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ORIGINAL ARTICLE

Predictor–corrector for non-linear differential and integral equation with fractal–fractional operators

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Abstract
Fractal–fractional differential and integral operators have been recognized recently as superior operators as they are able to depict physical problem with both memory effect and self-similar properties. Therefore, differential and integral equations constructed from these new operators are of great importance. In this paper, we extend the method of predictor–corrector to obtain numerical solution of non-linear differential and integral equations. Some examples are presented to illustrate the efficiency of the new method for solving these new equations.

Keywords Fractal–fractional derivative · Fractal–fractional integral · Predictor–corrector · Non-linear equations

1 Introduction

Due to the capability of differential and integral operators to model different real world problems, researchers in the past years have devoted their attention in suggesting new differential and integral operators that could possibly be used in terms of modelling more complex real-world problems. In the last decades, quite a few have been suggested, fractional differential and integral operators with power law kernel, which was the first for non-local operators, fractional differential and integral operators with exponential decay law, which was suggested by Caputo and Fabrizio, fractional differential and integral operators with the generalized Mittag-Leffler function, which was suggested by Atangana and Baleanu [1–7]. For each one of these operators, their associate variable orders have been suggested as they were found suitable for modelling anomalous problems. While these differential and integral operators with their associated variable order have been in great results in the last decades, researchers have found many physical problems that could not be modelled by them. For instance, one can find in nature real-world problems exhibiting either power law and self-similar behavior, or exponential decay with self-similar behavior or even more complicate one, crossover and self-similar behavior [8–18]. None of the above-mentioned differential and integral operators can be used for these purposes. Until very recently, Atangana suggested new differential and integral operators where the differential operator is the convolution of the fractal derivative with fractional kernel, including power law, exponential decay law, and the generalized Mittag-Leffler function. Due to the novelty and their capability of modelling complex real-world problems, new numerical scheme was needed to handle these new equations. New operators called as fractal–fractional differential and integral operators were introduced by Atangana in [19]. This new operators aim to attract more non-local natural problems that display at the same time fractal behaviors. In this paper, the corresponding predictor–corrector will be suggested.

Definition 1 [19] Suppose that $f(t)$ be continuous and fractal differentiable on an open interval (a, b) with order τ, then the fractal–fractional derivative of $f(t)$ with order α in the

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Riemann–Liouville sense having power law type kernel is given by

$$^{\mathrm{FFP}}D_{0,t}^{\alpha,\beta}(f(t)) = \frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t^{\beta}}\int_0^t (t-s)^{m-\alpha-1}f(s)\mathrm{d}s,$$

where $m-1<\alpha,\beta\leq m\in\mathbb{N}$ and

$$\frac{\mathrm{d}f(s)}{\mathrm{d}t^{\beta}} = \lim_{t\to s}\frac{f(t)-f(s)}{t^{\beta}-s^{\beta}}.$$

Definition 2 [19] Suppose that $f(t)$ be continuous on an open interval (a,b), then the fractal–fractional integral of $f(t)$ with order α having power law-type kernel is given by

$$^{\mathrm{FFP}}J_{0,t}^{\alpha,\beta}(f(t)) = \frac{\beta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\beta-1}f(s)\mathrm{d}s.$$

Definition 3 [19] Suppose that $f(t)$ be continuous and fractal differentiable on an open interval (a,b) with order β, then the fractal–fractional derivative of $f(t)$ with order α in the Riemann–Liouville sense having Mittag-Leffler kernel is given by

$$^{\mathrm{FFM}}_{0}D_t^{\alpha}(f(t)) = \frac{AB(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right]f(s)\mathrm{d}s,$$

where $\alpha>0,\beta\leq m\in\mathbb{N}$ and $AB(0)=AB(1)=1$.

Definition 4 [19] Suppose that $f(t)$ be continuous on an open interval (a,b), then the fractal–fractional integral of $f(t)$ with order α having Mittag-Leffler type kernel is given by

$$^{\mathrm{FFM}}J_{0,t}^{\alpha,\beta}(f(t)) = \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\int_0^t \tau^{\beta-1}f(\tau)(t-\tau)^{\alpha-1}\mathrm{d}\tau + \frac{\beta(1-\alpha)t^{\beta-1}f(t)}{AB(\alpha)}.$$

Definition 5 [19] Suppose that $f(t)$ be continuous and fractal differentiable on an open interval (a,b) with order β, then the fractal–fractional derivative of $f(t)$ with order α in the Riemann–Liouville sense having exponentially decaying type kernel is given by

$$^{\mathrm{FFE}}_{0}D_t^{\alpha}(f(t)) = \frac{M(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \exp\left[-\frac{\alpha}{1-\alpha}(t-s)\right]f(s)\mathrm{d}s,$$

where $\alpha>0,\beta\leq m\in\mathbb{N}$ and $M(0)=M(1)=1$ [19].

Definition 6 [19] The fractional integral associate to the new fractional derivative with exponential decay kernel is defined as

$$^{\mathrm{FFE}}_{0}I_t^{\alpha}(f(t)) = \frac{(1-\alpha)\alpha}{M(\alpha)}\beta t^{\beta-1}f(t) + \frac{\alpha}{M(\alpha)}\int_0^t \tau^{\beta-1}f(\tau)\mathrm{d}\tau.$$

2 New method with Atangana–Baleanu fractal–fractional derivative

Let us consider the following Cauchy problem

$$\begin{aligned} ^{\mathrm{FFM}}_{0}D_t^{\alpha,\beta}y(t) &= f(t,y(t)),\\ y(0) &= y_0, \end{aligned} \tag{1}$$

where the derivative is Atangana–Baleanu fractal–fractional derivative. Integrating above equation, we obtain the following

$$y(t)-y(0) = \frac{1-\alpha}{AB(\alpha)}\beta t^{\beta-1}f(t,y(t)) + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\int_0^t \tau^{\beta-1}(t-\tau)^{\alpha-1}f(\tau,y(\tau))\mathrm{d}\tau. \tag{2}$$

We have with the initial condition

$$y(t) = y_0 + \frac{1-\alpha}{AB(\alpha)}F(t,y(t)) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t F(\tau,y(\tau))(t-\tau)^{\alpha-1}\mathrm{d}\tau. \tag{3}$$

At the point $t_{n+1}=(n+1)\Delta t$, we have

$$y(t_{n+1}) = y_0 + \frac{1-\alpha}{AB(\alpha)}\beta t_{n+1}^{\beta-1}f(t_{n+1},y(t_{n+1})) + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}\tau^{\beta-1}\times(t_{n+1}-\tau)^{\alpha-1}f(\tau,y(\tau))\mathrm{d}\tau. \tag{4}$$

For the approximation of the function $f(\tau,y(\tau))$, we write the Newton polynomial which is given by

$$\begin{aligned} P_2(\tau) &= f(t_{k+1},y_{k+1}) + \frac{f(t_{k+1},y_{k+1})-f(t_k,y_k)}{h}(\tau-t_{k+1})\\ &+ \frac{f(t_{k+1},y_{k+1})-2f(t_k,y_k)+f(t_{k-1},y_{k-1})}{2h^2}\\ &\times(\tau-t_{k+1})(\tau-t_k). \end{aligned} \tag{5}$$

Here if we put this polynomial into above equation, we write as follows:

$$
\begin{aligned}
y_{n+1} = {} & y_0 + \frac{1-\alpha}{AB(\alpha)}\beta t_{n+1}^{\beta-1} f\left(t_{n+1}, y_{n+1}^p\right) \\
& + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \\
& \times \sum_{k=0}^{n-1} \left\{ \begin{array}{c} f\left(t_{k+1}, y_{k+1}\right) J_{1,k}^{\alpha,\beta} \\ + \frac{f\left(t_{k+1}, y_{k+1}\right) - f\left(t_k, y_k\right)}{h} J_{2,k}^{\alpha,\beta} \\ + \frac{f\left(t_{k+1}, y_{k+1}\right) - 2f\left(t_k, y_k\right) + f\left(t_{k-1}, y_{k-1}\right)}{2h^2} J_{3,k}^{\alpha,\beta} \end{array} \right\} \\
& + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} f\left(t_{n+1}, y_{n+1}^p\right) J_{1,n}^{\alpha,\beta} \\
& + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \frac{f\left(t_{n+1}, y_{n+1}^p\right) - f\left(t_n, y_n\right)}{h} J_{2,n}^{\alpha,\beta} \\
& + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \\
& \frac{f\left(t_{n+1}, y_{n+1}^p\right) - 2f\left(t_n, y_n\right) + f\left(t_{n-1}, y_{n-1}\right)}{2h^2} J_{3,n}^{\alpha,\beta}.
\end{aligned} \tag{6}
$$

Here

$$
\begin{aligned}
y_{n+1}^p = {} & y_0 + \frac{1-\alpha}{AB(\alpha)}\beta t_{n+1}^{\beta-1} f\left(t_n, y_n\right) \\
& + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \sum_{k=0}^{n} \\
& \int_{t_k}^{t_{k+1}} \tau^{\beta-1}\left(t_{n+1}-\tau\right)^{\alpha-1} f(\tau, y(\tau))\mathrm{d}\tau.
\end{aligned} \tag{7}
$$

Also we can write

$$
\begin{aligned}
y_{n+1}^p = {} & y_0 + \frac{1-\alpha}{AB(\alpha)}\beta t_{n+1}^{\beta-1} f\left(t_n, y_n\right) \\
& + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \sum_{k=0}^{n} f\left(t_k, y_k\right) \\
& \int_{t_k}^{t_{k+1}} \tau^{\beta-1}\left(t_{n+1}-\tau\right)^{\alpha-1}\mathrm{d}\tau
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
y_{n+1}^p = {} & y_0 + \frac{1-\alpha}{AB(\alpha)}\beta t_{n+1}^{\beta-1} f\left(t_n, y_n\right) \\
& + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} \sum_{k=0}^{n} f\left(t_k, y_k\right) J_{1,k}^{\alpha,\beta}.
\end{aligned} \tag{9}
$$

We write the first iteration

$$
\begin{aligned}
y_1 = {} & y_0 + \frac{1-\alpha}{AB(\alpha)}\beta h^{\beta-1} f\left(t_0, y_0\right) \\
& + \frac{\alpha\beta}{AB(\alpha)\Gamma(\alpha)} f\left(t_0, y_0\right) J_{0,0}^{\alpha,\beta},
\end{aligned} \tag{10}
$$

where

$$
J_{0,0}^{\alpha,\beta} = \frac{\Gamma(1+\beta)\Gamma(\alpha)h^{\alpha+\beta-1}}{\beta\Gamma(\alpha+\beta)}. \tag{11}
$$

3 New method with Caputo–Fabrizio fractal–fractional derivative

Let us consider the following Cauchy problem:

$$
\begin{aligned}
{}_0^{\mathrm{FFE}}D_t^{\alpha,\beta} y(t) &= f(t, y(t)) \\
y(0) &= y_0,
\end{aligned} \tag{12}
$$

where the derivative is Atangana–Baleanu fractal–fractional derivative. Integrating above equation, we obtain the following:

$$
\begin{aligned}
y(t) = {} & y_0 + \frac{(1-\alpha)\alpha\beta t^{\beta-1}}{M(\alpha)} f(t, y(t)) \\
& + \frac{\alpha\beta}{M(\alpha)} \int_0^t \tau^{\beta-1} f(\tau, y(\tau))\mathrm{d}\tau
\end{aligned} \tag{13}
$$

with the initial condition. At the point $t_{n+1} = (n+1)\Delta t$, we have

$$
\begin{aligned}
y\left(t_{n+1}\right) = {} & y_0 + \frac{(1-\alpha)\alpha\beta}{M(\alpha)}\beta t_{n+1}^{\beta-1} f\left(t_{n+1}, y_{n+1}^p\right) \\
& + \frac{\alpha\beta}{M(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \tau^{\beta-1} f(\tau, y(\tau))\mathrm{d}\tau.
\end{aligned} \tag{14}
$$

At interval $\left[t_k, t_{k+1}\right]$, the function $f(\tau, y(\tau))$ is given by

$$
\begin{aligned}
P_2(\tau) = {} & f\left(t_{k+1}, y_{k+1}\right) + \frac{f\left(t_{k+1}, y_{k+1}\right) - f\left(t_k, y_k\right)}{h}\left(\tau - t_{k+1}\right) \\
& + \frac{f\left(t_{k+1}, y_{k+1}\right) - 2f\left(t_k, y_k\right) + f\left(t_{k-1}, y_{k-1}\right)}{2h^2} \\
& \times \left(\tau - t_{k+1}\right)\left(\tau - t_k\right).
\end{aligned} \tag{15}
$$

Here if we put this polynomial into above equation, we write such as

$$
y_{n+1} = y_n + \frac{\alpha\beta}{M(\alpha)} \sum_{k=0}^{n-1} \left\{ \begin{array}{c} f\left(t_{k+1}, y_{k+1}\right) \int_{t_k}^{t_{k+1}} \tau^{\beta-1}\mathrm{d}\tau \\ + \frac{f\left(t_{k+1}, y_{k+1}\right) - f\left(t_k, y_k\right)}{h} \\ \times \int_{t_k}^{t_{k+1}} \tau^{\beta-1}\left(\tau - t_{k+1}\right)\mathrm{d}\tau \\ + \frac{f\left(t_{k+1}, y_{k+1}\right) - 2f\left(t_k, y_k\right) + f\left(t_{k-1}, y_{k-1}\right)}{2h^2} \\ \times \int_{t_k}^{t_{k+1}} \tau^{\beta-1}\left(\tau - t_{k+1}\right)\left(\tau - t_k\right)\mathrm{d}\tau \end{array} \right\}. \tag{16}
$$

Here

$$y_{n+1} = y_n + \frac{(1-\alpha)\alpha\beta}{M(\alpha)} t_{n+1}^{\beta-1} f\left(t_{n+1}, y_{n+1}^p\right) + \frac{\alpha\beta}{M(\alpha)} \sum_{k=0}^{n-1} \left\{ \begin{array}{c} f\left(t_{k+1}, y_{k+1}\right) \frac{h^\beta}{\beta}\left((k+1)^\beta - k^\beta\right) \\ + \frac{f\left(t_{k+1}, y_{k+1}\right) - f\left(t_k, y_k\right)}{h} \\ \times \left(\frac{h^{\beta+1}\left(k^\beta(k+1+\beta) - (k+1)^{\beta+1}\right)}{\beta(\beta+1)} \right) \\ + \frac{f\left(t_{k+1}, y_{k+1}\right) - 2f\left(t_k, y_k\right) + f\left(t_{k-1}, y_{k-1}\right)}{2h^2} \\ \times \left(\frac{2h^{\beta+2}\left(\left(k - \frac{\beta}{2}\right)(k+1)^{\beta+1} - k^{\beta+1}\left(k+1+\frac{\beta}{2}\right)\right)}{\beta(\beta+1)(\beta+2)} \right) \end{array} \right\}. \tag{17}$$

Also we can write

$$\begin{aligned} y_{n+1} = {} & y_n + \frac{(1-\alpha)\alpha\beta}{M(\alpha)} t_{n+1}^{\beta-1} f\left(t_{n+1}, y_{n+1}^p\right) \\ & + \frac{\alpha\beta}{M(\alpha)} \sum_{k=0}^{n-1} \left\{ \begin{array}{c} f\left(t_{k+1}, y_{k+1}\right) \frac{h^\beta}{\beta}\left((k+1)^\beta - k^\beta\right) \\ + \frac{f\left(t_{k+1}, y_{k+1}\right) - f\left(t_k, y_k\right)}{h} \\ \times \left(\frac{h^{\beta+1}\left(k^\beta(k+1+\beta) - (k+1)^{\beta+1}\right)}{\beta(\beta+1)} \right) \\ + \frac{f\left(t_{k+1}, y_{k+1}\right) - 2f\left(t_k, y_k\right) + f\left(t_{k-1}, y_{k-1}\right)}{2h^2} \\ \times \left(\frac{2h^{\beta+2}\left(\left(k - \frac{\beta}{2}\right)(k+1)^{\beta+1} - k^{\beta+1}\left(k+1+\frac{\beta}{2}\right)\right)}{\beta(\beta+1)(\beta+2)} \right) \end{array} \right\} \\ & + \frac{\alpha\beta}{M(\alpha)} f\left(t_{n+1}, y_{n+1}^p\right) \frac{h^\beta}{\beta}\left((n+1)^\beta - n^\beta\right) \\ & + \frac{\alpha\beta}{M(\alpha)} \left(\frac{f\left(t_{n+1}, y_{n+1}^p\right) - f\left(t_n, y_n\right)}{h} \right) \\ & \times \left(\frac{h^{\beta+1}\left(n^\beta(n+1+\beta) - (n+1)^{\beta+1}\right)}{\beta(\beta+1)} \right) \\ & + \frac{\alpha\beta}{M(\alpha)} \left(\frac{f\left(t_{k+1}, y_{k+1}\right) - 2f\left(t_k, y_k\right) + f\left(t_{k-1}, y_{k-1}\right)}{2h^2} \right) \\ & \times \left(\frac{2h^{\beta+2}\left(\left(n - \frac{\beta}{2}\right)(n+1)^{\beta+1} - n^{\beta+1}\left(n+1+\frac{\beta}{2}\right)\right)}{\beta(\beta+1)(\beta+2)} \right), \end{aligned} \tag{18}$$

where

$$\begin{aligned} y_{n+1}^p = {} & y_0 + \frac{(1-\alpha)\alpha\beta}{M(\alpha)} t_{n+1}^{\beta-1} f\left(t_n, y_n\right) \\ & + \frac{\alpha\beta}{M(\alpha)} \sum_{k=0}^{n} f\left(t_k, y_k\right) \frac{h^\beta}{\beta}\left((k+1)^\beta - k^\beta\right). \end{aligned} \tag{19}$$

We write the first iteration:

$$\begin{aligned} y_1 = {} & \frac{(1-\alpha)\alpha\beta}{M(\alpha)} h^{\beta-1} f\left(t_0, y_0\right) \\ & + \frac{\alpha\beta}{M(\alpha)} f\left(t_0, y_0\right) \frac{h^\beta}{\beta}. \end{aligned} \tag{20}$$

4 New method with Caputo fractal–fractional derivative

Let us consider the following Cauchy problem

$$\begin{aligned} {}_0^{FFP}D_t^{\alpha,\beta} y(t) &= f(t, y(t)), \\ y(0) &= y_0, \end{aligned} \tag{21}$$

where the derivative is Caputo fractal–fractional derivative. Integrating above equation, we obtain the following:

$$y(t) - y(0) = \frac{\beta}{\Gamma(\alpha)} \int_0^t \tau^{\beta-1}(t-\tau)^{\alpha-1} f(\tau, y(\tau)) \mathrm{d}\tau. \tag{22}$$

We have with the initial condition

$$y\left(t_{n+1}\right) = y_0 + \frac{\beta}{\Gamma(\alpha)} \int_0^{t_{n+1}} \tau^{\beta-1}\left(t_{n+1} - \tau\right)^{\alpha-1} f(\tau, y(\tau)) \mathrm{d}\tau. \tag{23}$$

at the point $t_{n+1} = (n+1)\Delta t$ and we can write as follows:

$$y_{n+1} = y_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \tau^{\beta-1}\left(t_{n+1} - \tau\right)^{\alpha-1} f(\tau, y(\tau)) \mathrm{d}\tau. \tag{24}$$

For the approximation of the function $f(\tau, y(\tau))$, we write the Newton polynomial which is given by

$$\begin{aligned} P_2(\tau) = {} & f\left(t_{k+1}, y_{k+1}\right) + \frac{f\left(t_{k+1}, y_{k+1}\right) - f\left(t_k, y_k\right)}{h}\left(\tau - t_{k+1}\right) \\ & + \frac{f\left(t_{k+1}, y_{k+1}\right) - 2f\left(t_k, y_k\right) + f\left(t_{k-1}, y_{k-1}\right)}{2h^2} \\ & \times \left(\tau - t_{k+1}\right)\left(\tau - t_k\right). \end{aligned} \tag{25}$$

Here if we put this polynomial into above equation, we write such as

$$y_{n+1} = y_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \left\{ \begin{array}{c} f\left(t_{k+1}, y_{k+1}\right) \\ \times \int_{t_k}^{t_{k+1}} \tau^{\beta-1}\left(t_{n+1} - \tau\right)^{\alpha-1} \mathrm{d}\tau \\ + \frac{f\left(t_{k+1}, y_{k+1}\right) - f\left(t_k, y_k\right)}{h} \\ \times \int_{t_k}^{t_{k+1}} \tau^{\beta-1}\left(t_{n+1} - \tau\right)^{\alpha-1}\left(\tau - t_{k+1}\right) \mathrm{d}\tau \\ + \frac{f\left(t_{k+1}, y_{k+1}\right) - 2f\left(t_k, y_k\right) + f\left(t_{k-1}, y_{k-1}\right)}{2h^2} \\ \times \int_{t_k}^{t_{k+1}} \tau^{\beta-1}\left(t_{n+1} - \tau\right)^{\alpha-1}\left(\tau - t_{k+1}\right)\left(\tau - t_k\right) \mathrm{d}\tau \end{array} \right\}. \tag{26}$$

Here

$$J_{1,k}^{\alpha,\beta} = \int_{t_k}^{t_{k+1}} \tau^{\beta-1}\left(t_{n+1}-\tau\right)^{\alpha-1}\mathrm{d}\tau,$$
$$J_{2,k}^{\alpha,\beta} = \int_{t_k}^{t_{k+1}} \tau^{\beta-1}\left(t_{n+1}-\tau\right)^{\alpha-1}\left(\tau-t_{k+1}\right)\mathrm{d}\tau, \tag{27}$$
$$J_{3,k}^{\alpha,\beta} = \int_{t_k}^{t_{k+1}} \tau^{\beta-1}\left(t_{n+1}-\tau\right)^{\alpha-1}\left(\tau-t_{k+1}\right)\left(\tau-t_k\right)\mathrm{d}\tau,$$

and we write

$$y_{n+1} = y_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{k=0}^{n} \left\{ \begin{array}{c} f\left(t_{k+1}, y_{k+1}\right) J_{1,k}^{\alpha,\beta} + \frac{f\left(t_{k+1},y_{k+1}\right)-f\left(t_k,y_k\right)}{h} J_{2,k}^{\alpha,\beta} \\ + \frac{f\left(t_{k+1},y_{k+1}\right)-2f\left(t_k,y_k\right)+f\left(t_{k-1},y_{k-1}\right)}{2h^2} J_{3,k}^{\alpha,\beta} \end{array} \right\}. \tag{28}$$

Here

$$J_{1,k}^{\alpha,\beta} = \frac{((n+1)h)^{\alpha-1}}{\beta}, \left[\begin{array}{c} ((k+1)h)^{\beta} \text{ hypergeom}\left([\beta, 1-\alpha], [1+\beta], \frac{k+1}{n+1}\right) \\ -(kh)^{\beta} \text{ hypergeom}\left([\beta, 1-\alpha], [1+\beta], \frac{k}{n}\right), \end{array} \right] \tag{29}$$

$$J_{2,k}^{\alpha,\beta} = \frac{((n+1)h)^{\alpha-1}}{\beta(\beta+1)} \left[\begin{array}{c} \beta((k+1)h)^{\beta+1} \text{ hypergeom}\left([1+\beta, 1-\alpha], [2+\beta], \frac{k+1}{n+1}\right) \\ -((k+1)h)^{\beta+1}(1+\beta) \text{ hypergeom}\left([\beta, 1-\alpha], [1+\beta], \frac{k+1}{n}\right) \\ -\beta(kh)^{\beta+1} \text{ hypergeom}\left([1+\beta, 1-\alpha], [2+\beta], \frac{k}{n+1}\right) \\ +h(kh)^{\beta}(1+\beta)(1+k) \text{ hypergeom}\left([\beta, 1-\alpha], [1+\beta], \frac{k}{n}\right) \end{array} \right] \tag{30}$$

and

$$J_{3,k}^{\alpha,\beta} = \frac{((n+1)h)^{\alpha-1}}{\beta(\beta+1)(\beta+2)} \left[\begin{array}{c} \beta(\beta+1)((k+1)h)^{\beta+2} \\ \times \text{ hypergeom}\left([2+\beta, 1-\alpha], [3+\beta], \frac{k+1}{n+1}\right) \\ -2\beta(\beta+2)\left(k+\frac{1}{2}\right)h((k+1)h)^{\beta+1} \\ \times \text{ hypergeom}\left([1+\beta, 1-\alpha], [2+\beta], \frac{k+1}{n+1}\right) \\ +kh(\beta+1)(\beta+2)((k+1)h)^{\beta+1} \\ \times \text{ hypergeom}\left([\beta, 1-\alpha], [1+\beta], \frac{k+1}{n}\right) \\ +2\beta(\beta+2)\left(k+\frac{1}{2}\right)h(kh)^{\beta+1} \\ \times \text{ hypergeom}\left([1+\beta, 1-\alpha], [2+\beta], \frac{k}{n+1}\right) \\ -\beta(\beta+1)(kh)^{\beta+2} \\ \times \text{ hypergeom}\left([2+\beta, 1-\alpha], [3+\beta], \frac{k}{n+1}\right) \\ -h(\beta+1)(\beta+2)(k+1)(kh)^{\beta+1} \\ \times \text{ hypergeom}\left([\beta, 1-\alpha], [1+\beta], \frac{k}{n+1}\right) \end{array} \right]. \tag{31}$$

So we write the following:

$$\begin{aligned} y_{n+1} &= y_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{k=0}^{n} \left\{ \begin{array}{c} f\left(t_{k+1}, y_{k+1}\right) J_{1,k}^{\alpha,\beta} + \frac{f\left(t_{k+1},y_{k+1}\right)-f\left(t_k,y_k\right)}{h} J_{2,k}^{\alpha,\beta} \\ + \frac{f\left(t_{k+1},y_{k+1}\right)-2f\left(t_k,y_k\right)+f\left(t_{k-1},y_{k-1}\right)}{2h^2} J_{3,k}^{\alpha,\beta} \end{array} \right\}, \\ &= y_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \left\{ \begin{array}{c} f\left(t_{k+1}, y_{k+1}\right) J_{1,k}^{\alpha,\beta} + \frac{f\left(t_{k+1},y_{k+1}\right)-f\left(t_k,y_k\right)}{h} J_{2,k}^{\alpha,\beta} \\ + \frac{f\left(t_{k+1},y_{k+1}\right)-2f\left(t_k,y_k\right)+f\left(t_{k-1},y_{k-1}\right)}{2h^2} J_{3,k}^{\alpha,\beta} \end{array} \right\} \\ &+ \frac{\beta}{\Gamma(\alpha)} f\left(t_{n+1}, y_{n+1}^{p}\right) J_{1,n}^{\alpha,\beta} \\ &+ \frac{\beta}{\Gamma(\alpha)} \frac{f\left(t_{n+1}, y_{n+1}^{p}\right) - f\left(t_n, y_n\right)}{h} J_{2,n}^{\alpha,\beta} \\ &+ \frac{\beta}{\Gamma(\alpha)} \frac{f\left(t_{n+1}, y_{n+1}^{p}\right) - 2f\left(t_n, y_n\right) + f\left(t_{n-1}, y_{n-1}\right)}{2h^2} J_{3,n}^{\alpha,\beta}, \end{aligned} \tag{32}$$

where

$$y_{n+1}^{p} = y_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \tau^{\beta-1} \left(t_{n+1} - \tau\right)^{\alpha-1} f(\tau, y(\tau)) d\tau,$$
$$= y_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{k=0}^{n} f\left(t_k, y_k\right) \int_{t_k}^{t_{k+1}} \tau^{\beta-1} \left(t_{n+1} - \tau\right)^{\alpha-1} d\tau, \tag{33}$$
$$= y_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{k=0}^{n} f\left(t_k, y_k\right) J_{1,k}^{\alpha,\beta}.$$

Thus we have the following:

$$y_{n+1} = y_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{k=0}^{n} \left\{ \begin{array}{l} f\left(t_{k+1}, y_{k+1}\right) J_{1,k}^{\alpha,\beta} + \frac{f\left(t_{k+1}, y_{k+1}\right) - f\left(t_k, y_k\right)}{h} J_{2,k}^{\alpha,\beta} \\ + \frac{f\left(t_{k+1}, y_{k+1}\right) - 2f\left(t_k, y_k\right) + f\left(t_{k-1}, y_{k-1}\right)}{2h^2} J_{3,k}^{\alpha,\beta} \end{array} \right\},$$
$$= y_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \left\{ \begin{array}{l} f\left(t_{k+1}, y_{k+1}\right) J_{1,k}^{\alpha,\beta} + \frac{f\left(t_{k+1}, y_{k+1}\right) - f\left(t_k, y_k\right)}{h} J_{2,k}^{\alpha,\beta} \\ + \frac{f\left(t_{k+1}, y_{k+1}\right) - 2f\left(t_k, y_k\right) + f\left(t_{k-1}, y_{k-1}\right)}{2h^2} J_{3,k}^{\alpha,\beta} \end{array} \right\}$$
$$+ \frac{\beta}{\Gamma(\alpha)} f\left(t_{n+1}, y_{n+1}^{p}\right) J_{1,n}^{\alpha,\beta} + \frac{\beta}{\Gamma(\alpha)} \frac{f\left(t_{n+1}, y_{n+1}^{p}\right) - f\left(t_n, y_n\right)}{h} J_{2,n}^{\alpha,\beta}$$
$$+ \frac{\beta}{\Gamma(\alpha)} \frac{f\left(t_{n+1}, y_{n+1}^{p}\right) - 2f\left(t_n, y_n\right) + f\left(t_{n-1}, y_{n-1}\right)}{2h^2} J_{3,n}^{\alpha,\beta}. \tag{34}$$

5 Numerical illustrations and simulation

Example 1 We first consider the following problem:

$${}_{0}^{FFE}D_t^{\alpha,\beta} y(t) = t^2,$$
$$y(0) = 0,$$

where $\alpha = 0.7\ \beta = 0.4$. The exact solution of such equation is

$$y_{\text{exact}} = \frac{(1-\alpha)\alpha\beta}{M(\alpha)} t^{\beta+1} + \frac{\alpha\beta}{M(\alpha)} \frac{t^{\beta+2}}{\beta+2}.$$

The error of the proposed method is given as

$$\left\| y_{\text{ex}} - y_{\text{prop}} \right\|_{\infty} = 0.00013498.$$

Second, we take as

$${}_{0}^{\text{FFP}}D_t^{\alpha,\beta} y(t) = t^2,$$
$$y(0) = 0.$$

The error of the proposed method is calculated as

$$\left\| y_{\text{ex}} - y_{\text{prop}} \right\|_{\infty} = 0.000068817.$$

where $\alpha = 0.3, \beta = 0.45$. The exact solution is

$$y_{\text{exact}} = \frac{\beta}{\Gamma(\alpha)} \frac{\Gamma(\beta+3)\Gamma(\alpha) t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)(\beta+2)}.$$

Finally, we handle the following problem:

$${}_{0}^{\text{FFM}}D_t^{\alpha,\beta} y(t) = t \exp(-t),$$
$$y(0) = 0.$$

where $\alpha = 0.65\ \beta = 0.25$. The exact solution is

$$y_{\text{exact}} = t^{\alpha}\Gamma(\alpha) \left(\begin{array}{l} \frac{t^{\frac{\beta}{2}-1-\frac{\alpha}{2}} \alpha e^{-\frac{t}{2}} \text{WhittakerM}\left(\frac{\beta}{2}-\frac{\alpha}{2}, \frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2}, t\right)\Gamma(\beta+1)}{(\alpha+\beta+1)\Gamma(\alpha+\beta+1)} \\ + \frac{t^{\frac{\beta}{2}-1-\frac{\alpha}{2}} (\beta+1) e^{-\frac{t}{2}} \text{WhittakerM}\left(\frac{\beta}{2}-\frac{\alpha}{2}+1, \frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2}, t\right)\Gamma(\beta+1)}{(\alpha+\beta+1)\Gamma(\alpha+\beta+1)} \end{array} \right).$$

The error of the proposed method is as follows:

$$\left\| y_{\text{ex}} - y_{\text{prop}} \right\|_{\infty} = 0.036376.$$

The numerical simulations are depicted in Figs. 1, 2, 3.

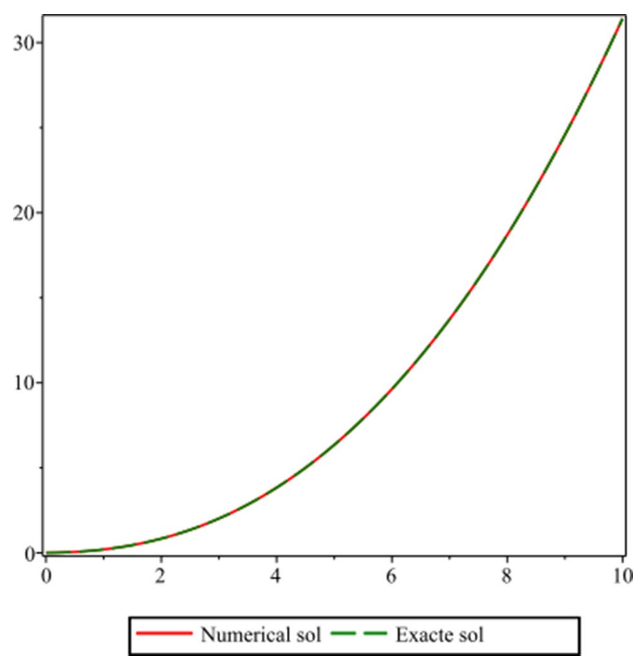

Fig. 1 Numerical solution of the considered problem with Caputo–Fabrizio fractal–fractional derivative

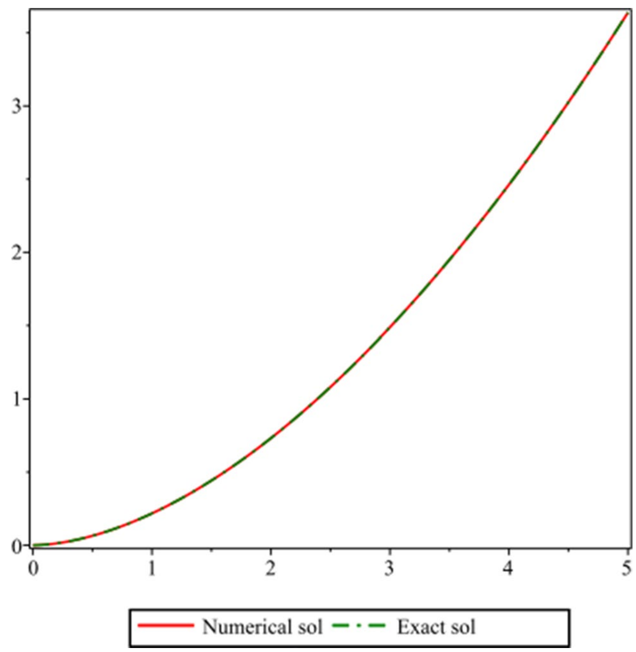

Fig. 2 Numerical solution of the considered problem with Caputo fractal–fractional derivative

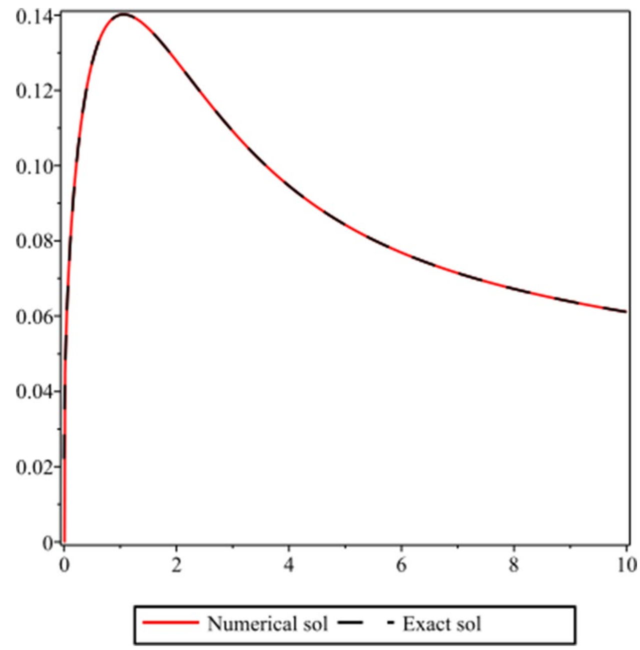

Fig. 3 Numerical solution of the considered problem with Atangana–Baleanu fractal–fractional derivative

Example 2 We consider the following problem:

$$
\begin{aligned}
{}_0^{\mathrm{FFE}}D_t^{\alpha,\beta}y(t) &= t\exp(-t),\\
y(0) &= 0,
\end{aligned}
$$

where $\alpha = 0.5, \beta = 0.7$. The exact solution is

$$
\begin{aligned}
{}_0^{\mathrm{FFP}}D_t^{\alpha,\beta}y(t) &= \cos(t),\\
y(0) &= 0,
\end{aligned}
$$

where $\alpha = 0.5, \beta = 0.7$. The exact solution of such equation is as follows:

$$
y_{\mathrm{exact}} = \frac{\beta}{\Gamma(\alpha)}\frac{t^{\alpha+\beta-1}\mathrm{hypergeom}\left(\left[\frac{\beta}{2},\frac{\beta}{2}+\frac{1}{2}\right],\left[\frac{1}{2},\frac{\alpha}{2}+\frac{\beta}{2},\frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2}\right],-\frac{t^2}{4}\right)\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.
$$

$$
y_{\mathrm{exact}} = \frac{(1-\alpha)\alpha\beta t^{\beta-1}}{M(\alpha)}t\exp(-t) + \frac{\alpha\beta}{M(\alpha)}\left(\frac{t^{\frac{\beta}{2}}e^{-\frac{t}{2}}\mathrm{WhittakerM}\left(\frac{\beta}{2},\frac{\beta}{2}+\frac{1}{2},t\right)}{\beta+1}\right).
$$

The error norm is calculated as

$$
\left\|y_{\mathrm{ex}} - y_{\mathrm{prop}}\right\|_\infty = 0.006979.
$$

We handle

The error norm is given by

$$
\left\|y_{\mathrm{ex}} - y_{\mathrm{prop}}\right\|_\infty = 2.22358 \times 10^{-6}.
$$

We solve

$$
\begin{aligned}
{}_0^{\mathrm{FFM}}D_t^{\alpha,\beta}y(t) &= t\cos(t),\\
y(0) &= 0,
\end{aligned}
$$

where $\alpha = 0.55, \beta = 0.7$. The exact solution is

$$
u_{\mathrm{exact}} = \frac{t^{\alpha+\beta}\mathrm{hypergeom}\left(\left[1+\frac{\beta}{2},\frac{\beta}{2}+\frac{1}{2}\right],\left[\frac{1}{2},1+\frac{\alpha}{2}+\frac{\beta}{2},\frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2}\right],-\frac{t^2}{4}\right)\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}.
$$

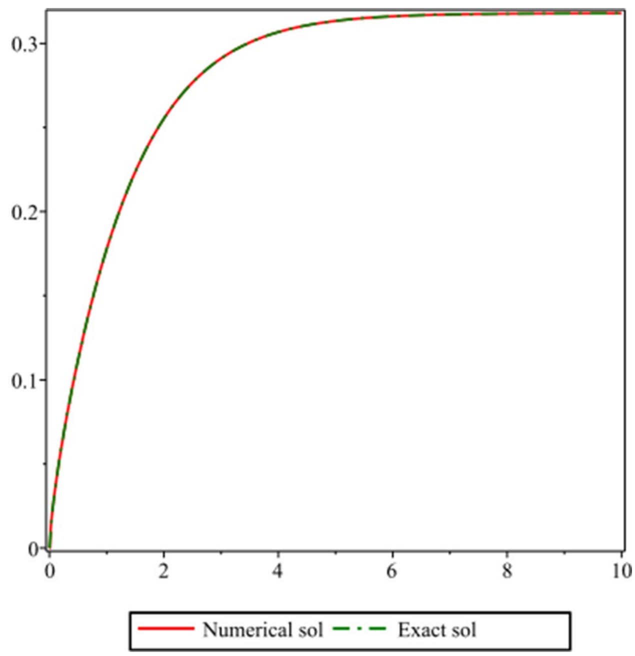

Fig. 4 Numerical solution of the considered problem with Caputo–Fabrizio fractal–fractional derivative

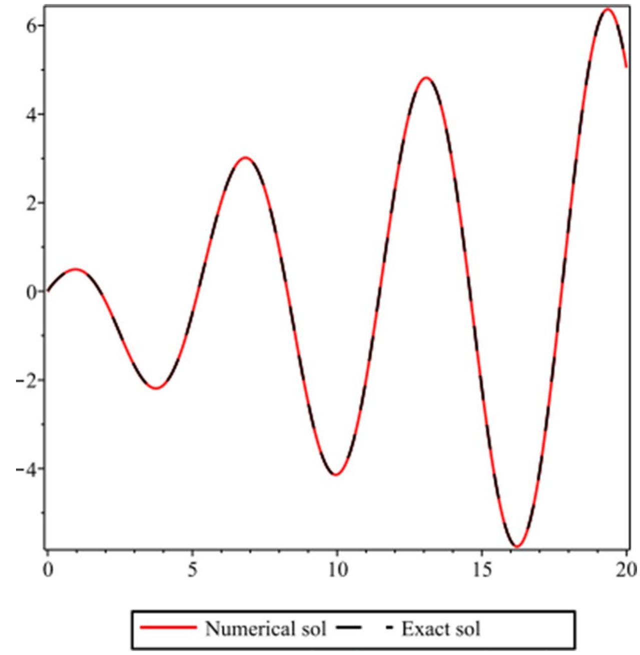

Fig. 6 Numerical solution of the considered problem with Atangana–Baleanu fractal–fractional derivative

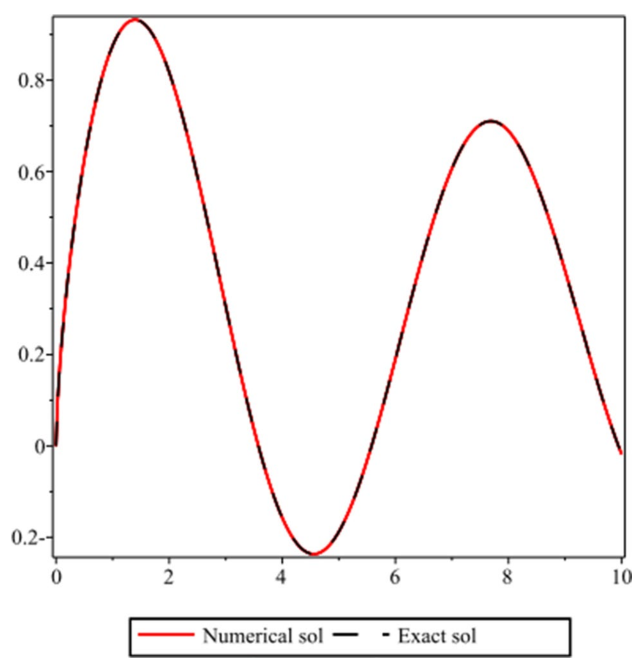

Fig. 5 Numerical solution of the considered problem with Caputo fractal–fractional derivative

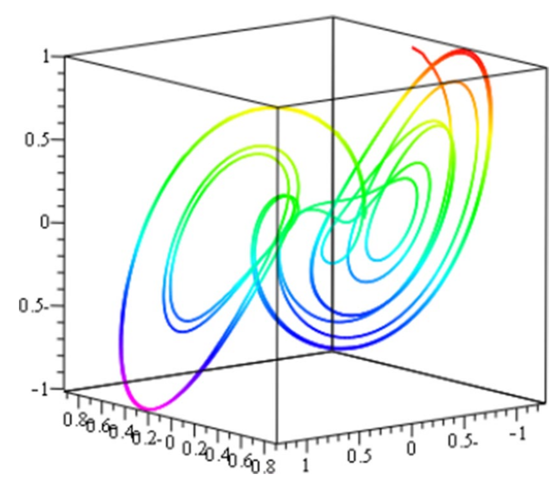

Fig. 7 Numerical solution of the Coullet system with Caputo–Fabrizio fractal–fractional derivative

The error norm is calculated as

$$\left\| y_{\text{ex}} - y_{\text{prop}} \right\|_{\infty} = 0.017102.$$

The numerical simulations are depicted in Figs. 4, 5, 6.

Example 3 We next consider the Coullet system:

$$\begin{aligned} {}_{0}^{\text{FFE}}D_{t}^{\alpha,\beta}x(t) &= y, \\ {}_{0}^{\text{FFE}}D_{t}^{\alpha,\beta}y(t) &= z, \\ {}_{0}^{\text{FFE}}D_{t}^{\alpha,\beta}z(t) &= 0.8x - 1.1y - 0.45z - x^{3} \end{aligned}$$

with the initial conditions

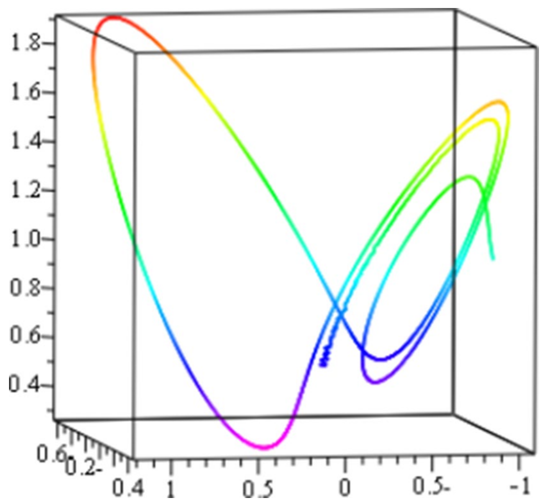

Fig. 8 Numerical simulation of the chaotic problem with Atangana–Baleanu fractal–fractional derivative

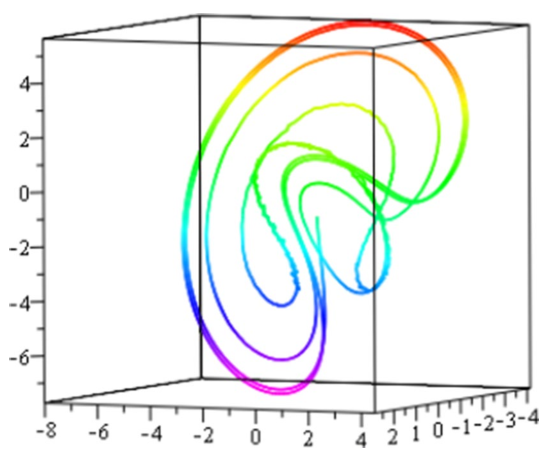

Fig. 9 Numerical solution of the chaotic problem with Caputo fractal–fractional derivative

$x(0) = -1, y(0) = 0, z(0) = 1.$

The numerical simulation is depicted in Fig. 7 for $\alpha = 0.96, \beta = 0.98$.

Example 4 We next consider the Shimizu–Morioka system

$$
\begin{aligned}
&{}_{0}^{\mathrm{FFM}}D_t^{\alpha,\beta}x(t) = y,\\
&{}_{0}^{\mathrm{FFM}}D_t^{\alpha,\beta}y(t) = (1-z)x - 0.75y,\\
&{}_{0}^{\mathrm{FFM}}D_t^{\alpha,\beta}z(t) = x^2 - 0.45z
\end{aligned}
$$

with the initial conditions

$x(0) = -1, y(0) = 0.1, z(0) = 1.$

The numerical simulation is depicted in Fig. 8 for $\alpha = 0.98, \beta = 0.75$.

Example 5 We next consider the following chaotic problem:

$$
\begin{aligned}
&{}_{0}^{FFP}D_t^{\alpha,\beta}x(t) = 0.1z + y - x,\\
&{}_{0}^{FFP}D_t^{\alpha,\beta}y(t) = -xz - x,\\
&{}_{0}^{FFP}D_t^{\alpha,\beta}z(t) = xy - 3
\end{aligned}
$$

with the initial conditions

$x(0) = 0.2, y(0) = 0.1, z(0) = -1.$

The numerical simulation is depicted in Fig. 9 for $\alpha = 0.96, \beta = 0.8$.

6 Conclusion

In the last past years, new differential and integral operators were introduced with the aim to capture more complex problems arising in many fields of science,technology, and engineering. Most have been applied with great success, nevertheless, none was able to depict at the same time problems displaying memory and self-similarities. Then, new differential and integral operators called fractal–fractional were introduced and are able to capture both scenarios. Due to the capabilities of these new operators to modeling complex real world problems, new numerical or adapted numerical schemes were needed. In this paper, we adapted the methodology used to derive the method of predictor–corrector which is efficient in solving the associated differential and integral equations.

Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interests regarding the publication of this paper.

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