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ORIGINAL ARTICLE

New collocation method for stochastic response surface reliability analyses

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Abstract
The stochastic response surface method (SRSM) is widely used in engineering reliability analyses due to its efficiency and accuracy. The selection of collocation points in the SRSM has great significance, as it may strongly affect the computed results. This paper investigates the performance of different selection strategies in SRSM, and proposes a new collocation method. First, two commonly used collocation methods—the regression-based collocation method and the linearly independent collocation method—are briefly reviewed; and their limitations in application to reliability analysis are discussed. Then, an improved collocation method that achieves a better tradeoff between efficiency and accuracy is proposed. Four examples are employed to test the performance of the proposed collocation method; and a comparative study is conducted to demonstrate its advantages with respect to some other existing collocation methods.

Keywords Reliability analysis · Stochastic response surface method · Polynomial chaos expansion · Collocation point · Probability of failure

List of symbols

p	Order of PCE
$\mathbf{X}$	A vector of random variables in physical space
$\mathbf{U}$	A vector of uncorrelated standard normal random variables
$\mathbf{y}$	Random output of the model
$\Gamma_p(\cdot)$	Multidimensional Hermite polynomials of order p
n	Number of random variables in PCE
$\mathbf{a}$	A vector of unknown coefficients
$\mathbf{T}$	Hermite polynomial information matrix
T	Transpose matrix operator
N_a	Number of unknown coefficients
$\mathbf{P}_i$	Selected collocation point
$\mathbf{P}_i^{'}$	Symmetric point of $\mathbf{P}_i$ with respect to the origin
ζ	Asymmetrical ratio of the selected collocation points
Δ	Relative error with respect to MCS or LHS
N_p	Number of selected collocation points or limit state function evaluations
COV	Coefficient of variation
P_f	Probability of failure
μ	Mean value
SD	Standard deviation
σ_{t}	Applied support pressure at the tunnel face
$\sigma_{\mathrm{c,partial}}$	Collapse pressure provided by partial collapse mechanism
$\sigma_{\mathrm{c,global}}$	Collapse pressure provided by global collapse mechanism
$\sigma_{\mathrm{c,max}}$	Maximum collapse pressure provided by partial collapse or global collapse
c	Cohesion
φ	Friction angle
ρ	Correlation coefficient

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1 Introduction

Reliability methods have been applied to many structural engineering studies, as they can account for the effects of uncertainties and provide a basis for risk assessments. Various methods have been proposed to estimate the reliability index or the probability of failure, such as the first order reliability method (FORM) (see e.g. [1, 2]), the second order reliability method (SORM) (see e.g. [3–5]), the response surface method (RSM) [6], and the sampling method [7], among others.

The RSM approximates an implicit limit state function (LSF) using a set of actual values of the LSF, and it has been proved to be effective to address complex engineering problems with inexplicit LSFs (see, e.g. [8–10]). However, RSMs can only fit the LSF well in the vicinity of the design point; i.e., in the vicinity of the point within the failure region with highest probability of occurrence. In contrast, the stochastic response surface method (SRSM)—an extension of classical deterministic RSMs to consider uncertain inputs and outputs through a series expansion of standard random variables [11]—can fit the LSF in the entire variable space, thus providing better predictions of the output response [12, 13].

Recently, considerable efforts have been put to solve reliability problems in structural and geotechnical engineering using SRSMs. For instance, Isukapalli et al. [11] proposed using the SRSM to quantify uncertainty propagation; Huang et al. [14] developed Excel add-ins to promote the application of SRSM without the need for advanced mathematical and programming skills; Li et al. [12] proposed an improved SRSM for reliability analysis of rock slopes involving correlated non-normal random variables; Mollon et al. [15] applied the collocation-based SRSM to estimate the probability of face collapse in tunnels driven by a compressed-air pressurized shield; and Wang and Li [16] employed the SRSM to estimate the reliability of a tunnel considering non-Gaussian dependent random variables under incomplete probability information.

However, one remaining challenge for a successful implementation of the SRSM in practice is how to reliably and efficiently select collocation points to estimate the unknown coefficients of the polynomial chaos expansion (PCE); this aspect is crucial, as it has a significant impact on the performance of the SRSM [12]. Although some advanced sampling methods have been developed in recent years—for instance, Blatman and Sudret [17] and Blatman and Sudret [18] proposed an adaptive sparse stochastic response surface method with least square regression and least angle regression; Xiong et al. [19] and Hampton and Doostan [20] suggested the use of weighted sampling method to improve the computational efficiency of SRSM. The conventional collocation method using the roots of next higher order Hermite polynomials is still widely used in engineering application (particularly in geotechnical engineering as mentioned above) for its simplicity. Therefore, many researchers put their efforts to improve the performance of conventional collocation method.

Isukapalli et al. [11] recommended, through a conventional regression-based collocation method (RBCM), that the collocation points should be close to the origin, and be twice the number of unknown coefficients. However, the information matrix produced by RBCM is not always full rank, resulting in unstable and ineffective estimations [12, 21, 22].

Alternatively, the linearly independent collocation method (LICM)-based SRSM ensures a full rank information matrix, because it rejects collocation points until such requirement is fulfilled; and the number of selected collocation points is equivalent to the number of unknown coefficients, thus being efficient [21, 23]. However, the LICM may select highly asymmetrical collocation points with respect to the origin, leading to an unexpected result: i.e., that higher odd-order SRSMs produce worse estimates of probability of failure than lower even-order SRSMs [24]. To overcome this drawback, but at the expense of efficiency, Xiao et al. [24] proposed a new collocation method that requires many more collocation points than LICM.

Therefore, a good strategy to select collocation points for SRSMs, with a good balance between computational efficiency and accuracy, is still needed. Building on the merits of LICM and RBCM, this paper proposes one new strategy to select collocation points, so that better estimates of probability of failure using SRSM can be obtained, while still maintaining a reasonable computational efficiency. Advantages and disadvantages of different alternatives are discussed, and several examples are employed to illustrate the computational efficiency and the accuracy of the suggested approach.

2 Stochastic response surface method

As a conceptual extension of the classical (deterministic) RSM, the SRSM represents inputs and outputs of an uncertain system through a series expansion of standard random variables. Then, for a given SRSM order and based on a limited number of observed data, the unknown coefficients of such series expansion can be estimated. The procedure is well reviewed by Isukapalli et al. [11], Li et al. [12], Xiong et al. [19] and many others. A brief summary of the steps to use the SRSM with Hermite polynomials is given below.

Step 1: Transformation of random variables: random variables with Gaussian distribution considered in the

SRSM are often transformed first from their original physical space, **X**, into the uncorrelated standard normal variables, **U**.

Step 2: Functional representation of the outputs: to express the response of the deterministic model using a PCE. For instance, for normal variables, the output can be represented in terms of a Hermite PCE with a series of standard normal random variables [11, 12]

$$\mathbf{y} = a_0\Gamma_0 + \sum_{i_1=1}^{n} a_{i_1}\Gamma_1(U_{i_1}) + \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1} a_{i_1 i_2}\Gamma_2(U_{i_1}, U_{i_2}) + \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3}\Gamma_3(U_{i_1}, U_{i_2}, U_{i_3}) + \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2}\sum_{i_4=1}^{i_3} a_{i_1 i_2 i_3 i_4}\Gamma_4(U_{i_1}, U_{i_2}, U_{i_3}, U_{i_4}) + \cdots, \tag{1}$$

where **y** is a random output of the model; $a_{i_1,i_2,\ldots,i_n}$ are unknown coefficients in the expansion to be estimated; and $\Gamma_p(U_{i_1}, U_{i_2}, \ldots, U_{i_p})$ are multidimensional Hermite polynomials of order p given by

$$\Gamma_p(U_{i_1}, U_{i_2}, \ldots, U_{i_p}) = (-1)^p e^{\frac{1}{2}\mathbf{U}^T\mathbf{U}} \frac{\partial^p}{\partial U_{i_1}\partial U_{i_2}\ldots\partial U_{i_p}} e^{\frac{1}{2}\mathbf{U}^T\mathbf{U}}. \tag{2}$$

Considering a pth order Hermite PCE with n random variables, the number of the unknown coefficients, N_a, can be calculated as [15]

$$N_a = \frac{(n+p)!}{n!p!}. \tag{3}$$

Step 3: Estimation of the unknown coefficient in the PCE: once the Hermite PCE for the output response has been established, the vector of unknown coefficients in Eq. (1), **a**, can be determined using the selected collocation points and the least square regression-based probabilistic collocation method [25], as

$$\mathbf{a} = (\mathbf{T}\mathbf{T}^T)^{-1}\mathbf{T}^T\mathbf{y}, \tag{4}$$

where **T** is a Hermite polynomial information matrix of dimension $N \times N_a$; N is the number of selected collocation points; and T stands for the transpose matrix operator.

Step 4: Hermite PCE-based reliability analysis: using the obtained coefficients of the Hermite PCE, conventional sampling methods (e.g., Monte Carlo simulation) can be directly employed to evaluate the probability of failure of the analytical expression that approximates the original LSF, and hence to evaluate the corresponding reliability index.

The reader should note that the main goal of this work is to improve the method to select optimal collocation points employed in Step 3; our proposals towards this objective are discussed in the next section.

3 Selection of the collocation points

One key issue for successful application of the SRSM is associated with the selection of collocation points. To that end, Webster et al. [25] proposed that the roots of the next higher order Hermite polynomials can be used as input variables to construct the collocation points. Then, the number of collocation points becomes $(p+1)^n$ for Hermite polynomials in which one of its roots is 0; otherwise, an extra collocation point, **0**, should be included, as such collocation point is located in the region of maximum probability [26]. But, since the number of available collocation points is usually larger than the number of unknown coefficients, **a**, an effective strategy to select collocation points may increase the computational efficiency and accuracy of SRSMs. Two widely used strategies will be discussed first; then, a new collocation method that may improve the existing ones is proposed.

3.1 Regression-based collocation method

Isukapalli et al. [11] proposed a regression-based collocation method (RBCM), in which the collocation points located closer to the origin—i.e., in regions of higher probability—are preferred, and selecting the collocation points so that they are symmetrically distributed with respect to the origin. To obtain good estimates of the unknown coefficients, Isukapalli et al. [11] recommended that the number of selected points should be twice the number of unknown coefficients. However, Li et al. [12] observed that the rank of the information matrix, **T**, does not linearly increase with the number of collocation points, suggesting that some collocation points selected are not mutually independent; for instance, for a rock slope problem with five random variables, the 3rd order SRSM with 56 unknown coefficients requires the first 248 collocation points to provide a full rank information matrix. This indicates that the selection scheme proposed by Isukapalli et al. [11] may not always result in robust estimates, since the Hermite polynomial information matrix, **T**, may not always keep a full rank and be close to singular.

3.2 Linearly independent collocation method

To circumvent this problem associated to the RBCM, Li and Zhang [27] observed that each new set of proposed collocation points should fulfill that the $(j+1)$th row of **T** is linearly independent to its previous j rows; otherwise, it should be rejected. Sudret [28] and Mao et al. [23] emphasized that the sample points selected should lead to an invertible information matrix. Based on these observations, Jiang et al. [21] and Jiang et al. [13] proposed the linearly independent collocation method (LICM), in which collocation points that can increase the rank of the information matrix are reserved, being otherwise rejected. In this way, only N_a collocation points that produce a full rank information matrix will be selected; this is equal to the number of unknown coefficients, N_a, thus significantly reducing the number of deterministic function evaluations, and improving computational efficiency.

However, according to Xiao et al. [24], the collocation points selected in LICM could be asymmetric, particularly for odd-orders; and this could lead to large computational errors in reliability analyses. To overcome this drawback, Xiao et al. [24] proposed an improved collocation method, but it requires many more collocation points (about 2–4 times higher than LICM for a few random variables, and even more times higher for more random variables). Thus, such method has limited possibilities for application due to its high computational cost.

3.3 A proposed symmetric full rank collocation method

In our proposed symmetric full rank collocation method (SFRCM), two types of collocation methods—RBCM and LICM—are combined to achieve a better balance between computational efficiency and accuracy. In particular, the new method combines the merits of RBCM (i.e., to obtain collocation points closer to the origin, and to maintain the symmetry of selected collocation points) with the full rank criterion used in LICM (i.e., to ensure that the information matrix, **T**, is always invertible). A detailed implementation procedure is illustrated below to further explain the details of our proposed method (see also the flowchart in Fig. 1). The steps are:

1. Determine the order of the Hermite PCE, and compute the roots of the next higher order Hermite polynomial. For instance, for a 2nd order Hermite PCE, the three roots of the 3rd order Hermite polynomial are 0, $\sqrt{3}$ and $-\sqrt{3}$.
2. Generate a group of collocation points ("Group A") associated to the roots of the next higher order Hermite polynomial. If the Hermite polynomial has not a 0 root, an additional collocation point located on the origin, **0**, should be included. Thus, Group A will contain $(p+1)^n$ or $(p+1)^n+1$ points.
3. Compute the norms of all the collocation points, and sort them with increasing norm values. Make $i=0$.
4. Make $i=i+1$, select the associated collocation point, $\mathbf{P}_i$, from the available collocation points in Group A, and put it into the group of selected collocation points ("Group B").
5. Compute the rank of the new information matrix, **T**. If the rank of the information matrix does not increase, reject $\mathbf{P}_i$; otherwise, accept it.
6. If $\mathbf{P}_i$ is kept within Group B, its symmetric point with respect to the origin, $\mathbf{P}_i^{'}$, should also be included into Group B (except the point located on the origin, **0**). Then, update the rank of the new information matrix, and delete $\mathbf{P}_i^{'}$ from Group A.
7. Repeat Steps 4–6 until the rank of the information matrix, **T**, equals the number of unknown coefficient, N_a.

In this way, our proposed SFRCM can guarantee that all the selected collocation points are symmetric with respect to the origin, and that the information matrix is always invertible. The computational cost and accuracy associated to this proposed procedure will be illustrated in the following sections.

4 Comparison of collocation points provided by different methods

To further analyze the performance of our proposed symmetric full rank collocation method (SFRCM), and its comparison to the regression-based collocation method (RBCM) and to the linearly independent collocation method (LICM), Table 1 lists the required number of collocation points, and the rank of the associated information matrix, obtained for PCEs of order 1st to 6th, and with up to ten random variables. It is observed that LICM requires the least number of collocation points while RBCM needs the most. Specifically, the exact numbers of collocation points associated to them are N_a and $2N_a$, respectively. The number of selected collocation points with the proposed SFRCM ranges between (N_a, $2N_a$): i.e., it is larger than the number required for LICM, but smaller than the number required for RBCM. (Note also that, for most of the cases considered, the required number is about $1.5N_a$, or the average between LICM and RBCM.)

The ranks of the information matrices obtained with LICM and SFRCM are always identical for the same PCE order and the same number of random variables; and, since the rank is equal to the number of unknown coefficients (i.e., N_a), this ensures that the information matrix is invertible. However, when the RBCM involves more than four random variables, or when the PCEs of 3rd, 5th or 6th order with four random variables are considered, the rank of the information matrix will be less than N_a, possibly leading to inadequate reliability estimations. Therefore, care must be taken when the RBCM is applied in reliability analyses involving a relatively larger number of random variables.

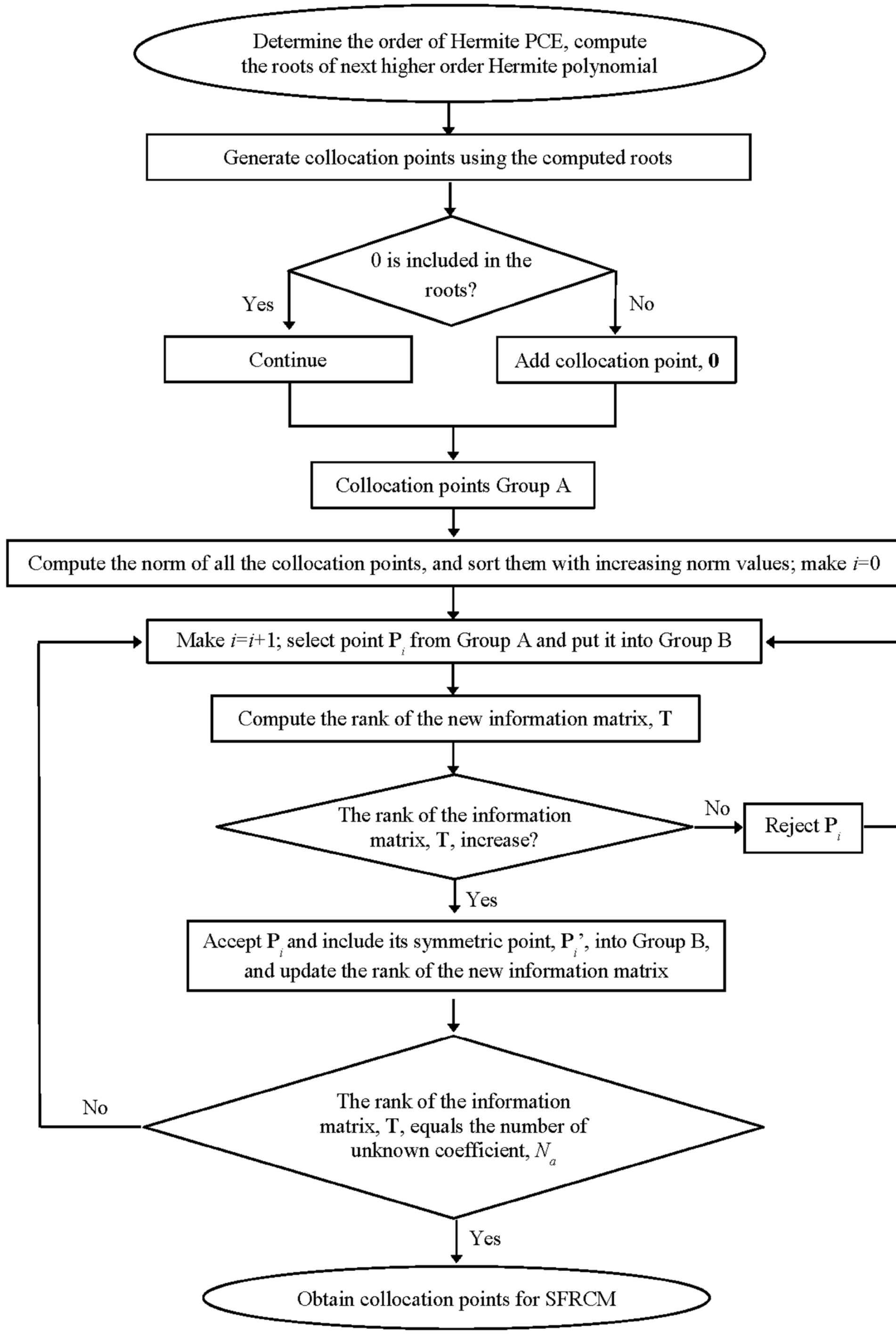


Fig. 1 A flowchart to illustrate the implementation procedure of the proposed SFRCM

More importantly, Table 1 can also guide readers, so that they can pre-evaluate the computational cost of their own cases: the total computation cost can be roughly estimated multiplying the computational time of a single deterministic model evaluation times the required number of collocation points. Moreover, as indicated by Jiang et al. [21], the process to select collocation points might be complex and expensive; therefore, to ease the application of the SRSM in practice, all the collocation points for the three collocation methods considered herein (considering up to 6th order and up to ten random variables) are provided as Supplementary materials.

To better assess the three collocation methods considered, we defined a ratio, ζ, to quantify the asymmetry of the collocation points: ζ is defined as the number of asymmetric collocation points divided by the number of selected collocation points. Table 2 visualizes the distribution pattern of the selected collocation points for the three collocation methods

Table 1 Number of collocation points and its corresponding rank of information matrix for different collocation methods

Order	Methods	Number of random variables									
		1	2	3	4	5	6	7	8	9	10
1	LICM[a]	2/2[d]	3/3	4/4	5/5	6/6	7/7	8/8	9/9	10/10	11/11
	SFRCM[b]	3/2	5/3	7/4	9/5	11/6	13/7	15/8	17/9	19/10	21/11
	RBCM[c]	3/2	5/3	8/4	10/5	12/5[e]	14/5	16/5	18/6	20/6	22/6
2	LICM	3/3	6/6	10/10	15/15	21/21	28/28	36/36	45/45	55/55	66/66
	SFRCM	3/3	7/6	13/10	21/15	31/21	43/28	57/36	73/45	91/55	111/66
	RBCM	3/3	9/6	20/10	30/15	42/19	56/24	72/30	90/36	110/42	132/49
3	LICM	4/4	10/10	20/20	35/35	56/56	84/84	120/120	165/165	220/220	286/286
	SFRCM	5/4	13/10	27/20	49/35	81/56	125/84	183/120	257/165	349/220	461/286
	RBCM	5/4	17/10	40/20	70/34	112/44	168/57	240/73	330/102	440/130	572/153
4	LICM	5/5	15/15	35/35	70/70	126/126	210/210	330/330	495/495	715/715	1001/1001
	SFRCM	5/5	17/15	43/35	91/70	171/126	295/210	477/330	733/495	1081/715	1541/1001
	RBCM	5/5	25/15	70/35	140/70	252/106	420/168	660/258	990/381	1430/538	2002/736
5	LICM	6/6	21/21	56/56	126/126	252/252	462/462	792/792	1287/1287	2002/2002	N.A.[f] /N.A.
	SFRCM	7/6	25/21	69/56	161/126	333/252	629/462	1107/792	1841/1287	2002/2923	N.A./N.A.
	RBCM	7/6	37/21	112/56	252/114	504/223	924/388	1584/620	2574/1003	4004/1293	6006/1939
6	LICM	7/7	28/28	84/84	210/210	462/462	924/924	1716/1716	N.A./N.A.	N.A./N.A.	N.A. /N.A.
	SFRCM	7/7	31/28	99/84	259/210	591/462	1219/924	1716/2325	3003/4165	N.A. /N.A.	N.A. /N.A.
	RBCM	7/7	49/28	168/84	420/190	924/422	1848/839	3432/1513	6006/2195	10010/3332	N.A. /N.A.[g]

[a]Linearly independent collocation method

[b]Symmetric full rank collocation method

[c]Regression-based collocation method

[d]The value in front of the slash indicates the number of selected collocation points; the value behind the slash represents the rank of the information matrix with respect to the specified collocation method. Note also that, the number of the selected collocation points by LICM is also the number of unknown coefficients, N_a

[e]Cells in gray indicate that the rank of the information matrix is not full

[f]Not available due to large computational cost

[g]It is predicted that the rank of the information matrix is not full

considered herein, using two random variables and PCEs of order 1st to 6th. Note that all the points selected by RBCM and SLICM are perfectly symmetric with respect to the origin (i.e., with ζ equal to 0); and that the SFRCM requires less points closer to the origin than RBCM. Similarly, note that the collocation points of LICM are asymmetric with respect to the origin for all the orders considered herein. It is also evident that the asymmetrical ratios for odd-orders are generally larger than for even orders, which might explain the relatively poor performance of LICM when it deals with odd order PCEs [24]. The influence of asymmetric collocation points provided by LICM on the computed reliability results is further illustrated in the case studies below.

5 Case studies

The reliability of four mathematical and engineering problems is considered in this study as benchmark tests of the proposed symmetric full rank collocation method (SFRCM). Other existing collocation methods—RBCM and LICM—are also employed for comparison. To measure accuracy of the reliability results, Monte Carlo simulation (MCS) or Latin hypercube sampling (LHS) analyses are conducted using the true limit state functions (LSFs). (As they provide unbiased estimates of the reliability results, they are considered as the 'reference' or 'exact' solution for comparison.) Additionally, note that the RBCM in this study uses collocation points that are twice the number of the unknown coefficients, as recommended by Isukapalli et al. [11].

5.1 Example 1

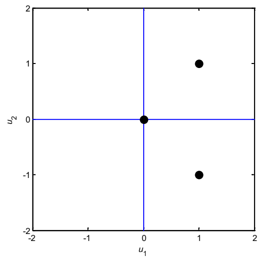
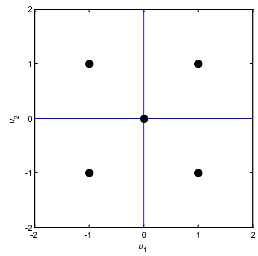
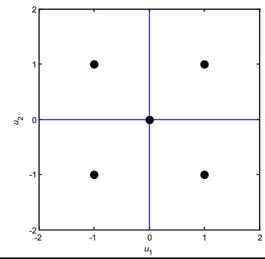
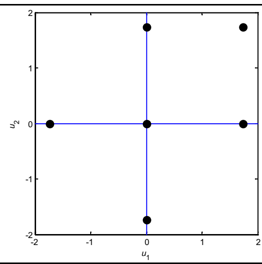
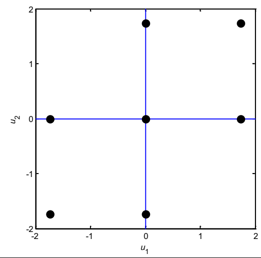
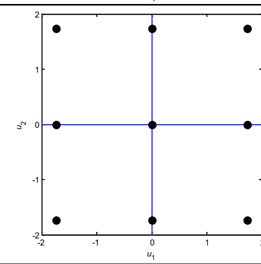
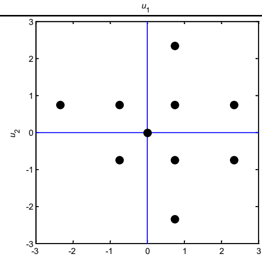
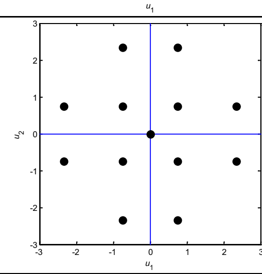
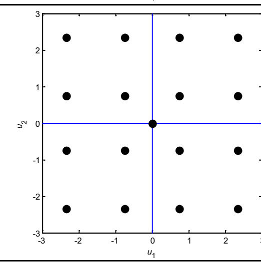
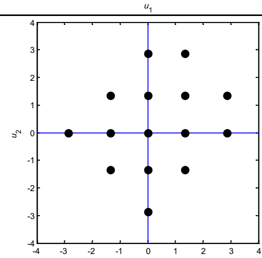
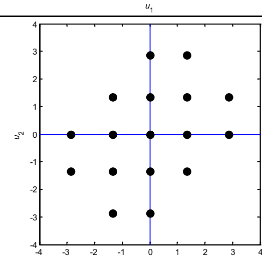
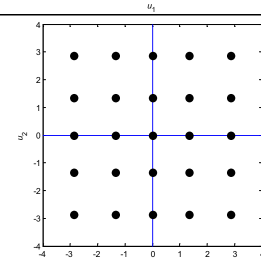
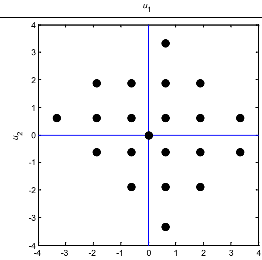
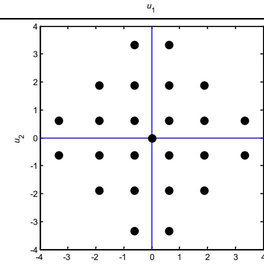
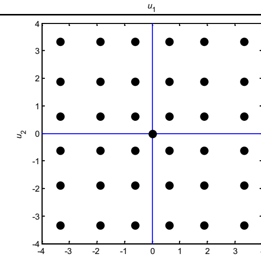
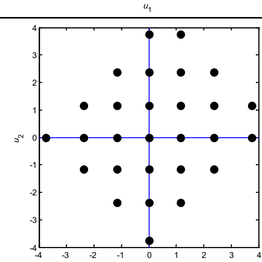
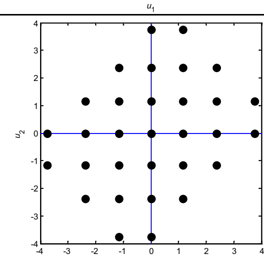
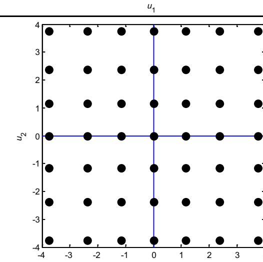
The first case is a mathematical problem previously employed by Grooteman [29] and Periçaro et al. [30] in reliability analyses. The LSF is given as

$$g(\mathbf{x}) = 2 + 0.0.15 \sum_{i=1}^{9} x_i^2 - x_{10}, \tag{5}$$

and ten independent random variables with a standard normal distribution are considered. Table 3 lists the number of collocation points and the corresponding probability of failure results computed using different collocation methods.

According to Table 3, LICM only needs N_a collocation points to fulfill the requirement of linearly independency, whereas RBCM deploys $2N_a$ points. In comparison, SFRCM employs an intermediate number between N_a and $2N_a$, with relatively larger number (close to $2N_a$) for first order PCEs

Table 2 Distribution pattern of selected collocation points for three different collocation methods considering two random variables

Order	LICM	SFRCM	RBCM
1			
2			
3			
4			
5			
6			

The asymmetrical ratios, ζ, of LICM for orders of 1st to 6th are 2/3, 1/6, 3/10, 2/15, 4/21, 3/28

Table 3 Computed results by different collocation methods for Example 1

$N=10$	Order	N_p^a	$P_f\,(\times 10^{-3})$	Δ (%)[b]
LICM	1	11	79.08	1432.56
	2	66	13.37	159.11
	3	286	103.44	1904.65
	4	1001	5.16	0.00
SFRCM	1	21	2.22	−56.98
	2	111	13.84	168.22
	3	461	6.03	16.86
	4	1541	5.15	−0.19
RBCM	1	22	0	−100.00
	2	132	0	−100.00
	3	572	0	−100.00
	4	2002	0	−100.00
MCS	–	5,000,000	5.16[c]	–

[a] N_p = number of limit state function evaluations

[b] Δ = relative error in relation to MCS

[c] $\mathrm{COV}_{P_f}=0.62\%$

Table 4 Computed results by different collocation methods for Example 2

$N=3$	Order	N_p^a	$P_f\,(\times 10^{-2})$	Δ (%)[b]
LICM	1	4	0.08	−94.63
	2	10	1.32	−11.41
	3	20	1.37	−8.05
	4	35	1.49	0.00
	5	56	1.48	−0.67
SFRCM	1	7	2.32	55.70
	2	13	1.32	−11.41
	3	27	1.50	0.67
	4	43	1.49	0.00
	5	69	1.49	0.00
RBCM	1	8	2.14	43.62
	2	20	1.32	−11.41
	3	40	1.50	0.67
	4	70	1.48	−0.67
	5	112	1.49	0.00
MCS	–	5,000,000	1.49[c]	–

[a] N_p = number of limit state function evaluations

[b] Δ = relative error in relation to MCS

[c] $\mathrm{COV}_{P_f}=1.16\%$

and relatively smaller numbers (tending to $1.5N_a$) for higher orders PCEs (i.e., 2nd, 3rd and 4th orders). This implies that some effort is needed to preserve the symmetrical properties of the collocation points, compared to LICM.

In terms of accuracy, it is interesting to observe that, for the 3rd order PCE and under a full rank information matrix, LICM produces a probability of failure with a large relative error ($\Delta = 1904.65\%$) in this case; our proposed SFRCM, on the other hand, obtains a much better estimate, with a relative error of 16.86% with respect to the MCS result. For the 4th order PCE, both LICM and SFRCM produce accurate reliability results, with relative errors of 0.00% and −0.19%, respectively. Similarly, information matrices for the conventional RBCM (with 1st to 4th order) fail to meet the requirement of full rank (see Table 1), thus providing useless results.

5.2 Example 2

The second example was originally proposed by Ranganathan [31], and later employed by Santosh et al. [32] and Periçaro et al. [30] to test the performance of Hasofer–Lind–Rackwitz–Fiessler based algorithms. This example is related to a cantilever beam with a performance function written as

$$g(\mathbf{x}) = x_1 x_2 - 2000x_3, \tag{6}$$

where x_1 and x_2 are normally distributed random variables and x_3 is a lognormal random variable. The mean values and standard deviations of the three random variables are (0.32, 0.032), (1,400,000, 70,000) and (100, 40). All random variables are considered independently. Table 4 compares probabilities of failure computed using SRSMs with different collocation methods.

As in Example 1, the proposed SFRCM requires a number of collocation points, N_p, that is between those required by LICM and RBCM, so that N_p is below $1.5N_a$ for higher order PCEs (i.e., 2nd, 3rd and 4th orders). This is different from what was observed in Example 1, suggesting that a relatively larger number of collocation points (i.e., $N_p > 1.5N_a$) will be required in SFRCM for a larger number of random variables, and vice versa.

In addition, all three methods applied produce similar reliability results in this case. However, LICM produces a relative error of −8.05% with the 3rd order PCE, which is obviously larger than relative errors with SFRCM and RBCM (both 0.67%). Again, this suggests that the asymmetry of the collocation points obtained with the 3rd order LICM introduces errors to the computed reliability result. Moreover, and although the LICM can finally produce a comparable accuracy using the 4th and 5th order PCEs, our proposed SFRCM improves the computational efficiency of the other two methods for the same level of accuracy.

5.3 Example 3

Next, a problem of stress distribution in a steel joint, previously analyzed by Nguyen et al. [33] and Jiang et al. [21], was employed herein to test the ability of our proposed method in dealing with a highly nonlinear LSF. The LSF, which is related to elevated temperatures and fatigue, is written as:

$$g(\mathbf{x}) = x_1 - 10^4 \left[\frac{x_2 (x_4 x_5)^{1.71}}{x_3} + \frac{(1 - x_2)(x_4 x_5)^{1.88}}{x_6} \right]. \quad (7)$$

The statistical information about the six random variables considered is listed in Table 5.

Table 6 lists the required number of collocation points and the computed probabilities of failure for the three collocation methods considered. As expected, the number of collocation points, N_p, required by our proposed SFRCM is between those employed by LICM (N_a) and RBCM ($2N_a$), indicating that N_p varies from above $1.5N_a$ to below $1.5N_a$ with increasing order of the PCE for this case.

In addition, the proposed SFRCM can often outperform the reliability results of LICM. In particular, when the 3rd order PCE is used, the probability of failure computed by LICM has a relative error of −16.11% with respect to the MCS result; however, for SFRCM, the relative error is only −4.53%. This, again, reveals the influence of the asymmetry of the collocation points on the reliability result; and that the RBCM fails to generate reliable results due to information matrices that do not fulfill the full rank requirement.

Table 6 Computed results by different collocation methods for example 3

$N=6$	Order	N_p^a	$P_f\,(\times 10^{-3})$	Δ (%)[b]
LICM	1	7	8.48	−14.60
	2	28	6.82	−31.32
	3	84	8.33	−16.11
	4	210	9.94	0.10
	5	462	10.03	1.01
SFRCM	1	13	11.44	15.21
	2	43	7.24	−27.09
	3	125	9.48	−4.53
	4	295	9.99	0.60
	5	629	10.00	0.70
RBCM	1	14	0.00	−100.00
	2	56	0.00	−100.00
	3	168	0.00	−100.00
	4	420	499.00	4925.18
	5	924	220.84	2123.97
MCS	–	5,000,000	9.93[c]	–

[a] N_p = number of limit state function evaluations

[b] Δ = relative error in relation to MCS

[c] $\mathrm{COV}_{P_f} = 0.45\%$

5.4 Example 4

Finally, to illustrate the applicability of the proposed approach with implicit LSFs, a practical engineering case concerning the face stability of a circular tunnel driven in a layered soil by a compressed-air pressurized shield is employed. Based on the 3D limit analysis mechanism developed by Mollon et al. [34], the collapse mechanism proposed by Senent and Jimenez [35] allows one to compute the critical face pressure in a layered ground (with a softer-top and a stronger-bottom). The mechanism can consider cases in which the contact between layers intersects the tunnel face, and it can also consider the possibility of partial and global collapse (see Fig. 2). Additional details can be found in Senent and Jimenez [35].

The LSF can be constructed according to the tunnel face collapse mechanism, comparing the maximum collapse pressure and the applied support pressure; it results in:

$$g(\mathbf{x}) = \sigma_t - \sigma_{c,\max}, \quad (8)$$

where $\sigma_t = 10$ kPa is the support pressure that is supposed to be applied at the tunnel face, $\sigma_{c,\max}$ is the maximum collapse pressure due to partial collapse or global collapse [i.e., $\sigma_{c,\max} = \max(\sigma_{c,\text{partial}}, \sigma_{c,\text{global}})$], as obtained from the 3D mechanism, and where vector, $\mathbf{x}$, are the random variables in the model.

To represent uncertainty, the Mohr–Coulomb strength parameters (i.e., c_1 and φ_1 for the top layer, and c_2 and φ_2 for the bottom layer) are considered as random variables with lognormal distributions. Their mean values and standard deviations are determined from the literature using reasonable assumptions (see, e.g. [35, 36]); see Table 7. The cohesion, c, and friction angle, φ, of each layer are assumed to be negatively correlated, and the strength parameters of one layer are assumed to be independent of those of the other layers (i.e., $\rho_{c_i,\varphi_i} = -0.5$ and $\rho_{c_i,c_j} = \rho_{\varphi_i,\varphi_j} = \rho_{c_i,\varphi_j} = 0$ for $i, j = 1, 2$ and $i \neq j$). In addition, for simplicity, the unit weight of soil and the circular tunnel diameter are considered to be deterministic, with

Table 5 Statistical information of random variables considered example 3

Random variable	x_1	x_2	x_3	x_4	x_5	x_6
μ	1.0440	0.7000	0.2391	1.0110	0.0005	1.8020
SD	0.31320	0.07000	0.95640	0.15165	0.00008	0.72080
Distribution type	Lognormal	Lognormal	Lognormal	Lognormal	Gumbel Max	Lognormal

μ Mean value, *SD* standard deviation

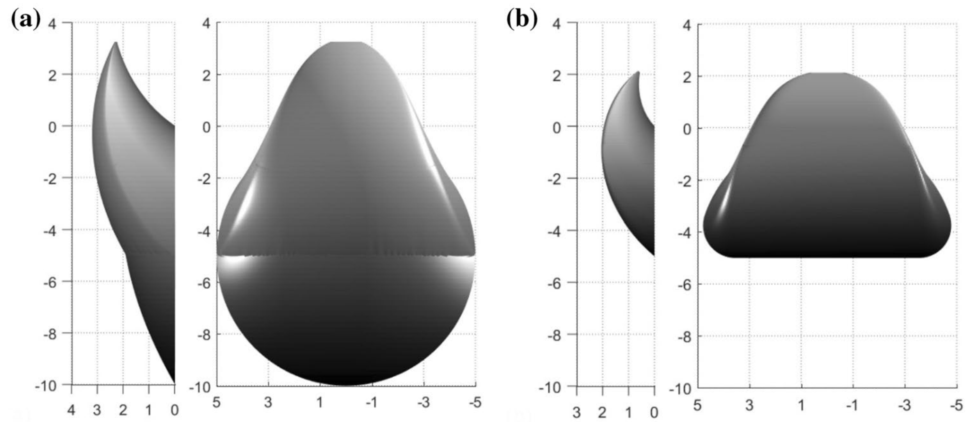


Fig. 2 Examples of geometries computed with the mechanism for the stability of the tunnel face: **a** global collapse and **b** partial collapse

Table 7 Statistical information of random variables considered in example 4

Layer	Random variable	μ	SD	Distribution type
1 (top)	c_1 (kPa)	8	2.4	Lognormal
	φ_1 (°)	30	3	Lognormal
2 (bottom)	c_2 (kPa)	20	6	Lognormal
	φ_2 (°)	40	4	Lognormal

μ Mean value, *SD* standard deviation

values of 20 kN/m^3 and 10 m, respectively. The inter-layer position measured from the crown of the circular tunnel to the bottom is set to be 5 m.

Table 8 illustrates the performance of SRSMs using various collocation methods. [The Latin hypercube sampling (LHS) is employed in this case to estimate the probability of failure, due to the higher computational cost of the LSF.] Results show that the required number of collocation points follow a similar pattern as in previous cases. And the reliability results show that the proposed SFRCM and RBCM provide adequate estimates of the probability of failure when the 4th order PCE is employed. (The SFRCM provides a relatively better estimate in this case.) Note also that RBCM provides good results with the 2nd order PCE, although this may be a "chance" results, as its performance does not improve, as it should, for higher order PCEs. Results also show that LICM is unable to maintain its symmetry, producing large deviations with respect to the LHS result, even for the 4th and 5th order PCEs.

The reduced performance of SRSM in this case might be attributed to two possible reasons: (1) the LSF of the tunnel face stability case is highly nonlinear and unsmooth, (2) the reference value of the probability of failure computed by LHS (8.67×10^{-3}) may be not accurate enough for comparison, since it is associated to a relatively large coefficient of variation (COV) of 17.08%; this problem will be further analyzed in our future studies.

Table 8 Computed results by different collocation methods for example 4

$N=4$	Order	N_p^{a}	$P_f\,(\times10^{-3})$	Δ (%)[b]
LICM	1	5	14.25	64.34
	2	15	23.45	170.51
	3	35	66.82	670.65
	4	70	22.11	155.03
	5	126	55.22	536.91
SFRCM	1	9	12.02	38.66
	2	21	26.00	199.93
	3	49	11.87	36.90
	4	91	7.94	−8.37
	5	161	8.02	−7.50
RBCM	1	10	20.02	130.89
	2	30	9.24	6.61
	3	70	5.15	−40.65
	4	140	7.24	−16.51
	5	252	361.98	4075.09
LHS[c]	–	5000	8.67[d]	–

[a] N_p = number of limit state function evaluations
[b] Δ = relative error in relation to LHS
[c] LHS represents the Latin hypercube sampling
[d] COV_{P_f} = 17.08%

6 Summary and conclusions

This paper proposes a new symmetric full rank collocation method (SFRCM) to select collocation points in structural and geotechnical reliability analyses using the stochastic response surface method (SRSM). For a better balance between computational efficiency and accuracy, the proposed method builds on two conventional collocation methods; in particular, it combines the merits of: (1) the regression-based collocation method (RBCM) (i.e., to obtain collocation points closer to the origin, and to maintain their symmetrical property) and (2) of the linearly independent collocation method (LICM) (i.e., to use a full rank criterion to assure that the information matrix that is always invertible).

Four illustrative examples are used to demonstrate the advantages of the proposed method in terms of its computational efficiency and accuracy, and properties of the collocation points provided by the collocation methods considered, such as the numbers of required collocation points, their symmetry features, and the ranks of the associated information matrices, are analyzed in detail. Results of the analyses conducted, and the main conclusions obtained, are summarized as follows.

1. To ease the application of SRSM in engineering practice, the collocation points corresponding to the three collocation methods discussed herein—i.e., LICM, SFRCM, RBCM—, considering PCEs of up to 6th order and up to ten random variables, are provided as Supplementary materials.
2. The RBCM uses two times the number of unknown coefficients as its collocation points. However, it cannot always ensure robust reliability results, since the resulting information matrix is not always a full rank matrix, particularly when a relative large numbers of random variables are involved (e.g., $n \geq 4$). But, when it is full rank, the reliability results of RBCM are quite accurate, and often even slightly better than those of LICM and SFRCM. (This is probably partly due to the fact that it deploys more collocation points.)
3. The LICM only requires an equal number of collocation points than the number of unknown coefficients in the PCE. In addition, it is designed so as to ensure a full rank information matrix, thus avoiding the robustness problem encountered with RBCM. However, in the four examples considered in this study, when the 3rd or 5th order PCE is used, LICM sometimes computes probabilities of failure with larger errors, than those computed with the proposed SFRCM or with RBCM. This agrees well with the observations by Xiao et al. [24] in their analysis of a layered soil slope considering spatial variability. The poor performance of LICM for the odd-order PCEs is attributed to the asymmetry of the selected collocation points. In addition, and as revealed by the tunnel face stability problem in Example 4, the LICM may also introduce huge errors with highly nonlinear and unsmooth LSFs, even when even-order PCEs are employed.
4. Results from the four illustrative examples suggest that our proposed method achieves the best overall performance, thereby providing a robust and efficient estimation of the probability of failure. The reason is that the proposed SFRCM employs symmetrical collocation points that assure a full rank information matrix; and using just a few more collocation points than LICM. At the same time, it provides reliability results with accuracies equivalent to those of the RBCM (when the latter can generate reliable results), but with a reduced computational effort.

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