac{\partial^{\alpha_i}V}{\partial|x_i''|^{\alpha_i}}d\alpha_i \\
&+ \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}\int_1^2 Q_{m+j}(\alpha_{m+j})\frac{\partial^{\alpha_{m+j}}V}{\partial|x_{m+j}''|^{\alpha_{m+j}}}d\alpha_{m+j} + \sum_{i=0}^{m}\omega_i' \int_0^1 Q_i'(\alpha_i')\frac{\partial^{\alpha_i'}V}{\partial|x_i''|^{\alpha_i'}}d\alpha_i' \\
&+ \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}'\int_0^1 Q_{m+j}'(\alpha_{m+j}')\frac{\partial^{\alpha_{m+j}'}V}{\partial|x_{m+j}''|^{\alpha_{m+j}'}}d\alpha_{m+j}'\Bigg)\Bigg| + J_{\theta'}|F(X_i', X_{m+j}', \theta')| - J_{\theta''}|F(X_i'', X_{m+j}'', \theta'')| \\
&\le \Theta\Bigg(\Bigg(\sum_{i=0}^{m}|\omega_i|\chi_{\alpha_i}\mu_{\alpha_i}(|x_i'| - |x_i''|) + \sum_{\substack{j=1\\ i=j}}^{n} |\omega_{m+j}|\chi_{\alpha_{m+j}}\mu_{\alpha_{m+j}}(|x_{m+j}'| - |x_{m+j}''|) \\
&+ \sum_{i=0}^{m}|\omega_i'|\chi_{\alpha_i'}\mu_{\alpha_i'}(|x_i'| - |x_i''|) + \sum_{\substack{j=1\\ i=j}}^{n} |\omega_{m+j}'|\chi_{\alpha_{m+j}'}\mu_{\alpha_{m+j}'}(|x_{m+j}'| - |x_{m+j}''|)\Bigg).
\end{aligned} \tag{34}
$$

This implies that $\|\Sigma V(X_i', X_{m+j}', \theta') - \Sigma V(X_i'', X_{m+j}'', \theta'')\|_\infty \to 0$ is as $X_i' \to X_i''$, $X_{m+j}' \to X_{m+j}''$, $\theta' \to \theta''$, the right-hand side of the above inequality tends to zero. As a direct consequence of the steps mentioned above along with the Arzelà-Ascoli Theorem, we conclude that $\Sigma : (\psi, \mathbb{R}) \to C(\psi, \mathbb{R})$ is continuous and completely continuous.

Step 4: The map Σ is priori bound.

Finally, we show that the set

$O := \left\{ V \in C(\psi, \mathbb{R}); V = \varepsilon\Sigma(V) \text{ for some } 0 < \varepsilon < 1 \right\}$

is bounded.

Let $V \in O$, then

$$\begin{aligned} |V(X_i, X_{m+j}, \theta)| &= |\varepsilon\Sigma(V(X_i, X_{m+j}, \theta))| \\ &\le \varepsilon \times \Bigg| V_0(X_i, X_{m+j}) \\ &\quad + J_\theta\Bigg(\sum_{i=0}^{m} \omega_i \int_1^2 Q_i(\alpha_i) \frac{\partial^{\alpha_i} V}{\partial |x_i|^{\alpha_i}} d\alpha_i \\ &\quad + \sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j} \int_1^2 Q_{m+j}(\alpha_{m+j}) \frac{\partial^{\alpha_{m+j}} V}{\partial |x_{m+j}|^{\alpha_{m+j}}} d\alpha_{m+j} \\ &\quad + \sum_{i=0}^{m} \omega_i' \int_0^1 Q_i'(\alpha_i') \frac{\partial^{\alpha_i'} V}{\partial |x_i|^{\alpha_i'}} d\alpha_i' \\ &\quad + \sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j}' \int_0^1 Q_{m+j}'(\alpha_{m+j}') \frac{\partial^{\alpha_{m+j}'} V}{\partial |x_{m+j}|^{\alpha_{m+j}'}} d\alpha_{m+j}' \\ &\quad + F(X_i, X_{m+j}, \theta) \Bigg) \Bigg|. \end{aligned} \tag{35}$$

From the Eq. (32), we obtain the following expression:

$$\begin{aligned} |V(X_i, X_{m+j}, \theta)| &\le \varepsilon \times \Bigg(B_2 + \Theta\Bigg(\sum_{i=0}^{m} |\omega_i| \chi_{\alpha_i} \rho_{\alpha_i} \\ &\quad + \sum_{\substack{j=1 \\ i=j}}^{n} |\omega_{m+j}| \chi_{\alpha_{m+j}} \rho_{\alpha_{m+j}} + \sum_{i=0}^{m} |\omega_i'| \chi_{\alpha_i \alpha_i'} \rho_{\alpha_i \alpha_i'} \\ &\quad + \sum_{\substack{j=1 \\ i=j}}^{n} |\omega_{m+j}'| \chi_{\alpha_{m+j}'} \rho_{\alpha_{m+j}'} \Bigg) \epsilon + \Theta B_1 \Bigg). \end{aligned} \tag{36}$$

Hence, $\|V\|_\infty < \infty$.

The above proof shows that O is bounded. As a consequence of the Schaefer fixed point theorem, we deduce that V has a fixed point which is a solution of the problem on $C(\psi, \mathbb{R})$. □

4 Description of the TSADM

Considering the Eqs. (15, 16), we describe the TSADM as follows:

The operator form of the Eq. (15) can be written as

$$\begin{aligned} D_\theta V(X_i, X_{m+j}, \theta) &= \sum_{i=0}^{m} \omega_i \int_1^2 Q_i(\alpha_i) \frac{\partial^{\alpha_i} V}{\partial |x_i|^{\alpha_i}} d\alpha_i \\ &\quad + \sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j} \int_1^2 Q_{m+j}(\alpha_{m+j}) \frac{\partial^{\alpha_{m+j}} V}{\partial |x_{m+j}|^{\alpha_{m+j}}} d\alpha_{m+j} \\ &\quad + \sum_{i=0}^{m} \omega_i' \int_0^1 Q_i'(\alpha_i') \frac{\partial^{\alpha_i'} V}{\partial |x_i|^{\alpha_i'}} d\alpha_i' \\ &\quad + \sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j}' \int_0^1 Q_{m+j}'(\alpha_{m+j}') \frac{\partial^{\alpha_{m+j}'} V}{\partial |x_{m+j}|^{\alpha_{m+j}'}} d\alpha_{m+j}' \\ &\quad + F(X_i, X_{m+j}, \theta). \end{aligned} \tag{37}$$

Applying the inverse operator J_θ of D_θ into Eq. (37), we obtain

$$\begin{aligned} V(X_i, X_{m+j}, \theta) &= V_0(X_i, X_{m+j}) \\ &\quad + J_\theta\Bigg(\sum_{i=0}^{m} \omega_i \int_1^2 Q_i(\alpha_i) \frac{\partial^{\alpha_i} V}{\partial |x_i|^{\alpha_i}} d\alpha_i \\ &\quad + \sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j} \int_1^2 Q_{m+j}(\alpha_{m+j}) \frac{\partial^{\alpha_{m+j}} V}{\partial |x_{m+j}|^{\alpha_{m+j}}} d\alpha_{m+j} \\ &\quad + \sum_{i=0}^{m} \omega_i' \int_0^1 Q_i'(\alpha_i') \frac{\partial^{\alpha_i'} V}{\partial |x_i|^{\alpha_i'}} d\alpha_i' \\ &\quad + \sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j}' \int_0^1 Q_{m+j}'(\alpha_{m+j}') \frac{\partial^{\alpha_{m+j}'} V}{\partial |x_{m+j}|^{\alpha_{m+j}'}} d\alpha_{m+j}' \\ &\quad + F(X_i, X_{m+j}, \theta) \Bigg). \end{aligned} \tag{38}$$

The recursion formula for the TSADM from the Eq. (38) is given as

$$\begin{aligned} V_0(X_i, X_{m+j}, \theta) &= V_0(X_i, X_{m+j}) \\ &\quad + J_\theta\Big(F(X_i, X_{m+j}, \theta) \Big), \end{aligned} \tag{39}$$

and

$$V_{p+1}(X_i, X_{m+j}, \theta) = J_\theta\Bigg(\sum_{i=0}^{m} \omega_i \int_1^2 Q_i(\alpha_i) \frac{\partial^{\alpha_i} V_p}{\partial |x_i|^{\alpha_i}} d\alpha_i + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j} \int_1^2 Q_{m+j}(\alpha_{m+j}) \frac{\partial^{\alpha_{m+j}} V_p}{\partial |x_{m+j}|^{\alpha_{m+j}}} d\alpha_{m+j} + \sum_{i=0}^{m} \omega_i' \int_0^1 Q_i'(\alpha_i') \frac{\partial^{\alpha_i'} V_p}{\partial |x_i|^{\alpha_i'}} d\alpha_i' + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}' \int_0^1 Q_{m+j}'(\alpha_{m+j}') \frac{\partial^{\alpha_{m+j}'} V_p}{\partial |x_{m+j}|^{\alpha_{m+j}'}} d\alpha_{m+j}' \Bigg), \tag{40}$$

where $p = 0, 1, 2, \cdots$.

The first iteration for the Eq. (39) can be split into several components as

$$V_0 = \delta_0 + \delta_1 + \delta_2 + \ldots + \delta_M = \delta, \tag{41}$$

where $\delta_0, \delta_1, \delta_2, \cdots, \delta_M$ are the terms obtained from integrating the source term F and from the given conditions.

If we choose any of the terms from Eq. (41) as V_0 and it satisfies the main Eq. (15) and the given conditions, then we get the exact solution and terminate the process. If V_0 does not satisfy the Eq. (15), then we choose another term and repeat the process until we get the exact solution. If any term in the Eq. (41) does not satisfy either the given condition or the Eq. (15), then we will go to the next step of applying the ADM to obtain the solution by choosing $V_0 = \delta$. Since this method involves two steps, this method is known as the two-step Adomian decomposition method (TSADM).

If the present method is applicable to the problem, then the obtained solution is an analytical solution of the problem. The only limitation in this method is that the first term (zeroth term) of the series contains verifying the term, which satisfies the considered equation and associated with the initial and boundary conditions. If there is no such term involved in the zeroth term of the series, then we will obtain a semi-analytical solution with its next step. For the next step, we apply the ADM on the considered problem and obtain the approximate solution without use of linearization and discrtization. Moreover, the TSADM is a powerful and efficient method for such types of the problem in caparison with other numerical methods without linearization, discretization and Adomain polynomial.

5 Applications

In this section, we consider a generalized example to show the efficiency and applicability of the method in the present article. We compare the obtained results with the other existing numerical methods and draw a conclusion on the basis of the obtained result.

Example Consider the multi-dimensional Riesz space distributed-order advection–diffusion equation, described as follows:

$$\frac{\partial V(X_i, X_{m+j}, \theta)}{\partial \theta} = \sum_{i=0}^{m} \omega_i \int_1^2 Q_i(\alpha_i) \frac{\partial^{\alpha_i} V}{\partial |x_i|^{\alpha_i}} d\alpha_i + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j} \int_1^2 Q_{m+j}(\alpha_{m+j}) \frac{\partial^{\alpha_{m+j}} V}{\partial |x_{m+j}|^{\alpha_{m+j}}} d\alpha_{m+j} + \sum_{i=0}^{m} \omega_i' \int_0^1 Q_i'(\alpha_i') \frac{\partial^{\alpha_i'} V}{\partial |x_i|^{\alpha_i'}} d\alpha_i' + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}' \int_0^1 Q_{m+j}'(\alpha_{m+j}') \frac{\partial^{\alpha_{m+j}'} V}{\partial |x_{m+j}|^{\alpha_{m+j}'}} d\alpha_{m+j}' + F(X_i, X_{m+j}, \theta), (X_i, X_{m+j}, \theta) \in \Omega \times I, \tag{42}$$

with the boundary and initial conditions

$$V(X_i, X_{m+j}, \theta) = 0, (X_i, X_{m+j}, \theta) \in \partial\Omega \times I,$$
$$V(X_i, X_{m+j}, 0) = \prod_{i=0}^{m} x_i^{2}(1 - x_i)^2 \times \prod_{\substack{j=1\\ i=j}}^{n} x_{m+j}^2 (1 - x_{m+j})^2, (X_i, X_{m+j}) \in \Omega\ , \tag{43}$$

where $I = (0, 1]$, $\Omega = \prod_{i=0}^{m}(0, 1) \times \prod_{\substack{j=1\\ i=j}}^{n} (0, 1)$,

$Q_i(\alpha_i) = -2\Gamma(5 - \alpha_i)\cos\frac{\pi\alpha_i}{2}$, $Q_i'(\alpha_i') = -2\Gamma(5 - \alpha_i')\cos\frac{\pi\alpha_i'}{2}$, $Q_{m+j}(\alpha_{m+j}) = -2\Gamma(5 - \alpha_{m+j})\cos\frac{\pi\alpha_{m+j}}{2}$, $Q_{m+j}'(\alpha_{m+j}') = -2\Gamma(5 - \alpha_{m+j}')\cos\frac{\pi\alpha_{m+j}'}{2}$.

The source term is given by

$$F(X_i, X_{m+j}, \theta) = e^{\theta} \prod_{i=0}^{m} {x_i}^2 (1-x_i)^2 \times \prod_{\substack{j=1\\ i=j}}^{n} x_{m+j}^2 (1-x_{m+j})^2 - e^{\theta} \prod_{\substack{j=1\\ i=j}}^{n} x_{m+j}^2 (1-x_{m+j})^2 \left[\sum_{i=0}^{m} \omega_i \Big(G_1(x_i) + G_1(1-x_i) \Big) + \sum_{i=0}^{m} \omega_i' \Big(G_2(x_i) + G_2(1-x_i) \Big) \right] - e^{\theta} \prod_{i=0}^{m} {x_i}^2 (1-x_i)^2 \left[\sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j} \Big(G_1(x_{m+j}) + G_1(1-x_{m+j}) \Big) + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}' \Big(G_2(x_{m+j}) + G_2(1-x_{m+j}) \Big) \right], \tag{44}$$

where

$$G_1(s) = \Gamma(5)\frac{s^3 - s^2}{\ln s} - 2\Gamma(4)\left[\frac{3s^2 - 2s}{\ln s} - \frac{s^2 - s}{(\ln s)^2}\right] + \frac{\Gamma(3)}{\ln s}\left[6s - 2 - \frac{5s}{\ln s} + \frac{3}{\ln s} + \frac{2s}{(\ln s)^2} - \frac{2}{(\ln s)^2}\right],$$

and

$$G_2(s) = \Gamma(5)\frac{s^4 - s^3}{\ln s} - 2\Gamma(4)\left[\frac{4s^3 - 3s^2}{\ln s} - \frac{s^3 - s^2}{(\ln s)^2}\right] + \frac{\Gamma(3)}{\ln s}\left[12s^2 - 6s - \frac{1}{\ln s}\left(7s^2 - 5s - \frac{2s^2}{\ln s} + \frac{2s}{\ln s}\right)\right].$$

The exact solution of the Eq. (42) is $e^{\theta} \prod_{i=0}^{m} {x_i}^2 (1-x_i)^2 \times \prod_{\substack{j=1\\ i=j}}^{n} x_{m+j}^2 (1-x_{m+j})^2$.

The operator form of the Eq. (42) can be written as

$$D_\theta V(X_i, X_{m+j}, \theta) = \sum_{i=0}^{m} \omega_i \int_1^2 Q_i(\alpha_i) \frac{\partial^{\alpha_i} V}{\partial |x_i|^{\alpha_i}} d\alpha_i + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j} \int_1^2 Q_{m+j}(\alpha_{m+j}) \frac{\partial^{\alpha_{m+j}} V}{\partial |x_{m+j}|^{\alpha_{m+j}}} d\alpha_{m+j} + \sum_{i=0}^{m} \omega_i' \int_0^1 Q_i'(\alpha_i') \frac{\partial^{\alpha_i'} V}{\partial |x_i|^{\alpha_i'}} d\alpha_i' + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}' \int_0^1 Q_{m+j}'(\alpha_{m+j}') \frac{\partial^{\alpha_{m+j}'} V}{\partial |x_{m+j}|^{\alpha_{m+j}'}} d\alpha_{m+j}' + F(X_i, X_{m+j}, \theta). \tag{45}$$

On applying the inverse operator J_θ of D_θ into the Eq. (45), we obtain

$$V(X_i, X_{m+j}, \theta) = V_0(X_i, X_{m+j}) + J_\theta\Bigg(\sum_{i=0}^{m} \omega_i \int_1^2 Q_i(\alpha_i) \frac{\partial^{\alpha_i} V}{\partial |x_i|^{\alpha_i}} d\alpha_i + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j} \int_1^2 Q_{m+j}(\alpha_{m+j}) \frac{\partial^{\alpha_{m+j}} V}{\partial |x_{m+j}|^{\alpha_{m+j}}} d\alpha_{m+j} + \sum_{i=0}^{m} \omega_i' \int_0^1 Q_i'(\alpha_i') \frac{\partial^{\alpha_i'} V}{\partial |x_i|^{\alpha_i'}} d\alpha_i' + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}' \int_0^1 Q_{m+j}'(\alpha_{m+j}') \frac{\partial^{\alpha_{m+j}'} V}{\partial |x_{m+j}|^{\alpha_{m+j}'}} d\alpha_{m+j}' + F(X_i, X_{m+j}, \theta)\Bigg). \tag{46}$$

The recursion formula for the TSADM from Eq. (46) is

$$V_0(X_i, X_{m+j}, \theta) = V_0(X_i, X_{m+j}) + J_\theta\Big(F(X_i, X_{m+j}, \theta) \Big), \tag{47}$$

and

$$V_{p+1}(X_i, X_{m+j}, \theta) = J_\theta\Bigg(\sum_{i=0}^{m} \omega_i \int_1^2 Q_i(\alpha_i) \frac{\partial^{\alpha_i} V_p}{\partial |x_i|^{\alpha_i}} d\alpha_i + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j} \int_1^2 Q_{m+j}(\alpha_{m+j}) \frac{\partial^{\alpha_{m+j}} V_p}{\partial |x_{m+j}|^{\alpha_{m+j}}} d\alpha_{m+j} + \sum_{i=0}^{m} \omega_i' \int_0^1 Q_i'(\alpha_i') \frac{\partial^{\alpha_i'} V_p}{\partial |x_i|^{\alpha_i'}} d\alpha_i' + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}' \int_0^1 Q_{m+j}'(\alpha_{m+j}') \frac{\partial^{\alpha_{m+j}'} V_p}{\partial |x_{m+j}|^{\alpha_{m+j}'}} d\alpha_{m+j}' \Bigg), \tag{48}$$

where $p = 0, 1, 2, \cdots$.

By solving Eq. (47), we obtain

$$V_0 = \prod_{i=0}^{m} {x_i}^2(1-x_i)^2 \times \prod_{\substack{j=1\\ i=j}}^{n} x_{m+j}^2(1-x_{m+j})^2 + e^\theta \prod_{i=0}^{m} {x_i}^2(1-x_i)^2 \times \prod_{\substack{j=1\\ i=j}}^{n} {x_{m+j}}^2(1-x_{m+j})^2 - e^\theta \prod_{\substack{j=1\\ i=j}}^{n} x_{m+j}^2(1-x_{m+j})^2 \Bigg[\sum_{i=0}^{m} \omega_i \Big(G_1(x_i) + G_1(1-x_i) \Big) + \sum_{i=0}^{m} \omega_i' \Big(G_2(x_i) + G_2(1-x_i) \Big) \Bigg] - e^\theta \prod_{i=0}^{m} {x_i}^2(1-x_i)^2 \Bigg[\sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j} \Big(G_1(x_{m+j}) + G_1(1-x_{m+j}) \Big) + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}' \Big(G_2(x_{m+j}) + G_2(1-x_{m+j}) \Big) \Bigg]. \tag{49}$$

From the above Eq. (49), the first iteration (zeroth term) of the TSADM can be split into two parts as follows:

$$V_0 = \Lambda_0 + \Lambda_1, \tag{50}$$

where

$$\Lambda_0 = e^\theta \prod_{i=0}^{m} {x_i}^2(1-x_i)^2 \times \prod_{\substack{j=1\\ i=j}}^{n} x_{m+j}^2(1-x_{m+j})^2, \tag{51}$$

and

$$\Lambda_1 = \prod_{i=0}^{m} {x_i}^2(1-x_i)^2 \times \prod_{\substack{j=1\\ i=j}}^{n} x_{m+j}^2(1-x_{m+j})^2 - e^\theta \prod_{\substack{j=1\\ i=j}}^{n} x_{m+j}^2(1-x_{m+j})^2 \Bigg[\sum_{i=0}^{m} \omega_i \Big(G_1(x_i) + G_1(1-x_i) \Big) + \sum_{i=0}^{m} \omega_i' \Big(G_2(x_i) + G_2(1-x_i) \Big) \Bigg] - e^\theta \prod_{i=0}^{m} {x_i}^2(1-x_i)^2 \Bigg[\sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j} \Big(G_1(x_{m+j}) + G_1(1-x_{m+j}) \Big) + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}' \Big(G_2(x_{m+j}) + G_2(1-x_{m+j}) \Big) \Bigg]. \tag{52}$$

Here, we generalize our problem and discover the general analytical solution of the multi-dimensional Riesz space distributed-order advection–diffusion equation.

According to the TSADM process, we choose the first iteration as the term involved in the Eq. (50) and the term satisfies the problem and the given conditions, so we terminate the process and we obtain the solution of the problem.

Let us take $V_0 = \Lambda_0$ and check that the assumption of V_0 satisfying the Eq. (42) and also the given conditions. If this choice of V_0 is approved, then this implies that the chosen term is a solution to the problem.

Let us take $V_0 = \Lambda_0$ as a solution of Eq. (42), so that it satisfies the Eq. (42).

To prove $V_0 = \Lambda_0$ is the exact solution of Eq. (42), we substitute $V_0 = \Lambda_0$ in the left-hand side of the Eq. (42) and hence we obtain

$$\frac{\partial \Lambda_0}{\partial \theta} = e^{\theta} \prod_{i=0}^{m} {x_i}^2 (1-x_i)^2 \times \prod_{\substack{j=1 \\ i=j}}^{n} x_{m+j}^2 (1-x_{m+j})^2. \tag{53}$$

Now, we calculate the terms on the right-hand side of the Eq. (42) for Λ_0, by using Definition 5, as follows:

$$\begin{aligned} &\omega_i \int_1^2 Q_i(\alpha_i) \frac{\partial^{\alpha_i} \Lambda_0}{\partial |x_i|^{\alpha_i}} d\alpha_i + \omega_i' \int_0^1 Q_i'(\alpha_i') \frac{\partial^{\alpha_i'} \Lambda_0}{\partial |x_i|^{\alpha_i'}} d\alpha_i' \\ &= e^{\theta} x_{m+j}^2 (1-x_{m+j})^2 \left[\omega_i \Big(G_1(x_i) + G_1(1-x_i) \Big) + \omega_i' \Big(G_2(x_i) + G_2(1-x_i) \Big) \right], \end{aligned} \tag{54}$$

and

$$\begin{aligned} &\omega_{m+j} \int_1^2 Q_{m+j}(\alpha_{m+j}) \frac{\partial^{\alpha_{m+j}} \Lambda_0}{\partial |x_{m+j}|^{\alpha_{m+j}}} d\alpha_{m+j} + \omega_{m+j}' \int_0^1 Q_{m+j}'(\alpha_{m+j}') \frac{\partial^{\alpha_{m+j}'} \Lambda_0}{\partial |x_{m+j}|^{\alpha_{m+j}'}} d\alpha_{m+j}' \\ &= e^{\theta} {x_i}^2 (1-x_i)^2 \left[\omega_{m+j} \Big(G_1(x_{m+j}) + G_1(1-x_{m+j}) \Big) + \omega_{m+j}' \Big(G_2(x_{m+j}) + G_2(1-x_{m+j}) \Big) \right]. \end{aligned} \tag{55}$$

Taking summation on the both sides of the above equations, we get

$$\begin{aligned} &\sum_{i=0}^{m} \omega_i \int_1^2 Q_i(\alpha_i) \frac{\partial^{\alpha_i} \Lambda_0}{\partial |x_i|^{\alpha_i}} d\alpha_i + \sum_{i=0}^{m} \omega_i' \int_0^1 Q_i'(\alpha_i') \frac{\partial^{\alpha_i'} \Lambda_0}{\partial |x_i|^{\alpha_i'}} d\alpha_i' \\ &= e^{\theta} \prod_{\substack{j=1 \\ i=j}}^{n} x_{m+j}^2 (1-x_{m+j})^2 \left[\sum_{i=0}^{m} \omega_i \Big(G_1(x_i) + G_1(1-x_i) \Big) + \sum_{i=0}^{m} \omega_i' \Big(G_2(x_i) + G_2(1-x_i) \Big) \right], \end{aligned} \tag{56}$$

and

$$\begin{aligned} &\sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j} \int_1^2 Q_{m+j}(\alpha_{m+j}) \frac{\partial^{\alpha_{m+j}} \Lambda_0}{\partial |x_{m+j}|^{\alpha_{m+j}}} d\alpha_{m+j} + \sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j}' \int_0^1 Q_{m+j}'(\alpha_{m+j}') \frac{\partial^{\alpha_{m+j}'} \Lambda_0}{\partial |x_{m+j}|^{\alpha_{m+j}'}} d\alpha_{m+j}' \\ &= e^{\theta} \prod_{i=0}^{m} {x_i}^2 (1-x_i)^2 \left[\sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j} \Big(G_1(x_{m+j}) + G_1(1-x_{m+j}) \Big) + \sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j}' \Big(G_2(x_{m+j}) + G_2(1-x_{m+j}) \Big) \right]. \end{aligned} \tag{57}$$

On adding the above equations, we obtain

$$\begin{aligned} &e^{\theta} \prod_{\substack{j=1 \\ i=j}}^{n} x_{m+j}^2 (1-x_{m+j})^2 \left[\sum_{i=0}^{m} \omega_i \Big(G_1(x_i) + G_1(1-x_i) \Big) + \sum_{i=0}^{m} \omega_i' \Big(G_2(x_i) + G_2(1-x_i) \Big) \right] \\ &+ e^{\theta} \prod_{i=0}^{m} {x_i}^2 (1-x_i)^2 \left[\sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j} \Big(G_1(x_{m+j}) + G_1(1-x_{m+j}) \Big) + \sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j}' \Big(G_2(x_{m+j}) + G_2(1-x_{m+j}) \Big) \right] \\ &= \sum_{i=0}^{m} \omega_i \int_1^2 Q_i(\alpha_i) \frac{\partial^{\alpha_i} \Lambda_0}{\partial |x_i|^{\alpha_i}} d\alpha_i + \sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j} \int_1^2 Q_{m+j}(\alpha_{m+j}) \frac{\partial^{\alpha_{m+j}} \Lambda_0}{\partial |x_{m+j}|^{\alpha_{m+j}}} d\alpha_{m+j} \\ &+ \sum_{i=0}^{m} \omega_i' \int_0^1 Q_i'(\alpha_i') \frac{\partial^{\alpha_i'} \Lambda_0}{\partial |x_i|^{\alpha_i'}} d\alpha_i' + \sum_{\substack{j=1 \\ i=j}}^{n} \omega_{m+j}' \int_0^1 Q_{m+j}'(\alpha_{m+j}') \frac{\partial^{\alpha_{m+j}'} \Lambda_0}{\partial |x_{m+j}|^{\alpha_{m+j}'}} d\alpha_{m+j}'. \end{aligned} \tag{58}$$

By adding the source term $F(X_i, X_{m+j}, \theta)$ in the Eq. (58), we finally obtain the expression for the right side of the Eq. (42) as

$$e^{\theta} \prod_{i=0}^{m} x_i^2(1-x_i)^2 \times \prod_{\substack{j=1\\ i=j}}^{n} x_{m+j}^2(1-x_{m+j})^2 = \sum_{i=0}^{m} \omega_i \int_1^2 Q_i(\alpha_i) \frac{\partial^{\alpha_i}\Lambda_0}{\partial|x_i|^{\alpha_i}} d\alpha_i + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j} \int_1^2 Q_{m+j}(\alpha_{m+j}) \frac{\partial^{\alpha_{m+j}}\Lambda_0}{\partial|x_{m+j}|^{\alpha_{m+j}}} d\alpha_{m+j} + \sum_{i=0}^{m} \omega_i' \int_0^1 Q_i'(\alpha_i') \frac{\partial^{\alpha_i'}\Lambda_0}{\partial|x_i|^{\alpha_i'}} d\alpha_i' + \sum_{\substack{j=1\\ i=j}}^{n} \omega_{m+j}' \int_0^1 Q_{m+j}'(\alpha_{m+j}') \frac{\partial^{\alpha_{m+j}'}\Lambda_0}{\partial|x_{m+j}|^{\alpha_{m+j}'}} d\alpha_{m+j}' + F(X_i, X_{m+j}, \theta). \tag{59}$$

As we see that Eqs. (53) and (59) are the same, we conclude that the left-hand side of the Eq. (53) is equal to the right-hand side of the Eq. (59). The result obtained proves that $V_0 = \Lambda_0$ satisfies the Eq. (42) and the given conditions. Hence, the obtained solution is the exact solution of the problem by using the TSADM. For the solution of the considered problem, using numerical techniques is possible only on dimensions up to five. For higher dimension, we face difficulties because the process of the techniques needs approximation and discretization. The Riesz operator is involved in Eq. (42) with distributed integral, and the approximation method firstly deals with this operator by converting the multi-term Riesz operator after the solution with the help of approximation. All procedures involved in these methods increased the computation effort and required a large size of memory space with time complexity.

6 Conclusion

The two main objectives achieved in this article are described as follows:

1. Obtained the analytical solution of the multi-dimensional RSDOAD equation.
2. Established the new existence and uniqueness results for the multi-dimensional RSDOAD equation.

To address the applicability of the proposed method, we considered the generalized example and solved analytically. We obtained new results of existence and uniqueness employing with the fixed point theorem and the Banach contraction principle.

In this work, the proposed method was successfully applied to the problem. The obtained results prove that the TSADM is highly efficient and convenient for solving such types of problems. The TSADM was used for the first time to solve the multi-dimensional RSDOAD equation and the new condition for the existence and uniqueness for the problem was developed. If we use the numerical method [17] for solving such types of problems, firstly, we have to approximate/discretize the Riesz fractional operator, and it causes the roundoff error in the obtained approximate solution. Furthermore, the adopted method does not use approximation/discretization and even gives the analytical solution to the problem in one iteration.

Compliance with ethical standards

Conflict of interest The authors state that there is no conflict of interest.

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