



Graded mesh discretization for coupled system of nonlinear multi-term time-space fractional diffusion equations

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Abstract

In this paper, we develop an efficient finite difference/spectral method to solve a coupled system of nonlinear multi-term time-space fractional diffusion equations. In general, the solutions of such equations typically exhibit a weak singularity at the initial time. Based on the $L1$ formula on nonuniform meshes for time stepping and the Legendre–Galerkin spectral method for space discretization, a fully discrete numerical scheme is constructed. Taking into account the initial weak singularity of the solution, the convergence of the method is proved. The optimal error estimate is obtained by providing a generalized discrete form of the fractional Grönwall inequality which enables us to overcome the difficulties caused by the sum of Caputo time-fractional derivatives and the positivity of the reaction term over the nonuniform time mesh. The error estimate reveals how to select an appropriate mesh parameter to obtain the temporal optimal convergence. Furthermore, numerical experiments are presented to confirm the theoretical claims.

Keywords Galerkin–Legendre spectral approximation · $L1$ -type schemes · Multi-term time-space fractional diffusion · Nonsmooth solution · Discrete fractional Grönwall inequality

1 Introduction

In the last decades, fractional differential equations have been extensively studied because of their powerful potential to depict many processes in science and engineering. Compared with the integer-order differential equations, the fractional differential equations can provide a much more accurate and adequate description of the physical processes involving memory and hereditary properties. Though some fractional differential equations with special form, e.g., linear equations, can be solved by analytical methods, e.g., the

Laplace transform method, the Mellin transform method or the Fourier transform method, the analytical solutions of many generalized fractional differential equations (e.g., nonlinear fractional differential equations and multi-term fractional differential equations) are rather difficult to obtain. Hence, developing numerical methods is of great importance in practical applications. The specific nature of the fractional operators involves computational challenges which, if not properly addressed, may lead to unreliable or even wrong results. The numerical methods for solving fractional differential equations can be broadly divided into two classes.

- (a) *Fractional differential equations with smooth solutions*
Unfortunately, The majority of numerical methods developed in the literature are just methods that were devised originally for standard integer-order operators then applied in a naive way to their fractional-order counterparts without a proper knowledge of the features of fractional-order problems leading unexpected results. There have been various numerical methods to solve fractional differential equations, such as finite element methods [17], finite difference methods [7, 25], and spectral methods [3, 11].

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(b) *Fractional differential equations with nonsmooth solutions* Solutions of fractional differential equations typically exhibit weak singularities [8]. This topic is discussed in more detail in the survey chapter [35]. This singularity is a consequence of the weakly singular behavior of the kernels of fractional integral and derivatives. From a physical perspective, its importance is related to the natural emergence of completely monotone relaxation functions in models whose dynamics is governed by these operators. It is shown in [34] that imposing excessive smoothness requirements on the solutions to a partial differential equation (e.g., for the sake of simplifying the error analysis or for obtaining a higher convergence order) has drastic implications regarding the class of admissible problems. Up to now, a few numerical methods have been proposed for fractional differential equations with non-smooth solutions, such as the use of nonuniform grids (see, e.g. [15, 44]), the nonpolynomial/singular basis (see, e.g. [2, 41]), nonsmooth transformations [43], and Lubich's correction method (see, e.g. [45, 48]). Lubich's method consists of adding suitable correction terms to the corresponding numerical approximation, which makes the new approximation exact for low regularity terms of the solutions while still maintaining high accuracy for high regularity terms.

Multi-term time fractional differential equations are motivated by their flexibility to describe complex multirate physical processes, see, e.g. [6, 26, 36]. They are proposed to improve the modeling accuracy in depicting the anomalous diffusion process, successfully capturing power-law frequency dependence [18], adequately modeling various types of viscoelastic damping [4], properly simulating the unsteady flow of a fractional Maxwell fluid [37] and Oldroyd-B fluid [12]. Coupled systems of fractional-order differential equations equipped with received considerable attention in anomalous diffusion [33]. To gain insights into the behavior of the solution of such models, there has been substantial interest in deriving a closed form solution. Daftardar-Gejji et al. [5] obtained the linear and non-linear diffusion-wave equations of fractional order by Adomian decomposition method. Existence, uniqueness and a priori estimates for a class of these equations were obtained by Luchko [27] based on an appropriate maximum principle and the Fourier method. Ding et al. [9] presented the analytical solutions for the multi-term time-space fractional advection–diffusion equations with mixed boundary conditions. Ming et al. [28] proposed Analytical solutions of multi-term time fractional differential equations and application to unsteady flows of generalized viscoelastic fluid. Li et al. [22] investigated the well-posedness and the long-time asymptotic behaviour

for initial-boundary value problems of multi-term time-fractional diffusion equations. Using Luchko's Theorem and the equivalent relationship between the Laplacian operator and the Riesz fractional derivative, Jiang et al. [16] proposed some new analytic techniques to solve three types of multi-term time-space Caputo–Riesz fractional advection diffusion equations with nonhomogeneous Dirichlet boundary conditions.

The analytical solutions obtained in these studies are for the linear case and involve special functions and infinite series, and thus are inconvenient for numerical evaluation. Hence, efficient and accurate numerical approaches are demanded to deal with this model, especially the nonlinear case. Dehghan et al. [7] developed high-order finite difference method and Galerkin spectral method for the numerical solution of multi-term time-fractional differential equations. Alikhanov [1] investigated the finite difference methods for the Dirichlet and Robin boundary value problems of a multi-term variable-distributed order diffusion equation. Wang and Wen [39] proposed high-order compact exponential finite difference method for the multi-term time-fractional diffusion equation, based on the $L1$ formula on a general nonuniform time mesh. Wei [40] proposed and analyzed a fully discrete local discontinuous Galerkin method for a class of multi-term time fractional diffusion equations. Wang et al. [38] proposed an efficient spectral Galerkin method using the $L2 - 1_\sigma$ formula for time discretization and the Legendre–Galerkin spectral method for space discretization to solve the three-dimensional multi-term time-space fractional diffusion equation. Zaky [42] developed a Legendre spectral tau method to handle the multi-term time-fractional diffusion equations. Fu et al. [13] presented a semi-analytical boundary-only collocation technique for solving multi-term time-fractional diffusion-wave equations. Shi et al. [32] studied nonconforming quasi-Wilson finite element fully-discrete approximation for two dimensional multi-term time fractional diffusion-wave equation on regular and anisotropic meshes. Hendy [14] constructed a fully implicit difference scheme for the numerical solution of the multi-term time-space fractional advection–diffusion model.

The $L1$ -type schemes have a wide range of applicability in solving fractional differential equations in time [29]. Most reports on $L1$ -type methods refer to the efficiency analysis of numerical schemes for linear fractional differential equations [24], whereas only few works have discussed the stability and the convergence of $L1$ -type schemes for nonlinear time-fractional differential equations [14, 20, 21]. These results are for the single-order fractional problems. It is observed that a direct application of the discrete fractional Grönwall inequality proposed in [24] for the analysis of nonlinear multi-term time fractional diffusion equations discretization on nonuniform graded mesh cannot be achieved, motivating us to consider this case of study.

Our aim in this paper is twofold. First, we provide a novel discrete fractional Grönwall inequality for the numerical analysis of nonlinear multi-term time fractional differential equations using $L1$ -type schemes on a nonuniform graded mesh. Second, we continue our work [15, 44] and propose and analyze an efficient numerical method for the following coupled system of nonlinear multi-term time-space fractional diffusion equations:

$$\sum_{r=1}^m q_r \frac{\partial^{\beta_r} \psi_1}{\partial t^{\beta_r}} - \gamma \frac{\partial^\alpha \psi_1}{\partial |x|^\alpha} = \rho F_1(\psi_1, \psi_2) + f_1(x, t), \quad x \in \Omega, t \in I, \tag{1.1}$$

$$\sum_{r=1}^m q_r \frac{\partial^{\beta_r} \psi_2}{\partial t^{\beta_r}} - \gamma \frac{\partial^\alpha \psi_2}{\partial |x|^\alpha} = \rho F_2(\psi_1, \psi_2) + f_2(x, t), \quad x \in \Omega, t \in I, \tag{1.2}$$

$$\psi_i(x, 0) = \phi_i(x), \quad x \in \Omega, \quad i \in \{1, 2\}, \tag{1.3}$$

$$\psi_i(x, t) = 0, \quad x \in \partial\Omega, t \in I, \tag{1.4}$$

where F_1 and F_2 are nonlinear functions, and $\Omega = (a, b) \subset \mathbb{R}$ and $I = (0, T] \subset \mathbb{R}$ are the space and time domains, respectively. The parameters γ, ρ are positive constants. the parameter q_m is strictly positive. The boundary of Ω is denoted by $\partial\Omega$, $\phi_i(x)$ is a known sufficiently smooth function, and $\frac{\partial^{\beta_r} \psi}{\partial t^{\beta_r}}$ ($1 > \beta_m > \dots > \beta_2 > \beta_1 > 0$) represents the Caputo fractional derivative. The Caputo fractional derivative $\frac{\partial^{\beta_r} \psi}{\partial t^{\beta_r}}$ is defined as [29, 30]

$$\frac{\partial^{\beta_r} \psi}{\partial t^{\beta_r}} f(x, t) = \frac{1}{\Gamma(1 - \beta_r)} \int_0^t (t - s)^{-\beta_r} \frac{\partial f(x, s)}{\partial s} ds, \tag{1.5}$$

where $\Gamma(\cdot)$ denotes the Gamma function. The Riesz space fractional derivative of order α with respect to $x \in \Omega$, namely $\frac{\partial^\alpha}{\partial |x|^\alpha}$, is defined as [29, 30]

$$\frac{\partial^\alpha f(x)}{\partial |x|^\alpha} := c_\alpha ({}_a D_x^\alpha f(x) + {}_x D_b^\alpha f(x)), \tag{1.6}$$

$c_\alpha = \frac{-1}{2 \cos \frac{\pi\alpha}{2}}$, ${}_a D_x^\alpha \psi$ and ${}_x D_b^\alpha \psi$ are the left- and right-Riemann–Liouville derivatives of order α , defined as [29, 30]

$${}_a D_x^\alpha f(x) = \frac{1}{\Gamma(2 - \alpha)} \frac{\partial^2}{\partial x^2} \int_a^x \frac{f(\xi) d\xi}{(x - \xi)^{\alpha-1}}, \tag{1.7}$$

$${}_x D_b^\alpha f(x) = \frac{1}{\Gamma(2 - \alpha)} \frac{\partial^2}{\partial x^2} \int_x^b \frac{f(\xi) d\xi}{(\xi - x)^{\alpha-1}}. \tag{1.8}$$

The paper is organized as follows. In Sect. 2, some definitions and lemmas on the spaces of fractional derivatives are introduced. In Sect. 3, we develop the $L1$ spectral Galerkin

scheme for a coupled system of nonlinear multi-term time-space fractional diffusion equations. The convergence is rigorously proved in Sect. 4 by providing a generalized discrete form for the fractional Grönwall inequality which enables us to overcome the difficulties caused by the sum of Caputo time-fractional derivatives and the positivity of the reaction term over the nonuniform time mesh. In Sect. 5, the numerical experiments are shown to confirm the theoretical analysis, and the conclusions follow in Sect. 6.

2 Preliminaries

In this section, based on Ervin and Roop [10], we present some definitions and lemmas on the spaces of fractional derivatives, which are useful for the rigorous analysis of convergence.

We write (\cdot, \cdot) for the inner product on the space $L^2(\Omega)$ with the L^2 norm $\|\cdot\|_{L^2(\Omega)}$. For convenience, we denote $\|\cdot\|_{L^2(\Omega)}$ as $\|\cdot\|$.

Definition 1 (*Left fractional derivative space*) For $\eta > 0$, we define the semi-norm

$$|\psi|_{J_L^\eta(\Omega)} = \|{}_a D_x^\eta \psi\|, \tag{2.1}$$

and the norm

$$\|\psi\|_{J_L^\eta(\Omega)} = \left(|\psi|_{J_L^\eta(\Omega)}^2 + \|\psi\|^2 \right)^{1/2}, \tag{2.2}$$

and denote $J_L^\eta(\Omega)$ and $J_{L,0}^\eta(\Omega)$ as the closure of $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{J_L^\eta(\Omega)}$, respectively.

Definition 2 (*Right fractional space*) For $\eta > 0$, we define the semi-norm

$$|\psi|_{J_R^\eta(\Omega)} = \|{}_x D_b^\eta \psi\|, \tag{2.3}$$

and the norm

$$\|\psi\|_{J_R^\eta(\Omega)} = \left(|\psi|_{J_R^\eta(\Omega)}^2 + \|\psi\|^2 \right)^{1/2}, \tag{2.4}$$

and denote J_R^η and $J_{R,0}^\eta$ as the closure of $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{J_R^\eta(\Omega)}$, respectively.

Definition 3 (*Symmetric fractional derivative space*) Let $\eta > 0$ and $\eta \neq n - \frac{1}{2}$, $n \in \mathbb{N}$, we define the semi-norm

$$|\psi|_{J_S^\eta(\Omega)} = |({}_a D_x^\eta \psi, {}_x D_b^\eta \psi)|^{1/2}, \tag{2.5}$$

and the norm

$$\|\psi\|_{J_s^n(\Omega)} = \left(\|\psi\|^2 + |\psi|_{J_s^n(\Omega)}^2 \right)^{1/2}, \tag{2.6}$$

and denote $J_s^n(\Omega)$ and $J_{s,0}^n(\Omega)$ as the closure of $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{J_s^n(\Omega)}$, respectively.

Definition 4 (Fractional Sobolev space) For $\eta > 0$, we define the semi-norm

$$|\psi|_{H^\eta(\Omega)} = \| |\xi|^\eta \mathcal{F}(\hat{\psi})(\xi) \|_{L^2(\mathbb{R})}, \tag{2.7}$$

and the norm

$$\|\psi\|_{H^\eta(\Omega)} = \left(\|\psi\|^2 + |\psi|_{H^\eta(\Omega)}^2 \right)^{\frac{1}{2}}, \tag{2.8}$$

and denote $H^\eta(\Omega)$ and $H_0^\eta(\Omega)$ as the closure of $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{H^\eta(\Omega)}$, respectively. Here, $\mathcal{F}(\hat{\psi})(\xi)$ is the Fourier transformation of the function $\hat{\psi}$, and $\hat{\psi}$ is the zero extension of ψ outside Ω .

Lemma 1 ([10]) The spaces J_L^n, J_R^n, J_s^n and H^n are equivalent with equivalent semi-norms and norms if $\eta \neq n - \frac{1}{2}, n \in \mathbb{N}$.

Lemma 2 (Adjoint property) Let $1 < \eta < 2$, then for any $\psi \in H_0^\eta(\Omega)$ and $v \in H_0^{\eta/2}(\Omega)$, we get

$$\begin{aligned} ({}_a D_x^\eta \psi, v) &= ({}_a D_x^{\eta/2} \psi, {}_x D_b^{\eta/2} v), \quad ({}_x D_b^\eta \psi, v) \\ &= ({}_x D_b^{\eta/2} \psi, {}_a D_x^{\eta/2} v). \end{aligned} \tag{2.9}$$

The following lemma and remark summarize the properties of the interpolation operator I_N .

Lemma 3 (see [31]) Let $s \geq 1$. If $\psi_j \in H^s(\Omega)$ then there exists a constant $C > 0$ independent of N , such that $\|\psi_j - I_N \psi_j\|_l \leq CN^{l-s} \|\psi_j\|_s$, for any $0 \leq l \leq 1$.

Remark 1 A smooth solution of a fractional differential equation does not mean a smooth source term and vice versa. Therefore, the regularity order s of the solution ψ_i is not the same as the regularity order r of the source term f_j , i.e. $\|I_N f_j - f_j\| \leq CN^{-r} \|\psi_j\|_r, \forall f_j \in H^r(\Omega)$.

Lemma 4 (see [10]) Let $\eta > 0$ and $u \in J_{L,0}^\eta(\Omega) \cap J_{R,0}^\eta(\Omega)$. Then

$$\begin{aligned} ({}_a D_x^\eta u, {}_x D_b^\eta u) &= \cos(\pi\eta) \| -_\infty D_x^\eta \hat{u} \|_{L^2(\mathbb{R})}^2 \\ &= \cos(\pi\eta) \| {}_x D_\infty^\eta \hat{u} \|_{L^2(\mathbb{R})}^2. \end{aligned} \tag{2.10}$$

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n with $\alpha, \beta > -1$ defined on $\Lambda = (-1, 1)$. The Jacobi polynomials are defined by the three-term recurrence (cf. [31]):

$$\begin{aligned} P_{n+1}^{(\alpha,\beta)}(x) &= (a_n^{(\alpha,\beta)} x - b_n^{(\alpha,\beta)}) P_n^{(\alpha,\beta)}(x) - c_n^{(\alpha,\beta)} P_{n-1}^{(\alpha,\beta)}(x), \quad n \geq 1, \\ P_0^{(\alpha,\beta)}(x) &= 1, \quad P_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha + \beta + 2)x + \frac{1}{2}(\alpha - \beta), \end{aligned} \tag{2.11}$$

where

$$a_n^{(\alpha,\beta)} = \frac{(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)}{2(n + 1)(n + \alpha + \beta + 1)}, \tag{2.12}$$

$$b_n^{(\alpha,\beta)} = \frac{(\beta^2 - \alpha^2)(2n + \alpha + \beta + 1)}{2(n + 1)(n + \alpha + \beta + 1)(2n + \alpha + \beta)}, \tag{2.13}$$

$$c_n^{(\alpha,\beta)} = \frac{(n + \alpha)(n + \beta)(2n + \alpha + \beta + 2)}{(n + 1)(n + \alpha + \beta + 1)(2n + \alpha + \beta)}. \tag{2.14}$$

We have

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n P_n^{(\beta,\alpha)}(1), \quad P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)}. \tag{2.15}$$

For $\alpha, \beta > -1$, the Jacobi polynomials are orthogonal with respect to the Jacobi weight function: $\omega^{(\alpha,\beta)}(x) = (1 - x)^\alpha (1 + x)^\beta$, namely,

$$\int_{-1}^1 P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) \omega^{(\alpha,\beta)}(x) dx = \gamma_n^{(\alpha,\beta)} \delta_{mn}, \tag{2.16}$$

where δ_{mn} is the Kronecker symbol, and

$$\gamma_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1) n! \Gamma(n + \alpha + \beta + 1)}. \tag{2.17}$$

In particular, the Legendre polynomial is defined as $L_i(x) = J_i^{0,0}(x)$. Let us denote by I_N the Legendre–Gauss–Lobatto interpolation operator $I_N : C(\bar{\Omega}) \rightarrow \mathcal{W}_N$ as $\psi(x_k) = I_N \psi(x_k) \in \mathcal{P}_N, \quad k = 0, 1, \dots, N$.

In what follows, we shall use the Legendre–Gauss–Lobatto interpolation and quadrature. Let $\{x_j\}_{j=0}^N$ be the zeros of $(1 - x^2)L'_i(x)$, and let $\{\omega_j\}_{j=0}^N$ be the corresponding weights (cf. [31]). Then, the Legendre–Gauss–Lobatto quadrature has the exactness

$$\int_{-1}^1 f(x) dx = \sum_{j=0}^N f(x_j) \omega_j, \quad \forall f \in \mathcal{P}_{2N-1}, \tag{2.18}$$

where \mathcal{P}_N denotes the set of all polynomials of degree less than or equal to N .

3 The numerical algorithm

In this section, we shall develop a robust scheme for the problem (1.1)–(1.4) with a nonsmooth solution in time direction. The scheme employs the Legendre–Galerkin spectral method in space and an $L1$ -type approximation with graded mesh for the time-fractional derivative. In spite of the assumption that the initial conditions and source terms are smooth, the solutions of the fractional differential equation can be nonsmooth and even have a strong singularity at $t = 0$. Hence, the rate of convergence may be deteriorate significantly. To make up for the lost of accuracy near $t = 0$, we refer to the nonuniform $L1$ approach in [23].

Definition 5 For a positive integer M , let $\{t_i = T(i/M)^\gamma\}_{i=0}^M$ be the graded mesh with a grading parameter $\gamma > 1$ and $\tau_i := t_i - t_{i-1}$ for $i = 1, \dots, M$. Let $\{\psi^n\}_{n=0}^M$ be a sequence of real functions defined on Ω . The nonuniform discrete time-fractional difference Caputo operator Δ_M^β is given by

$$\Delta_M^\beta \psi_j^n = \sum_{i=1}^n a_{n-i}^{n,\beta} \delta_t \psi_j^i = \sum_{i=0}^n b_{n-i}^{n,\beta} \psi_j^i, \quad \forall n = 1, \dots, M, \tag{3.1}$$

where $a_{n-i}^{n,\beta} = \frac{\omega_{2-\beta}(t_n - t_{i-1}) - \omega_{2-\beta}(t_n - t_i)}{\tau_i}$, $1 \leq i \leq n$, $\omega_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, $t > 0$, $b_0^{n,\beta} = a_0^{n,\beta}$, $b_n^{n,\beta} = -a_{n-1}^{n,\beta}$, $b_{n-i}^{n,\beta} = a_{n-i}^{n,\beta} - a_{n-i-1}^{n,\beta}$, $\forall i = 1, \dots, n-1$ and $\delta_t \psi_j^i = \psi_j^i - \psi_j^{i-1}$.

The integral mean-value theorem says that the nonuniform $L1$ coefficient (3.1) satisfies

$$a_{n-i+1}^{n,\beta} < \omega_{1-\beta}(t_n - t_{i-1}) < a_{n-i}^{n,\beta}, \quad 1 \leq i \leq n. \tag{3.2}$$

Let ψ_j^n be the discrete approximation of solution $\psi_j(x, t_n) \forall j \in \{1, 2\}$, for $x \in \Omega$. At each time level n , the fractional temporal component of (1.1)–eqrefus12v can be approximated using (3.1), i.e.,

$$\sum_{r=0}^m q_r \Delta_M^{\beta_r} \psi_1^n - \gamma \frac{\partial^\alpha \psi_1^n}{\partial |x|^\alpha} = \rho F_1(\psi_1^n, \psi_2^n) + f_1^n, \tag{3.3}$$

$$\sum_{r=0}^m q_r \Delta_M^{\beta_r} \psi_2^n - \gamma \frac{\partial^\alpha \psi_2^n}{\partial |x|^\alpha} = \rho F_2(\psi_1^n, \psi_2^n) + f_2^n.$$

The approximation space \mathcal{W}_N is defined as

$$\mathcal{W}_N = \mathcal{P}_N(\Omega) \cap H_0^1(\Omega).$$

Then the $L1$ /spectral Galerkin scheme of (1.1)–(1.4) can be expressed as follows: for $n = 1, 2, \dots, M$, $j = 1, 2$, find $\psi_{j,N}^n \in \mathcal{W}_N$ such that

$$\sum_{r=0}^m q_r a_0^{n,\beta_r} (\psi_{1,N}^n, v_N) - \gamma \left(\frac{\partial^\alpha \psi_{1,N}^n}{\partial |x|^\alpha}, v_N \right) = \sum_{r=0}^m q_r a_{n-1}^{n,\beta_r} (\psi_{1,N}^0, v_N) - \sum_{r=0}^m q_r \sum_{i=1}^{n-1} b_{n-i}^{n,\beta_r} (\psi_{1,N}^i, v_N) + \rho (I_N(F_1(\psi_{1,N}^n, \psi_{2,N}^n)), v_N) + (I_N f_1^n, v_N),$$

$$\sum_{r=0}^m q_r a_0^{n,\beta_r} (\psi_{2,N}^n, v_N) - \gamma \left(\frac{\partial^\alpha \psi_{2,N}^n}{\partial |x|^\alpha}, v_N \right) = \sum_{r=0}^m q_r a_{n-1}^{n,\beta_r} (\psi_{2,N}^0, v_N) - \sum_{r=0}^m q_r \sum_{i=1}^{n-1} b_{n-i}^{n,\beta_r} (\psi_{2,N}^i, v_N) + \rho (I_N(F_2(\psi_{1,N}^n, \psi_{2,N}^n)), v_N) + (I_N f_2^n, v_N), \tag{3.4}$$

$$\psi_{1,N}^0 = P_N \phi_1, \quad \psi_{2,N}^0 = P_N \phi_2, \quad n = 1, \dots, M, \quad v_N \in \mathcal{W}_N^0,$$

where P_N is an appropriate projection operator, its related properties are given in the following theorem.

Lemma 5 (see [47]) *Let ϵ and s be real numbers satisfying $s > \frac{\alpha}{2}$. For any function $\psi_j \in H_0^{\frac{s}{2}}(\Omega) \cap H^s(\Omega)$, there exists a projector operator P_N such that the following error estimates hold:*

$$\|\psi_j - P_N \psi_j\| \leq CN^{-s} \|\psi_j\|_s, \quad \text{if } \alpha \neq \frac{3}{2}, \tag{3.5}$$

and

$$\|\psi_j - P_N \psi_j\| \leq CN^{\epsilon-s} \|\psi_j\|_s, \quad \text{if } \alpha = \frac{3}{2} \ \& \ 0 < \epsilon < \frac{1}{2}. \tag{3.6}$$

The function space \mathcal{W}_N can be expressed as

$$\mathcal{W}_N = \text{span}\{\varphi_n(x), n = 0, 1, \dots, N-2\}, \tag{3.7}$$

in which $\varphi_n(x)$ is defined as follows:

$$\varphi_n(x) = L_n(\hat{x}) - L_{n+2}(\hat{x}), \quad \hat{x} \in [-1, 1], x = \frac{(b-a)\hat{x} + a + b}{2} \in [a, b], \tag{3.8}$$

where $L_n(\hat{x})$ is the Legendre polynomial of order n . We expand the approximate solution as

$$\psi_{j,N}^n = \sum_{i=0}^{N-2} \hat{\psi}_{j,i}^n \varphi_i(x), \quad j \in \{1, 2\}, \tag{3.9}$$

where $\hat{\psi}_{j,i}^n$ are unknown coefficients. Plugging the above expression into (3.4) and taking $v_N = \varphi_k, 0 \leq k \leq N - 2$, we obtain the following matrix form representation using Lemma 2:

$$\left(\sum_{r=0}^m q_r a_0^{n,\beta_r} \bar{M} - \gamma c_\alpha (S + S^T) \right) \Psi_j^n = K_j^{n-1} + \rho H_j^n + F_j^n, \quad j \in \{1, 2\}, \tag{3.10}$$

where

$$\begin{aligned} K_j^{n-1} &= - \sum_{r=0}^m q_r \sum_{q=0}^{n-1} b^{n,\beta_r} \bar{M} \Psi_j^q \\ h_{j,i}^n &= \int_{\Omega} \varphi_i(x) I_N F_j (\psi_{1,N}^n, \psi_{2,N}^n) dx, \\ s_{ij} &= \int_{\Omega} {}_a D_x^{\frac{\alpha}{2}} \varphi_i(x) {}_x D_b^{\frac{\alpha}{2}} \varphi_j(x) dx, \\ S &= (s_{ij})_{i,j=0}^{N-2}, \\ m_{ij} &= \int_{\Omega} \varphi_i(x) \varphi_j(x) dx, \\ \bar{M} &= (m_{ij})_{i,j=0}^{N-2}, \\ H_j^n &= (h_{j,0}^n, h_{j,1}^n, \dots, h_{j,N-2}^n), \\ \Psi_j^n &= (\hat{\psi}_{j,0}^n, \hat{\psi}_{j,1}^n, \dots, \hat{\psi}_{j,N-2}^n)^T, \\ f_{i,j}^n &= \int_{\Omega} \varphi_i(x) I_N f_j^n dx, \\ F_j^n &= (f_{0,j}^n, f_{1,j}^n, \dots, f_{N-2,j}^n)^T. \end{aligned} \tag{3.11}$$

Lemma 6 ([31, 46]) *The elements of the stiffness matrix S are given as*

$$s_{ij} = a_i^j - a_i^{j+2} - a_{i+2}^j + a_{i+2}^{j+2}, \quad \forall i, j = 0, 1, \dots, N - 2, \tag{3.12}$$

where

$$\begin{aligned} a_i^j &= \int_{\Lambda} {}_a D_x^{\frac{\alpha}{2}} L_i(\hat{x}) {}_x D_b^{\frac{\alpha}{2}} L_j(\hat{x}) dx \\ &= \left(\frac{b-a}{2} \right)^{1-\alpha} \frac{\Gamma(i+1)\Gamma(j+1)}{\Gamma\left(i - \frac{\alpha}{2} + 1\right)\Gamma\left(j - \frac{\alpha}{2} + 1\right)} \\ &\quad \times \sum_{r=0}^N \varpi_r^{-\frac{\alpha}{2}, -\frac{\alpha}{2}} P_i^{\left(\frac{\alpha}{2}, -\frac{\alpha}{2}\right)} \\ &\quad \left(x_r^{-\frac{\alpha}{2}, -\frac{\alpha}{2}} \right) P_j^{\left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right)} \left(x_r^{-\frac{\alpha}{2}, -\frac{\alpha}{2}} \right), \end{aligned} \tag{3.13}$$

and $\left\{ \varpi_r^{-\frac{\alpha}{2}, -\frac{\alpha}{2}}, x_r^{-\frac{\alpha}{2}, -\frac{\alpha}{2}} \right\}_{i=0}^N$ are the set of Jacobi–Gauss weights and nodes corresponding to the Jacobi parameters $-\frac{\alpha}{2}, -\frac{\alpha}{2}$. The mass matrix \bar{M} is symmetric whose nonzero elements are

$$m_{ij} = m_{ji} = \begin{cases} \frac{b-a}{2j+1} + \frac{b-a}{2j+5}, & i = j, \\ -\frac{b-a}{2j+5}, & i = j + 2. \end{cases} \tag{3.14}$$

Noticing that $H_r^n = H_r^n(\psi_{1,N}^n, \psi_{2,N}^n)$, we can solve the non-linear system (3.10) by the following iteration algorithm:

Algorithm 1: Nonuniform L_1 - Galerkin-Legendre spectral scheme for problem (1.1)-(1.4)

Set $\Psi_s^{n,0} = \Psi_s^{n-1}, \psi_{s,N}^{n,0} = \sum_{j=0}^{N-2} \hat{\psi}_{s,j}^{n,0} \varphi_j(x), s = 1, 2;$

for $r = 0 : Q$ **do**

Solve $\begin{cases} \left(\sum_{r=0}^m q_r a_0^{n,\beta_r} \bar{M} - \gamma c_\alpha (S + S^T) \right) \Psi_1^{n,r+1} = \\ K_1^{n-1} - \rho H_1^{n,r} (\psi_{1,N}^n, \psi_{2,N}^n) + F_1^n, \\ \left(\sum_{r=0}^m q_r a_0^{n,\beta_r} \bar{M} - \gamma c_\alpha (S + S^T) \right) \Psi_2^{n,r+1} = \\ K_2^{n-1} - \rho H_2^{n,r} (\psi_{1,N}^n, \psi_{2,N}^n) + G^n, \end{cases}$ to get $\Psi_1^{n,r+1}$

and $\Psi_2^{n,r+1};$

Compute $\psi_{1,N}^{n,r+1} = \sum_{j=0}^{N-2} \hat{\psi}_{1,j}^{n,r+1} \varphi_j(x)$ and $\psi_{2,N}^{n,r+1} = \sum_{j=0}^{N-2} \hat{\psi}_{2,j}^{n,r+1} \varphi_j(x);$

if $\left\| \psi_{1,N}^{n,r+1} - \psi_{1,N}^{n,r} \right\| \leq \epsilon$ & $\left\| \psi_{2,N}^{n,r+1} - \psi_{2,N}^{n,r} \right\| \leq \epsilon$ **then**

| break;

end

end

Set $\Psi_1^n = \Psi_1^{n,r+1}$ and $\Psi_2^n = \Psi_2^{n,r+1}.$

4 Theoretical analysis

The present section is devoted to present sharp error estimates in the following theorem for the nonuniform $L1$ -Galerkin spectral scheme (3.4). In the sequel, C and C_ψ will denote generic positive constants independent of M, N and n , and may be different under different circumstances. We define the following space of functions:

$$C^\beta([0, T]; X) := \left\{ f \in C^\beta([0, T]; X) : \sup_{0 < \tau < T} \frac{\|u(\cdot, t + \tau) - u(\cdot, t)\|_{C([0, T - \tau]; X)}}{\tau^\beta} \leq \infty \right\},$$

and fix the following notation

$$A(\psi_j, w) = -\gamma c_\alpha \left[\left({}_a D_x^{\alpha/2} \psi_j, {}_x D_b^{\alpha/2} w \right) + \left({}_x D_b^{\alpha/2} \psi_j, {}_a D_x^{\alpha/2} w \right) \right]. \tag{4.1}$$

Define the orthogonal projection operator $P_N : H_0^2(\Omega) \rightarrow \mathcal{W}_N$ such that

$$A(\psi_j - P_N \psi_j, w) = 0, \quad \forall w \in \mathcal{W}_N. \tag{4.2}$$

For convenience of theoretical analysis, we give the following semi-norm and norm, respectively,

$$|\psi_j|_{\alpha/2} := A(\psi_j, \psi_j)^{1/2},$$

$$\|\psi_j\|_{\alpha/2} := \left(\|\psi_j\|^2 + |\psi_j|_{\alpha/2}^2 \right)^{1/2}. \tag{4.3}$$

The following two lemmas is deduced from [19] and used to obtain the truncation error for the multi-term fractional operator approximation

Lemma 7 Let $\{t_n = T(n/M)^\gamma\}_{n=0}^M$ be the graded mesh with a grading parameter $\gamma \geq 1$. Then for $\psi_j^n, j \in \{1, 2\}$, satisfies (1.1)–(1.4), the the truncating error in time can be represented by

$$\left| R_t^{n, \beta_r} \right| = \left| \frac{\partial^{\beta_r}}{\partial t^{\beta_r}} \psi_j(t_n) - \Delta_M^{\beta_r} \psi(t_n) \right|$$

$$\leq t_n^{-\beta_r} \left((\tau_1/t_n)^{1-\beta_r} + \max_{p=2, \dots, n} f^{p, \beta_r} \right), \tag{4.4}$$

$$\forall n = 1, \dots, M, \quad r = 1, \dots, m,$$

where

$$f^{1, \beta_r} := \tau_1^{\beta_r} \sup_{s \in (0, t_1)} \left(s^{1-\beta_r} \left| \delta_t(\psi_j(t_1)) - \partial_s \psi_j(s) \right| \right),$$

$$f^{p, \beta_r} := \tau_j^{2-\beta_r} t_p^{\beta_r} \sup_{s \in (t_{p-1}, t_p)} \left(s^{1-\beta_r} \left| \partial_s^2 \psi_j(s) \right| \right), \quad \forall p \geq 2. \tag{4.5}$$

Lemma 8 ([19]) Suppose that $|\partial_t^l \psi_j(t)| \leq 1 + t^{\beta_m - l}$, for $l = 1, 2$ and $t \in (0, T]$, then

$$\left| R_t^{n, \beta_r} \right| \leq t_n^{-\beta_m} \left((\tau_1/t_n)^{1-\beta_m} + \max_{p=2, \dots, n} f^{p, \beta_1} \right)$$

$$\leq t_n^{-\beta_m} M^{-\min\{\gamma\beta_m, 2-\beta_m\}},$$

$$\forall n = 1, \dots, M,$$

and the consistency error for the multi term fractional derivative in time given as

$$\left| R_t^n \right| = \sum_{r=1}^m q_r \left| R_t^{n, \beta_r} \right| \leq \mathbf{c} t_n^{-\beta_m} M^{-\min\{\gamma\beta_m, 2-\beta_m\}},$$

$$\mathbf{c} = \sum_{r=1}^m q_r, \quad \forall n = 1, \dots, M. \tag{4.7}$$

Now, the following lemma is devoted to introduce a proper discrete fractional Grönwall inequality.

Lemma 9 (Nonuniform discrete fractional Grönwall inequality [23, 24]) For any finite time $t_M = T > 0$, and a given nonnegative sequence $(\lambda_l)_{l=0}^{M-1}$, assume that there exists a constant λ , independent of time steps, such that $\lambda \geq \sum_{l=0}^{M-1} \lambda_l$ and let $\{v^n\}_{n=1}^M$ and $\{\zeta^n, g^n\}_{n=1}^M$ be sequences of non-negative numbers satisfying

$$\Delta_M^\beta (v^n)^2 \leq \sum_{l=1}^n \lambda_{n-l} (v^l)^2 + v^n (\zeta^n + g^n),$$

$$\forall n = 1, \dots, M. \tag{4.8}$$

If the time grids satisfy

$$\tau_{i-1} \leq \tau_i, \quad 2 \leq i \leq M, \tag{4.9}$$

then with the maximum time step

$$\tau_M \leq \sqrt[\beta]{\frac{1}{2\Gamma(2-\beta)\lambda}},$$

the following inequality holds:

$$\begin{aligned}
 v^n &\leq 2 \left(v^0 + \max_{1 \leq j \leq n} \sum_{l=1}^j P_{j-l}^{(j)} \zeta^l \right. \\
 &\quad \left. + v_{1+\beta}(t_n) \max_{1 \leq j \leq n} g^j \right) E_\beta(2\lambda t_n^\beta), \\
 &\quad \forall 1 \leq n \leq M,
 \end{aligned}
 \tag{4.10}$$

where $P_{n-j}^{(n)}$ is the discrete convolution kernel defined in [23] and $E_\beta(t)$ is the Mittag-Leffler function. Moreover, the convolution summation $\sum_{l=1}^j P_{j-l}^{(j)} \zeta^l$ in (4.10) is meaningful since it can be bounded by $\omega_{1+\beta}(t_j) \max_{1 \leq l \leq j} \zeta^l$.

Remark 2 The boundedness estimate (4.10) is also valid if the condition (4.8) is replaced by

$$\Delta_M^\beta v^n \leq \sum_{l=1}^n \lambda_{n-l} v^l + \zeta^n + g^n, \quad \forall n = 1, \dots, M.$$

In the following lemma, we clarify how to extend the Grönwall Lemma 9 for the multi-term fractional operators.

Theorem 1 For any time $\{t_M = T > 0\}$, and a given non-negative constant λ . Suppose that the grid function $\{v^n | n \geq 0\}$, such that $v^0 \geq 0$, satisfies

$$\sum_{r=1}^m q_r \Delta_M^{\beta_r} v^k \leq \lambda v^k + (\zeta^k + \eta^k), \quad k \geq 1,
 \tag{4.11}$$

where $\Delta_M^{\beta_r}$ is the fractional difference operator defined in (3.1), $q_r > 0$, and $\{\zeta^k, \eta^k, 1 \leq k \leq M\}$, are non-negative sequences. If the time grid fulfills $\tau_{k-1} \leq \tau_k, 2 \leq k \leq M$, with the maximum time-step restriction

$$\tau_M \leq \sqrt{\frac{q_m}{2\Gamma(2 - \beta_m)\lambda}},$$

it holds that

$$v^k \leq \frac{2}{q_m} \omega_{1+\beta_m}(t_k) \left(\sum_{l=1}^k (\zeta^l + \eta^l) + v^0 W \right) E_{\beta_m} \left(2 \frac{\lambda}{q_m} t_k^{\beta_m} \right), \quad \forall 1 \leq k \leq M.
 \tag{4.12}$$

Proof Multiply both sides of (4.11) by τ_M and summing over k from $1 \rightarrow j$ for both sides of (4.11), we conclude

$$\begin{aligned}
 \tau_M \sum_{r=1}^m q_r \sum_{k=1}^j \sum_{s=1}^k a_{k-s}^{k, \beta_r} (v^s - v^{s-1}) \\
 \leq \lambda \tau_M \sum_{k=1}^j v^k + \tau_M \sum_{k=1}^j (\zeta^k + \eta^k), \quad k \geq 1.
 \end{aligned}
 \tag{4.13}$$

Since

$$\begin{aligned}
 &\tau_M \sum_{k=1}^j \sum_{s=1}^k a_{k-s}^{k, \beta_r} (v^s - v^{s-1}) \\
 &= \tau_M \sum_{s=1}^j (v^s - v^{s-1}) \sum_{k=s}^j a_{k-s}^{k, \beta_r} \\
 &= \tau_M \sum_{s=1}^j v^s \sum_{k=s}^j a_{k-s}^{k, \beta_r} - \tau_M \sum_{s=1}^j v^{s-1} \sum_{k=s}^j a_{k-s}^{k, \beta_r} \\
 &= \tau_M \sum_{s=2}^{j+1} v^{s-1} \sum_{k=s-1}^j a_{k-s+1}^{k, \beta_r} \\
 &\quad - \tau_M \sum_{s=2}^j v^{s-1} \sum_{k=s}^j a_{k-s}^{k, \beta_r} - \tau_M v^0 \sum_{k=1}^j a_{k-1}^{k, \beta_r} \\
 &= \tau_M v^j a_0^{j, \beta_r} - \tau_M v^0 \sum_{k=1}^j a_{k-1}^{k, \beta_r} \\
 &\quad + \tau_M \sum_{s=2}^j v^{s-1} \left[\sum_{k=s-1}^j a_{k-s+1}^{k, \beta_r} - \sum_{k=s}^j a_{k-s+1}^{k, \beta_r} \right] \\
 &= \tau_M \sum_{s=1}^j a_{j-s}^{j, \beta_r} v^s - \tau_M v^0 \sum_{k=1}^j a_{k-1}^{k, \beta_r},
 \end{aligned}
 \tag{4.14}$$

then

$$\begin{aligned}
 \tau_M \sum_{r=1}^m q_r \sum_{k=1}^j \sum_{s=1}^k a_{k-s}^{k, \beta_r} (v^s - v^{s-1}) \\
 = \tau_M \sum_{r=1}^m q_r \sum_{s=1}^j a_{j-s}^{j, \beta_r} v^s \\
 - \tau_M v^0 \sum_{r=1}^m q_r \sum_{k=1}^j a_{k-1}^{k, \beta_r}.
 \end{aligned}
 \tag{4.15}$$

Denote

$$\sum_{r=1}^m q_r \sum_{k=1}^j a_{k-1}^{k, \beta_r} := W > 0,$$

then according to (4.13) and (4.15), we deduce

$$\begin{aligned}
 \tau_M \sum_{r=1}^m q_r \sum_{s=1}^j a_{j-s}^{j, \beta_r} v^s \\
 \leq \lambda \tau_M \sum_{k=1}^j v^k \\
 + \tau_M \sum_{k=1}^j (\zeta^k + \eta^k) + \tau_M v_0 W, \quad k \geq 1.
 \end{aligned}
 \tag{4.16}$$

Assume that

$$\tau_M \sum_{k=1}^j v^k = y^j, \quad y^0 = 0$$

then

$$\frac{y^k - y^{k-1}}{\tau_M} = v^k \geq 0, \quad k \geq 1.$$

According to that, (4.16) can have the following form:

$$\begin{aligned} & \sum_{r=1}^m q_r \sum_{s=1}^j a_{j-s}^{i,\beta_r} (y^s - y^{s-1}) \\ &= \sum_{r=1}^m q_r \Delta_M^{\beta_r} y^j \leq \lambda y^j \\ &+ \tau_M \sum_{k=1}^j (\zeta^k + \eta^k) + \tau_M v_0 W. \end{aligned} \tag{4.17}$$

Now, as

$$(y^s - y^{s-1}) \geq 0,$$

after using the assumption of the positivity of q_m , we get

$$\begin{aligned} & \Delta_M^{\beta_m} y^j \\ & \leq \frac{1}{q_m} \left[\lambda y^j + \tau_M \sum_{k=1}^j (\zeta^k + \eta^k) + \tau_M v_0 W \right], \end{aligned} \tag{4.18}$$

$1 \leq j \leq M.$

Now, applying Lemma 9, we get

$$\begin{aligned} y^j &= \tau_M \sum_{k=1}^j v^k \\ &\leq \frac{2\tau_M}{q_m} \omega_{1+\beta_m}(t_j) \\ &\left(\sum_{k=1}^j (\zeta^k + \eta^k) + v_0 W \right) E_{\beta_m} \left(2 \frac{\lambda}{q_m} t_j^{\beta_m} \right), \end{aligned} \tag{4.19}$$

then the proof is fulfilled. □

The following theorem is devoted to study the convergence analysis.

Theorem 2 Let $\{\psi_1, \psi_2\}$ and $\{\psi_{1,N}^n, \psi_{2,N}^n\}$ be solutions of (1.1) – (1.4) and (3.4), and $\psi_i \in C^\beta([0, T]; L^2(\Omega)) \cap C([0, T; H_0^{\frac{\alpha}{2}}(\Omega) \cap H^s(\Omega))$, and suppose that $\{\psi_1, \psi_2\}$ is sufficiently regular in spatial directions and has a regularity hypothesis on temporal derivatives of $\{\psi_1, \psi_2\}$ reads $\left\| \frac{\partial^2 \psi_j}{\partial t^2} \right\|_{L^\infty(I)} \leq C(1 + t^{\beta_m - 2})$. Assume also that $\frac{\partial^\beta \psi_j}{\partial t^\beta} \in C([0, T; H_0^{\frac{\alpha}{2}}(\Omega) \cap H^s(\Omega))$. Also, the functions F_j in (1.1)–(1.4) are Lipschitz, $f_j \in C(0, T; H^r(\Omega))$ and $\phi_i \in H_0^{\frac{\alpha}{2}}(\Omega) \cap H^s(\Omega)$, then, if $\tau_M \leq \sqrt{\frac{q_m}{2\Gamma(2-\beta_m)\lambda}}$ holds, the Galerkin spectral scheme (3.4) admits a unique solution

$\{\psi_{1,N}^n, \psi_{2,N}^n\}$ under smoothly graded meshes $t_n = T(n/M)^r$ satisfying

$$\begin{aligned} & \left\| \psi_{1,N}^n - \psi_1(x, t_n) \right\| + \left\| \psi_{2,N}^n - \psi_2(x, t_n) \right\| \\ & \leq \mathbf{C} (M^{-\min\{\gamma\beta_m, 2-\beta_m\}} + N^{-s} + N^{-r}), \end{aligned} \tag{4.20}$$

if $\alpha \neq \frac{3}{2}$,

$$\begin{aligned} & \left\| \psi_{1,N}^n - \psi_1(x, t_n) \right\| + \left\| \psi_{2,N}^n - \psi_2(x, t_n) \right\| \\ & \leq \mathbf{C} (M^{-\min\{\gamma\beta_m, 2-\beta_m\}} + N^{\epsilon-s} + N^{\epsilon-r}), \end{aligned} \tag{4.21}$$

if $\alpha = \frac{3}{2}$ and $0 < \epsilon < \frac{1}{2}$,

such that \mathbf{C} is a positive constant independent of N, M and n . Particularly, the numerical solution $\{\psi_{1,N}^n, \psi_{2,N}^n\}$ achieves an optimal time convergence order $O(M^{\beta_m - 2})$ for the optimal grid parameter $\gamma_{opt} = (2 - \beta_m)/\beta_m$.

Proof The following variational form can be given thanks to (7), (3.3) at $i = 1$ and using the notation (4.1),

$$\begin{aligned} & \left(\sum_{r=1}^m q_r \Delta_M^{\beta_r} \psi_1^n, v_1 \right) + A(\psi_1^n, v_1) \\ & - \rho (F_1(\psi_1^n, \psi_2^n), v_1) - (f_1^n, v_1) + (R_1^n, v_1) = 0. \end{aligned} \tag{4.22}$$

Let $e_1 = \psi_1 - \psi_{1,N}$, $\zeta_1 = \psi_1 - P_N \psi_1$ and $\eta_1 = P_N \psi_1 - \psi_{1,N}$, we get $e_1^n = \zeta_1^n + \eta_1^n$. Using Lemma 5, we have the following estimate in the case of $\alpha \neq \frac{3}{2}$,

$$\|e_1^n\| \leq \|\zeta_1^n\| + \|\eta_1^n\| \leq CN^{-s} \|\psi_1\|_s + \|\eta_1^n\|. \tag{4.23}$$

The full discretization of the first equation in (3.4) can be rewritten generally as

$$\begin{aligned} & \left(\sum_{r=1}^m q_r \Delta_M^{\beta_r} \psi_{1,N}^n, v_1 \right) \\ & + A(\psi_{1,N}^n, v_1) - \rho \left(I_N (F_1(\psi_{1,N}^n, \psi_{2,N}^n)), v_1 \right) \\ & - (I_N f_1^n, v_1) + (R_1^n, v_1) = 0. \end{aligned} \tag{4.24}$$

Then the error weak form is achieved by taking $v_1 = \eta_1^n$,

$$\begin{aligned} & \left(\sum_{r=1}^m q_r \Delta_M^{\beta_r} e_1^n, \eta_1^n \right) + A(e_{1,N}^n, \eta_1^n) \\ & - \rho \left(I_N (F_1(\psi_{1,N}^n, \psi_{2,N}^n)) - F_1(\psi_1^n, \psi_2^n), \eta_1^n \right) \\ & - (I_N f_1^n - f_1^n, \eta_1^n) + (R_1^n, \eta_1^n) = 0. \end{aligned} \tag{4.25}$$

Starting from the first term in the left-hand side of (4.25) and according to [24],

$$\begin{aligned} (\Delta_M^{\beta_r} e_1^n, \eta_1^n) &= (\Delta_M^{\beta_r} \eta_1^n, \eta_1^n) + (\Delta_M^{\beta_r} \zeta_1^n, \eta_1^n) \\ &\geq \frac{1}{2} \Delta_M^{\beta_r} \|\eta_1^n\|^2 + (\Delta_M^{\beta_r} \zeta_1^n, \eta_1^n). \end{aligned} \tag{4.26}$$

For the second term in the left-hand side of (4.25) and by considering (4.1) and (4.3), we get

$$A(e_1^n, \eta_1^n) = A(\zeta_1^n, \eta_1^n) + A(\eta_1^n, \eta_1^n) = |\eta_1^n|_{\alpha/2}^2. \tag{4.27}$$

Substituting those results into (4.25), we deduce

$$\begin{aligned} &\sum_{r=1}^m q_r \Delta_M^{\beta_r} \|\eta_1^n\|^2 + 2|\eta_1^n|_{\alpha/2}^2 \\ &\leq 2 \sum_{r=1}^m q_r (\Delta_M^{\beta_r} \zeta_1^n, \eta_1^n) \\ &\quad + 2\rho \left(I_N \left(F_1(\psi_{1,N}^n, \psi_{2,N}^n) \right) - F_1(\psi_1^n, \psi_2^n), \eta_1^n \right) \\ &\quad + 2(I_N f_1^n - f_1^n, \eta_1^n) + 2(R_t^n, \eta_1^n). \end{aligned} \tag{4.28}$$

Neglecting the second positive term in the left-hand side of (4.28) and using Lemma 5 to bound the first term in the right-hand side of (4.28) at $\alpha \neq \frac{3}{2}$, we obtain

$$\begin{aligned} &\sum_{r=1}^m q_r \Delta_M^{\beta_r} \|\eta_1^n\|^2 \\ &\leq 2CN^{-s} \sum_{r=1}^m q_r \|\Delta_M^{\beta_r} \psi_1^n\|_s \|\eta_1^n\| \\ &\quad + 2\rho \left(I_N \left(F_1(\psi_{1,N}^n, \psi_{2,N}^n) \right) - F_1(\psi_1^n, \psi_2^n), \eta_1^n \right) \\ &\quad + 2(I_N f_1^n - f_1^n, \eta_1^n) + 2(R_t^n, \eta_1^n). \end{aligned} \tag{4.29}$$

On the other hand, we use (4.23), Lemma 3 and the Lipschitz condition on F to reach

$$\begin{aligned} &\|I_N F_1(\psi_{1,N}^n, \psi_{2,N}^n) - F_1(\psi_1^n, \psi_2^n)\| \\ &\leq \|I_N (F_1(\psi_{1,N}^n, \psi_{2,N}^n) - F_1(\psi_1^n, \psi_2^n))\| \\ &\quad + \|I_N F_1(\psi_1^n, \psi_2^n) - F_1(\psi_1^n, \psi_2^n)\| \\ &\leq C \left(\|F_1(\psi_{1,N}^n, \psi_{2,N}^n) - F_1(\psi_1^n, \psi_2^n)\| \right) \\ &\quad + CN^{-s} \left(\|\psi_{1,N}^n\|_s + \|\psi_{2,N}^n\|_s \right) \\ &\leq CN^{-s} \left(\|\psi_{1,N}^n\|_s + \|\psi_{2,N}^n\|_s \right) \\ &\quad + C(\|\eta_1^n\| + \|\eta_2^n\|). \end{aligned} \tag{4.30}$$

In the following, rearrange terms conveniently, take the absolute value and use the triangle inequality. Next, use

Young’s inequality and Lemma 3 side by side to (4.29) to obtain

$$\begin{aligned} &\sum_{r=1}^m q_r \Delta_M^{\beta_r} \|\eta_1^n\|^2 \\ &\leq 2CN^{-s} \sum_{r=1}^m q_r \|\Delta_M^{\beta_r} \psi_1^n\|_s \|\eta_1^n\| \\ &\quad + 2\rho \|I_N (F_1(\psi_{1,N}^n, \psi_{2,N}^n)) - F_1(\psi_1^n, \psi_2^n)\| \|\eta_1^n\| \\ &\quad + 2\|I_N f_1^n - f_1^n\| \|\eta_1^n\| + (R_t^n, \eta_1^n) \\ &\leq \tilde{C} \left((N^{-s} + N^{-r})^2 + \|\eta_1^n\|^2 + \|\eta_1^n\| \|\eta_2^n\| + (R_t^n)^2 \right), \end{aligned} \tag{4.31}$$

such $\tilde{C} = 6 \times \max\{2C \sum_{r=1}^m q_r \|\Delta_M^{\beta_r} \psi_{1,N}^n\|_s, 2C \|\psi_{1,N}^n\|_s, 2C \|\psi_{1,N}^n\|_r, 2\rho C (\|\psi_{1,N}^n\|_s + \|\psi_{2,N}^n\|_s), 2\rho C, 4\}$.

Similarly, by following the same steps for (1.1) at $i = 2$ and its numerical form in (3.4), we conclude

$$\begin{aligned} &\sum_{r=1}^m q_r \Delta_M^{\beta_r} \|\eta_2^n\|^2 \\ &\leq \tilde{C} \left((N^{-s} + N^{-r})^2 + \|\eta_2^n\|^2 + \|\eta_1^n\| \|\eta_2^n\| + (R_t^n)^2 \right), \end{aligned} \tag{4.32}$$

such $\tilde{C} = 6 \times \max\{2C \sum_{r=1}^m q_r \|\Delta_M^{\beta_r} \psi_{2,N}^n\|_s, 2C \|\psi_{2,N}^n\|_s, 2C \|\psi_{2,N}^n\|_r, 2\rho C (\|\psi_{2,N}^n\|_s + \|\psi_{1,N}^n\|_s), 2\rho C, 4\}$. Adding (4.31) and (4.32) together, we obtain

$$\begin{aligned} &\sum_{r=1}^m q_r \Delta_M^{\beta_r} (\|\eta_1^n\|^2 + \|\eta_2^n\|^2) \\ &\leq 4\hat{C} \left(\|\eta_1^n\|^2 + \|\eta_2^n\|^2 + (R_t^n)^2 + (N^{-s} + N^{-r})^2 \right), \end{aligned} \tag{4.33}$$

where $\hat{C} = \max\{\tilde{C}, \tilde{C}\}$. A direct application of Theorem 1 and Lemma 8 yields the result. Then the proof is fulfilled when $\alpha \neq \frac{3}{2}$ and the desired result is obtained. The proof when $\alpha = \frac{3}{2}$ and $0 < \epsilon < \frac{1}{2}$ can be derived similarly. \square

5 Numerical results

Consider the following coupled system of ten-term fractional diffusion equations:

$$\begin{aligned} &\sum_{r=1}^{10} \frac{\partial^{\beta_r} \psi_1}{\partial t^{\beta_r}} - \frac{\partial^\alpha \psi_1}{\partial |x|^\alpha} \\ &= \sin(\psi_1) + \psi_2^2 + f_1(x, t), \quad x \in (0, 1), t \in (0, 1], \end{aligned} \tag{5.1}$$

Table 1 Numerical temporal accuracy of ψ_1 and ψ_2 with $N = 100$

M	$\gamma = 2$		$\gamma = 2$		$\gamma = 3$		$\gamma = 3$	
	Error (ψ_1)	Order	Error (ψ_2)	Order	Error (ψ_1)	Order	Error (ψ_2)	Order
50	2.87×10^{-4}	–	5.85×10^{-5}	–	7.39×10^{-5}	–	1.48×10^{-5}	–
100	1.46×10^{-4}	0.97	3.04×10^{-5}	0.94	2.76×10^{-5}	1.42	5.69×10^{-6}	1.39
200	7.48×10^{-5}	0.97	1.56×10^{-5}	0.96	1.02×10^{-5}	1.43	2.13×10^{-6}	1.42
400	3.79×10^{-5}	0.98	7.97×10^{-6}	0.97	3.76×10^{-6}	1.45	7.89×10^{-7}	1.44
800	1.92×10^{-5}	0.98	4.05×10^{-6}	0.98	1.37×10^{-6}	1.45	2.89×10^{-7}	1.45
1600	9.72×10^{-6}	0.98	2.05×10^{-6}	0.98	4.99×10^{-7}	1.46	1.05×10^{-7}	1.46
$\min \{\gamma\beta_m, 2 - \beta_m\}$		1.00		1.00		1.50		1.50

Table 2 Numerical spatial accuracy of ψ_1 and ψ_2 with $\gamma = 3$ and $M = 1600$

N	Error (ψ_1)	Order	Error (ψ_2)	Order
10	2.90×10^{-5}	$N^{-4.53}$	1.24×10^{-5}	$N^{-4.90}$
20	1.94×10^{-6}	$N^{-4.39}$	6.08×10^{-7}	$N^{-4.77}$
40	4.99×10^{-7}	$N^{-3.93}$	1.05×10^{-7}	$N^{-4.35}$
60	4.99×10^{-7}	$N^{-3.54}$	1.05×10^{-7}	$N^{-3.92}$
80	4.99×10^{-7}	$N^{-3.31}$	1.05×10^{-7}	$N^{-3.67}$
100	4.99×10^{-7}	$N^{-3.15}$	1.05×10^{-7}	$N^{-3.49}$

$$\sum_{r=1}^{10} \frac{\partial^{\beta_r} \psi_2}{\partial t^{\beta_r}} - \frac{\partial^\alpha \psi_2}{\partial |x|^\alpha} = \sin(\psi_2) + \psi_1^2 + f_2(x, t), x \in (0, 1), t \in (0, 1]. \tag{5.2}$$

We choose the fractional orders $\beta_r = \frac{1+2m-r}{4m}$ and $\alpha = 1.5$. The nonsmooth solutions $\psi_1(x, t) = t^{\beta_m} x(1-x)^2 \sin(\pi x)$ and $\psi_2(x, t) = (t^{\beta_m} + t)x^2(1-x)^2 \cos(\pi x)$ are used to generate the source functions f_1 and f_2 and the initial-boundary conditions. Tables 1 and 2 show the convergence rates of the approximate solutions of ψ_1 and ψ_2 in temporal and spatial directions, respectively.

6 Conclusion

Due to the lack of a discrete fractional Grönwall-type inequality, the techniques of analyzing the $L1$ difference schemes can not be applied directly to multi-term fractional diffusion equations that contains positive reaction term on a nonuniform time mesh, especially when the maximum order of the fractional derivatives is not an integer. This limitation was overcome in Theorem 1 by exploiting a novel discrete fractional Grönwall inequality for the $L1$ scheme over a nonuniform time mesh. This inequality was a main result. Then we proposed a finite difference/spectral method to solve coupled systems of nonlinear multi-term time-space fractional diffusion

equations with non-smooth solutions in the time direction. The proposed method combined the strength of the $L1$ scheme with temporal nonuniform mesh and the Galerkin–Legendre method. Finally, we illustrated the use of Theorem 1 by outlining some convergence estimates of the scheme. The numerical results accompanying our analysis supported our theoretical contributions. The proposed approach extended our earlier work [44] that uses the $L1$ approximation to the single-order fractional derivatives.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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