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ORIGINAL ARTICLE

Chebyshev polynomials for the numerical solution of fractal–fractional model of nonlinear Ginzburg–Landau equation

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Abstract

This paper introduces a new version for the nonlinear Ginzburg–Landau equation derived from fractal–fractional derivatives and proposes a computational scheme for their numerical solutions. The fractal–fractional derivative is defined in the Atangana–Riemann–Liouville sense with Mittage–Leffler kernel. The proposed approach is based on the shifted Chebyshev polynomials (S-CPs) and the collocation scheme. Through the way, a new operational matrix (OM) of fractal–fractional derivative is derived for the S-CPs and used in the presented method. More precisely, the unknown solution is separated into their real and imaginary parts, and then, these parts are expanded in terms of the S-CPs with undetermined coefficients. These expansions are substituted into the main equation and the generated operational matrix is utilized to extract a system of nonlinear algebraic equations. Thereafter, the yielded system is solved to obtain the approximate solution of the problem. The accuracy of the proposed approach is examined through some numerical examples. Numerical results confirm the suggested approach is very accurate to provide satisfactory results.

Keywords Fractal–fractional Ginzburg–Landau equation · Shifted Chebyshev polynomials (SCPs) · Operational matrix (OM) · OM of fractal–fractional derivative

1 Introduction

Over the last decades, fractional calculus (the theory of integral and derivative operators of arbitrary orders) has been extensively studied to express various phenomena in Engineering and Physics, e.g. [1–4]. This concept has been usefully utilized in electromagnetism [5], fluid mechanics [6], visco-elastic materials [7], propagation of spherical flames [8] and dynamics of viscoelastic materials [9]. Note that the main reason of using fractional operators in modeling physical systems is their non-local property (which it means that the present state and all the previous states of a dynamical system affect on the its next state) [10, 11]. Some useful results about theoretical analysis of such operators can be found in [12–14]. It should be noted that such operators are often singular. So, it is very difficult to get analytic solutions for problems involving fractional operators. In recent years, several numerical methods have been proposed for solving such problems, for instance, see [15–19].

One of the most-investigated nonlinear partial differential equations in Physics and Engineering is the Ginzburg–Landau equation. This equation describes various types of phenomena, such as nonlinear waves, second-order phase transitions, superfluidity, superconductivity, Bose–Einstein condensation, strings in field theory and liquid crystals [20]. Therefore, it is very necessary to solve this equation. However, there are many numerical and analytical methods for the numerical solution of this equation, for instance, see [21, 21–25]. The fractional version of the Ginzburg–Landau

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equation has been well examined from various aspects, for instance, see [26–29] and references therein.

In recent years, the orthogonal polynomials have been widely utilized for the numerical solution of various types of problems in Engineering and Science, for instance, see [30, 31]. The main reason for such wide applications is that solving the original problem is changed to solving an algebraic system of linear/nonlinear equations. It is worth noting that eigenfunctions of the singular Sturm–Liouville problems can be utilized with high order of accuracy to approximate any smooth function [32]. This useful property of such functions is often called the (exponential) spectral accuracy. The Chebyshev polynomials are one of the main classes of orthogonal polynomials which have been successfully used in various areas, e.g. [33–37].

Since, as far as we know, there is no previous study related to the nonlinear Ginzburg–Landau equation involved with fractal–fractional derivative operator. The main aims of this work is to introduce nonlinear time fractal–fractional Ginzburg–Landau equation and to propose a computational method based on the shifted Chebyshev polynomials (S-CPs) for its numerical solution. Therefore, we focus on the following problem:

$$\begin{aligned} &{}^{FFM}_{\ 0}D_t^{\alpha,\beta}\Theta(x,t) - (\vartheta + i\eta)\Theta_{xx}(x,t) + (\kappa + i\xi)\left|\Theta(x,t)\right|^2 \\ &\Theta(x,t) - (\psi(x) + i\sigma(x))\Theta(x,t) = f(x,t), \end{aligned} \tag{1.1}$$

on the domain $(x,t) \in [0,1]\times[0,1]$ with the initial condition

$$\Theta(x,0) = g(x), \tag{1.2}$$

and the boundary conditions

$$\Theta(0,t) = h(t), \quad \Theta(1,t) = z(t), \tag{1.3}$$

where $i = \sqrt{-1}$ is the unit imaginary number, Θ is an undetermined complex function, g, h and z are complex functions, ψ and σ are real functions, and ϑ, η, κ and ξ are known constants. Here, ${}^{FFM}0D_t^{\alpha,\beta}$ denotes the fractal–fractional partial differentiation operator of order (α,β) (where $\alpha,\beta \in (0,1)$) in the Atangana–Riemann–Liouville sense with Mittag-Leffler non-singular kernel [38, 39].

In the proposed method, solving the above fractal–fractional problem is changed to solving a system of algebraic equations. To this end, first, the function $\Theta(x,t)$ is decomposed into its real and imaginary parts. Then, these parts are expanded by the S-CPs with undetermined coefficients and substituted into the nonlinear fractal–fractional differential equation introduced in Eq. (1.1) and the conditions expressed in Eqs. (1.2)–(1.3). Finally, the operational matrix (OM) of fractal–fractional derivative and the collocation scheme are utilized to extract an algebraic system of nonlinear equations. The method is mainly privileged, because of the special properties of the S-CPs. Note that the above-mentioned OM is obtained for the first time in the present paper, which can also be used on other kinds of fractal–fractional differential equations.

This work includes the following sections: Sect. 2 briefly reviews the fractal–fractional calculus. Sect. 3 provides the S-CPs and some relevant results. The OM of fractal–fractional differentiation of the S-CPs is derived in Sect. 4. The presented approach is formulated in Sect. 5. Some numerical examples are examined in Sect. 6. Finally, in Sect. 7, the main conclusions of the study are highlighted.

2 Fractal–fractional calculus

In this section, some essential notions of the fractal–fractional calculus are briefly reviewed.

Definition 2.1 [10] The one- and two-parameter Mittag-Leffler functions are defined, respectively, by

$$\mathbf{E}_\mu(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(j\mu+1)}, \mu \in \mathbb{R}^+, t \in \mathbb{R}, \tag{2.1}$$

and

$$\mathbf{E}_{\mu,\nu}(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(j\mu+\nu)}, \mu,\ \nu \in \mathbb{R}^+, t \in \mathbb{R}. \tag{2.2}$$

Definition 2.2 [38, 39] The fractal–fractional derivative of order (α,β) of the continuous function $g(t)$ in the Atangana–Riemann–Liouville sense with Mittag-Leffler kernel is defined by

$${}^{FFM}_{\ 0}D_t^{\alpha,\beta}g(t) = \frac{\mathbf{C}(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t^\beta}\int_0^t \mathbf{E}_\alpha\left(\frac{-\alpha(t-s)^\alpha}{1-\alpha}\right)g(s)\mathrm{d}s, \tag{2.3}$$

where $\alpha,\beta \in (0,1)$, $\mathbf{C}(\alpha) = 1-\alpha+\dfrac{\alpha}{\Gamma(\alpha)}$ and

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t^\beta} = \lim_{\tau\to t}\frac{g(\tau)-g(t)}{\tau^\beta - t^\beta}. \tag{2.4}$$

Remark 1 [38, 39] The above definition can be represented as follows:

$${}^{FFM}_{\ 0}D_t^{\alpha,\beta}g(t) = \frac{\mathbf{C}(\alpha)t^{1-\beta}}{\beta(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathbf{E}_\alpha\left(\frac{-\alpha(t-s)^\alpha}{1-\alpha}\right)g(s)\mathrm{d}s. \tag{2.5}$$

Lemma 2.3 *Let $\alpha,\beta \in (0,1)$ and $k \in \mathbb{N}$. Then, we have*

$${}^{FFM}_{\ 0}D_t^{\alpha,\beta}t^k = \frac{\mathbf{C}(\alpha)\,k!\,t^{k-\beta+1}}{\beta(1-\alpha)}\mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha\,t^\alpha}{1-\alpha}\right). \tag{2.6}$$

Proof According to Remark 1, the proof is straightforward. □

3 Shifted Chebyshev polynomials (S-CPs)

An $(n+1)$-set of the S-CPs can be defined over [0, 1] by the following formula [40]:

$$\varphi_i(t) = \begin{cases} 1, & i=0, \\ \sum_{k=0}^{i} a_{ik} t^k, & i=1,2,\dots,n, \end{cases} \tag{3.1}$$

where

$$a_{ik} = (-1)^{i-k} \frac{i(i+k-1)!\, 2^{2k}}{(i-k)!(2k)!}. \tag{3.2}$$

These basis polynomials are orthogonal with respect to the weight functions $w(t) = \frac{1}{\sqrt{t-t^2}}$ with the following condition:

$$\int_0^1 \varphi_i(t)\varphi_j(t)w(t)\mathrm{d}t = \frac{\pi\gamma_i}{2}\delta_{ij}, \tag{3.3}$$

where δ is Kronecker's delta, and $\gamma_0 = 2$ and $\gamma_i = 1$ for $i \geq 1$. So, any function $g(t) \in L_w^2[0,1]$ can be expressed by these polynomials as follows:

$$g(t) \simeq \sum_{i=0}^{n} g_i\varphi_i(t) \triangleq G^{\mathrm{T}}\Phi_n(t), \tag{3.4}$$

where

$$G = \left[g_0\ g_1\ \cdots\ g_n\right]^{\mathrm{T}}, \quad \Phi_n(t) = \left[\varphi_0(t)\ \varphi_1(t)\ \cdots\ \varphi_n(t)\right]^{\mathrm{T}}, \tag{3.5}$$

and

$$g_i = \frac{2}{\pi\gamma_i}\int_0^1 g(t)\varphi_i(t)w(t)\mathrm{d}t. \tag{3.6}$$

In a like way, any function $u(x,t) \in L_w^2([0,1]\times[0,1])$ can be expressed by the S-CPs as follows:

$$u(x,t) \simeq \sum_{i=0}^{n}\sum_{j=0}^{n} u_{ij}\varphi_i(x)\varphi_j(t) \triangleq \Phi_n(x)^{\mathrm{T}}\mathbf{U}\Phi_n(t), \tag{3.7}$$

where $\mathbf{U} = [u_{ij}]$ is an $(n+1)$-order square matrix with entries

$$u_{ij} = \frac{4}{\pi^2\gamma_{i-1}\gamma_{j-1}}\int_0^1\int_0^1 u(x,t)\varphi_{i-1}(x) \varphi_{j-1}(t)w(x)w(t)\mathrm{d}x\mathrm{d}t, \quad i,j=1,2,\dots,n+1. \tag{3.8}$$

Note that the derivative of the vector $\Phi_n(t)$ can be expressed by

$$\frac{\mathrm{d}\Phi_n(t)}{\mathrm{d}t} = \mathbf{D}^{(1)}\Phi_n(t), \tag{3.9}$$

where $\mathbf{D}^{(1)} = [d_{ij}^{(1)}]$ is an $(n+1)$-order matrix (called the differentiation OM of the S-CPs), and has the following entries:

$$d_{ij}^{(1)} = \begin{cases} \frac{4(i-1)}{\gamma_{j-1}}, & i=2,3,\dots,n+1,\ \ j=1,2,\dots,i-1,\ \ i+j \text{ is odd}, \\ 0, & \text{otherwise}. \end{cases}$$

Generally, the OM of r times differentiation of $\Phi_n(t)$ can be expressed as follows:

$$\frac{\mathrm{d}^r\Phi_n(t)}{\mathrm{d}t^r} = \mathbf{D}^{(r)}\Phi_n(t), \tag{3.10}$$

where $\mathbf{D}^{(r)}$ is obtained by r times multiplying $\mathbf{D}^{(1)}$ in itself.

4 Operational matrix (OM) of fractal–fractional derivative

In this section, we derive the OM of fractal–fractional derivative for the S-CPs.

Theorem 4.1 *Let $\Phi_n(t)$ be the vector expressed in Eq. (3.5) and $\alpha,\ \beta \in (0,1)$ be two real constants. The fractal–fractional derivative of order (α,β) of this vector in the Atangana–Riemann–Liouville sense can be expressed by*

$$^{\mathrm{FFM}}_{\ \ 0}D_t^{\alpha,\beta}\Phi_n(t) \simeq \mathbf{P}^{(\alpha,\beta)}\Phi_n(t), \tag{4.1}$$

where $\mathbf{P}^{(\alpha,\beta)} = [p_{ij}^{(\alpha,\beta)}]$ is an $(n+1)$-order matrix (called the fractal–fractional derivative OM of the S-CPs), and its elements are given by

$$p_{1j}^{(\alpha,\beta)} = \begin{cases} \frac{\mathbf{C}(\alpha)}{\sqrt{\pi}\beta(1-\alpha)}\sum_{r=0}^{\infty}\frac{1}{\Gamma(r\alpha+1)}\left(\frac{-\alpha}{1-\alpha}\right)^r\frac{\Gamma\left(\alpha r-\beta+\frac{3}{2}\right)}{\Gamma(\alpha r-\beta+2)}, & j=1, \\ \frac{2\,\mathbf{C}(\alpha)}{\sqrt{\pi}\beta(1-\alpha)}\sum_{l=0}^{j-1}\sum_{r=0}^{\infty}\frac{a_{(j-1)l}}{\Gamma(r\alpha+1)}\left(\frac{-\alpha}{1-\alpha}\right)^r\frac{\Gamma\left(\alpha r+l-\beta+\frac{3}{2}\right)}{\Gamma(\alpha r+l-\beta+2)}, & j=2,3,\dots,n+1, \end{cases}$$

and for $i = 2, 3, \ldots, n+1$,

$$p_{ij}^{(\alpha,\beta)} = \begin{cases} \frac{\mathbf{C}(\alpha)}{\sqrt{\pi}\beta(1-\alpha)} \sum_{k=0}^{i-1}\sum_{r=0}^{\infty} \frac{k!\,a_{(i-1)k}}{\Gamma(\alpha r+k+1)}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{\Gamma\left(\alpha r+k-\beta+\frac{3}{2}\right)}{\Gamma(\alpha r+k-\beta+2)}, & j=1,\\ \frac{2\,\mathbf{C}(\alpha)}{\sqrt{\pi}\beta(1-\alpha)} \sum_{k=0}^{i-1}\sum_{l=0}^{j-1}\sum_{r=0}^{\infty} \frac{k!\,a_{(i-1)k}a_{(j-1)l}}{\Gamma(r\alpha+k+1)}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{\Gamma\left(\alpha r+k+l-\beta+\frac{3}{2}\right)}{\Gamma(\alpha r+k+l-\beta+2)}, & j=2,3\ldots,n+1, \end{cases}$$

in which the coefficients $a_{(i-1)k}$ *and* $a_{(j-1)l}$ *are previously introduced in Eq. (3.2).*

Proof From Eq. (3.1) and Lemma 2.3, we have

$$ {}^{\mathrm{FFM}}_{\;\;0}D_t^{\alpha,\beta}\varphi_0(t) = \frac{\mathbf{C}(\alpha)\,t^{1-\beta}}{\beta(1-\alpha)}\mathbf{E}_\alpha\left(\frac{-\alpha\,t^\alpha}{1-\alpha}\right). \tag{4.2}$$

The above relation can be expressed by the S-CPs as follows:

$$\frac{\mathbf{C}(\alpha)\,t^{1-\beta}}{\beta(1-\alpha)}\mathbf{E}_\alpha\left(\frac{-\alpha\,t^\alpha}{1-\alpha}\right) \simeq \sum_{\hat{j}=0}^{n}\hat{p}_{0\hat{j}}^{(\alpha,\beta)}\varphi_{\hat{j}}(t), \tag{4.3}$$

where

$$\hat{p}_{0\hat{j}}^{(\alpha,\beta)} = \frac{2\,\mathbf{C}(\alpha)}{\pi\beta(1-\alpha)\gamma_{\hat{j}}}\int_0^1 t^{1-\beta}\mathbf{E}_\alpha\left(\frac{-\alpha\,t^\alpha}{1-\alpha}\right)\varphi_{\hat{j}}(t)w(t)\mathrm{d}t.$$

From Eq. (3.1) and the above relation, we have

$$\int_0^1 t^{1-\beta}\mathbf{E}_\alpha\left(\frac{-\alpha t^\alpha}{1-\alpha}\right)\varphi_{\hat{j}}(t)w(t)\mathrm{d}t = \begin{cases} \int_0^1 t^{1-\beta}\mathbf{E}_\alpha\left(\frac{-\alpha t^\alpha}{1-\alpha}\right)w(t)\mathrm{d}t, & \hat{j}=0,\\ \sum_{l=0}^{\hat{j}} a_{\hat{j}l}\int_0^1 t^{l-\beta+1}\mathbf{E}_\alpha\left(\frac{-\alpha t^\alpha}{1-\alpha}\right)w(t)\mathrm{d}t, & \hat{j}=1,2,\ldots,n. \end{cases} \tag{4.4}$$

Definition 2.1 and Eq. (4.4) result in

$$\int_0^1 t^{1-\beta}\mathbf{E}_\alpha\left(\frac{-\alpha t^\alpha}{1-\alpha}\right)\varphi_{\hat{j}}(t)w(t)\mathrm{d}t = \begin{cases} \sum_{r=0}^{\infty}\frac{1}{\Gamma(r\alpha+1)}\left(\frac{-\alpha}{1-\alpha}\right)^r\frac{\sqrt{\pi}\Gamma\left(\alpha r-\beta+\frac{3}{2}\right)}{\Gamma(\alpha r-\beta+2)}, & \hat{j}=0,\\ \sum_{l=0}^{\hat{j}}\sum_{r=0}^{\infty}\frac{a_{\hat{j}l}}{\Gamma(r\alpha+1)}\left(\frac{-\alpha}{1-\alpha}\right)^r\frac{\sqrt{\pi}\Gamma\left(\alpha r+l-\beta+\frac{3}{2}\right)}{\Gamma(\alpha r+l-\beta+2)}, & \hat{j}=1,2,\ldots,n. \end{cases} \tag{4.5}$$

Hence, from Eqs. (4.2)–(4.5), we obtain

$$ {}^{\mathrm{FFM}}_{\;\;0}D_t^{\alpha,\beta}\varphi_0(t) \simeq \sum_{\hat{j}=0}^{n}\hat{p}_{0\hat{j}}^{(\alpha,\beta)}\varphi_{\hat{j}}(t), \tag{4.6}$$

where

$$\hat{p}_{0\hat{j}}^{(\alpha,\beta)} = \begin{cases} \frac{\mathbf{C}(\alpha)}{\sqrt{\pi}\beta(1-\alpha)}\sum_{r=0}^{\infty}\frac{1}{\Gamma(r\alpha+1)}\left(\frac{-\alpha}{1-\alpha}\right)^r\frac{\Gamma\left(\alpha r-\beta+\frac{3}{2}\right)}{\Gamma(\alpha r-\beta+2)}, & \hat{j}=0,\\ \frac{2\,\mathbf{C}(\alpha)}{\sqrt{\pi}\beta(1-\alpha)}\sum_{l=0}^{\hat{j}}\sum_{r=0}^{\infty}\frac{a_{\hat{j}l}}{\Gamma(r\alpha+1)}\left(\frac{-\alpha}{1-\alpha}\right)^r\frac{\Gamma\left(\alpha r+l-\beta+\frac{3}{2}\right)}{\Gamma(\alpha r+l-\beta+2)}, & \hat{j}=1,2,\ldots,n. \end{cases} \tag{4.7}$$

Besides, from Eq. (3.1) and the linear property of the fractal–fractional derivative operator, we have

$$\begin{aligned} {}^{\mathrm{FFM}}_{\;\;0}D_t^{\alpha,\beta}\varphi_{\hat{i}}(t) &= {}^{\mathrm{FFM}}0D_t^{\alpha,\beta}\left(\sum_{k=0}^{\hat{i}}a_{\hat{i}k}\,t^k\right)\\ &= \sum_{k=0}^{\hat{i}}a_{\hat{i}k}\,{}^{\mathrm{FFM}}0D_t^{\alpha,\beta}\,t^k, \qquad \hat{i}=1,2,\ldots,n. \end{aligned} \tag{4.8}$$

Lemma 2.3 and the above relation yield

$$\sum_{k=0}^{\hat{i}}a_{\hat{i}k}\,{}^{\mathrm{FFM}}_{\;\;0}D_t^{\alpha,\beta}\,t^k = \sum_{k=0}^{\hat{i}}a_{\hat{i}k}\frac{\mathbf{C}(\alpha)\,k!\,t^{k-\beta+1}}{\beta(1-\alpha)}\mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha\,t^\alpha}{1-\alpha}\right), \qquad \hat{i}=1,2,\ldots,n.$$

Approximating the elements of the above relation by the S-CPs yields

$$\frac{\mathbf{C}(\alpha)}{\beta(1-\alpha)}\sum_{k=0}^{\hat{i}}a_{\hat{i}k}k!\,t^{k-\beta+1}\,\mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha\,t^\alpha}{1-\alpha}\right) \simeq \sum_{\hat{j}=0}^{n}\hat{p}_{\hat{i}\hat{j}}^{(\alpha,\beta)}\varphi_{\hat{j}}(t), \qquad \hat{i}=1,2,\ldots,n,$$

where

$$\hat{p}^{(\alpha,\beta)}_{\hat{i}\hat{j}} = \begin{cases} \frac{\mathbf{C}(\alpha)}{\pi\beta(1-\alpha)} \sum_{k=0}^{\hat{i}} k!\, a_{\hat{i}k} \int_0^1 t^{k-\beta+1}\, \mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha\, t^{\alpha}}{1-\alpha}\right) w(t)\mathrm{d}t, & \hat{j}=0,\\ \frac{2\,\mathbf{C}(\alpha)}{\pi\beta(1-\alpha)} \sum_{k=0}^{\hat{i}} k!\, a_{\hat{i}k} \int_0^1 t^{k-\beta+1}\, \mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha\, t^{\alpha}}{1-\alpha}\right) \varphi_{\hat{j}}(t) w(t)\mathrm{d}t, & \hat{j}=1,2,\dots,n. \end{cases}$$

From Eq. (3.1), we have

$$\int_0^1 t^{k-\beta+1}\, \mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha\, t^{\alpha}}{1-\alpha}\right) \varphi_{\hat{j}}(t) w(t)\mathrm{d}t = \sum_{l=0}^{\hat{j}} a_{\hat{j}l} \int_0^l \left(t^{k+l-\beta+1} \mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha\, t^{\alpha}}{1-\alpha}\right)\right) w(t)\mathrm{d}t.$$

On the other hand, by considering Definition 2.1, we obtain

$$\int_0^1 t^{k-\beta+1}\, \mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha\, t^{\alpha}}{1-\alpha}\right) w(t)\mathrm{d}t = \sum_{r=0}^{\infty} \frac{1}{\Gamma(r\alpha+k+1)} \left(\frac{-\alpha}{1-\alpha}\right)^r \frac{\sqrt{\pi}\,\Gamma\left(\alpha r+k-\beta+\frac{3}{2}\right)}{\Gamma(\alpha r+k-\beta+2)},$$

and

$$\int_0^l \left(t^{k+l-\beta+1} \mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha\, t^{\alpha}}{1-\alpha}\right)\right) w(t)\mathrm{d}t = \sum_{r=0}^{\infty} \frac{1}{\Gamma(r\alpha+k+1)} \left(\frac{-\alpha}{1-\alpha}\right)^r \frac{\sqrt{\pi}\,\Gamma\left(\alpha r+k+l-\beta+\frac{3}{2}\right)}{\Gamma(\alpha r+k+l-\beta+2)}. \tag{4.9}$$

Therefore, from Eqs. (4.8)–(4.9), we obtain

$${}^{\mathrm{FFM}}_{\ 0}D_t^{\alpha,\beta} \varphi_{\hat{i}}(t) \simeq \sum_{\hat{j}=0}^{n} \hat{p}^{(\alpha,\beta)}_{\hat{i}\hat{j}} \varphi_{\hat{j}}(t), \qquad \hat{i}=1,2,\dots,n,$$

where

$$\hat{p}^{(\alpha,\beta)}_{\hat{i}\hat{j}} = \begin{cases} \frac{\mathbf{C}(\alpha)}{\sqrt{\pi}\beta(1-\alpha)} \sum_{k=0}^{\hat{i}} \sum_{r=0}^{\infty} \frac{k!\, a_{\hat{i}k}}{\Gamma(r\alpha+k+1)} \left(\frac{-\alpha}{1-\alpha}\right)^r \frac{\Gamma\left(\alpha r+k-\beta+\frac{3}{2}\right)}{\Gamma(\alpha r+k-\beta+2)}, & \hat{j}=0,\\ \frac{2\,\mathbf{C}(\alpha)}{\sqrt{\pi}\beta(1-\alpha)} \sum_{k=0}^{\hat{i}} \sum_{l=0}^{\hat{j}} \sum_{r=0}^{\infty} \frac{k!\, a_{\hat{i}k} a_{\hat{j}l}}{\Gamma(r\alpha+k+1)} \left(\frac{-\alpha}{1-\alpha}\right)^r \frac{\Gamma\left(\alpha r+k+l-\beta+\frac{3}{2}\right)}{\Gamma(\alpha r+k+l-\beta+2)}, & \hat{j}=1,2,\dots,n. \end{cases} \tag{4.10}$$

Thus, by change of the indexes $i=\hat{i}+1$ and $j=\hat{j}+1$ in Eqs. (4.7)–(4.10), the proof is completed. □

Remark 2 Note that to do numerical computations often a few terms of the infinite series expressing two parameters Mittag-Leffler function, and consequently, a few terms of the infinite series in the above theorem is utilized. Throughout the paper, the first 30 terms of this series is used.

As an illustrative example, for $n=5$ and $(\alpha,\beta)=(\frac{1}{2},\frac{1}{4})$, we have

$$\mathbf{P}^{(\frac{1}{2},\frac{1}{4})} = \begin{pmatrix} 1.7083682 & 1.1713126 & -0.2763249 & 0.1048443 & -0.0506811 & 0.0284990\\ 1.2077200 & 2.3833525 & 0.8185425 & -0.1734736 & 0.0674740 & -0.0342160\\ 0.3913427 & 1.8674590 & 2.6812790 & 0.8831868 & -0.1617650 & 0.0593137\\ 0.1743604 & 0.3532049 & 1.7363979 & 2.8066474 & 0.9403543 & -0.1592480\\ 0.1166231 & 0.2851773 & 0.2589298 & 1.6869550 & 2.8830557 & 0.9845545\\ 0.0621188 & 0.1241162 & 0.2276172 & 0.2071576 & 1.6613565 & 2.9369075 \end{pmatrix}$$

5 The proposed method

To solve the fractal–fractional problem introduced in Eqs. (1.1)–(1.3), we first decompose the complex functions of the problem in their real and imaginary parts as follows:

$$\Theta(x,t)=u(x,t)+i\,v(\zeta,\tau), \quad f(x,t)=f_1(x,t)+i\,f_2(x,t), \quad g(x)=g_1(x)+i\,g_2(x),$$
$$h(t)=h_1(t)+i\,h_2(t), \quad z(t)=z_1(t)+i\,z_2(t), \tag{5.1}$$

where $u(x,t)$, $v(x,t)$ and $f_i(x,t)$, $g_i(x)$, $h_i(t)$, $z_i(t)$ for $i = 1, 2$ are real functions. Therefore, the mentioned problem can be represented into a coupled system of nonlinear fractal–fractional differential equations as follows:

$$\begin{aligned}
&{}^{FFM}_{0}D_t^{\alpha,\beta}u(x,t) - \vartheta u_{xx}(x,t) + \eta v_{xx}(x,t) \\
&\quad + \kappa\left(u^2(x,t) + v^2(x,t)\right)u(x,t) - \xi\left(u^2(x,t) + v^2(x,t)\right)v(x,t) \\
&\quad - \psi(x)u(x,t) + \sigma(x)v(x,t) = f_1(x,t), \\
&{}^{FFM}_{0}D_t^{\alpha,\beta}v(x,t) - \vartheta v_{xx}(x,t) - \eta u_{xx}(x,t) \\
&\quad + \kappa\left(u^2(x,t) + v^2(x,t)\right)v(x,t) + \xi\left(u^2(x,t) + v^2(x,t)\right)u(x,t) \\
&\quad - \psi(x)v(x,t) - \sigma(x)u(x,t) = f_2(x,t)
\end{aligned} \tag{5.2}$$

with the initial conditions

$$u(x,0) = g_1(x), \quad v(x,0) = g_2(x), \tag{5.3}$$

and the boundary conditions

$$\begin{aligned}
&\left[\Phi_n(0)^T\mathbf{U} - H_1^T\right]\Phi_n(t) \triangleq \Lambda_1^T\Phi_n(t) \simeq 0, \quad \left[\Phi_n(0)^T\mathbf{V} - H_2^T\right]\Phi_n(t) \triangleq \Lambda_2^T\Phi_n(t) \simeq 0, \\
&\left[\Phi_n(1)^T\mathbf{U} - Z_1^T\right]\Phi_n(t) \triangleq \Lambda_3^T\Phi_n(t) \simeq 0, \quad \left[\Psi_n(1)^T\mathbf{V} - Z_2^T\right]\Phi_n(t) \triangleq \Lambda_4^T\Phi_n(t) \simeq 0.
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
&u(0,t) = h_1(t), \quad v(0,t) = h_2(t), \\
&u(1,t) = z_1(t), \quad v(1,t) = z_2(t).
\end{aligned} \tag{5.4}$$

Now, we approximate the real and imaginary parts of the solution of the problem by the S-CPs as follows:

$$\begin{aligned}
&u(x,t) \simeq \Phi_n(x)^T\mathbf{U}\Phi_n(t), \\
&v(x,t) \simeq \Phi_n(x)^T\mathbf{V}\Phi_n(t),
\end{aligned} \tag{5.5}$$

where $\mathbf{U} = [u_{ij}]$ and $\mathbf{V} = [v_{ij}]$ are (n+1)-order undetermined square matrices, and $\Phi_n(.)$ is in accordance with Eq. (3.5). From Theorem 4.1, we have

$$\begin{aligned}
&{}^{FFM}_{0}D_t^{\alpha,\beta}u(x,t) \simeq \Phi_n(x)^T\mathbf{U}\mathbf{P}^{(\alpha,\beta)}\Phi_n(t), \\
&{}^{FFM}_{0}D_t^{\alpha,\beta}v(x,t) \simeq \Phi_n(x)^T\mathbf{V}\mathbf{P}^{(\alpha,\beta)}\Phi_n(t).
\end{aligned} \tag{5.6}$$

Also, two times derivative with respect to x on both sides of Eq. (5.5) yields

$$\begin{aligned}
&u_{xx}(x,t) \simeq \Phi_n(x)^T\left(\mathbf{D}^{(2)}\right)^T\mathbf{U}\Phi_n(t), \\
&v_{xx}(x,t) \simeq \Phi_n(x)^T\left(\mathbf{D}^{(2)}\right)^T\mathbf{V}\Phi_n(t).
\end{aligned} \tag{5.7}$$

Substituting Eqs. (5.5)–(5.7) into Eq. (5.2) gives

Meanwhile, the functions given in Eqs. (5.3) and (5.4) can be expressed by the S-CPs as follows:

$$g_1(x) \simeq \Phi_n(x)^T G_1, \quad g_2(x) \simeq \Phi_n(x)^T G_2, \tag{5.9}$$

and

$$\begin{aligned}
&h_1(t) \simeq H_1^T\Phi_n(t), \quad h_2(t) \simeq H_2^T\Phi_n(t), \\
&z_1(t) \simeq Z_1^T\Phi_n(t), \quad z_2(t) \simeq Z_2^T\Phi_n(t),
\end{aligned} \tag{5.10}$$

where G_i, H_i and Z_i for $i = 1, 2$ are known vectors. Hence, from Eqs. (5.5), (5.9) and (5.10), and the initial and boundary conditions expressed in Eqs. (5.3) and (5.4), the following relations can be extracted:

$$\begin{aligned}
&\Phi_n(x)^T\left[\mathbf{U}\Phi_n(0) - G_1\right] \triangleq \Phi_n(x)^T\Pi_1 \simeq 0, \\
&\Phi_n(x)^T\left[\mathbf{V}\Phi_n(0) - G_2\right] \triangleq \Phi_n(x)^T\Pi_2 \simeq 0,
\end{aligned} \tag{5.11}$$

and

Eventually, from Eqs. (5.8), (5.11) and (5.12), a system of $2(n+1)^2$ algebraic equations can be extracted as follows:

$$\begin{cases}
\mathbf{R}_l\left(\zeta_i, \zeta_j\right) = 0, & l = 1,2, \ i = 2,3,\dots,n, \ j = 2,3,\dots,n+1, \\
\left[\Pi_l\right]_j = 0, & l = 1,2, \ j = 1,2,\dots,n+1, \\
\left[\Lambda_l\right]_j = 0, & l = 1,2,3,4, \ j = 2,3,\dots,n+1,
\end{cases} \tag{5.13}$$

where $\zeta_k = \frac{1}{2}\left(1 - \cos\left(\frac{(2k-1)\pi}{2(n+1)}\right)\right)$ for $k = 1, 2, \dots, n+1$ is the k-th root of the shifted Chebyshev polynomial of $(n+1)$-th degree on [0, 1]. The above algebraic system should be solved due to compute the unknown matrices $\mathbf{U}$ and $\mathbf{V}$ in Eq. (5.5), and consequently to obtain an approximate solution for the problem.

6 Numerical examples

In this section, some numerical examples are solved using the method presented in Sect. 5. Note that Maple 17 via 20 digits precision is utilized for numerical implementations.

$$\begin{aligned}
\mathbf{R}_1(x,t) &\triangleq \Phi_n(x)^T\left[\mathbf{U}\mathbf{P}^{(\alpha,\beta)} - \left(\mathbf{D}^{(2)}\right)^T(\vartheta\mathbf{U} - \eta\mathbf{V}) - \psi(x)\mathbf{U} + \sigma(x)\mathbf{V}\right]\Phi_n(t) \\
&\quad + \left(\Phi_n(x)^T[\kappa\mathbf{U} - \xi\mathbf{V}]\Phi_n(t)\right)\left(\left(\Phi_n(x)^T\mathbf{U}\Phi_n(t)\right)^2 + \left(\Phi_n(x)^T\mathbf{V}\Phi_n(t)\right)^2\right) - f_1(x,t) \simeq 0, \\
\mathbf{R}_2(x,t) &\triangleq \Phi_n(x)^T\left[\mathbf{V}\mathbf{P}^{(\alpha,\beta)} - \left(\mathbf{D}^{(2)}\right)^T(\vartheta\mathbf{V} + \eta\mathbf{U}) - \psi(x)\mathbf{V} - \sigma(x)\mathbf{U}\right]\Phi_n(t) \\
&\quad + \left(\Phi_n(x)^T[\kappa\mathbf{V} + \xi\mathbf{U}]\Phi_n(t)\right)\left(\left(\Phi_n(x)^T\mathbf{U}\Phi_n(t)\right)^2 + \left(\Phi_n(x)^T\mathbf{V}\Phi_n(t)\right)^2\right) - f_2(x,t) \simeq 0.
\end{aligned} \tag{5.8}$$

Table 1 The MA-error and the C-order of the proposed scheme for two values α where $\beta = 0.25$ in Example 1

	$\alpha = 0.25$				$\alpha = 0.75$			
	Real part		Imaginary part		Real part		Imaginary part	
n	MA-error	C-order	MA-error	C-order	MA-error	C-order	MA-error	C-order
4	5.3668E−06	–	3.3837E−06	–	1.3779E−05	–	6.0241E−06	–
5	8.5699E−07	05.0311	5.3485E−07	05.0590	2.7489E−06	04.4206	1.3135E−06	04.1768
6	1.0097E−07	06.9367	6.2538E−08	06.9614	5.4083E−07	05.2735	2.4755E−07	05.4130
7	2.5311E−08	05.1807	1.5492E−08	05.2251	1.4216E−07	05.0031	6.7865E−08	04.8456
8	6.7845E−09	05.5890	4.0775E−09	05.6665	3.6959E−08	05.7187	1.5936E−08	06.1508

Table 2 The MA-error and the C-order of the proposed scheme for two values β where $\alpha = 0.65$ in Example 1

	$\beta = 0.15$				$\beta = 0.45$			
	Real part		Imaginary part		Real part		Imaginary part	
n	MA-error	C-order	MA-error	C-order	MA-error	C-order	MA-error	C-order
4	2.1794E−05	–	8.7717E−06	–	4.3864E−06	–	2.7811E−06	–
5	2.8986E−06	05.5325	1.3386E−06	05.1554	8.7380E−07	04.4246	5.3574E−07	04.5166
6	4.5991E−07	05.9712	2.0290E−07	06.1195	1.8610E−07	05.0164	1.0910E−07	05.1617
7	1.0661E−07	05.4738	4.9486E−08	05.2834	5.3702E−08	04.6537	3.1473E−08	04.6548
8	2.5951E−08	05.9981	1.0901E−08	06.4221	1.5580E−08	05.2531	8.7981E−09	05.4107

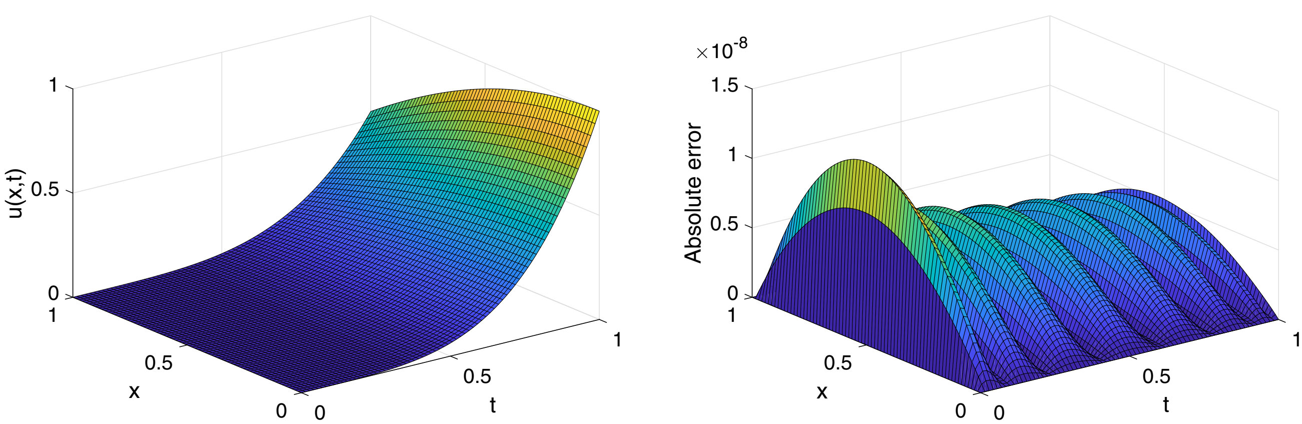

Fig. 1 Plots of the S-CPs solution and the AE function (respectively left and right) for the real part where ($\alpha = 0.25, \beta = 0.25$) and $n = 8$ in Example 1

The convergence order (C-order) of the proposed scheme is calculated by

$$\text{C-order} = \left|\log\left(\frac{\varepsilon_2}{\varepsilon_1}\right)\right| \Big/ \log\left(\frac{(n_2+1)^2}{(n_1+1)^2}\right),$$

where ε_1 and ε_2 are, respectively, the first and the second values of the maximum absolute error (MA-error) appearing in the presented method. Also, $(n_i + 1)^2$ for $i = 1, 2$ is the S-CPs number utilized in the ith implementation.

Example 1 Consider the nonlinear time fractal–fractional Ginzburg–Landau equation

$$ {}^{\mathrm{FFM}}_{0}D_t^{\alpha,\beta}\Theta(x,t) - (1+2i)\Theta_{xx}(x,t) + (1+i)\left|\Theta(x,t)\right|^2\Theta(x,t) - x^2(1+ix)\Theta(x,t) = f(x,t), $$

where

$$ f(x,t) = \Bigg(\frac{\mathbf{C}(\alpha)\,4!\,t^{5-\beta}}{\beta(1-\alpha)}\,\mathbf{E}_{\alpha,5}\Big(\frac{-\alpha\,t^{\alpha}}{1-\alpha}\Big) + (1+2i)t^4 + (1+i)t^{12} - x^2(1+ix)t^4\Bigg)\mathrm{e}^{ix}, $$

with the homogeneous initial condition and the following boundary conditions:

$$ \Theta(0,t) = t^4, \quad \Theta(1,t) = t^4\mathrm{e}^{i}. \tag{6.1} $$

The analytic solution of this example is

$$\Theta(x,t) = t^4 e^{ix}.$$

The method presented in Sect. 5 with some values n is used for solving this example. The values of the MA-error and the C-order of the real and imaginary parts of the solution for some selections (α, β) are summarized in Tables 1 and 2. Figures of the approximate solution (AS) and the corresponding absolute error (AE) function for the real and imaginary parts in the case of $n = 8$, where $(\alpha = 0.25, \beta = 0.25)$ are illustrated, respectively, in Figs. 1 and 2. The reported results clarify that one can get excellent results by applying only a few number of the S-CPs. Moreover, applying more basis functions improves the accuracy rapidly.

Example 2 Consider the nonlinear time fractal–fractional Ginzburg–Landau equation

$$\begin{aligned} {}^{FFM}_{0}D_t^{\alpha,\beta}\Theta(x,t) - 5i\Theta_{xx}(x,t) + 2\left|\Theta(x,t)\right|^2\Theta(x,t) \\ - i\,e^{-ix}\Theta(x,t) = f(x,t), \end{aligned}$$

where

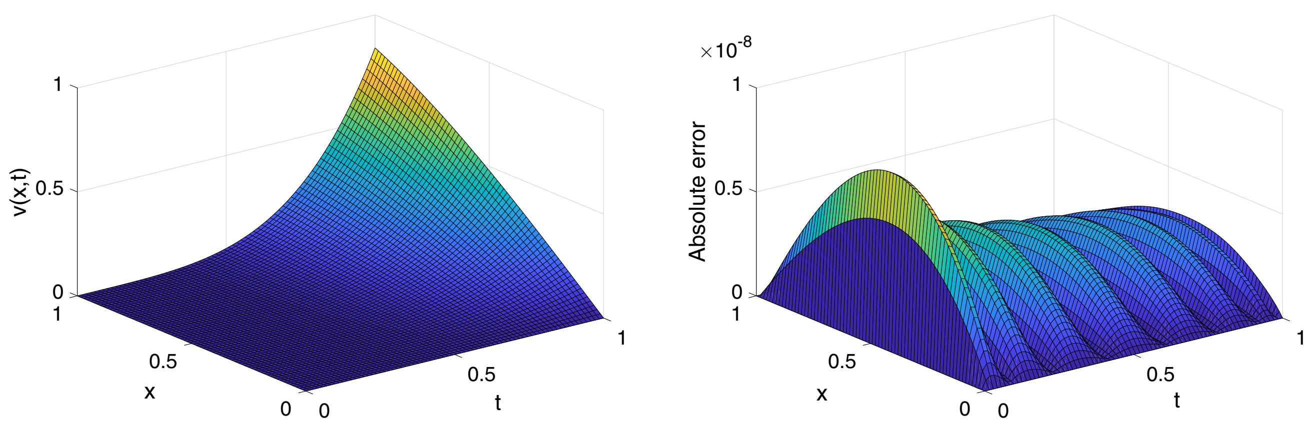

Fig. 2 Plots of the S-CPs solution and the AE function (respectively, left and right) for the imaginary part where $(\alpha = 0.25, \beta = 0.25)$ and $n = 8$ in Example 1

Table 3 The MA-error and the C-order of the proposed scheme for two values α where $\beta = 0.35$ in Example 2

	$\alpha = 0.35$				$\alpha = 0.45$			
	Real part		Imaginary part		Real part		Imaginary part	
n	MA-error	C-order	MA-error	C-order	MA-error	C-order	MA-error	C-order
4	4.8430E−04	–	4.0752E−04	–	4.8430E−04	–	4.0752E−04	–
5	2.5319E−05	08.0932	2.1306E−05	08.0931	2.5319E−05	08.0932	2.1306E−05	08.0931
6	1.6214E−06	08.9142	1.3643E−06	08.9144	1.6214E−06	08.9142	1.3643E−06	08.9144
7	1.4746E−08	17.5991	4.9919E−08	12.3865	1.8627E−08	16.7242	5.7703E−08	11.8439
8	6.7368E−09	03.3255	1.4299E−08	05.3072	8.6838E−09	03.2396	1.7407E−08	05.0874

Table 4 The MA-error and the C-order of the proposed scheme for two values α where $\alpha = 0.75$ in Example 2

	$\beta = 0.15$				$\beta = 0.35$			
	Real part		Imaginary part		Real part		Imaginary part	
n	MA-error	C-order	MA-error	C-order	MA-error	C-order	MA-error	C-order
4	4.8430E−04	–	4.0752E−04	–	4.8430E−04	–	4.0752E−04	–
5	2.5319E−05	08.0932	2.1306E−05	08.0931	2.5319E−05	08.0932	2.1306E−05	08.0931
6	1.6214E−06	08.9142	1.3643E−06	08.9144	1.6214E−06	08.9142	1.3643E−06	08.9144
7	1.4038E−07	09.1614	2.1680E−07	06.8875	8.1855E−08	11.1812	1.5911E−07	08.0460
8	5.2070E−08	04.2101	6.5141E−08	05.1043	3.4380E−08	03.6825	5.3468E−08	04.6293

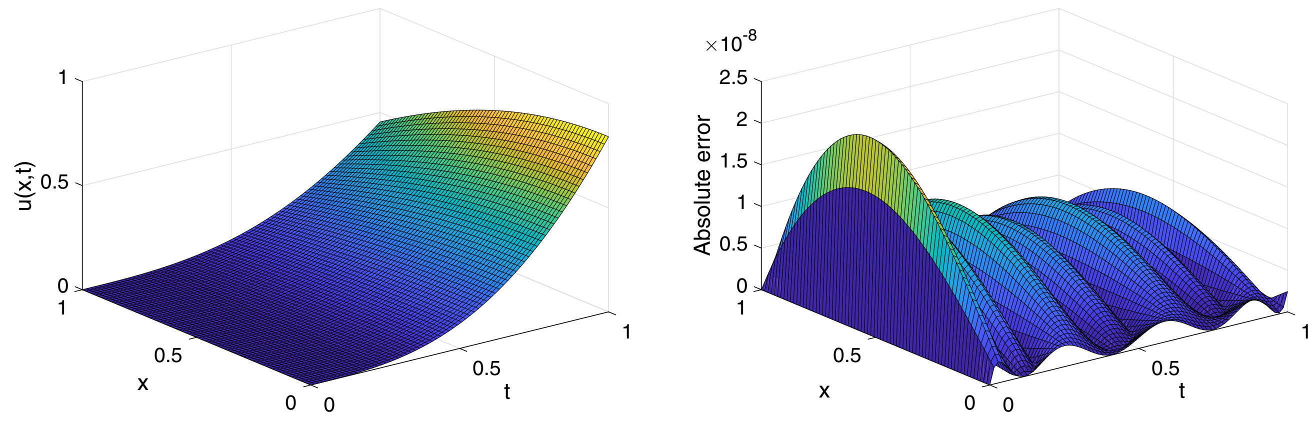

Fig. 3 Plots of the S-CPs solution and the AE function (respectively, left and right) for the real part where ($\alpha = 0.45, \beta = 0.35$) and $n = 8$ in Example 2

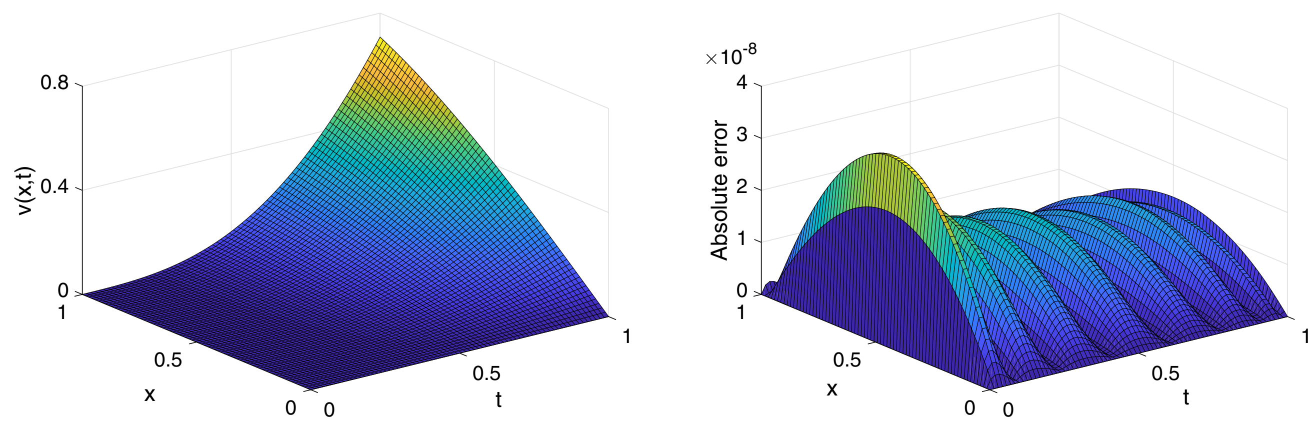

Fig. 4 Plots of the S-CPs solution and the AE function (respectively, left and right) for the imaginary part where ($\alpha = 0.45, \beta = 0.35$) and $n = 8$ in Example 2

$$f(x,t) = \left(\frac{\mathbf{C}(\alpha)\, t^{4-\beta}}{\beta(1-\alpha)} \sum_{k=0}^{\infty} (-1)^k (2k+3)(2k+2)\, t^{2k}\, \mathbf{E}_{\alpha,2k+4} \left(\frac{-\alpha\, t^{\alpha}}{1-\alpha} \right) + 5i\, t^2 \sin(t) + 2t^6 \sin^3(t) - i\, t^2 \sin(t) \mathrm{e}^{-ix} \right) \mathrm{e}^{ix},$$

with the homogeneous initial condition and the following boundary conditions:

$$\Theta(0,t) = t^2 \sin(t), \quad \Theta(1,t) = t^2 \sin(t) \mathrm{e}^{i}. \tag{6.2}$$

The analytic solution of this example is

$$\Theta(x,t) = t^2 \sin(t) \mathrm{e}^{ix}.$$

The established method with some values n is applied for the numerical solution of this example. The values of the MA-error and the C-order of the real and imaginary parts of the solution for some selections (α, β) are given in Tables 3 and 4. Figures of the AS and the corresponding AE function for the real and imaginary parts in the case of $n = 8$, where ($\alpha = 0.45, \beta = 0.35$) are shown, respectively, in Figs. 3 and 4. From the reported results, it can be seen that applying more terms of the S-CPs provides numerical results with high accuracy. Moreover, it can be seen that by increasing the number of the S-CPs the approximate solutions tend to the exact solutions of the problem with high order of accuracy. Note that the first ten terms of the series appeared in the right-hand side are used for the numerical simulations. This assumption also is utilized in the next example.

Example 3 Consider the nonlinear time fractal–fractional Ginzburg–Landau equation

$$ {}^{\mathrm{FFM}}_{\ \ 0}D_t^{\alpha,\beta} \Theta(x,t) - 2\Theta_{xx}(x,t) + 3i \left| \Theta(x,t) \right|^2 \Theta(x,t) - \left(2x + 1 + 3ix^2\right) \Theta(x,t) = f(x,t),$$

where

Table 5 The MA-error and the C-order of the proposed scheme for two values α where $\beta = 0.25$ in Example 3

	$\alpha = 0.25$				$\alpha = 0.65$			
	Real part		Imaginary part		Real part		Imaginary part	
n	MA-error	C-order	MA-error	C-order	MA-error	C-order	MA-error	C-order
4	6.0705E−04	–	7.2142E−04	–	6.0705E−04	–	7.2142E−04	–
5	5.8568E−05	06.4129	6.9601E−05	06.4129	5.8568E−05	06.4129	6.9601E−05	06.4129
6	3.9669E−06	08.7323	4.7143E−06	08.7322	3.9669E−06	08.7323	4.7390E−06	08.7153
7	4.6791E−08	16.6254	8.5917E−08	14.9963	1.5745E−07	12.0819	3.0937E−07	10.2187
8	2.4332E−08	02.7758	4.7284E−08	02.5352	6.0521E−08	04.0588	1.1770E−07	04.1024

Table 6 The MA-error and the C-order of the proposed scheme for two values β where $\alpha = 0.80$ in Example 3

	$\beta = 0.35$				$\beta = 0.55$			
	Real part		Imaginary part		Real part		Imaginary part	
n	MA-error	C-order	MA-error	C-order	MA-error	C-order	MA-error	C-order
4	6.0705E−04	–	7.2142E−04	–	6.0705E−04	–	7.2142E−04	–
5	5.8568E−05	06.4129	6.9682E−05	06.4097	5.8568E−05	06.4129	6.9604E−05	06.4128
6	3.9669E−06	08.7323	5.2297E−06	08.3995	3.9669E−06	08.7323	4.8242E−06	08.6576
7	3.1877E−07	09.4407	6.2439E−07	07.9581	1.9748E−07	11.2336	3.8721E−07	09.4450
8	1.1715E−07	04.2494	2.2605E−07	04.3130	9.0209E−08	03.3260	1.7542E−07	03.3611

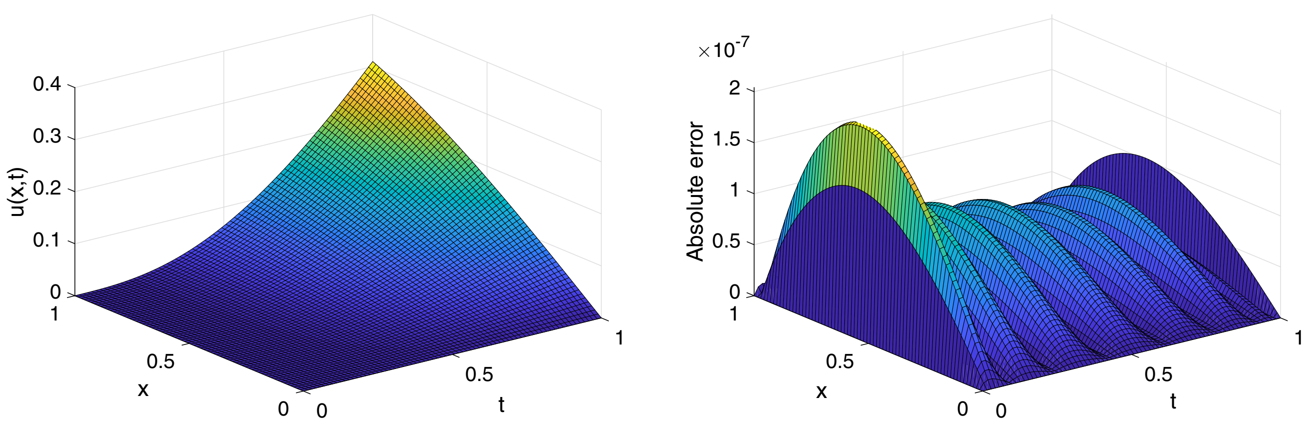

Fig. 5 Plots of the S-CPs solution and the AE function (respectively left and right) for the real part where ($\alpha = 0.80, \beta = 0.55$) and $n = 8$ in Example 3

$$f(x,t) = i\left(\frac{\mathbf{C}(\alpha)\, t^{4-\beta}}{\beta(1-\alpha)} \sum_{k=0}^{\infty} (-1)^k (k+3)(k+2)(k+1)\, t^k \, \mathbf{E}_{\alpha,k+4}\left(\frac{-\alpha\, t^{\alpha}}{1-\alpha}\right) + 2t^3 e^{-t} + 3i\, t^9 e^{-3t} - \left(2x + 1 + 3ix^2\right) t^3 e^{-t}\right) e^{-ix},$$

with the homogeneous initial condition and the following boundary conditions:

$$\Theta(0,t) = it^3 e^{-t}, \quad \Theta(1,t) = it^3 e^{-(t+i)}. \tag{6.3}$$

The analytic solution of this example is

$$\Theta(x,t) = it^3 e^{-(t+ix)}.$$

The presented method with some values n is used for solving this example. The values of the MA-error and the C-order of the real and imaginary parts of the solution for some selections (α, β) are reported in Tables 5 and 6. Plots of the AS

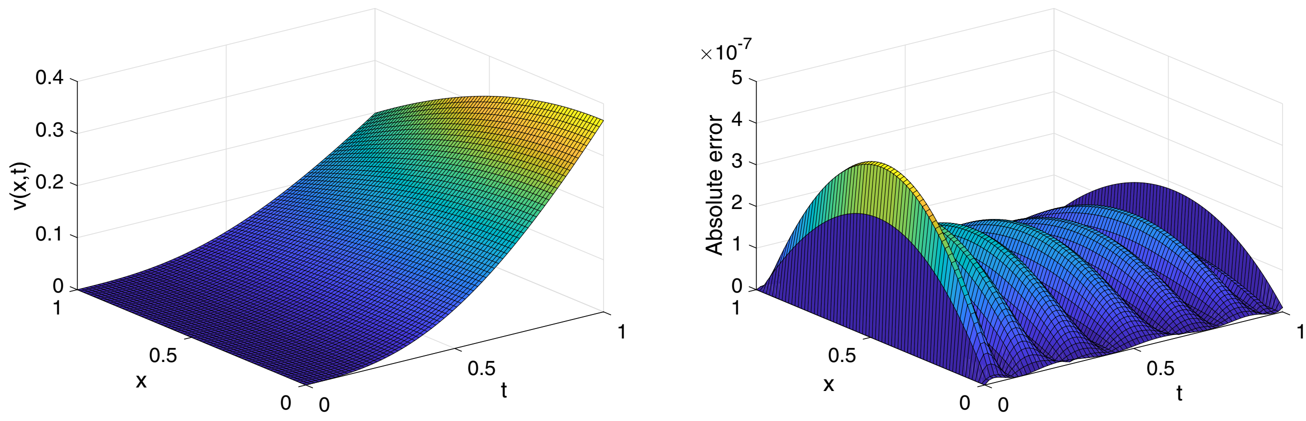

Fig. 6 Plots of the S-CPs solution (left) and the AE function (right) for the imaginary part where ($\alpha = 0.80, \beta = 0.55$) and $n = 8$ in Example 3

and the corresponding AE function for the real and imaginary parts in the case of $n = 8$, where ($\alpha = 0.80, \beta = 0.55$) are shown in Figs. 5 and 6, respectively. The achieved results evidently show that the approach is efficient and reliable for this example.

7 Conclusion

In this paper, a novel class of nonlinear Ginzburg–Landau equation has been introduced. The fractal–fractional derivative in the Atangana–Riemann–Liouville sense with Mittag-Leffler non-singular kernel utilized to express this new class. An accurate scheme based on the shifted Chebyshev polynomials (S-CPs) proposed for the numerical solution of this class of problems. To design the proposed method, a novel operational matrix of fractal–fractional differentiation constructed for the S-CPs. This matrix and the collocation method have been mutually applied to change the original problem to a system of nonlinear algebraic equations. The established method applied on several numerical examples. The yielded results confirm the high accuracy of the proposed approach.

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