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ORIGINAL ARTICLE

Numerical results of Emden–Fowler boundary value problems with derivative dependence using the Bernstein collocation method

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Abstract
In this paper, we propose an efficient numerical technique based on the Bernstein polynomials for the numerical solution of the equivalent integral form of the derivative dependent Emden–Fowler boundary value problems which arises in various fields of applied mathematics, physical and chemical sciences. The Bernstein collocation method is used to convert the integral equation into a system of nonlinear equations. This system is then solved efficiently by suitable iterative method. The error analysis of the present method is discussed. The accuracy of the proposed method is examined by calculating the maximum absolute error and the L_2 error of four examples. The obtained numerical results are compared with the results obtained by the other known techniques.

Keywords Derivative dependence · Singular differential equation · Bernstein polynomials · Functional approximation · Green's function.

1 Introduction

We consider the following derivative dependent Emden–Fowler boundary value problems as

$$\begin{cases} \left(r(t)y'(t)\right)' = s(t)f\left(t, y(t), r(t)y'(t)\right), & t \in (0,1), \\ y(0) = \gamma_4, \text{ or } \lim_{t\to 0^+} r(t)\, y'(t) = 0, & \gamma_1 y(1) + \gamma_2 y'(1) = \gamma_3, \end{cases} \quad (1)$$

where $\gamma_1 > 0$, γ_2, γ_3, and γ_4 are real constants. The Emden-Fowler type Eq. (1) arises in many fields of mathematical sciences and astrophysics such as in the study of heat explosion [1], in calculation of oxygen concentration inside a spherical cell [2], to measure heat sources in human head [3], in shallow membrane cap theory [4], in modeling thermal explosion in a rectangular slab [5, 6].

Note that the Eq. (1) is called doubly singular boundary value problem [8], where $r(t) = t^a v(t)$, $v(0) \neq 0$, $s(t) = t^b z(t)$, $z(0) \neq 0$ with $r(0) = 0$ and $s(t)$ is allowed to be discontinuous at $t = 0$. The existence and uniqueness results of solution of these problems can be found in [7–11].

Finding numerical solution of such problems is very challenging due to singularity at the origin and strong nonlinearity of the form $f(t, y(t), r(t)y'(t))$. Numerous numerical methods for solving (1) when $f(t, y(t), r(t)y'(t)) = f(t, y(t))$ have been developed like the finite difference method [12–14], the spline finite difference method [15], the parametric-spline method [16], the cubic spline method [17], the optimal parametric iteration method [18], the B-spline collocation method [19], the Adomian decomposition method (ADM) with Green's function [20–25], the Laguerre wavelets collocation method [26], the classical polynomial approximation method [27], the modified variational iteration method [28], the Mickens' type non-standard finite difference schemes [29], the homotopy analysis method [30, 31], the homotopy perturbation method [32], the Haar-wavelet collocation method [33, 34], the Haar wavelet quasi-linearization method [35, 36], the advanced Adomian decomposition method [37] and the Bernstein collocation method [38].

To the best of our knowledge there is very few methods provided so far for numerical solution of the derivative dependent Emden–Fowler boundary value problems such as the modified Adomian decomposition method [24, 25], the improved homotopy analysis method [39] and the B-spline collocation method [40].

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In this paper, we propose an efficient collocation method based on the Bernstein polynomials for the numerical solution of the equivalent integral form of the derivative dependent Emden–Fowler boundary value problems (1). The Bernstein collocation method (BCM) is used to convert the integral equations into a system of nonlinear equations. Then a suitable iterative technique is used to find numerical solutions of the system of nonlinear equations. The error analysis of the proposed method is provided. The accuracy of the proposed method is examined by calculating the maximum absolute error L_∞ and the L_2 error of some numerical examples. To check the efficiency of the present method the obtained numerical results are compared with the results obtained by the other known techniques.

2 Integral form of derivative dependent Emden-Fowler BVPs

2.1 For Dirichlet-Robin boundary conditions

Consider the following derivative dependent Emden-Fowler equation with Dirichlet-Robin BCs

$$\begin{cases} \big(r(t)y'(t)\big)' = s(t)f\big(t, y(t), r(t)y'(t)\big), & t \in (0,1), \\ y(0) = \gamma_4,\ \gamma_1 y(1) + \gamma_2 y'(1) = \gamma_3. \end{cases} \tag{2}$$

Integrating the Eq. (2) from t to 1 and then from 0 to t and changing the order of integration and applying the boundary conditions, we obtain the equivalent integral equation

$$y(t) = \gamma_4 + \frac{(\gamma_3 - \gamma_1\gamma_4)}{\gamma_1 h(1) + \gamma_2 h'(1)} h(t) + \int_0^1 G(t,\xi)\, s(\xi)\, f\big(\xi, y(\xi), r(\xi)\, y'(\xi)\big) d\xi, \quad t \in (0,1), \tag{3}$$

where $G(t,\xi)$ is given by

$$G(t,\xi) = \begin{cases} h(t) - \frac{\gamma_1 h(\xi) h(t)}{\gamma_1 h(1) + \gamma_2 h'(1)}, & t \le \xi, \\ h(\xi) - \frac{\gamma_1 h(t) h(\xi)}{\gamma_1 h(1) + \gamma_2 h'(1)}, & \xi \le t, \end{cases} \tag{4}$$

where $h(t) = \int_0^t \frac{1}{r(\xi)} d\xi$, $h(1) = \int_0^1 \frac{1}{r(\xi)} d\xi$ and $h'(1) = \frac{1}{r(1)}$.

2.2 For Neumann-Robin boundary conditions

Similarly, consider the derivative dependent Emden-Fowler equation with Neumann-Robin BCs

$$\begin{cases} \big(r(t)y'(t)\big)' = s(t)f\big(t, y(t), r(t)y'(t)\big), & t \in (0,1), \\ \lim\limits_{t\to 0^+} r(t)\, y'(t) = 0, \quad \gamma_1\, y(1) + \gamma_2\, y'(1) = \gamma_3. \end{cases} \tag{5}$$

Integrating the Eq. (5) from t to 1 and then from 0 to t and changing the order of integration and applying the boundary conditions, we obtain an integral equation

$$y(t) = \frac{\gamma_3}{\gamma_1} + \int_0^1 G(t,\xi)\, s(\xi)\, f\big(\xi, y(\xi), r(\xi)\, y'(\xi)\big) d\xi, \quad t \in (0,1), \tag{6}$$

with

$$G(t,\xi) = \begin{cases} \int_\xi^1 \frac{1}{r(t)} dt + \frac{\gamma_2}{\gamma_1 r(1)}, & t \le \xi, \\ \int_\xi^1 \frac{1}{r(t)} dt - \int_\xi^t \frac{1}{r(t)} dt + \frac{\gamma_2}{\gamma_1 r(1)}, & \xi \le t. \end{cases} \tag{7}$$

3 The Bernstein collocation method

he Bernstein polynomials play a prominent role in many areas of mathematical sciences. One of the important property of these polynomials is that they all vanish, except at the end points of the interval [0, 1]. This gives more flexibility in which to impose boundary conditions at the end points of the interval. These polynomials have several other useful properties, such as the continuity, the positivity and complete basis formation over the interval [0, 1]. These polynomials have frequently been used to solve various differential and integral equations [41–51].

Definition 1 The Bernstein polynomials [41] of degree n are defined as

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, 1, 2, \cdots n, \quad t \in [0,1], \tag{8}$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$.

A recursive definition can also be used to generate these polynomials,

$$B_{i,n}(t) = (1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t).$$

The derivative of the Bernstein polynomials is given by

$$\frac{dB_{i,n}(t)}{dt} = n[B_{i-1,n-1}(t) - B_{i,n-1}(t)],$$

and their finite integral is

$$\int_0^1 B_{i,n}(t) dt = \frac{1}{n+1}.$$

Definition 2 The Bernstein polynomials form a complete basis with the following properties

(i) $B_{i,n}(t) = 0$, when $i < 0$ or $i > n$,
(ii) $B_{i,n}(0) = B_{i,n}(1) = 0$, when $i = 1, 2, \cdots n-1$,
(iii) They form the partition of unity:

$$\sum_{i=0}^{n} B_{i,n}(t) = 1.$$

and their derivative verify the partition of nullity:

$$\sum_{i=0}^{n} \frac{d^p B_{i,n}(t)}{dt^p} = 1,\ p \geq 0.$$

This property is closely related to the capability of an approximation to reproduce exactly a polynomial solution [52].

Note that an excellent performance in terms of error can be reached with Bernstein expansion for relatively low order approximations, but for a higher degree of the Bernstein polynomial there may be an increase in the numerical dissipation due to the evaluation of binomial terms and powers of a very high order. This drawback can be relieved by using the binomial multiplicative formula:

$$\binom{n}{i} = \prod_{l=1}^{i} \frac{n-l+1}{l},$$

which allows a more efficient computation of binomial terms [47].

Any function $v(t) \in L^2[0,1]$ can be approximated by the Bernstein basis polynomials as

$$v(t) = \sum_{i=0}^{\infty} a_i B_{i,n}(t). \tag{9}$$

For numerical purpose, we consider the first $(n+1)$ terms of the above expansion as

$$v(t) \approx \sum_{i=0}^{n} a_i B_{i,n}(t). \tag{10}$$

The collocation points on an interval [0, 1] is defined as

$$t_j = t_0 + \frac{j}{n}, \quad j = 0, 1, 2, \ldots n, \quad 0 \leq t_0 < 1. \tag{11}$$

Such collocation points are considered for which maxima are reached for the Bernstein polynomial.

In next subsection, we establish a collocation method based on Bernstein polynomials for finding numerical solution of the integral Eqs. (3) and (6).

3.1 Dirichlet-Robin boundary conditions

To establish a numerical algorithm, we reconsider Eq. (3) as follows:

$$y(t) = \gamma_4 + \left(\frac{\gamma_3 - \gamma_1\gamma_4}{\gamma_1 h(1) + \gamma_2 h'(1)} \right) h(t) + \int_0^1 G(t,\xi)\ s(\xi)\ f(\xi, y(\xi), r(\xi)\ y'(\xi)) d\xi, \quad t \in (0,1). \tag{12}$$

In Eq. (12), we consider

$$\psi(t) = f(t, y(t), r(t)y'(t)). \tag{13}$$

On approximating $y(t)$, $y'(t)$ and $\psi(t)$ by the Bernstein basis polynomials, we get

$$y(t) \approx \sum_{i=0}^{n} a_i B_{i,n}(t), \tag{14}$$

$$y'(t) \approx \sum_{i=0}^{n} a_i B'_{i,n}(t), \quad \text{where } ' = \frac{d}{dt}, \tag{15}$$

$$\psi(t) \approx \sum_{i=0}^{n} b_i B_{i,n}(t). \tag{16}$$

Substituting the expression from (14) and (15) into (12), we obtain

$$\sum_{i=0}^{n} a_i B_{i,n}(t) = \gamma_4 + \left(\frac{\gamma_3 - \gamma_1\gamma_4}{\gamma_1 h(1) + \gamma_2 h'(1)} \right) h(t) + \sum_{i=0}^{n} b_i \int_0^1 G(t,\xi)\ s(\xi)\ B_{i,n}(\xi) d\xi, \tag{17}$$

which can be written as

$$\sum_{i=0}^{n} a_i B_{i,n}(t) = \gamma_4 + \left(\frac{\gamma_3 - \gamma_1\gamma_4}{\gamma_1 h(1) + \gamma_2 h'(1)} \right) h(t) + \sum_{i=0}^{n} b_i K_i(t), \tag{18}$$

where

$$K_i(t) = \int_0^1 G(t,\xi)\ s(\xi)\ B_{i,n}(\xi) d\xi, \quad i = 0, 1, 2, \ldots n. \tag{19}$$

On differentiating (18) w.r.t. t, we get

$$\sum_{i=0}^{n} a_i B'_{i,n}(t) = \left(\frac{\gamma_3 - \gamma_1\gamma_4}{\gamma_1 h(1) + \gamma_2 h'(1)} \right) h'(t) + \sum_{i=0}^{n} b_i K'_i(t). \tag{20}$$

where

$$K_i'(t) = \frac{d}{dt}\left(\int_0^1 G(t,\xi)\ s(\xi)\ B_{i,n}(\xi)d\xi \right), \qquad i = 0,1,2,\dots n.$$

Using the expressions of $y(t)$, $y'(t)$ and $\psi(t)$ from Eqs. (14), (15) and (16), Eq. (13) takes form

$$\sum_{i=0}^{n} b_i B_{i,n}(t) = f\left(t,\ \sum_{i=0}^{n} a_i B_{i,n}(t),\ r(t)\sum_{i=0}^{n} a_i B'_{i,n}(t) \right). \tag{21}$$

Upon substituting the expressions from Eqs. (18) and (20) into (21) and inserting the collocation points t_j defined in (11), we obtain the nonlinear system of equations as

$$\sum_{i=0}^{n} b_i B_{i,n}(t_j) - f\left[t_j,\ \gamma_4 + \frac{(\gamma_3-\gamma_1\gamma_4)}{\gamma_1 h(1)+\gamma_2 h'(1)}\ h(t_j) + \sum_{i=0}^{n} b_i K_i(t_j),\right.$$
$$\left. r(t_j)\left(\frac{(\gamma_3-\gamma_1\gamma_4)}{\gamma_1 h(1)+\gamma_2 h'(1)}\ h'(t_j) + \sum_{i=0}^{n} b_i K_i'(t_j) \right) \right] = 0,$$
$$j = 0,1,2,\dots n, \tag{22}$$

where $b_0, b_1, \dots, b_n$ are the unknowns. The nonlinear system of Eq. (22) is solved numerically by the Newton's iteration method to get the unknowns b_i, which are then substituted in Eq. (18) to get the numerical solution of (12).

3.2 Neumann-Robin boundary conditions

Let us reconsider the integral Eq. (6) as

$$y(t) = \frac{\gamma_3}{\gamma_1} + \int_0^1 G(t,\xi)s(\xi)f(\xi, y(\xi), r(\xi)y'(\xi))d\xi, \quad t\in(0,1). \tag{23}$$

Following similar steps of previous subsection, we substitute the expressions from Eqs. (13), (14), (15) and (16) into Eq. (23) and get

$$\sum_{i=0}^{n} a_i B_{i,n}(t) = \frac{\gamma_3}{\gamma_1} + \sum_{i=0}^{n} b_i \int_0^1 G(t,\xi)\ s(\xi)\ B_{i,n}(\xi)d\xi, \tag{24}$$

which can further be written as

$$\sum_{i=0}^{n} a_i B_{i,n}(t) = \frac{\gamma_3}{\gamma_1} + \sum_{i=0}^{n} b_i K_i(t), \tag{25}$$

and

$$\sum_{i=0}^{n} a_i B'_{i,n}(t) = \sum_{i=0}^{n} b_i K_i'(t). \tag{26}$$

Using (25) and (26) into (21) and inserting the collocation points t_j, we obtain the nonlinear system of equations as

$$\sum_{i=0}^{n} b_i B_{i,n}(t_j) - f\left(t_j,\ \frac{\gamma_3}{\gamma_1} + \sum_{i=0}^{n} b_i K_i(t_j),\ r(t_j)\sum_{i=0}^{n} b_i K_i'(t_j) \right) = 0,$$
$$j = 0,1,2\dots,n, \tag{27}$$

with the unknowns $b_0, b_1, \cdots, b_n$. Solving the nonlinear system of Eqs. (27) by Newton's iteration method, we obtain the unknown coefficients which will be substituted in Eq. (25) to get the numerical solution of (23).

Remark 1 In the present analysis, the nonlinear systems of Eqs. (22) and (27) lead to full matrices which are generally computationally demanding. But in this method, we have need solve a very small sized matrix to reach the desired accuracy. So, it is computationally efficient to use the Bernstein collocation method for solving these nonlinear systems of equations.

4 Error analysis

Let $\mathbb{X} = C[0,1]\bigcap C^1(0,1]$ be the Banach space with the norm [7, 8] defined as

$$\|y\| = \max\{\|y\|_0, \|y\|_1\},\ y\in\mathbb{X}, \tag{28}$$

where $\|y\|_0$ and $\|y\|_1$ are defined as

$$\|y\|_0 = \max_{t\in[0,1]} |y(t)|, \tag{29}$$

and

$$\|y\|_1 = \max_{t\in[0,1]} |r(t)y'(t)|. \tag{30}$$

We consider the following integral equation

$$y(t) = g(t) + \int_0^1 G(t,\xi)\ s(\xi)\ f(\xi, y(\xi), r(\xi)y'(\xi))d\xi, \qquad t\in(0,1). \tag{31}$$

Note that the integral Eqs. (3) and (6) are special cases of (31) when $g(t) = \gamma_4 + \frac{\gamma_3-\gamma_1\gamma_4}{\gamma_1 h(1)+\gamma_2 h'(1)}h(t)$ and $g(t) = \frac{\gamma_3}{\gamma_1}$, respectively.

Theorem 1 (See [53]) *If $v(t)\in C[0,1]$, the sequence $\{B_n(v)\}$ converges uniformly to v, where $B_n(v) = \sum_{i=0}^{n} a_i\ B_i^n(t)$ is the Bernstein approximation function. In other words for any $\epsilon > 0$ there exists a number $n\in\mathbb{N}$ such that $\|B_n(v) - v\| < \epsilon$.*

Theorem 2 (See [54]) *If $v(t)$ is bounded and $v''(t)$ exists in* $[0,1]$, *then the error bound for Bernstein's approximation function is obtained as*

$$\|B_n(v) - v\| \le \frac{\|v''\|}{2n} \max_{t\in[0,1]} \big(t(1-t)\big) = \frac{\|v''\|}{8n}, \tag{32}$$

and the rate of convergence of Bernstein's approximation function is precisely $1/n$ [55], *provided* $v''(t) \neq 0$.

Theorem 3 *Let* $y(t)$ *and* $y_n(t)$ *be the exact and the approximate solutions of the integral Eq.* (31). *Assume that the nonlinear function* $f(t, y, ry')$ *satisfies the Lipschitz condition*

$$|f(t,y,ry') - f(t,y_n,ry'_n)| \le l_1|y-y_n| + l_2|r(y'-y'_n)|, \tag{33}$$

where l_1 *and* l_2 *are the Lipschitz constants. Then the error bound for Bernstein collocation method is estimated as*

$$\|y - y_n\| \le \frac{wlm}{4n}, \tag{34}$$

where $l = \max\{l_1, l_2\}$, $m = \max\{m_1, m_2\}$, $w = \|y''\|$,

$$m_1 = \max_{t\in[0,1]} \int_0^1 \big|G(t,\xi)\, s(\xi)\big| d\xi < \infty,$$

$$m_2 = \max_{t\in[0,1]} \int_0^1 \big|r(t)G_t(t,\xi)\, s(\xi)\big| d\xi < \infty.$$

Proof Consider

$$\begin{aligned}\|y-y_n\|_0 &= \max_{t\in[0,1]} \left| \int_0^1 G(t,\xi)\, s(\xi) \Big(f\big(\xi, y(\xi), r(\xi)y'(\xi)\big) - f\big(\xi, y_n(\xi), r(\xi)y'_n(\xi)\big)\Big) d\xi \right| \\ &\le \max_{t\in[0,1]} \left| \int_0^1 G(t,\xi)\, s(\xi) d\xi \right| \times \max_{\xi\in[0,1]} \left| f\big(\xi, y(\xi), r(\xi)y'(\xi)\big) - f\big(\xi, y_n(\xi), r(\xi)y'_n(\xi)\big) \right|.\end{aligned}$$

Applying the Lipschitz condition, the above inequality becomes

$$\begin{aligned}\|y-y_n\|_0 &\le m_1 \max_{\xi\in[0,1]} \Big\{ l_1|y(\xi)-y_n(\xi)| + l_2|r(\xi)(y'(\xi)-y'_n(\xi))| \Big\} \\ &\le 2lm_1 \max\Big\{ \|y-y_n\|_0, \|y-y_n\|_1 \Big\} \\ &= 2lm_1 \|y-y_n\|.\end{aligned} \tag{35}$$

In the same way, we obtain

$$\begin{aligned}\|y-y_n\|_1 &= \max_{t\in[0,1]} \left| \int_0^1 r(t)G_t(t,\xi)\, s(\xi) \Big(f\big(\xi, y(\xi), r(\xi)y'(\xi)\big) - f\big(\xi, y_n(\xi), r(\xi)y'_n(\xi)\big)\Big) d\xi \right| \\ &\le \max_{t\in[0,1]} \left| \int_0^1 r(t)G_t(t,\xi)\, s(\xi) d\xi \right| \times \max_{\xi\in[0,1]} \left| f\big(\xi, y(\xi), r(\xi)y'(\xi)\big) - f\big(\xi, y_n(\xi), r(\xi)y'_n(\xi)\big) \right|.\end{aligned}$$

Using the Lipschitz condition, we get

$$\begin{aligned}\|y-y_n\|_1 &\le m_2 \max_{\xi\in[0,1]} \Big\{ l_1|y(\xi)-y_n(\xi)| + l_2|r(y'(\xi)-y'_n(\xi))| \Big\} \\ &\le 2lm_2 \max\Big\{ \|y-y_n\|_0, \|y-y_n\|_1 \Big\} \\ &= 2lm_2 \|y-y_n\|.\end{aligned} \tag{36}$$

From Eqs. (35) and (36), we obtain

$$\begin{aligned}\|y-y_n\| &= \max\Big\{ \|y-y_n\|_0, \|y-y_n\|_1 \Big\} \\ &\le \max\Big\{ 2lm_1\|y-y_n\|, 2lm_2\|y-y_n\| \Big\} \\ &\le 2lm\|y-y_n\| = 2lm \max_{\xi\in[0,1]} |y(\xi)-y_n(\xi)|.\end{aligned} \tag{37}$$

Replacing $y_n(\xi)$ by the Bernstein solution $B_n\big(y(\xi)\big)$, Eq. (37) reduces to

$$\|y-y_n\| \le 2lm \max_{\xi\in[0,1]} |y(\xi) - B_n(y(\xi))|. \tag{38}$$

Using the result from Eq. (32), the Eq. (38) becomes

$$\|y-y_n\| \le 2lm\frac{w}{2n} \max_{\xi\in[0,1]} \big(\xi(1-\xi)\big) = \frac{wlm}{4n}. \tag{39}$$

□

5 Numerical results

We examine the accuracy of the proposed method by solving the several derivative dependent Emden-Fowler type singular BVPs. For comparison purpose, we define the maximum absolute error as

$$L_\infty = \max_{t\in[0,1]} |y(t) - y_n(t)|, \quad n = 1, 2, \dots,$$

and the L_2 error as

Table 1 Comparison of numerical results of maximum absolute errors of Example 1 when $l = 1$

n	$k = 0.25$		$k = 0.5$		$k = 0.75$	
	L_∞	E_n [24]	L_∞	E_n [24]	L_∞	E_n [24]
4	3.38E-05	9.47E-04	3.99E-05	6.55E-04	5.04E-05	8.36E-04
5	1.66E-06	1.59E-04	2.08E-06	1.41E-04	2.58E-06	2.32E-04
6	1.66E-06	3.09E-05	9.25E-08	6.27E-05	1.13E-07	3.62E-05
7	3.02E-09	3.98E-06	3.68E-09	1.22E-05	4.48E-09	1.79E-05
8	1.07E-10	9.46E-07	1.30E-10	1.47E-06	1.57E-10	3.86E-06
9	3.50E-12	9.53E-08	4.24E-12	3.58E-07	5.11E-12	2.67E-07
10	1.04E-13	1.91E-08	1.25E-13	3.26E-08	1.51E-13	8.04E-08

Table 2 Comparison of numerical results of maximum absolute errors of Example 1 when $l = 2.5$

n	$k = 0.25$		$k = 0.5$		$k = 0.75$	
	L_∞	E_n [24]	L_∞	E_n [24]	L_∞	E_n [24]
4	2.10E-03	9.21E-04	2.17E-03	8.65E-04	2.23E-03	6.55E-04
5	3.88E-04	1.75E-04	3.99E-04	2.14E-04	4.10E-04	1.41E-04
6	6.91E-05	8.36E-05	7.13E-05	6.14E-05	7.34E-05	6.27E-05
7	1.25E-05	2.08E-05	1.28E-05	1.40E-05	1.32E-05	1.21E-05
8	2.20E-06	5.11E-06	2.27E-06	5.95E-06	2.34E-06	1.54E-06
9	3.61E-07	5.56E-07	3.70E-07	9.38E-07	3.79E-07	4.07E-07
10	7.16E-08	6.88E-08	7.94E-08	6.74E-08	9.53E-08	4.40E-08

$$L_2 = \left(\sum_{j=1}^{n} |y(t_j) - y_n(t_j)|^2 \right)^{1/2}.$$

Here $y(t)$ is the exact solution and $y_n(t)$ is the Bernstein solution. The maximum absolute error is defined as

$$E_n = \max_{t\in[0,1]} |y(t) - \psi_n(t)|, \quad n = 1, 2, \dots ,$$

where $\psi_n(t) = \sum_{j=0}^{n} y_j(t)$ denotes Adomian decomposition method solution [24].

Example 1 Consider the following derivative dependent Emden-Fowler BVP [24] as

$$\begin{cases} \left(t^k y'(t)\right)' = t^{k+l-2}\left(lty'(t) + l\left(k+l-1\right)y(t)\right), & t \in (0,1), \quad l > 0, \\ y(0) = 1, \quad y(1) = e. \end{cases} \tag{40}$$

Its exact solution is given by $y(t) = e^{t^l}$. The equivalent integral form of (40) is

$$y(t) = 1 + \frac{(e-1)}{(1-k)^2} t^{1-k} + \int_0^1 G(t,\xi)\, \xi^{k+l-2} \left(l\xi y' + l\left(k+l-1\right)y(\xi) \right) d\xi, \tag{41}$$

where $G(t,\xi)$ is

$$G(t,\xi) = \begin{cases} \frac{t^{1-k}}{1-k}(1-\xi^{1-k}), & t \le \xi, \\ \frac{\xi^{1-k}}{1-k}(1-t^{1-k}), & \xi \le t. \end{cases} \tag{42}$$

Table 3 Numerical results of L_2 error of Example 1 when $l = 1$

n	$k = 0.25$	$k = 0.5$	$k = 0.75$
3	1.23E-03	1.42E-03	1.63E-03
4	6.46E-05	7.40E-05	8.42E-05
5	3.22E-06	3.66E-06	4.14E-06
6	3.22E-06	1.55E-07	1.74E-07
7	5.52E-09	6.21E-09	6.94E-09
8	1.93E-10	2.16E-10	2.41E-10
9	6.43E-12	7.17E-12	7.97E-12
10	1.90E-13	2.11E-13	2.34E-13

We compare the numerical results of maximum absolute errors L_∞ and E_n obtained by BCM and the ADM [24] of Example 1 for $l = 1$ and $l = 2.5$ with different values of $k = 0.25, 0.50, 0.75$ in Tables 1 and 2. In addition, the numerical results of the L_2 error are shown in Tables 3 and 4. From the numerical results, it is observed that the BCM

Table 4 Numerical results of L_2 error of Example 1 when $l = 2.5$

n	$k = 0.25$	$k = 0.5$	$k = 0.75$
3	2.02E-02	2.18E-02	2.37E-02
4	2.70E-03	2.84E-03	2.99E-03
5	5.27E-04	5.72E-04	6.29E-04
6	8.16E-05	8.60E-05	9.06E-05
7	1.78E-05	1.93E-05	2.13E-05
8	2.78E-06	2.93E-06	3.10E-06
9	5.18E-07	5.52E-07	5.93E-07
10	1.20E-07	1.33E-07	1.48E-07

converges faster than the ADM [24]. It can be seen that as the degree of the Bernstein polynomial increases, the numerical errors decreases significantly.

Example 2 Consider the following Emden-Fowler equation with derivative dependence [24]

$$\begin{cases} \left(t^k y'(t)\right)' = t^{k-1}\left(-ty'(t)e^{y(t)} - ke^{y(t)}\right), \quad t \in (0,1), \\ y(0) = \ln\left(\frac{1}{2}\right), \quad y(1) = \ln\left(\frac{1}{3}\right). \end{cases} \tag{43}$$

Its exact solution is given by $y(t) = \ln\left(\frac{1}{2+t}\right)$. The equivalent integral form of (43) is

$$y(t) = \ln\left(\frac{1}{2}\right) + \frac{\left(\ln\left(\frac{1}{3}\right) - \ln\left(\frac{1}{2}\right)\right)}{(1-k)^2} t^{1-k} + \int_0^1 G(t,\xi)\, \xi^{k-1}\left(-\xi y(\xi)' e^{y(\xi)} - ke^{y(\xi)}\right) d\xi, \tag{44}$$

where $G(t,\xi)$ is

$$G(t,\xi) = \begin{cases} \frac{t^{1-k}}{1-k}(1-\xi^{1-k}), & t \le \xi, \\ \frac{\xi^{1-k}}{1-k}(1-t^{1-k}), & \xi \le t. \end{cases} \tag{45}$$

Comparison of the numerical results of L_∞ and E_n obtained by the BCM and the ADM [24] of Example 2 is given in Table 5. In addition, the numerical results of the L_2 error is shown in Table 6.

Example 3 Consider the following derivative dependent Emden-Fowler BVP [24]

$$\begin{cases} \left(t^k y'(t)\right)' = t^{k+l-2}\left(lte^{y(t)}y'(t) + l\left(k+l-1\right)e^{y(t)}\right), \quad t \in (0,1), \\ y(0) = \ln\left(\frac{1}{4}\right), \quad y(1) = \ln\left(\frac{1}{5}\right). \end{cases} \tag{46}$$

Its exact solution is given by $y(t) = \ln\left(\frac{1}{4+t^l}\right)$. The equivalent integral form of (46) is

$$y(t) = \ln\left(\frac{1}{4}\right) + \frac{\left(\ln\left(\frac{1}{5}\right) - \ln\left(\frac{1}{4}\right)\right)}{(1-k)^2} t^{1-k} + \int_0^1 G(t,\xi)\, \xi^{k+l-2}\left(l\xi e^{y(\xi)} y'(\xi) + l\left(k+l-1\right)e^{y(\xi)}\right) d\xi, \tag{47}$$

where $G(t,\xi)$ is

$$G(t,\xi) = \begin{cases} \frac{t^{1-k}}{1-k}(1-\xi^{1-k}), & t \le \xi, \\ \frac{\xi^{1-k}}{1-k}(1-t^{1-k}), & \xi \le t. \end{cases} \tag{48}$$

In Tables 7 and 8, we provide the numerical results of maximum absolute errors obtained by the BCM and the ADM [24] of Example 3 for $l = 1$ and $l = 3.5$ with

Table 6 Numerical results of L_2 error of Example 2

n	$k = 0.25$	$k = 0.5$	$k = 0.75$
3	1.25E-04	1.62E-04	2.17E-04
4	1.42E-05	1.82E-05	2.40E-05
5	1.79E-06	2.27E-06	2.98E-06
6	2.21E-07	2.78E-07	3.62E-07
7	2.92E-08	3.66E-08	4.75E-08
8	3.76E-09	4.67E-09	6.04E-09
9	5.09E-10	6.33E-10	8.17E-10
10	6.71E-11	8.31E-11	1.07E-10

Table 5 Comparison of numerical results of maximum absolute error of Example 2

	$k = 0.25$		$k = 0.5$		$k = 0.75$	
n	L_∞	E_n [24]	L_∞	E_n [24]	L_∞	E_n [24]
4	7.82E-06	5.79E-04	1.06E-05	9.71E-04	1.52E-05	7.73E-04
5	1.03E-06	2.18E-04	1.38E-06	3.53E-04	1.94E-06	3.49E-04
6	1.34E-07	5.07E-05	1.77E-07	4.57E-05	2.43E-07	8.71E-05
7	1.76E-08	8.74E-06	2.31E-08	2.14E-05	3.17E-08	2.48E-05
8	2.30E-09	2.24E-06	3.00E-09	4.03E-06	4.07E-09	8.27E-06
9	3.06E-10	6.14E-07	3.99E-10	1.29E-06	5.42E-10	1.15E-06
10	4.05E-11	5.88E-08	5.27E-11	3.34E-07	7.14E-11	4.82E-07

Table 7 Comparison of numerical results of maximum absolute error of Example 3 when $l = 1$

	$k = 0.25$		$k = 0.5$		$k = 0.75$	
n	L_∞	E_n[24]	L_∞	E_n[24]	L_∞	E_n[24]
4	2.37E-07	4.84E-05	3.13E-07	9.43E-05	4.38E-07	1.79E-04
5	1.69E-08	1.11E-05	2.25E-08	1.52E-05	3.09E-08	1.16E-05
6	1.21E-09	1.12E-06	1.58E-09	1.52E-06	2.13E-09	2.78E-06
7	8.82E-11	1.23E-07	1.14E-10	2.49E-07	1.53E-10	3.10E-07
8	6.35E-12	2.30E-08	8.17E-12	2.69E-08	1.09E-11	4.91E-08
9	4.66E-13	2.16E-09	6.00E-13	4.01E-09	8.00E-13	6.76E-09
10	3.41E-14	3.54E-10	4.38E-14	5.40E-10	5.81E-14	9.22E-10

Table 8 Comparison of numerical results of maximum absolute error of Example 3 when $l = 3.5$

	$k = 0.25$		$k = 0.5$		$k = 0.75$	
n	L_∞	E_n[24]	L_∞	E_n[24]	L_∞	E_n[24]
4	1.95E-05	1.88E-04	2.08E-05	2.19E-04	2.21E-05	2.38E-04
5	5.09E-06	1.51E-05	5.28E-06	1.58E-05	5.47E-06	2.89E-05
6	1.30E-06	3.04E-06	1.36E-06	3.51E-06	5.47E-06	3.76E-06
7	1.96E-08	3.13E-07	1.90E-08	3.27E-07	1.83E-08	6.25E-07
8	6.44E-08	4.78E-08	6.71E-08	6.53E-08	7.00E-08	6.93E-08
9	6.78E-09	6.47E-09	7.13E-09	6.96E-09	7.49E-09	1.30E-09
10	6.78E-09	1.13E-09	1.60E-09	1.33E-09	1.65E-09	1.39E-09

Table 9 Numerical results of L_2 error of Example 3 when $l = 1$

n	$k = 0.25$	$k = 0.5$	$k = 0.75$
3	7.01E-06	5.75E-14	1.16E-05
4	4.34E-07	5.44E-07	7.00E-07
5	3.01E-08	3.74E-08	4.80E-08
6	2.04E-09	2.52E-09	3.19E-09
7	1.49E-10	1.83E-10	2.32E-10
8	1.05E-11	1.29E-11	1.62E-11
9	7.90E-13	9.65E-13	1.21E-12
10	5.75E-14	7.00E-14	8.78E-14

Table 10 Numerical results of L_2 error of Example 3 when $l = 3.5$

n	$k = 0.25$	$k = 0.5$	$k = 0.75$
3	4.32E-04	4.98E-04	5.92E-04
4	3.17E-05	3.71E-05	4.49E-05
5	8.88E-06	1.02E-05	1.21E-05
6	2.15E-06	2.49E-06	1.21E-05
7	3.44E-08	3.53E-08	3.69E-08
8	1.17E-07	1.36E-07	1.63E-07
9	1.28E-08	1.50E-08	1.81E-08
10	1.28E-08	3.64E-09	4.33E-09

$k = 0.25,\ 0.50,\ 0.75$, respectively. We also present the L_2 error in Tables 9 and 10. It has been observed that the BCM converges faster and as the value of n increases, the L_∞ and L_2 errors decrease rapidly.

Example 4 Consider the following derivative dependent Emden-Fowler BVP [25]

$$\begin{cases} \left(t^k y'(t)\right)' = t^{k+l-2}\Big(ty'(t) + y(t)(k+l-1)\Big), \quad t \in (0,1), \quad l > 0, \\ \lim\limits_{t \to 0^+} r(t)\, y'(t) = 0, \quad y(1) = e. \end{cases} \tag{49}$$

Its exact solution is given by $y(t) = e^{t^l}$. The equivalent integral form of (49) is

$$y(t) = e + \int_0^1 G(t,\xi)\, \xi^{k+l-2}\Big(\xi y'(\xi) + y(\xi)\big(k+l-1\big)\Big) ds, \tag{50}$$

where $G(t,\xi)$ is

$$G(t,\xi) = \begin{cases} \frac{1}{1-k}(1-\xi^{1-k}), & t \le \xi, \\ \frac{1}{1-k}(1-t^{1-k}), & \xi \le t. \end{cases} \tag{51}$$

The numerical results of maximum absolute error L_∞ and L_2 error of Example 4 are provided in Table 11 for $k = 2$

Table 11 Numerical results of maximum absolute error L_∞ and L_2 error of Example 4 for $k = 2$

n	$l = 1$		$l = 2$		$l = 4$	
	L_∞	L_2	L_∞	L_2	L_∞	L_2
3	3.96E-03	6.63E-03	1.83E-02	3.51E-02	2.57E-01	5.10E-01
4	7.11E-04	1.34E-03	5.30E-03	9.17E-03	6.59E-02	1.18E-01
5	5.20E-04	8.82E-04	2.26E-03	4.02E-03	2.37E-02	4.04E-02
6	1.71E-04	3.13E-04	1.15E-03	2.00E-03	1.03E-02	1.77E-02
7	1.38E-04	2.56E-04	6.55E-04	1.19E-03	5.17E-03	9.12E-03
8	6.20E-05	1.15E-04	4.02E-04	7.34E-04	2.94E-03	5.33E-03
9	4.84E-05	1.02E-04	2.61E-04	4.95E-04	1.83E-03	3.39E-03
10	2.79E-05	5.33E-05	1.78E-04	3.40E-04	1.21E-03	2.30E-03

and different values of $l = 1,\ 2,\ 4$. It can be seen that as the degree of the Bernstein polynomial increases, the numerical errors decreases significantly.

Remark 2 From the numerical results, it can be seen that a smaller value of $n \le 10$ is sufficient for obtaining an excellent approximation. Also increasing the value of n results in an increment of computational time and the numerical results of L_∞ and ˇ$_2$ errors are increased or become constant because a problem of truncation error occurs when the degree of the Bernstein polynomial is raised. So, in this case, it is numerically advisable to use a smaller value of n. However, for a high-order collocation scheme, if no truncation of decimal positions could be achieved, the solution would be better than those coming from a lower number of evaluation points.

6 Conclusion

We have considered the derivative dependent Emden–Fowler boundary value problems, which arise in various mathematical modeling such as heat conduction problem [56], the unsteady poiseuille flow in a pipe [57], and electroelastic dynamic problem [58]. We have proposed the Bernstein collocation approach for the numerical solution of the equivalent integral form of the derivative dependent Emden-Fowler equation with two sets of boundary conditions. The error analysis of the Bernstein collocation method has been established under quite general conditions. The proposed method gives better numerical results which can be seen from Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. The accuracy and efficiency of the present method have been checked by evaluating the maximum absolute error and the L_2 error of several numerical examples.

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