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ORIGINAL ARTICLE

Mixed element algorithm based on a second-order time approximation scheme for a two-dimensional nonlinear time fractional coupled sub-diffusion model

Ruihan Feng[1] · Yang Liu[1] · Yaxin Hou[1] · Hong Li[1] · Zhichao Fang[1]

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Abstract
In this article, a numerical algorithm is presented to solve a two-dimensional nonlinear time fractional coupled sub-diffusion problem, where the second-order Crank–Nicolson scheme with a second-order WSGD formula is used in the time direction, and a mixed element method is applied in the space direction. The existence and uniqueness of the mixed element solution and the stability for fully discrete scheme are proven. In addition, the optimal a priori error estimates for unknown scalar function u and v in L^2 and H^1-norms and a priori error estimates for their fluxes σ and λ in $(L^2)^2$-norm are obtained. Finally, some numerical calculations are presented to illustrate the validity for the proposed numerical algorithm.

Keywords Nonlinear time fractional coupled sub-diffusion model · Second-order Crank–Nicolson scheme · Mixed element method

1 Introduction

Fractional calculus and models [1–12] have been studied and developed by a large number of authors. Especially, fractional diffusion models including time, space and space-time fractional cases have attracted a lot of attention. Time fractional diffusion models have many different forms of expression based on different application areas, such as the fractional Cable model [13–17], fractional mobile/immobile transport equation [18], fractional reaction-diffusion model [19–29], fractional fourth-order diffusion system [30, 31] and multi-term fractional subdiffusion model [32]. Space fractional diffusion problems can be expressed as fractional Allen–Cahn models [33–36], space-fractional reaction-diffusion models [37–41], space-time fractional diffusion problems [42–46]. In these fractional diffusion models, the coupled time-fractional diffusion systems are a kind of important mathematical models, which can reflect the interaction between different substances. Recently, some authors considered this kind of fractional coupled diffusion system. Hou et al. [47] developed a mixed finite element method for a coupled time fractional diffusion equation with a nonlinear term. Authors studied the $L1$-approximation for the Caputo time fractional derivative and obtained the temporal error result with $O(\tau^{2-\alpha}+\tau^{2-\beta})$. In [48], Kumar et al. considered the Galerkin finite element schemes based on the Crank–Nicolson algorithm for a coupled time-fractional nonlinear diffusion model.

In this paper, we consider a mixed element method to solve the following time fractional coupled sub-diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t}+k_1\dfrac{\partial^\alpha u}{\partial t^\alpha}=a\Delta u+f(u,v)+\bar{f}(\mathbf{x},t), \quad (\mathbf{x},t)\in\Omega\times J,\\ \dfrac{\partial v}{\partial t}+k_2\dfrac{\partial^\beta v}{\partial t^\beta}=c\Delta v+g(u,v)+\bar{g}(\mathbf{x},t), \quad (\mathbf{x},t)\in\Omega\times J,\\ u(\mathbf{x},t)=v(\mathbf{x},t)=0, \quad (\mathbf{x},t)\in\partial\Omega\times\bar{J},\\ u(\mathbf{x},0)=u_0(\mathbf{x}), v(\mathbf{x},0)=v_0(\mathbf{x}), \quad \mathbf{x}\in\bar{\Omega}, \end{cases} \tag{1}$$

✉ Yang Liu
mathliuyang@imu.edu.cn

Ruihan Feng
imath_frh@163.com

Yaxin Hou
15947413053@163.com

Hong Li
smslh@imu.edu.cn

Zhichao Fang
zcfang@imu.edu.cn

1 School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China

where $\Omega \subset R^d (d \leqslant 2)$ is a bounded convex polygonal region satisfying Lipschitz continuous boundary $\partial\Omega$, and $J = (0, T]$ is the time interval with $0 < T < \infty$. $u_0(\mathbf{x}), v_0(\mathbf{x}), \bar{f}(\mathbf{x}, t)$ and $\bar{g}(\mathbf{x}, t)$ are known functions. Parameters k_1, k_2, a and c are positive constants. The nonlinear terms $f(u, v)$ and $g(u, v)$ are two different quadratic polynomials without constant terms about u and v. The notation $\frac{\partial^\gamma u}{\partial t^\gamma}$ is the Caputo fractional derivative defined by

$$\frac{\partial^\gamma u(\mathbf{x}, t)}{\partial t^\gamma} = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\partial u(\mathbf{x}, s)}{\partial s} \frac{\mathrm{d}s}{(t-s)^\gamma}, \quad (0 < \gamma < 1), \tag{2}$$

where $\gamma = \alpha$ or β.

In this article, our main purpose is to present a second-order Crank–Nicolson mixed element algorithm for solving the two-dimensional nonlinear time fractional coupled sub-diffusion system, where the time fractional derivative is approximated by a weighted and shifted Grünwald difference (WSGD) formula, which was proposed by Tian et al. [49] and were also developed by Wang and Vong [50], Liu et al. [13], Liu et al. [15], Du et al. [51]. There exist some researches on the mixed element methods, see [47, 52, 53, 56]. In [47], Hou et al. developed a mixed element method for a one-dimensional time fractional coupled system, and obtained a lower time convergence rate. In [52], authors considered the mixed element method with a first-order time approximation scheme for a linear fractional sub-diffusion problem. In [53], Zhao et al. used the $L1$ approximation with the mixed element method for a linear fractional diffusion model. In [54], Li et al. solved a fractional diffusion wave model by using a mixed element method. In [56], Abbaszadeh and Dehghan solved a linear fractional reaction-diffusion model by using a mixed element method, derived the error estimate, and implemented the numerical tests. Here, we will show the detailed numerical analyses and calculations of the mixed element method with a time second-order convergence order for solving the two-dimensional nonlinear coupled time fractional diffusion model covering initial-boundary conditions. Based on these considerations, our main results are as the following:

- Compared with the standard finite element method, the approximation solutions for unknown scalar functions and their fluxes can be obtained by the mixed element method.
- The existence and uniqueness of the mixed element solution, the stability of the considered mixed element system and the a priori error estimates for unknown scalar functions and their fluxes are proven or derived.
- A two-dimensional example for verifying the spatial convergence rate and a one-dimensional example for checking the second-order time convergence order are provided, respectively.

Throughout the full text, we will use $C > 0$ as a positive constant, which is independent of space mesh h, time step τ and fractional parameters α, β. The structure of the article is as follows: In Sect. 2, based on the definition of the Caputo fractional derivative, some approximation formulas for the time fractional derivatives are provided and the mixed finite element scheme is proposed. In Sect. 3, the existence and uniqueness of the mixed element solution are proved and the stability is derived. In Sect. 4, some a priori error estimates are obtained. In Sect. 5, the numerical examples are provided to verify the theory results. In Sect. 6, the remarking conclusions are summarized.

2 Mixed element approximation

By introducing two auxiliary variables $\sigma = \nabla u$ and $\lambda = \nabla v$, the coupled system (1) can be rewritten as

$$\begin{cases} \dfrac{\partial u}{\partial t} + k_1 \dfrac{\partial^\alpha u}{\partial t^\alpha} = a\nabla \cdot \sigma + f(u, v) + \bar{f}(\mathbf{x}, t), \\ \dfrac{\partial v}{\partial t} + k_2 \dfrac{\partial^\beta v}{\partial t^\beta} = c\nabla \cdot \lambda + g(u, v) + \bar{g}(\mathbf{x}, t), \\ \sigma - \nabla u = 0, \\ \lambda - \nabla v = 0. \end{cases} \tag{3}$$

In the following content, we will give the fully discrete scheme.

Let $0 = t_0 < t_1 < t_2 < \cdots < t_M = T$ be a grid partition for the temporal interval $[0, T]$, where $t_n = n\tau, \tau = \frac{T}{M}$ is the time step and M is a positive integer. With $u^n = u(\mathbf{x}, t_n)$, we introduce the following notations:

$$\begin{aligned} &D_t l^{n+\frac{1}{2}} = \frac{l^{n+1} - l^n}{\tau}, \quad l^{n+\frac{1}{2}} = \frac{l^{n+1} + l^n}{2}, \\ &\sum_{i=0}^{n} p_\gamma(i) l^{n+\frac{1}{2}-i} = \frac{1}{2}\left(\sum_{i=0}^{n+1} p_\gamma(i) l^{n+1-i} + \sum_{i=0}^{n} p_\gamma(i) l^{n-i}\right), \\ &l(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}) = \begin{cases} \frac{3}{2} l(u^n, v^n) - \frac{1}{2} l(u^{n-1}, v^{n-1}), & n \geqslant 1, \\ l(u^0, v^0), & n = 0. \end{cases} \end{aligned} \tag{4}$$

Lemma 1 *Following Refs.*[15, 50], *we can get the discrete formulation of the fractional derivative* (2) *as follows*

$$\frac{\partial^\gamma l(\mathbf{x}, t_{n+\frac{1}{2}})}{\partial t^\gamma} = \sum_{i=0}^{n} \frac{p_\gamma(i)}{\tau^\gamma} l^{n+\frac{1}{2}-i} + O(\tau^2), \tag{5}$$

where

$$p_\gamma(i) = \begin{cases} \frac{\gamma+2}{2} g_0^\gamma, & i = 0, \\ \frac{\gamma+2}{2} g_i^\gamma + \frac{-\gamma}{2} g_{i-1}^\gamma, & i \geqslant 1, \end{cases}$$

$$g_0^\gamma = 1, g_i^\gamma = \frac{\Gamma(i-\gamma)}{\Gamma(-\gamma)\Gamma(i+1)}, g_i^\gamma = \left(1 - \frac{\gamma+1}{i}\right) g_{i-1}^\gamma, \quad i \geqslant 1.$$

Based on Lemma 1 and Refs. [30, 50], we can get the following time discrete formulation of the coupled system (3) at time $t_{n+\frac{1}{2}}$

Case 1: $n \geqslant 1$

$$\begin{cases} (a)D_t u^{n+\frac{1}{2}} + k_1\tau^{-\alpha}\sum_{i=0}^{n} p_\alpha(i)u^{n+\frac{1}{2}-i} - a\nabla\cdot\sigma^{n+\frac{1}{2}} \\ \quad = f(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}) + \bar{f}^{n+\frac{1}{2}} + R_1, \\ (b)\sigma^{n+\frac{1}{2}} - \nabla u^{n+\frac{1}{2}} = R_2, \\ (c)D_t v^{n+\frac{1}{2}}, + k_2\tau^{-\beta}\sum_{i=0}^{n} p_\beta(i)v^{n+\frac{1}{2}-i} - c\nabla\cdot\lambda^{n+\frac{1}{2}} \\ \quad = g(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}) + \bar{g}^{n+\frac{1}{2}} + R_3, \\ (d)\lambda^{n+\frac{1}{2}} - \nabla v^{n+\frac{1}{2}} = R_4, \end{cases} \tag{6}$$

Case 2: $n = 0$

$$\begin{cases} (a)D_t u^{\frac{1}{2}} + k_1\tau^{-\alpha}\sum_{i=0}^{0} p_\alpha(i)u^{\frac{1}{2}-i} - a\nabla\cdot\sigma^{\frac{1}{2}} = f(u^{\frac{1}{2}}, v^{\frac{1}{2}}) + \bar{f}^{\frac{1}{2}} + R_5, \\ (b)\sigma^{\frac{1}{2}} - \nabla u^{\frac{1}{2}} = R_6, \\ (c)D_t v^{\frac{1}{2}}, + k_2\tau^{-\beta}\sum_{i=0}^{0} p_\beta(i)v^{\frac{1}{2}-i} - c\nabla\cdot\lambda^{\frac{1}{2}} = g(u^{\frac{1}{2}}, v^{\frac{1}{2}}) + \bar{g}^{\frac{1}{2}} + R_7, \\ (d)\lambda^{\frac{1}{2}} - \nabla v^{\frac{1}{2}} = R_8, \end{cases} \tag{7}$$

where

$$R_1 = E_1^{n+\frac{1}{2}} + E_3^{n+\frac{1}{2}} + E_5^{n+\frac{1}{2}} + E_{11}^{n+\frac{1}{2}} + E_{13}^{n+\frac{1}{2}} = O(\tau^2), R_2 = E_7^{n+\frac{1}{2}} + E_9^{n+\frac{1}{2}} = O(\tau^2),$$

$$R_3 = E_2^{n+\frac{1}{2}} + E_4^{n+\frac{1}{2}} + E_6^{n+\frac{1}{2}} + E_{12}^{n+\frac{1}{2}} + E_{14}^{n+\frac{1}{2}} = O(\tau^2), R_4 = E_8^{n+\frac{1}{2}} + E_{10}^{n+\frac{1}{2}} = O(\tau^2),$$

$$R_5 = E_1^{\frac{1}{2}} + E_3^{\frac{1}{2}} + E_5^{\frac{1}{2}} + E_{11}^{\frac{1}{2}} + E_{13}^{\frac{1}{2}} = O(\tau), R_6 = E_7^{\frac{1}{2}} + E_9^{\frac{1}{2}} = O(\tau^2),$$

$$R_7 = E_2^{\frac{1}{2}} + E_4^{\frac{1}{2}} + E_6^{\frac{1}{2}} + E_{12}^{\frac{1}{2}} + E_{14}^{\frac{1}{2}} = O(\tau), R_8 = E_8^{\frac{1}{2}} + E_{10}^{\frac{1}{2}} = O(\tau^2),$$

$$E_1^{n+\frac{1}{2}} = u(t_{n+\frac{1}{2}}) - D_t u^{n+\frac{1}{2}} = \begin{cases} O(\tau^2), n \geqslant 1, \\ O(\tau), n = 0, \end{cases}$$

$$E_2^{n+\frac{1}{2}} = v(t_{n+\frac{1}{2}}) - D_t v^{n+\frac{1}{2}} = \begin{cases} O(\tau^2), n \geqslant 1, \\ O(\tau), n = 0, \end{cases}$$

$$E_3^{n+\frac{1}{2}} = k_1\frac{\partial^\alpha u(t_{n+\frac{1}{2}})}{\partial t^\alpha} - k_1\sum_{i=0}^{n}\frac{p_\alpha(i)}{\tau^\alpha}u^{n+\frac{1}{2}-i} = O(\tau^2),$$

$$E_4^{n+\frac{1}{2}} = k_2\frac{\partial^\beta v(t_{n+\frac{1}{2}})}{\partial t^\beta} - k_2\sum_{i=0}^{n}\frac{p_\beta(i)}{\tau^\beta}v^{n+\frac{1}{2}-i} = O(\tau^2),$$

$$E_5^{n+\frac{1}{2}} = a\nabla\cdot\sigma(t_{n+\frac{1}{2}}) - a\nabla\cdot\sigma^{n+\frac{1}{2}} = O(\tau^2),$$

$$E_6^{n+\frac{1}{2}} = c\nabla\cdot\lambda(t_{n+\frac{1}{2}}) - c\nabla\cdot\lambda^{n+\frac{1}{2}} = O(\tau^2),$$

$$E_7^{n+\frac{1}{2}} = \sigma(t_{n+\frac{1}{2}}) - \sigma^{n+\frac{1}{2}} = O(\tau^2), E_8^{n+\frac{1}{2}} = \lambda(t_{n+\frac{1}{2}}) - \lambda^{n+\frac{1}{2}} = O(\tau^2),$$

$$E_9^{n+\frac{1}{2}} = \nabla u(t_{n+\frac{1}{2}}) - \nabla u^{n+\frac{1}{2}} = O(\tau^2), E_{10}^{n+\frac{1}{2}} = \nabla v(t_{n+\frac{1}{2}}) - \nabla v^{n+\frac{1}{2}} = O(\tau^2),$$

$$E_{11}^{n+\frac{1}{2}} = f(u(t_{n+\frac{1}{2}}), v(t_{n+\frac{1}{2}})) - f(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}) = \begin{cases} O(\tau^2), n \geqslant 1, \\ O(\tau), n = 0, \end{cases}$$

$$E_{12}^{n+\frac{1}{2}} = g(u(t_{n+\frac{1}{2}}), v(t_{n+\frac{1}{2}})) - g(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}) = \begin{cases} O(\tau^2), n \geqslant 1, \\ O(\tau), n = 0, \end{cases}$$

$$E_{13}^{n+\frac{1}{2}} = \bar{f}(t_{n+\frac{1}{2}}) - \bar{f}^{n+\frac{1}{2}} = O(\tau^2), E_{14}^{n+\frac{1}{2}} = \bar{g}(t_{n+\frac{1}{2}}) - \bar{g}^{n+\frac{1}{2}} = O(\tau^2).$$

Remark 1 Based on the fourth formula in (4), we easily see that the coupled system (6)–(7) is a linearized system. In actual calculation, we can also approximate the nonlinear term $l(u(t_{n+\frac{1}{2}}), v(t_{n+\frac{1}{2}}))$ by the linearized term $l(\frac{3}{2}u^n - \frac{1}{2}u^{n-1}, \frac{3}{2}v^n - \frac{1}{2}v^{n-1})$, where l is taken as f or g and $n \geqslant 1$ is a positive integer.

Based on the coupled system (6)–(7), we find $\{u, v;\sigma, \lambda\} : [0, T] \mapsto H_0^1 \times (L^2(\Omega))^2$ satisfying the following mixed weak formulation.

Case 1: $n \geqslant 1$

$$\begin{cases} (a)(D_t u^{n+\frac{1}{2}}, w) + k_1\tau^{-\alpha}\left(\sum_{i=0}^{n} p_\alpha(i)u^{n+\frac{1}{2}-i}, w\right) + (a\sigma^{n+\frac{1}{2}}, \nabla w) \\ \quad = (f(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}), w) + (\bar{f}^{n+\frac{1}{2}}, w) + (R_1, w), \quad \forall w \in H_0^1, \\ (b)(\sigma^{n+\frac{1}{2}}, z) - (\nabla u^{n+\frac{1}{2}}, z) = (R_2, z), \quad \forall z \in (L^2(\Omega))^2, \\ (c)(D_t v^{n+\frac{1}{2}}, w) + k_2\tau^{-\beta}\left(\sum_{i=0}^{n} p_\beta(i)v^{n+\frac{1}{2}-i}, w\right) + (c\lambda^{n+\frac{1}{2}}, \nabla w) \\ \quad = (g(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}), w) + (\bar{g}^{n+\frac{1}{2}}, w) + (R_3, w), \quad \forall w \in H_0^1, \\ (d)(\lambda^{n+\frac{1}{2}}, z) - (\nabla v^{n+\frac{1}{2}}, z) = (R_4, z), \quad \forall z \in (L^2(\Omega))^2. \end{cases} \tag{8}$$

Case 2: $n = 0$

$$\begin{cases} (a)(D_t u^{\frac{1}{2}}, w) + k_1\tau^{-\alpha}\left(\sum_{i=0}^{0} p_\alpha(i)u^{\frac{1}{2}-i}, w\right) + (a\sigma^{\frac{1}{2}}, \nabla w) \\ \quad = (f(u^{\frac{1}{2}}, v^{\frac{1}{2}}), w) + (\bar{f}^{\frac{1}{2}}, w) + (R_5, w), \quad \forall w \in H_0^1, \\ (b)(\sigma^{\frac{1}{2}}, z) - (\nabla u^{\frac{1}{2}}, z) = (R_6, z), \quad \forall z \in (L^2(\Omega))^2, \\ (c)(D_t v^{\frac{1}{2}}, w) + k_2\tau^{-\beta}\left(\sum_{i=0}^{0} p_\beta(i)v^{\frac{1}{2}-i}, w\right) + (c\lambda^{\frac{1}{2}}, \nabla w) \\ \quad = (g(u^{\frac{1}{2}}, v^{\frac{1}{2}}), w) + (\bar{g}^{\frac{1}{2}}, w) + (R_7, w), \quad \forall w \in H_0^1, \\ (d)(\lambda^{\frac{1}{2}}, z) - (\nabla v^{\frac{1}{2}}, z) = (R_8, z), \quad \forall z \in (L^2(\Omega))^2. \end{cases} \tag{9}$$

Then we introduce the mixed element space $(W_h, \boldsymbol{Z}_h)$ as follows

$$\begin{aligned} W_h &= \{w_h \in C^0(\Omega) \cap H_0^1 | w_h \in P_1(K), \forall K \in \mathcal{K}_h\}, \\ \boldsymbol{Z}_h &= \{z_h = (z_{1h}, z_{2h}) \in (L^2(\Omega))^2 | z_{1h}, z_{2h} \in P_0(K), \forall K \in \mathcal{K}_h\}. \end{aligned} \tag{10}$$

Considering the above mixed element space, the fully discrete scheme of (8)–(9) is to find $\{u_h, v_h;\sigma_h, \lambda_h\} : [0, T] \mapsto W_h \times \boldsymbol{Z}_h$ such that

Case 1: $n \geqslant 1$

$$\begin{cases} (a)(D_t u_h^{n+\frac{1}{2}}, w_h) + k_1\tau^{-\alpha}\left(\sum_{i=0}^{n} p_\alpha(i)u_h^{n+\frac{1}{2}-i}, w_h\right) + (a\sigma_h^{n+\frac{1}{2}}, \nabla w_h) \\ \quad = (f(u_h^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}}), w_h) + (\bar{f}^{n+\frac{1}{2}}, w_h), \quad \forall w_h \in W_h, \\ (b)(\sigma_h^{n+\frac{1}{2}}, z_h) - (\nabla u_h^{n+\frac{1}{2}}, z_h) = 0, \quad \forall z_h \in \boldsymbol{Z}_h, \\ (c)(D_t v_h^{n+\frac{1}{2}}, w_h) + k_2\tau^{-\beta}\left(\sum_{i=0}^{n} p_\beta(i)v_h^{n+\frac{1}{2}-i}, w_h\right) + (c\lambda_h^{n+\frac{1}{2}}, \nabla w_h) \\ \quad = (g(u_h^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}}), w_h) + (\bar{g}^{n+\frac{1}{2}}, w_h), \quad \forall w_h \in W_h, \\ (d)(\lambda_h^{n+\frac{1}{2}}, z_h) - (\nabla v_h^{n+\frac{1}{2}}, z_h) = 0, \quad \forall z_h \in \boldsymbol{Z}_h. \end{cases} \tag{11}$$

Case 2: $n = 0$

$$\begin{cases} (a)(D_t u_h^{\frac{1}{2}}, w_h) + k_1\tau^{-\alpha}\left(\sum_{i=0}^{0} p_\alpha(i)u_h^{\frac{1}{2}-i}, w_h\right) + (a\sigma_h^{\frac{1}{2}}, \nabla w_h) \\ \quad = (f(u_h^{\frac{1}{2}}, v_h^{\frac{1}{2}}), w_h) + (\bar{f}^{\frac{1}{2}}, w_h), \quad \forall w_h \in W_h, \\ (b)(\sigma_h^{\frac{1}{2}}, z_h) - (\nabla u_h^{\frac{1}{2}}, z_h) = 0, \quad \forall z_h \in \boldsymbol{Z}_h, \\ (c)(D_t v_h^{\frac{1}{2}}, w_h) + k_2\tau^{-\beta}\left(\sum_{i=0}^{0} p_\beta(i)v_h^{\frac{1}{2}-i}, w_h\right) + (c\lambda_h^{\frac{1}{2}}, \nabla w_h) \\ \quad = (g(u_h^{\frac{1}{2}}, v_h^{\frac{1}{2}}), w_h) + (\bar{g}^{\frac{1}{2}}, w_h), \quad \forall w_h \in W_h, \\ (d)(\lambda_h^{\frac{1}{2}}, z_h) - (\nabla v_h^{\frac{1}{2}}, z_h) = 0, \quad \forall z_h \in \boldsymbol{Z}_h. \end{cases} \tag{12}$$

3 Existence, uniqueness and stability

3.1 Existence and uniqueness of the mixed element solution

Here, we will discuss the existence and uniqueness of the mixed element solution.

Theorem 1 *There exists a unique solution for the coupled mixed element system* (11)–(12).

Proof Now take FE space $W_h = span\{\phi_i\}_{i=1}^{M_h}$, $\boldsymbol{Z}_h = span\{\boldsymbol{\psi}_j\}_{j=1}^{N_h}$. For any $u_h, v_h \in W_h$, and any $\boldsymbol{\sigma}_h, \lambda_h \in \boldsymbol{Z}_h$, we have

$$u_h^n = \sum_{i=1}^{M_h} u_i^n \phi_i, v_h^n = \sum_{i=1}^{M_h} v_i^n \phi_i, \boldsymbol{\sigma}_h^n = \sum_{j=1}^{N_h} \sigma_j^n \boldsymbol{\psi}_j, \lambda_h^n = \sum_{j=1}^{N_h} \lambda_j^n \boldsymbol{\psi}_j.$$

Choosing $w_h = \phi_m$, $z_h = \boldsymbol{\psi}_l$ $(1 \leqslant m \leqslant M_h, 1 \leqslant l \leqslant N_h)$, and substituting them into the mixed element system (11), we have for $n \geqslant 1$

$$
\begin{cases}
(a)\ \sum_{i=1}^{M_h} D_t u_i^{n+\frac{1}{2}}(\phi_i,\phi_m) + k_1\tau^{-\alpha}\sum_{k=0}^{n} p_\alpha(k)\left(\sum_{i=1}^{M_h} u_i^{n+\frac{1}{2}-k}(\phi_i,\phi_m)\right) + a\sum_{j=1}^{N_h}\sigma_j^{n+\frac{1}{2}}(\boldsymbol{\psi}_j,\nabla\phi_m) \\
\quad = \left(\frac{3}{2}f(\sum_{i=1}^{M_h} u_i^n\phi_i,\sum_{i=1}^{M_h} v_i^n\phi_i) - \frac{1}{2}f(\sum_{i=1}^{M_h} u_i^{n-1}\phi_i,\sum_{i=1}^{M_h} v_i^{n-1}\phi_i),\phi_m\right) + (f^{n+\frac{1}{2}},\phi_m), \\
(b)\ \sum_{j=1}^{N_h}\sigma_j^{n+\frac{1}{2}}(\boldsymbol{\psi}_j,\boldsymbol{\psi}_l) - \sum_{i=1}^{M_h} u_i^{n+\frac{1}{2}}(\nabla\phi_i,\boldsymbol{\psi}_l) = 0, \\
(c)\ \sum_{i=1}^{M_h} D_t v_i^{n+\frac{1}{2}}(\phi_i,\phi_m) + k_2\tau^{-\beta}\sum_{k=0}^{n} p_\beta(k)\left(\sum_{i=1}^{M_h} v_i^{n+\frac{1}{2}-k}(\phi_i,\phi_m)\right) + c\sum_{j=1}^{N_h}\lambda_j^{n+\frac{1}{2}}(\boldsymbol{\psi}_j,\nabla\phi_m) \\
\quad = \left(\frac{3}{2}g(\sum_{i=1}^{M_h} u_i^n\phi_i,\sum_{i=1}^{M_h} v_i^n\phi_i) - \frac{1}{2}g(\sum_{i=1}^{M_h} u_i^{n-1}\phi_i,\sum_{i=1}^{M_h} v_i^{n-1}\phi_i),\phi_m\right) + (g^{n+\frac{1}{2}},\phi_m), \\
(d)\ \sum_{j=1}^{N_h}\lambda_j^{n+\frac{1}{2}}(\boldsymbol{\psi}_j,\boldsymbol{\psi}_l) - \sum_{i=1}^{M_h} v_i^{n+\frac{1}{2}}(\nabla\phi_i,\boldsymbol{\psi}_l) = 0.
\end{cases}
\tag{13}
$$

According to the above formula, we get the following matrix equations for $n \geqslant 1$

$$
\begin{cases}
(a)\ \left(\tau^{-1} + \frac{k_1}{2}p_\alpha(0)\tau^{-\alpha}\right)\boldsymbol{A}\boldsymbol{u}^{n+1} + \frac{a}{2}\boldsymbol{B}\boldsymbol{\sigma}^{n+1} \\
\quad = \tau^{-1}\boldsymbol{A}\boldsymbol{u}^n - \frac{k_1}{2}\tau^{-\alpha}\sum_{k=0}^{n}(p_\alpha(k+1)+p_\alpha(k))\boldsymbol{A}\boldsymbol{u}^{n-k} - \frac{a}{2}\boldsymbol{B}\boldsymbol{\sigma}^n + \frac{3}{2}\boldsymbol{F}^n - \frac{1}{2}\boldsymbol{F}^{n-1} + \bar{\boldsymbol{F}}^{n+\frac{1}{2}} \\
\quad \triangleq \boldsymbol{H}(\boldsymbol{u}^n,\boldsymbol{u}^{n-1},\cdots,\boldsymbol{u}^0;\boldsymbol{\sigma}^n;\boldsymbol{F}^n,\boldsymbol{F}^{n-1};\bar{\boldsymbol{F}}^{n+\frac{1}{2}}) \triangleq \boldsymbol{H}^n, \\
(b)\ \boldsymbol{C}\boldsymbol{\sigma}^{n+1} - \boldsymbol{D}\boldsymbol{u}^{n+1} = -\boldsymbol{C}\boldsymbol{\sigma}^n + \boldsymbol{D}\boldsymbol{u}^n \triangleq \boldsymbol{I}(\boldsymbol{u}^n,\boldsymbol{\sigma}^n) \triangleq \boldsymbol{I}^n, \\
(c)\ \left(\tau^{-1} + \frac{k_2}{2}p_\beta(0)\tau^{-\beta}\right)\boldsymbol{A}\boldsymbol{v}^{n+1} + \frac{c}{2}\boldsymbol{B}\lambda^{n+1} \\
\quad = \tau^{-1}\boldsymbol{A}\boldsymbol{v}^n - \frac{k_2}{2}\tau^{-\beta}\sum_{k=0}^{n}(p_\beta(k+1)+p_\beta(k))\boldsymbol{A}\boldsymbol{v}^{n-k} - \frac{c}{2}\boldsymbol{B}\lambda^n + \frac{3}{2}\boldsymbol{G}^n - \frac{1}{2}\boldsymbol{G}^{n-1} + \bar{\boldsymbol{G}}^{n+\frac{1}{2}} \\
\quad \triangleq \boldsymbol{J}(\boldsymbol{v}^n,\boldsymbol{v}^{n-1},\cdots,\boldsymbol{v}^0;\lambda^n;\boldsymbol{G}^n,\boldsymbol{G}^{n-1};\bar{\boldsymbol{G}}^{n+\frac{1}{2}}) \triangleq \boldsymbol{J}^n, \\
(d)\ \boldsymbol{C}\lambda^{n+1} - \boldsymbol{D}\boldsymbol{v}^{n+1} = -\boldsymbol{C}\lambda^n + \boldsymbol{D}\boldsymbol{v}^n \triangleq \boldsymbol{K}(\boldsymbol{v}^n,\lambda^n) \triangleq \boldsymbol{K}^n,
\end{cases}
\tag{14}
$$

where

$$\boldsymbol{A}=\left[(\phi_i,\phi_m)\right]^T_{1\leqslant i,m\leqslant M_h},\boldsymbol{B}=\left[(\boldsymbol{\psi}_j,\nabla\phi_m)\right]^T_{1\leqslant j\leqslant N_h,1\leqslant m\leqslant M_h},$$
$$\boldsymbol{C}=\left[(\boldsymbol{\psi}_j,\boldsymbol{\psi}_l)\right]^T_{1\leqslant j,l\leqslant N_h},\boldsymbol{D}=\left[(\nabla\phi_i,\boldsymbol{\psi}_l)\right]^T_{1\leqslant i\leqslant M_h,1\leqslant l\leqslant N_h},$$
$$\boldsymbol{F}^n=\left[\left(f(\sum_{i=0}^{M_h}u_i^n\phi_i,\sum_{i=0}^{M_h}v_i^n\phi_i),\phi_1\right),\cdots,\left(f(\sum_{i=0}^{M_h}u_i^n\phi_i,\sum_{i=0}^{M_h}v_i^n\phi_i),\phi_{M_h}\right)\right]^T,$$
$$\boldsymbol{G}^n=\left[\left(g(\sum_{i=0}^{M_h}u_i^n\phi_i,\sum_{i=0}^{M_h}v_i^n\phi_i),\phi_1\right),\cdots,\left(g(\sum_{i=0}^{M_h}u_i^n\phi_i,\sum_{i=0}^{M_h}v_i^n\phi_i),\phi_{M_h}\right)\right]^T,$$
$$\bar{\boldsymbol{F}}=\left[(\bar{f}^n,\phi_1),\cdots,(\bar{f}^n,\phi_{M_h})\right]^T,\bar{\boldsymbol{G}}=\left[(\bar{g}^n,\phi_1),\cdots,(\bar{g}^n,\phi_{M_h})\right]^T,$$
$$\boldsymbol{u}^n=[u_1^n,u_2^n,\cdots,u_{M_h}^n]^T,\boldsymbol{v}^n=[v_1^n,v_2^n,\cdots,v_{M_h}^n]^T,$$
$$\boldsymbol{\sigma}^n=[\sigma_1^n,\sigma_2^n,\cdots,\sigma_{N_h}^n]^T,\lambda^n=[\lambda_1^n,\lambda_2^n,\cdots,\lambda_{N_h}^n]^T.$$

We rewrite (14) as the following matrix formulation for $n\geqslant 1$

$$\boldsymbol{L}\boldsymbol{X}^{n+1}=\boldsymbol{Y}^n, \tag{15}$$

where

$$\boldsymbol{L}=\begin{bmatrix}(\tau^{-1}+\frac{k_1}{2}p_\alpha(0)\tau^{-\alpha})\boldsymbol{A} & \frac{a}{2}\boldsymbol{B} & & \\ -\boldsymbol{D} & \boldsymbol{C} & & \\ & & (\tau^{-1}+\frac{k_2}{2}p_\beta(0)\tau^{-\beta})\boldsymbol{A} & \frac{c}{2}\boldsymbol{B}\\ & & -\boldsymbol{D} & \boldsymbol{C}\end{bmatrix},$$

$$\boldsymbol{X}^{n+1}=\begin{bmatrix}\boldsymbol{u}^{n+1}\\ \boldsymbol{\sigma}^{n+1}\\ \boldsymbol{v}^{n+1}\\ \lambda^{n+1}\end{bmatrix},\qquad \boldsymbol{Y}^n=\begin{bmatrix}\boldsymbol{H}^n\\ \boldsymbol{I}^n\\ \boldsymbol{J}^n\\ \boldsymbol{K}^n\end{bmatrix}.$$

Next, we need to consider the boundary condition $w|_{\partial\Omega}=0,\forall w\in W_h$. Without loss of generality, we assume that all the boundary points are at the front, i.e.

$$\begin{aligned}\boldsymbol{u}^n&=[0,\cdots,0,u^n_{N+1},u^n_{N+2},\cdots,u^n_{M_h}]^T=[\boldsymbol{0},\tilde{\boldsymbol{u}}^n]^T,\\ \boldsymbol{v}^n&=[0,\cdots,0,v^n_{N+1},v^n_{N+2},\cdots,v^n_{M_h}]^T=[\boldsymbol{0},\tilde{\boldsymbol{v}}^n]^T.\end{aligned} \tag{16}$$

Based on (16), the p_i th row and the q_i th column of $\boldsymbol{L}$ should be crossed off, where p_i and q_i satisfy $1\leqslant p_1,q_1\leqslant N$ and $M_h+N_h+1\leqslant p_2,q_2\leqslant M_h+N_h+N$. At the same time, the p_i th row of $\boldsymbol{Y}^n$ and $\boldsymbol{Y}^n$ should be removed. By modifying the dimensions of total stiffness matrix $\boldsymbol{L}$ and total load vector $\boldsymbol{Y}^n$, we can get the following matrix equations for $n\geqslant 1$

$$\begin{bmatrix}(\tau^{-1}+\frac{k_1}{2}p_\alpha(0)\tau^{-\alpha})\tilde{\boldsymbol{A}} & \frac{a}{2}\tilde{\boldsymbol{B}} & & \\ -\tilde{\boldsymbol{D}} & \boldsymbol{C} & & \\ & & (\tau^{-1}+\frac{k_2}{2}p_\beta(0)\tau^{-\beta})\tilde{\boldsymbol{A}} & \frac{c}{2}\tilde{\boldsymbol{B}}\\ & & -\tilde{\boldsymbol{D}} & \boldsymbol{C}\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{u}}^{n+1}\\ \boldsymbol{\sigma}^{n+1}\\ \tilde{\boldsymbol{v}}^{n+1}\\ \lambda^{n+1}\end{bmatrix}=\begin{bmatrix}\tilde{\boldsymbol{H}}^n\\ \boldsymbol{I}^n\\ \tilde{\boldsymbol{J}}^n\\ \boldsymbol{K}^n\end{bmatrix}. \tag{17}$$

The form of matrix equations for $n=0$ is similar to the above. Let $\varepsilon_\alpha=\frac{1}{\tau^{-1}+\frac{k_1}{2}p_\alpha(0)\tau^{-\alpha}}$ and $\varepsilon_\beta=\frac{1}{\tau^{-1}+\frac{k_2}{2}p_\beta(0)\tau^{-\beta}}$ and (17) is equivalent to

$$\begin{cases}(a)\ \tilde{\boldsymbol{A}}\tilde{\boldsymbol{u}}^{n+1}+\dfrac{a\varepsilon_\alpha}{2}\tilde{\boldsymbol{B}}\boldsymbol{\sigma}^{n+1}=\varepsilon_\alpha\tilde{\boldsymbol{H}}^n,\\ (b)\ -\tilde{\boldsymbol{D}}\tilde{\boldsymbol{u}}^{n+1}+\boldsymbol{C}\boldsymbol{\sigma}^{n+1}=\boldsymbol{I}^n,\\ (c)\ \tilde{\boldsymbol{A}}\tilde{\boldsymbol{v}}^{n+1}+\dfrac{c\varepsilon_\beta}{2}\tilde{\boldsymbol{B}}\lambda^{n+1}=\varepsilon_\beta\tilde{\boldsymbol{J}}^n,\\ (d)\ -\tilde{\boldsymbol{D}}\tilde{\boldsymbol{v}}^{n+1}+\boldsymbol{C}\lambda^{n+1}=\boldsymbol{K}^n.\end{cases} \tag{18}$$

We know that $\tilde{\boldsymbol{A}}$ and $\boldsymbol{C}$ in (18) are invertible matrixes (Gram matrixes) which are formulated by the inner products of basis functions $\{\phi_i\}_{i=N+1}^{M_h}$ and $\{\boldsymbol{\psi}_j\}_{j=1}^{N_h}$, respectively. So, we can rewrite (18) as

$$\begin{cases}(a)\ \left(\boldsymbol{E}+\dfrac{a\varepsilon_\alpha}{2}\tilde{\boldsymbol{A}}^{-1}\tilde{\boldsymbol{B}}\boldsymbol{C}^{-1}\tilde{\boldsymbol{D}}\right)\tilde{\boldsymbol{u}}^{n+1}=\varepsilon_\alpha\tilde{\boldsymbol{A}}^{-1}\tilde{\boldsymbol{H}}^n-\dfrac{a\varepsilon_\alpha}{2}\tilde{\boldsymbol{A}}^{-1}\tilde{\boldsymbol{B}}\boldsymbol{C}^{-1}\boldsymbol{I}^n,\\ (b)\ \boldsymbol{\sigma}^{n+1}=\boldsymbol{C}^{-1}(\boldsymbol{I}^n+\tilde{\boldsymbol{D}}\tilde{\boldsymbol{u}}^{n+1}),\\ (c)\ \left(\boldsymbol{E}+\dfrac{c\varepsilon_\beta}{2}\tilde{\boldsymbol{A}}^{-1}\tilde{\boldsymbol{B}}\boldsymbol{C}^{-1}\tilde{\boldsymbol{D}}\right)\tilde{\boldsymbol{v}}^{n+1}=\varepsilon_\beta\tilde{\boldsymbol{A}}^{-1}\tilde{\boldsymbol{J}}^n-\dfrac{c\varepsilon_\beta}{2}\tilde{\boldsymbol{A}}^{-1}\tilde{\boldsymbol{B}}\boldsymbol{C}^{-1}\boldsymbol{K}^n,\\ (d)\ \lambda^{n+1}=\boldsymbol{C}^{-1}(\boldsymbol{K}^n+\tilde{\boldsymbol{D}}\tilde{\boldsymbol{v}}^{n+1}),\end{cases} \tag{19}$$

where $\boldsymbol{E}$ is an identity matrix. By taking two sufficiently small parameters ε_α and ε_β, we easily know that $\boldsymbol{E}+\frac{a\varepsilon_\alpha}{2}\tilde{\boldsymbol{A}}^{-1}\tilde{\boldsymbol{B}}\boldsymbol{C}^{-1}\tilde{\boldsymbol{D}}$ is an invertible matrix. So, we can obtain a unique iterative solution vector $[\tilde{\boldsymbol{u}}^{n+1},\boldsymbol{\sigma}^{n+1},\tilde{\boldsymbol{v}}^{n+1},\lambda^{n+1}]^T$ in (19) based on the computed solution vector $[\tilde{\boldsymbol{u}}^k,\boldsymbol{\sigma}^k,\tilde{\boldsymbol{v}}^k,\lambda^k]^T$, $k=0,1,2,\cdots,n$, where the solution vector with $k=1$ can be solved uniquely by adopting a similar process to the case above. Based on the above discussions, one can know that mixed element system has a unique solution.

3.2 Stability analysis

Next, we will give the stability analysis.

Lemma 2 (See [13, 50]) *Let* $\{p_\gamma(i)\}$ *be defined as in Lemma 1. Then for any positive integer L and real vector* $(w^0,w^1,\cdots,w^L)\in R^{L+1}$, *it holds that*

$$\sum_{n=0}^{L}\left(\sum_{i=0}^{n}p_\gamma(i)l^{n-i}\right)l^n\geqslant 0. \tag{20}$$

Theorem 2 *Let* $\{u_h^{n+1}\}_{n=0}^L$, $\{v_h^{n+1}\}_{n=0}^L$, $\{\sigma_h^{n+1}\}_{n=0}^L$, *and* $\{\lambda_h^{n+1}\}_{n=0}^L$ *be the solution of the fully discrete scheme (11)–(12). There holds true for* $\forall n=0,1,\ldots,L\,(0\leqslant L\leqslant M-1)$

$$\|u_h^{L+1}\|^2+\|v_h^{L+1}\|^2+\tau\sum_{n=0}^{L}(\|\sigma_h^{n+\frac12}\|^2+\|\lambda_h^{n+\frac12}\|^2)\leqslant C\Big(\|u_h^0\|^2+\|v_h^0\|^2+\max_{0\leqslant n\leqslant M-1}\{\|\bar f^{n+\frac12}\|^2+\|\bar g^{n+\frac12}\|^2\}\Big). \tag{21}$$

Proof Taking $w_h=2\tau u_h^{n+\frac12}$ in (11)(a) and $z_h=2a\tau\sigma_h^{n+\frac12}$ in (11)(b), we can get

$$\begin{cases}(a)\Big(\frac{u_h^{n+1}-u_h^n}{\tau},2\tau u_h^{n+\frac12}\Big)+k_1\tau^{-\alpha}\Big(\sum_{i=0}^{n}p_\alpha(i)u_h^{n+\frac12-i},2\tau u_h^{n+\frac12}\Big)\\ \quad+(a\sigma_h^{n+\frac12},2\tau\nabla u_h^{n+\frac12})=(f(u_h^{n+\frac12},v_h^{n+\frac12}),2\tau u_h^{n+\frac12})+(\bar f^{n+\frac12},2\tau u_h^{n+\frac12}),\\ (b)(\sigma_h^{n+\frac12},2a\tau\sigma_h^{n+\frac12})-(\nabla u_h^{n+\frac12},2a\tau\sigma_h^{n+\frac12})=0.\end{cases} \tag{22}$$

Adding (22)(a) and (22)(b), the above formula can be rewritten as

$$\|u_h^{n+1}\|^2-\|u_h^n\|^2+2k_1\tau^{1-\alpha}\Big(\sum_{i=0}^{n}p_\alpha(i)u_h^{n+\frac12-i},u_h^{n+\frac12}\Big)+2a\tau\|\sigma_h^{n+\frac12}\|^2=2\tau(f(u_h^{n+\frac12},v_h^{n+\frac12}),u_h^{n+\frac12})+2\tau(\bar f^{n+\frac12},u_h^{n+\frac12}). \tag{23}$$

Sum (23) for n from 1 to L to get

$$\sum_{n=1}^{L}(\|u_h^{n+1}\|^2-\|u_h^n\|^2)+2k_1\tau^{1-\alpha}\sum_{n=1}^{L}\Big(\sum_{i=0}^{n}p_\alpha(i)u_h^{n+\frac12-i},u_h^{n+\frac12}\Big)+2a\tau\sum_{n=1}^{L}\|\sigma_h^{n+\frac12}\|^2=2\tau\sum_{n=1}^{L}(f(u_h^{n+\frac12},v_h^{n+\frac12}),u_h^{n+\frac12})+2\tau\sum_{n=1}^{L}(\bar f^{n+\frac12},u_h^{n+\frac12}). \tag{24}$$

Using Hölder inequality inequality and Young inequality and noting that the fourth formula in (4), we have

$$\begin{aligned}&\|u_h^{L+1}\|^2+2a\tau\sum_{n=1}^{L}\|\sigma_h^{n+\frac12}\|^2+2k_1\tau^{1-\alpha}\sum_{n=1}^{L}\Big(\sum_{i=0}^{n}p_\alpha(i)u_h^{n+\frac12-i},u_h^{n+\frac12}\Big)\\ &\leqslant\|u_h^1\|^2+\frac32\tau\sum_{n=1}^{L}(\|u_h^n\|_\infty+\|v_h^n\|_\infty)(\|u_h^n\|+\|v_h^n\|)\|u_h^{n+\frac12}\|\\ &\quad+\frac12\tau\sum_{n=1}^{L}(\|u_h^{n-1}\|_\infty+\|v_h^{n-1}\|_\infty)(\|u_h^{n-1}\|+\|v_h^{n-1}\|)\|u_h^{n+\frac12}\|\\ &\quad+2\tau\sum_{n=1}^{L}\|u_h^{n+\frac12}\|^2+2\tau\sum_{n=1}^{L}\|\bar f^{n+\frac12}\|^2\\ &\leqslant\|u_h^1\|^2+C\tau\sum_{n=1}^{L}\|u_h^{n+\frac12}\|^2+C\tau\sum_{n=0}^{L}(\|u_h^n\|^2+\|v_h^n\|^2)+2\tau\sum_{n=1}^{L}\|\bar f^{n+\frac12}\|^2.\end{aligned} \tag{25}$$

Similarly, we take $w_h=2\tau v_h^{n+\frac12}$ in (11)(c) and $z_h=2\tau c\lambda_h^{n+\frac12}$ in (11)(d), and use Hölder inequality inequality and Young inequality to obtain

$$\|v_h^{L+1}\|^2+2c\tau\sum_{n=1}^{L}\|\lambda_h^{n+\frac12}\|^2+2k_2\tau^{1-\beta}\sum_{n=1}^{L}\Big(\sum_{i=0}^{n}p_\beta(i)v_h^{n+\frac12-i},v_h^{n+\frac12}\Big)\leqslant\|v_h^1\|^2+C\tau\sum_{n=1}^{L}\|v_h^{n+\frac12}\|^2+C\tau\sum_{n=0}^{L}(\|u_h^n\|^2+\|v_h^n\|^2)+2\tau\sum_{n=1}^{L}\|\bar g^{n+\frac12}\|^2. \tag{26}$$

Adding (25) and (26), we can get

$$\begin{aligned}&\|u_h^{L+1}\|^2+\|v_h^{L+1}\|^2+2\tau\sum_{n=1}^{L}(a\|\sigma_h^{n+\frac12}\|^2+c\|\lambda_h^{n+\frac12}\|^2)\\ &\quad+2k_1\tau^{1-\alpha}\sum_{n=1}^{L}\Big(\sum_{i=0}^{n}p_\alpha(i)u_h^{n+\frac12-i},u_h^{n+\frac12}\Big)\\ &\quad+2k_2\tau^{1-\beta}\sum_{n=1}^{L}\Big(\sum_{i=0}^{n}p_\beta(i)v_h^{n+\frac12-i},v_h^{n+\frac12}\Big)\\ &\leqslant\|u_h^1\|^2+\|v_h^1\|^2+C\tau\sum_{n=1}^{L}(\|u_h^{n+\frac12}\|^2+\|v_h^{n+\frac12}\|^2)\\ &\quad+C\tau\sum_{n=0}^{L}(\|u_h^n\|^2+\|v_h^n\|^2)\\ &\quad+2\tau\sum_{n=1}^{L}(\|\bar f^{n+\frac12}\|^2+\|\bar g^{n+\frac12}\|^2).\end{aligned} \tag{27}$$

By the similar way to (27), when $n=0$, we take $w_h=2\tau u_h^{\frac12}$ in (12)(a), $z_h=2a\tau\sigma_h^{\frac12}$ in (12)(b), $w_h=2\tau v_h^{\frac12}$ in (12)(c) and $z_h=2c\tau\lambda_h^{n+\frac12}$ in (12)(d) to easily get

$$\begin{aligned}&\|u_h^1\|^2+\|v_h^1\|^2+2\tau(a\|\sigma_h^{\frac12}\|^2+c\|\lambda_h^{\frac12}\|^2)\\ &\quad+2k_1\tau^{1-\alpha}\Big(\sum_{i=0}^{0}p_\alpha(i)u_h^{\frac12-i},u_h^{\frac12}\Big)+2k_2\tau^{1-\beta}\Big(\sum_{i=0}^{0}p_\beta(i)v_h^{\frac12-i},v_h^{\frac12}\Big)\\ &\leqslant C(\|u_h^0\|^2+\|v_h^0\|^2)+C\tau(\|u_h^{\frac12}\|^2+\|v_h^{\frac12}\|^2)+2\tau(\|\bar f^{\frac12}\|^2+\|\bar g^{\frac12}\|^2).\end{aligned} \tag{28}$$

By substituting (28) into (27), dropping two nonnegative terms, and using Lemma 2 and triangle inequality, we can get

$$\begin{aligned}&(1-C\tau)(\|u_h^{L+1}\|^2+\|v_h^{L+1}\|^2)+2\tau\sum_{n=0}^{L}(a\|\sigma_h^{n+\frac12}\|^2+c\|\lambda_h^{n+\frac12}\|^2)\\ &\leqslant C(\|u_h^0\|^2+\|v_h^0\|^2)+C\tau\sum_{n=0}^{L}(\|u_h^{n+\frac12}\|^2+\|v_h^{n+\frac12}\|^2)+\\ &\quad+C\tau\sum_{n=0}^{L}(\|u_h^n\|^2+\|v_h^n\|^2)+2\tau\sum_{n=0}^{L}(\|\bar f^{n+\frac12}\|^2+\|\bar g^{n+\frac12}\|^2)\\ &\leqslant C(\|u_h^0\|^2+\|v_h^0\|^2)+C\tau\sum_{n=0}^{L}(\|u_h^n\|^2+\|v_h^n\|^2)\\ &\quad+2\tau\sum_{n=0}^{L}(\|\bar f^{n+\frac12}\|^2+\|\bar g^{n+\frac12}\|^2).\end{aligned} \tag{29}$$

For the small enough τ, we can use Gronwall lemma to yield the result.

Remark 2 Here, based on the boundness assumptions for both $\|u^n\|_\infty$ and $\|v^n\|_\infty$, we can prove that $\|u_h^n\|_\infty$ and $\|v_h^n\|_\infty$ are bounded. For the related discussion, one can see Remark 3.3 in Ref. [31].

4 Error estimates of the mixed element scheme

In what follows, the error estimate theorem will be given. Firstly, we need to introduce two lemmas.

Lemma 3 (See [57]) *Let $(P_hu, P_h\sigma) : [0,T] \mapsto W_h \times \mathbf{Z}_h$ be given by the following mixed projections*

$$\begin{cases} (a)(\sigma - P_h\sigma, \nabla w_h) = 0, \quad \forall w_h \in W_h, \\ (b)(\sigma - P_h\sigma, z_h) - (\nabla(u - P_hu), z_h) = 0, \quad \forall z_h \in \mathbf{Z}_h, \end{cases} \tag{30}$$

then, there exists a constant C independent of h such that

$$\|u - P_hu\| + h(\|\sigma - P_h\sigma\| + \|u - P_hu\|_1) \leqslant Ch^2(\|u\|_{H^2} + \|\sigma\|_{(H^1)^2}), \tag{31}$$

$$\|u_t - P_hu_t\| \leqslant Ch^2(\|u_t\|_{H^2} + \|\sigma_t\|_{(H^1)^2}). \tag{32}$$

Lemma 4 *Referring to Ref.*[15], *we can easily get the following error inequality*

$$\left(\sum_{i=0}^{n} \frac{p_\alpha(i)}{\tau^\alpha}(u^{n+\frac{1}{2}} - P_hu^{n+\frac{1}{2}}), w_h\right) \leqslant C(h^4 + \tau^4) + \|w_h\|^2, \quad \forall w_h \in W_h. \tag{33}$$

Theorem 3 *If $\{u^{n+1}\}_{n=0}^L, \{v^{n+1}\}_{n=0}^L, \{\sigma^{n+1}\}_{n=0}^L$, and $\{\lambda^{n+1}\}_{n=0}^L$ are the exact solutions of (8)–(9), $\{u_h^{n+1}\}_{n=0}^L$, $\{v_h^{n+1}\}_{n=0}^L$, $\{\sigma_h^{n+1}\}_{n=0}^L$, and $\{\lambda_h^{n+1}\}_{n=0}^L$ are the numerical solutions of (11)–(12), then for $\forall n = 0, 1, \cdots, L (0 \leqslant L \leqslant M-1)$ there exists a positive constant C such that*

$$\begin{aligned} &\|u^{L+1} - u_h^{L+1}\| + \|v^{L+1} - v_h^{L+1}\| \\ &\leqslant C(h^2 + \tau^2), \left(\tau \sum_{n=1}^{L} \|\sigma^{n+\frac{1}{2}} - \sigma_h^{n+\frac{1}{2}}\|^2\right)^{\frac{1}{2}} + \left(\tau \sum_{n=1}^{L} \|\lambda^{n+\frac{1}{2}} - \lambda_h^{n+\frac{1}{2}}\|^2\right)^{\frac{1}{2}} \\ &\leqslant C(h + \tau^2), \left(\tau \sum_{n=1}^{L} \|u^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}\|_1^2\right)^{\frac{1}{2}} \\ &+ \left(\tau \sum_{n=1}^{L} \|v^{n+\frac{1}{2}} - v_h^{n+\frac{1}{2}}\|_1^2\right)^{\frac{1}{2}} \leqslant C(h + \tau^2). \end{aligned} \tag{34}$$

Proof For simplicity, we introduce the following notations

$$\begin{aligned} u(t_n) - u_h^n &= u(t_n) - P_hu^n + P_hu^n - u_h^n = \phi^n + \theta^n, \\ \sigma(t_n) - \sigma_h^n &= \sigma(t_n) - P_h\sigma^n + P_h\sigma^n - \sigma_h^n = \psi^n + \omega^n, \\ v(t_n) - v_h^n &= v(t_n) - P_hv^n + P_hv^n - v_h^n = \eta^n + \zeta^n, \\ \lambda(t_n) - \lambda_h^n &= \lambda(t_n) - P_h\lambda^n + P_h\lambda^n - \lambda_h^n = \rho^n + \xi^n. \end{aligned} \tag{35}$$

Combining (8), (11), (35) with Lemma 3, we can get the error equations for $n \geqslant 1$

$$\begin{cases} (a)(D_t\theta^{n+\frac{1}{2}}, w_h) + k_1\tau^{-\alpha}\left(\sum_{i=0}^{n} p_\alpha(i)\theta^{n+\frac{1}{2}-i}, w_h\right) + (a\omega^{n+\frac{1}{2}}, \nabla w_h) \\ \quad = -(D_t\phi^{n+\frac{1}{2}}, w_h) - k_1\tau^{-\alpha}\left(\sum_{i=0}^{n} p_\alpha(i)\phi^{n+\frac{1}{2}-i}, w_h\right) \\ \quad\quad + (f(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}) - f(u_h^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}}), w_h) + (R_1, w_h), \\ (b)(\omega^{n+\frac{1}{2}}, z_h) - (\nabla\theta^{n+\frac{1}{2}}, z_h) = (R_2, z_h), \\ (c)(D_t\zeta^{n+\frac{1}{2}}, w_h) + k_2\tau^{-\beta}\left(\sum_{i=0}^{n} p_\beta(i)\zeta^{n+\frac{1}{2}-i}, w_h\right) + (c\xi^{n+\frac{1}{2}}, \nabla w_h) \\ \quad = -(D_t\eta^{n+\frac{1}{2}}, w_h) - k_2\tau^{-\beta}\left(\sum_{i=0}^{n} p_\beta(i)\eta^{n+\frac{1}{2}-i}, w_h\right) \\ \quad\quad + (g(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}) - g(u_h^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}}), w_h) + (R_3, w_h), \\ (d)(\xi^{n+\frac{1}{2}}, z_h) - (\nabla\zeta^{n+\frac{1}{2}}, z_h) = (R_4, z_h). \end{cases} \tag{36}$$

Setting $w_h = 2\tau\theta^{n+\frac{1}{2}}$ in (36)(a), $z_h = 2a\tau\omega^{n+\frac{1}{2}}$ in (36)(b), we add the resulting equations to get

$$\begin{aligned} &\|\theta^{n+1}\|^2 - \|\theta^n\|^2 + 2k_1\tau^{1-\alpha}\left(\sum_{i=0}^{n} p_\alpha(i)\theta^{n+\frac{1}{2}-i}, \theta^{n+\frac{1}{2}}\right) + 2a\tau\|\omega^{n+\frac{1}{2}}\|^2 \\ &= -2\tau\left(\frac{\phi^{n+1} - \phi^n}{\tau}, \theta^{n+\frac{1}{2}}\right) - 2k_1\tau^{1-\alpha}\left(\sum_{i=0}^{n} p_\alpha(i)\phi^{n+\frac{1}{2}-i}, \theta^{n+\frac{1}{2}}\right) \\ &\quad + 2\tau(f(u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}) - f(u_h^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}}), \theta^{n+\frac{1}{2}}) + (R_1 + R_2, 2\tau\theta^{n+\frac{1}{2}}). \end{aligned} \tag{37}$$

Summing (37) for n from 1 to L, we arrive at

$$\|\theta^{L+1}\|^2 - \|\theta^1\|^2 + 2k_1\tau^{1-\alpha}\sum_{n=1}^{L}\left(\sum_{i=0}^{n}p_\alpha(i)\theta^{n+\frac{1}{2}-i},\theta^{n+\frac{1}{2}}\right) + 2a\tau\sum_{n=1}^{L}\|\omega^{n+\frac{1}{2}}\|^2 = -2\tau\sum_{n=1}^{L}\left(\frac{\phi^{n+1}-\phi^n}{\tau},\theta^{n+\frac{1}{2}}\right) - 2k_1\tau^{1-\alpha}\sum_{n=1}^{L}\left(\sum_{i=0}^{n}p_\alpha(i)\phi^{n+\frac{1}{2}-i},\theta^{n+\frac{1}{2}}\right) + 2\tau\sum_{n=1}^{L}(f(u^{n+\frac{1}{2}},v^{n+\frac{1}{2}}) - f(u_h^{n+\frac{1}{2}},v_h^{n+\frac{1}{2}}),\theta^{n+\frac{1}{2}}) + 2\tau\sum_{n=1}^{L}(R_1+R_2,\theta^{n+\frac{1}{2}}). \quad (38)$$

Noting that the nonlinear term $f(u, v)$ is a quadratic polynomial without constant terms about u and v, we use Hölder inequality, triangle inequality, (31) and Remark 2 to get

$$(f(u^{n+\frac{1}{2}},v^{n+\frac{1}{2}}) - f(u_h^{n+\frac{1}{2}},v_h^{n+\frac{1}{2}}),\theta^{n+\frac{1}{2}}) \leqslant \frac{3}{2}\Big[\sum_{k=n-1}^{n}(\|u^k\|_\infty + \|u_h^k\|_\infty + \|v_h^k\|_\infty)\|u^k-u_h^k\|\Big]\|\theta^{n+\frac{1}{2}}\| + \frac{3}{2}\Big[\sum_{k=n-1}^{n}(\|v^k\|_\infty + \|v_h^k\|_\infty + \|u^k\|_\infty)\|v^k-v_h^k\|\Big]\|\theta^{n+\frac{1}{2}}\| \leqslant C\Big[\sum_{k=n-1}^{n}(\|\phi^k+\theta^k\| + \|\eta^k+\zeta^k\|)\Big]\|\theta^{n+\frac{1}{2}}\| \leqslant C\Big[\sum_{k=n-1}^{n}(\|\theta^k\|^2 + \|\zeta^k\|^2 + \|\phi^k\|^2 + \|\eta^k\|^2) + \|\theta^{n+\frac{1}{2}}\|^2\Big]). \quad (39)$$

For (38), we use Cauchy–Schwarz inequality, Young inequality, Lemma 4 and inequality (39), we have

$$\|\theta^{L+1}\|^2 - \|\theta^1\|^2 + 2k_1\tau^{1-\alpha}\sum_{n=1}^{L}\left(\sum_{i=0}^{n}p_\alpha(i)\theta^{n+\frac{1}{2}-i},\theta^{n+\frac{1}{2}}\right) + 2a\tau\sum_{n=1}^{L}\|\omega^{n+\frac{1}{2}}\|^2 \leqslant C\int_{t_1}^{t_{L+1}}\|\phi_t\|^2\mathrm{d}s + C\tau\sum_{n=1}^{L}(h^4+\tau^4) + C\tau\sum_{n=1}^{L}(\|R_1\|^2+\|R_2\|^2) + C\tau\sum_{n=1}^{L}(\|\phi^{n+\frac{1}{2}}\|^2 + \|\theta^{n+\frac{1}{2}}\|^2 + \|\eta^{n+\frac{1}{2}}\|^2 + \|\zeta^{n+\frac{1}{2}}\|^2) + C\tau\sum_{n=0}^{L}(\|\phi^n\|^2 + \|\theta^n\|^2 + \|\eta^n\|^2 + \|\zeta^n\|^2). \quad (40)$$

We set $w_h = 2\tau\zeta^{n+\frac{1}{2}}$ in (36)(c) and $z_h = 2c\tau\xi^{n+\frac{1}{2}}$ in (36)(d), and use the similar process to the derivation of (37)–(40) to obtain

$$\|\zeta^{L+1}\|^2 - \|\zeta^1\|^2 + 2k_2\tau^{1-\beta}\sum_{n=1}^{L}\left(\sum_{i=0}^{n}p_\beta(i)\zeta^{n+\frac{1}{2}-i},\zeta^{n+\frac{1}{2}}\right) + 2c\tau\sum_{n=1}^{L}\|\xi^{k+\frac{1}{2}}\|^2 \leqslant C\int_{t_1}^{t_{L+1}}\|\eta_t\|^2\mathrm{d}s + C\tau\sum_{n=1}^{L}(h^4+\tau^4) + C\tau\sum_{n=1}^{L}(\|R_3\|^2+\|R_4\|^2) + C\tau\sum_{n=1}^{L}(\|\phi^{n+\frac{1}{2}}\|^2 + \|\theta^{n+\frac{1}{2}}\|^2 + \|\eta^{n+\frac{1}{2}}\|^2 + \|\zeta^{n+\frac{1}{2}}\|^2) + C\tau\sum_{n=0}^{L}(\|\phi^n\|^2 + \|\theta^n\|^2 + \|\eta^n\|^2 + \|\zeta^n\|^2). \quad (41)$$

Summing (40) and (41) and using triangle inequality, we have the following inequality

$$\|\theta^{L+1}\|^2 + \|\zeta^{L+1}\|^2 + 2\tau\left(a\sum_{n=1}^{L}\|\omega^{n+\frac{1}{2}}\|^2 + c\sum_{n=1}^{L}\|\xi^{n+\frac{1}{2}}\|^2\right) + 2k_1\tau^{1-\alpha}\sum_{n=1}^{L}\left(\sum_{i=0}^{n}p_\alpha(i)\theta^{n+\frac{1}{2}-i},\theta^{n+\frac{1}{2}}\right) + 2k_2\tau^{1-\beta}\sum_{n=1}^{L}\left(\sum_{i=0}^{n}p_\beta(i)\zeta^{n+\frac{1}{2}-i},\zeta^{n+\frac{1}{2}}\right) \leqslant \|\theta^1\|^2 + \|\zeta^1\|^2 + C\int_{t_1}^{t_{L+1}}(\|\phi_t\|^2 + \|\eta_t\|^2)\mathrm{d}s + C\tau\sum_{n=1}^{L}(h^4+\tau^4) + C\tau\sum_{n=1}^{L}(\|R_1\|^2+\|R_2\|^2+\|R_3\|^2+\|R_4\|^2) + C\tau\sum_{n=0}^{L+1}(\|\phi^n\|^2 + \|\theta^n\|^2 + \|\eta^n\|^2 + \|\zeta^n\|^2). \quad (42)$$

Subtracting (12) from (9), noticing (35) and Lemma 3, we obtain

$$
\begin{cases}
(a)(D_t\theta^{\frac{1}{2}}, w_h) + k_1\tau^{-\alpha}\left(\sum_{i=0}^{0} p_\alpha(i)\theta^{\frac{1}{2}-i}, w_h\right) + (a\omega^{\frac{1}{2}}, \nabla w_h) \\
= -(D_t\phi^{\frac{1}{2}}, w_h) - k_1\tau^{-\alpha}\left(\sum_{i=0}^{0} p_\alpha(i)\phi^{\frac{1}{2}-i}, w_h\right) + (f(u^{\frac{1}{2}}, v^{\frac{1}{2}}) - f(u_h^{\frac{1}{2}}, v_h^{\frac{1}{2}}), w_h) + (R_5, w_h), \\
(b)(\omega^{\frac{1}{2}}, z_h) - (\nabla\theta^{\frac{1}{2}}, z_h) = (R_6, z_h), \\
(c)(D_t\zeta^{\frac{1}{2}}, w_h) + k_2\tau^{-\beta}\left(\sum_{i=0}^{0} p_\beta(i)\zeta^{\frac{1}{2}-i}, w_h\right) + (c\xi^{\frac{1}{2}}, \nabla w_h) \\
= -(D_t\eta^{\frac{1}{2}}, w_h) - k_2\tau^{-\beta}\left(\sum_{i=0}^{0} p_\beta(i)\eta^{\frac{1}{2}-i}, w_h\right) + (g(u^{\frac{1}{2}}, v^{\frac{1}{2}}) - g(u_h^{\frac{1}{2}}, v_h^{\frac{1}{2}}), w_h) + (R_7, w_h), \\
(d)(\xi^{\frac{1}{2}}, z_h) - (\nabla\zeta^{\frac{1}{2}}, z_h) = (R_8, z_h).
\end{cases}
\tag{43}
$$

By the similar means to (42), we set $w_h = 2\tau\theta^{\frac{1}{2}}$ in (43)(a), $z_h = 2a\tau\omega^{\frac{1}{2}}$ in (43)(b), $w_h = 2\tau\zeta^{\frac{1}{2}}$ in (43)(c), $z_h = 2c\tau\xi^{\frac{1}{2}}$ in (43)(d), and consider $\|\theta^0\|^2 = 0$, $\|\zeta^0\|^2 = 0$ to have

$$
\begin{aligned}
&\|\theta^1\|^2 + \|\zeta^1\|^2 + 2\tau(a\|\omega^{\frac{1}{2}}\|^2 + c\|\xi^{\frac{1}{2}}\|^2) \\
&\quad + 2k_1\tau^{1-\alpha}\left(\sum_{i=0}^{0} p_\alpha(i)\theta^{\frac{1}{2}-i}, \theta^{\frac{1}{2}}\right) + 2k_2\tau^{1-\beta}\left(\sum_{i=0}^{0} p_\beta(i)\zeta^{\frac{1}{2}-i}, \zeta^{\frac{1}{2}}\right) \\
&\leqslant C\int_{t_0}^{t_1}(\|\phi_t\|^2 + \|\eta_t\|^2)\mathrm{d}s + C\tau(h^4 + \tau^4) + C(\tau^2\|R_5\|^2 + \|R_6\|^2 + \tau^2\|R_7\|^2 + \|R_8\|^2) \\
&\quad + C\tau(\|\phi^{\frac{1}{2}}\|^2 + \|\theta^{\frac{1}{2}}\|^2 + \|\eta^{\frac{1}{2}}\|^2 + \|\zeta^{\frac{1}{2}}\|^2).
\end{aligned}
\tag{44}
$$

By Lemma 2, substitute (44) into (42) to get

$$
\begin{aligned}
&(1 - C\tau)(\|\theta^{L+1}\|^2 + \|\zeta^{L+1}\|^2) + 2\tau\left(a\sum_{n=0}^{L}\|\omega^{n+\frac{1}{2}}\|^2 + c\sum_{n=0}^{L}\|\xi^{n+\frac{1}{2}}\|^2\right) \\
\leqslant &C\int_{t_0}^{t_{L+1}}(\|\phi_t\|^2 + \|\eta_t\|^2)\mathrm{d}s + C\tau\sum_{n=0}^{L}(h^4 + \tau^4) \\
&\quad + C\tau\sum_{n=1}^{L}(\|R_1\|^2 + \|R_2\|^2 + \|R_3\|^2 + \|R_4\|^2) \\
&\quad + C(\tau^2\|R_5\|^2 + \|R_6\|^2 + \tau^2\|R_7\|^2 + \|R_8\|^2) \\
&+ C\tau\sum_{n=0}^{L}(\|\phi^n\|^2 + \|\theta^n\|^2 + \|\eta^n\|^2 + \|\zeta^n\|^2).
\end{aligned}
\tag{45}
$$

Take $z_h = 2a\tau\nabla\theta^{n+\frac{1}{2}}$ in (36)(b), $z_h = 2c\tau\nabla\zeta^{n+\frac{1}{2}}$ in (36)(d), $z_h = 2a\tau\nabla\theta^{\frac{1}{2}}$ in (43)(b) and $z_h = 2c\tau\nabla\zeta^{\frac{1}{2}}$ in (43)(d) and use (45) to get

$$
\begin{aligned}
&2\tau\left(a\sum_{n=0}^{L}\|\nabla\theta^{n+\frac{1}{2}}\|^2 + c\sum_{n=0}^{L}\|\nabla\zeta^{n+\frac{1}{2}}\|^2\right) \\
\leqslant &C\tau\left(a\sum_{n=0}^{L}\|\omega^{n+\frac{1}{2}}\|^2 + c\sum_{n=0}^{L}\|\xi^{n+\frac{1}{2}}\|^2\right) + C\sum_{n=0}^{L+1}(\|\theta^n\|^2 + \|\zeta^n\|^2) + C\tau^4.
\end{aligned}
\tag{46}
$$

Table 1 Time convergence results with $h = 1/400$, $\alpha = 0.1$, $\beta = 0.1$

τ	1/10	1/20	1/30	1/40
$\|u - u_h\|_{L^\infty(L^2)}$	0.00140970	0.00038494	0.00017617	0.00010099
Order		1.8727	1.9277	1.9341
$\|v - v_h\|_{L^\infty(L^2)}$	0.00144093	0.00039550	0.00018262	0.00010566
Order		1.8653	1.9058	1.9019
$\|\sigma - \sigma_h\|_{L^\infty(L^2)}$	0.00482257	0.00130991	0.00060068	0.00034841
Order		1.8803	1.9229	1.8933
$\|\lambda - \lambda_h\|_{L^\infty(L^2)}$	0.00491705	0.00134216	0.00061799	0.00035663
Order		1.8732	1.9128	1.9110

Finally, for a small enough τ, combining (45)–(46), Gronwall lemma, Lemma 3 with triangle inequality, we complete the proof of Theorem 3.

5 Numerical tests

In this section, we will provide two numerical examples to verify the correctness of the theory results.

5.1 One-dimensional case

For calculating the time convergence order, we choose the following one-dimensional example

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial^\alpha u}{\partial t^\alpha} = u_{xx} + (u^2 - v) + \bar{f}(x,t), (x,t) \in (0,1) \times (0,1], \\ \frac{\partial v}{\partial t} + \frac{\partial^\beta v}{\partial t^\beta} = v_{xx} + (v - u^2) + \bar{g}(x,t), (x,t) \in (0,1) \times (0,1], \\ u(0,t) = u(1,t) = v(0,t) = v(1,t) = 0, t \in [0,1], \\ u(x,0) = 0, v(x,0) = 0, x \in [0,1], \end{cases} \tag{47}$$

where

$$\bar{f}(x,t) = \left(2t + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 4\pi^2 t^2\right)\sin(2\pi x) - t^4 \sin^2(2\pi x) + t^2 \sin(\pi x),$$
$$\bar{g}(x,t) = \left(2t + \frac{2t^{2-\beta}}{\Gamma(3-\beta)} + \pi^2 t^2\right)\sin(\pi x) + t^4 \sin^2(2\pi x) - t^2 \sin(\pi x).$$

Table 2 Time convergence results with $h = 1/400$, $\alpha = 0.5$, $\beta = 0.5$

τ	1/10	1/20	1/30	1/40
$\|u - u_h\|_{L^\infty(L^2)}$	0.00130587	0.00035798	0.00016418	0.00009435
Order		1.8671	1.9225	1.9257
$\|v - v_h\|_{L^\infty(L^2)}$	0.00142620	0.00039029	0.00018010	0.00010419
Order		1.8696	1.9074	1.9024
$\|\sigma - \sigma_h\|_{L^\infty(L^2)}$	0.00451851	0.00123194	0.00056722	0.00033099
Order		1.8749	1.9129	1.8724
$\|\lambda - \lambda_h\|_{L^\infty(L^2)}$	0.00486900	0.00132559	0.0006100018	0.00035197
Order		1.8770	1.9142	1.9115

Table 3 Time convergence results with $h = 1/400$, $\alpha = 0.9$, $\beta = 0.9$

τ	1/10	1/20	1/30	1/40
$\|u - u_h\|_{L^\infty(L^2)}$	0.00115468	0.00031938	0.00014693	0.00008465
Order		1.8541	1.9149	1.9166
$\|v - v_h\|_{L^\infty(L^2)}$	0.00122821	0.00033926	0.00015725	0.00009130
Order		1.8561	1.8964	1.8901
$\|\sigma - \sigma_h\|_{L^\infty(L^2)}$	0.00408988	0.00112064	0.00051724	0.00030305
Order		1.86774	1.9068	1.8584
$\|\lambda - \lambda_h\|_{L^\infty(L^2)}$	0.00429973	0.00117725	0.00054338	0.00031426
Order		1.8688	1.9068	1.9035

Table 4 Space convergence results with $\alpha = 0.01$, $\beta = 0.01$

(h, τ)	$\left(\frac{\sqrt{2}}{8}, \frac{1}{8}\right)$	$\left(\frac{\sqrt{2}}{12}, \frac{1}{12}\right)$	$\left(\frac{\sqrt{2}}{16}, \frac{1}{16}\right)$	$\left(\frac{\sqrt{2}}{20}, \frac{1}{20}\right)$
$\|u - u_h\|_{L^\infty(L^2)}$	0.10077404	0.04616411	0.02641138	0.01729676
Order		1.9254	1.9411	1.8969
$\|u - u_h\|_{L^\infty(H^1)}$	1.19060110	0.78336522	0.58510005	0.46735582
Order		1.0324	1.0144	1.0069
$\|v - v_h\|_{L^\infty(L^2)}$	0.42309826	0.173286912	0.10226602	0.06689381
Order		2.2016	1.8332	1.9022
$\|v - v_h\|_{L^\infty(H^1)}$	5.47961980	3.23900345	2.38217775	1.88930162
Order		1.2967	1.0680	1.03883
$\|\sigma - \sigma_h\|_{L^\infty((L^2)^2)}$	1.18632861	0.78200380	0.58450364	0.46703564
Order		1.0279	1.0119	1.0054
$\|\lambda - \lambda_h\|_{L^\infty((L^2)^2)}$	5.46326103	3.23436470	2.37998161	1.88811701
Order		1.2929	1.0662	1.0375

Table 5 Space convergence results with $\alpha = 0.5$, $\beta = 0.5$

(h, τ)	$\left(\frac{\sqrt{2}}{8}, \frac{1}{8}\right)$	$\left(\frac{\sqrt{2}}{12}, \frac{1}{12}\right)$	$\left(\frac{\sqrt{2}}{16}, \frac{1}{16}\right)$	$\left(\frac{\sqrt{2}}{20}, \frac{1}{20}\right)$
$\Vert u - u_h\Vert_{L^\infty(L^2)}$	0.10048551	0.04600208	0.02629754	0.01719557
Order		1.9270	1.9439	1.9038
$\Vert u - u_h\Vert_{L^\infty(H^1)}$	1.18985669	0.78310340	0.58496817	0.46726518
Order		1.0317	1.01399	1.0068
$\Vert v - v_h\Vert_{L^\infty(L^2)}$	0.42297575	0.17318831	0.10218722	0.06682943
Order		2.2022	1.8339	1.9031
$\Vert v - v_h\Vert_{L^\infty(H^1)}$	5.47920478	3.23828862	2.38178660	1.88907444
Order		1.2971	1.0678	1.0386
$\Vert \sigma - \sigma_h\Vert_{L^\infty((L^2)^2)}$	1.18560601	0.78175107	0.58437676	0.46694867
Order		1.0271	1.0115	1.0053
$\Vert \lambda - \lambda_h\Vert_{L^\infty((L^2)^2)}$	5.46285425	3.23365412	2.37959349	1.88789197
Order		1.2932	1.0660	1.0373

Table 6 Space convergence results with $\alpha = 0.99$, $\beta = 0.99$

(h, τ)	$\left(\frac{\sqrt{2}}{8}, \frac{1}{8}\right)$	$\left(\frac{\sqrt{2}}{12}, \frac{1}{12}\right)$	$\left(\frac{\sqrt{2}}{16}, \frac{1}{16}\right)$	$\left(\frac{\sqrt{2}}{20}, \frac{1}{20}\right)$
$\Vert u - u_h\Vert_{L^\infty(L^2)}$	0.10020384	0.04584098	0.02617323	0.01706952
Order		1.9287	1.9481	1.9155
$\Vert u - u_h\Vert_{L^\infty(H^1)}$	1.18913797	0.78284830	0.58482879	0.46715628
Order		1.0310	1.0137	1.0068
$\Vert v - v_h\Vert_{L^\infty(L^2)}$	0.42285473	0.17308931	0.10210530	0.06675830
Order		2.2030	1.8347	1.9043
$\Vert v - v_h\Vert_{L^\infty(H^1)}$	5.47879883	3.23758382	2.38139900	1.88884607
Order		1.2974	1.0677	1.0384
$\Vert \sigma - \sigma_h\Vert_{L^\infty((L^2)^2)}$	1.18490856	0.78150500	0.58424282	0.46684432
Order		1.0265	1.0112	1.0053
$\Vert \lambda - \lambda_h\Vert_{L^\infty((L^2)^2)}$	5.46245646	3.23295361	2.37920905	1.88766597
Order		1.2936	1.0659	1.0371

Table 7 Space convergence results with $\alpha = 0.25$, $\beta = 0.75$

(h, τ)	$\left(\frac{\sqrt{2}}{8}, \frac{1}{8}\right)$	$\left(\frac{\sqrt{2}}{12}, \frac{1}{12}\right)$	$\left(\frac{\sqrt{2}}{16}, \frac{1}{16}\right)$	$\left(\frac{\sqrt{2}}{20}, \frac{1}{20}\right)$
$\Vert u - u_h\Vert_{L^\infty(L^2)}$	0.10063713	0.04608708	0.02635726	0.01724866
Order		1.9262	1.9424	1.9002
$\Vert u - u_h\Vert_{L^\infty(H^1)}$	1.19024686	0.78324036	0.58503717	0.46731266
Order		1.0321	1.0142	1.0069
$\Vert v - v_h\Vert_{L^\infty(L^2)}$	0.42291107	0.17313566	0.10214415	0.06679278
Order		2.2026	1.8343	1.9037
$\Vert v - v_h\Vert_{L^\infty(H^1)}$	5.47898739	3.23791258	2.38158017	1.88895344
Order		1.2973	1.0677	1.0385
$\Vert \sigma - \sigma_h\Vert_{L^\infty((L^2)^2)}$	1.18598472	0.78188327	0.58444314	0.46699423
Order		1.0275	1.0117	1.0054
$\Vert \lambda - \lambda_h\Vert_{L^\infty((L^2)^2)}$	5.46264121	3.23328036	2.37938872	1.88777218
Order		1.2934	1.0659	1.0372

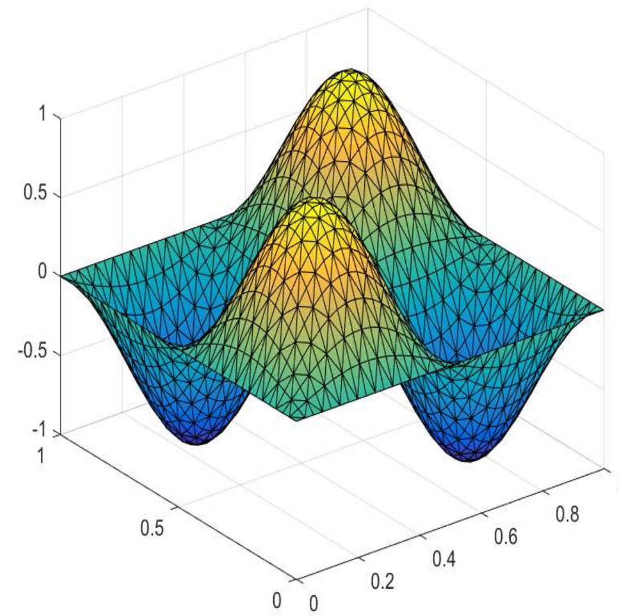

Fig. 1 Surface for exact solution u

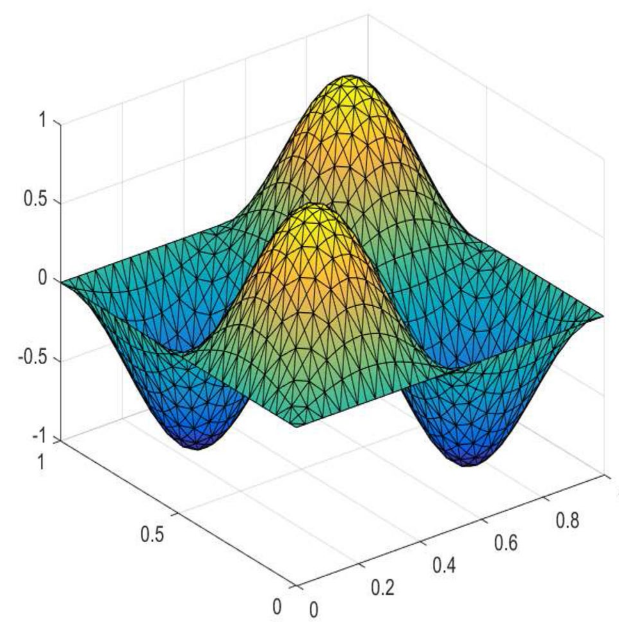

Fig. 2 Surface for numerical solution u_h

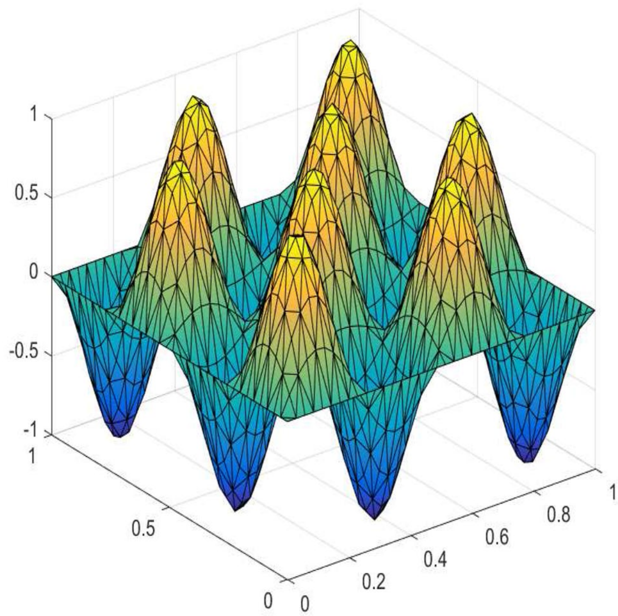

Fig. 3 Surface for exact solution v

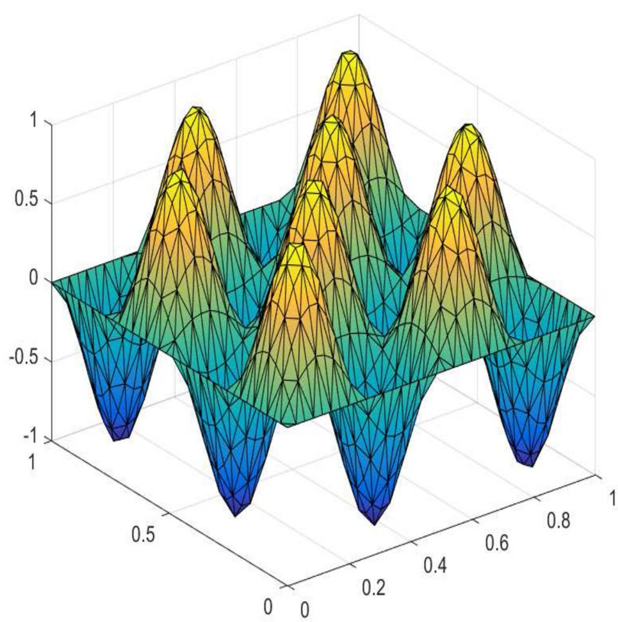

Fig. 4 Surface for numerical solution v_h

So we can easily obtain that the exact solution is

$$\begin{cases} u = t^2 \sin(2\pi x), \\ v = t^2 \sin(\pi x), \\ \sigma = 2\pi t^2 \cos(2\pi x), \\ \lambda = \pi t^2 \cos(\pi x). \end{cases} \tag{48}$$

By choosing a fixed space step $h = 1/400$, different time discrete parameters $\tau = 1/10$, 1/20, 1/30, 1/40, and changed fractional parameters $\alpha = \beta = 0.1, 0.5, 0.9$, we get some computing data in Tables 1, 2 and 3. From these computing results, one can easily see the convergence order in time for unknown functions and their derivatives are close to 2.

5.2 Two-dimensional example

We choose a two-dimensional nonlinear time fractional coupled sub-diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial^\alpha u}{\partial t^\alpha} = \Delta u + (u^2 - v) + \bar{f}(\mathbf{x}, t), (\mathbf{x}, t) \in \Omega \times J, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial^\beta v}{\partial t^\beta} = \Delta v + (v - u^2) + \bar{g}(\mathbf{x}, t), (\mathbf{x}, t) \in \Omega \times J, \\ u(\mathbf{x}, t) = v(\mathbf{x}, t) = 0, (\mathbf{x}, t) \in \partial\Omega \times \bar{J}, \\ u(\mathbf{x}, 0) = 0, v(\mathbf{x}, 0) = 0, \mathbf{x} \in \bar{\Omega}, \end{cases} \tag{49}$$

where $\Omega = (0, 1) \times (0, 1), J = (0, 1], k_1 = k_2 = a = c = 1$, and $\bar{f}(\mathbf{x}, t)$ and $\bar{g}(\mathbf{x}, t)$ are chosen as

$$\begin{aligned} \bar{f}(\mathbf{x}, t) = &\left(2t + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 8\pi^2 t^2\right) \sin(2\pi x) \sin(2\pi y) \\ &- t^4 \sin^2(2\pi x) \sin^2(2\pi y) + t^2 \sin(4\pi x) \sin(4\pi y), \bar{g}(\mathbf{x}, t) \\ = &\left(2t + \frac{2t^{2-\beta}}{\Gamma(3-\beta)} + 32\pi^2 t^2\right) \sin(4\pi x) \sin(4\pi y) \\ &+ t^4 \sin^2(2\pi x) \sin^2(2\pi y) - t^2 \sin(4\pi x) \sin(4\pi y). \end{aligned}$$

Considering the given function $\bar{f}(\mathbf{x}, t)$ and $\bar{g}(\mathbf{x}, t)$, we can easily test and verify the exact solution as follows

$$\begin{cases} u = t^2 \sin(2\pi x) \sin(2\pi y), \\ v = t^2 \sin(4\pi x) \sin(4\pi y), \\ \sigma = (2\pi t^2 \cos(2\pi x) \sin(2\pi y), 2\pi t^2 \sin(2\pi x) \cos(2\pi y)), \\ \lambda = (4\pi t^2 \cos(4\pi x) \sin(4\pi y), 4\pi t^2 \sin(4\pi x) \cos(4\pi y)). \end{cases} \tag{50}$$

For solving the equations above, we use triangular elements. Selecting any unit $e = \Delta P_i P_j P_k$, we choose

$$\begin{aligned} &N(x, y) = \{[N_i(x, y), N_j(x, y), N_k(x, y)]^T | \\ &\quad N_i = S_i/S, N_j = S_j/S, N_k = S_k/S\} \end{aligned}$$

as the linear interpolation basis function of finite element space V_h and choose the constant basis function of finite element space Z_h. So we can get $u_h = N_i u_i + N_j u_j + N_k u_k$, $v_h = N_i v_i + N_j v_j + N_k v_k$, $\sigma_h = [\sigma_{1h}, \sigma_{2h}]$ and $\lambda_h = [\lambda_{1h}, \lambda_{2h}]$. After unit analysis, we can synthesize stiffness matrices

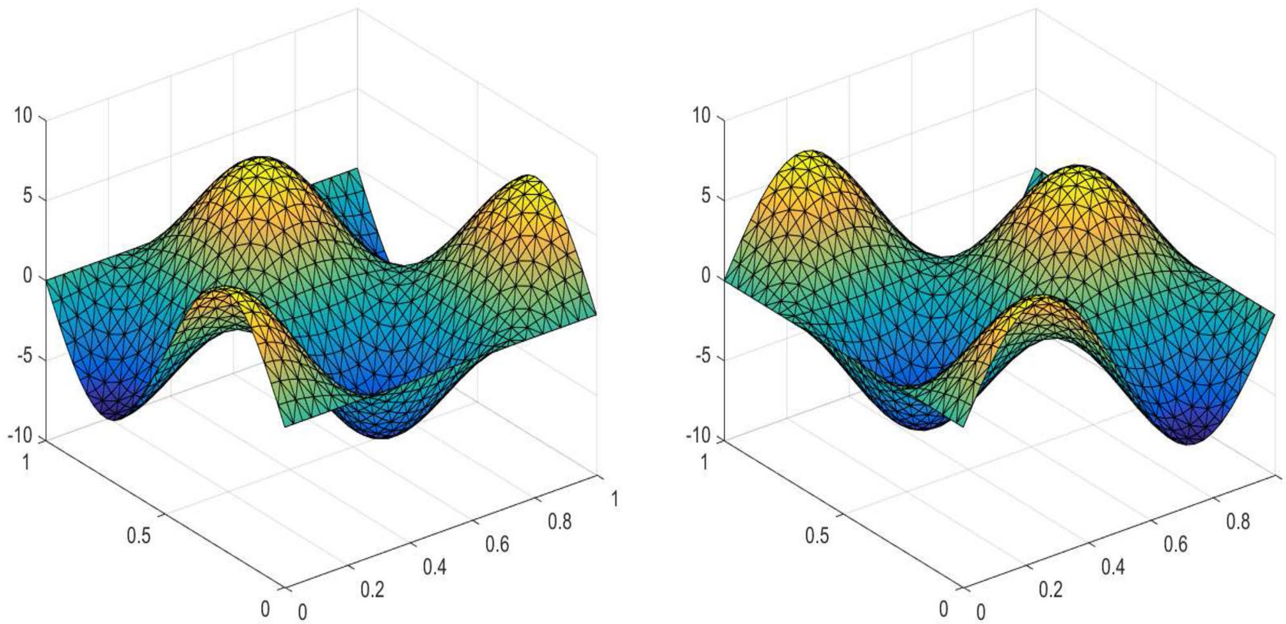

Fig. 5 Surface for exact solution $\sigma = (\sigma_1, \sigma_2)$

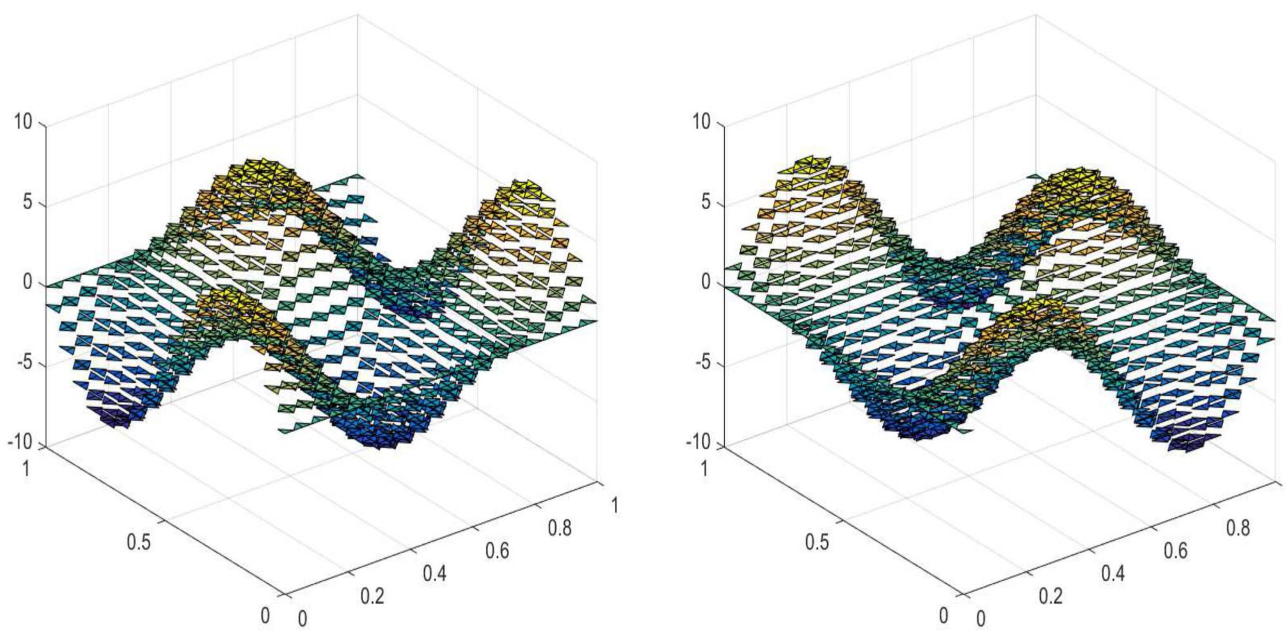

Fig. 6 Surface for numerical solution $\sigma_h = (\sigma_{1h}, \sigma_{2h})$

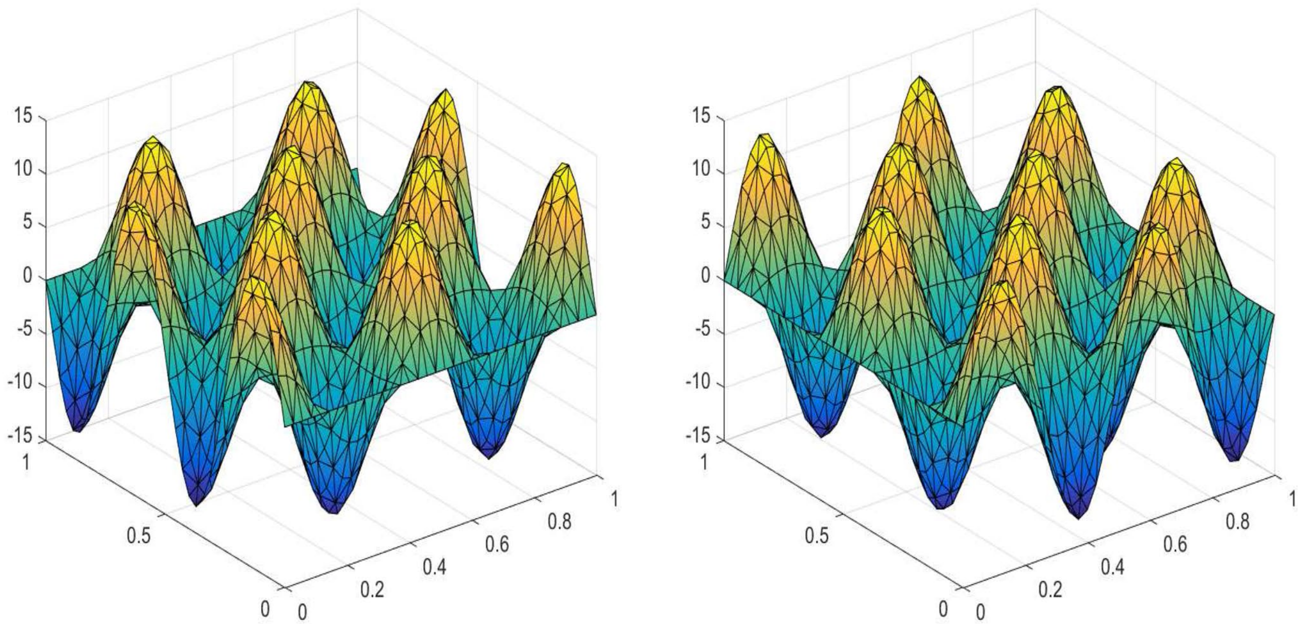

Fig. 7 Surface for exact solution $\lambda = (\lambda_1, \lambda_2)$

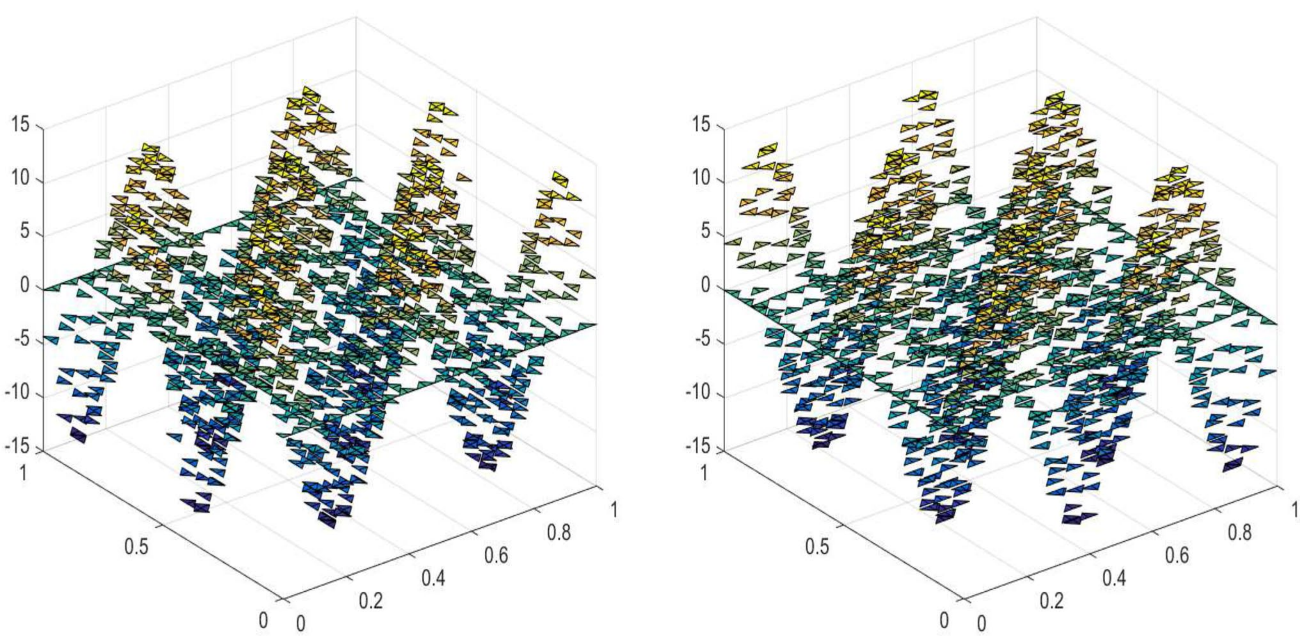

Fig. 8 Surface for numerical solution $\lambda_h = (\lambda_{1h}, \lambda_{2h})$

$$\Delta_e = \frac{1}{2}\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix}, A^{(e)} = \Delta_e \begin{bmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{6} \end{bmatrix},$$

$$C^{(e)} = \Delta_e \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B^{(e)} = D^{T(e)} = \frac{1}{2}\begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \end{bmatrix}, \quad (51)$$

$$a_i = \begin{vmatrix} y_j & 1 \\ y_k & 1 \end{vmatrix}, a_j = \begin{vmatrix} y_k & 1 \\ y_i & 1 \end{vmatrix}, a_k = \begin{vmatrix} y_i & 1 \\ y_j & 1 \end{vmatrix},$$

$$b_i = -\begin{vmatrix} x_j & 1 \\ x_k & 1 \end{vmatrix}, b_j = -\begin{vmatrix} x_k & 1 \\ x_i & 1 \end{vmatrix}, b_k = -\begin{vmatrix} x_i & 1 \\ x_j & 1 \end{vmatrix}.$$

For testing the space convergence order, we take changed step sizes $h = \sqrt{2}\tau = \frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{12}, \frac{\sqrt{2}}{16}, \frac{\sqrt{2}}{20}$, different fractional parameters $\alpha = \beta = 0.01, 0.5, 0.99$ to get the computing data in Tables 4, 5 and 6. We also show the computing data with $\alpha = 0.25, \beta = 0.75$ in Table 7. From Tables 4, 5, 6 and 7, one can find the space convergence orders of u, v in L^2-norm are close to 2. At the same time, the space convergence orders of u, v in H^1-norm and σ, λ in $(L^2)^2$-norm tend to 1. By the computed data, it is easy to check that the numerical convergence orders are in agreement with the theory results. Besides, by choosing $h = \frac{\sqrt{2}}{35}$, $\tau = \frac{1}{35}$ and $\alpha = \beta = 0.5$, the comparison surfaces between the numerical solution and the exact solution for unknown scalar functions and their fluxes are provided in Figs. 1, 2, 3, 4, 5, 6, 7 and 8, from which one can clearly see that the behaviors of the numerical solution under our mixed element spaces are almost consistent with the ones of exact solution which illustrates that our algorithm is effective for numerically solving nonlinear fractional coupled diffusion system.

Remark 3 In this article, we implement numerical computing by choosing triangular elements. One can also consider rectangular elements [55].

6 Conclusions

In this paper, we solve a time fractional coupled sub-diffusion system by a mixed element method with the second-order time approximation. By making use of this numerical algorithm, we can arrive at the approximation solutions of four functions. We provide the detailed proof of existence and uniqueness of the mixed element solution and stability analysis, and derive optimal a priori error estimates in L^2 and H^1-norm for the unknown u, v and a priori error estimates in $(L^2)^2$-norm for the fluxes σ and λ. Finally, via some numerical results, we illustrate the validity for the proposed numerical method.

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