



# Fractional-order Fibonacci-hybrid functions approach for solving fractional delay differential equations

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## Abstract

The aim of the current paper is to propose an efficient method for finding the approximate solution of fractional delay differential equations. This technique is based on hybrid functions of block-pulse and fractional-order Fibonacci polynomials. First, we define fractional-order Fibonacci polynomials. Next, using Fibonacci polynomials of fractional-order, we introduce a new set of basis functions. These new functions are called fractional-order Fibonacci-hybrid functions (FFHFs) which are appropriate for the approximation of smooth and piecewise smooth functions. The Riemann–Liouville integral operational matrix and delay operational matrix of the FFHFs are obtained. Then, using these matrices and collocation method, the problem is reduced to a system of algebraic equations. Using Newton’s iterative method, we solve this system. Some examples are proposed to test the efficiency and effectiveness of the present method. Given the application of these kinds of fractional equations in the modeling of many phenomena, a numerical example of this work includes the Hutchinson model which describes the rate of population growth.

**Keywords** Fractional-order Fibonacci-hybrid function · Fibonacci polynomial · Operational matrix · Collocation method

## 1 Introduction

In the last few decades, delay systems have been highly regarded by researchers. Because a lot of these time-delay systems appear in many systems and branches of various sciences such as engineering, chemistry, physics, disease models [1], traffic flow models [2], and so on.

Myshkis introduced the theory of a class of delay differential equations in 1949 [3]. In addition, Krasovski [4], Bellman and Cooke [5], El’sgol’c and Norkin [6], and Hale [7] studied in the field.

One of the branches of mathematical studies is the fractional calculus, which has attracted many researchers in

recent years, because this branch of mathematics emerges in the modeling of many phenomena. For example, in [8], the authors presented fractional Sturm–Liouville boundary value problems. In [9], the authors discussed the fractional rumor spreading dynamical model, and in [10], they regularized long-wave equation has been studied. To find out more about fractional calculus, we can refer to [11]. Fractional calculus is obtained by the change of the derivative and integral order from the integer to the non-integer. In fact, the fractional calculus is the classical mathematical development. Some advantage of this branch of mathematics is explained in [12].

The fractional differential equation is a kind of fractional equations that appears in the modeling of many phenomena in the various sciences. Recently, these equations have been considered by many researchers and mathematicians, and this has led to the improvement and expansion of this kind of equations. The examples of these works can be found in [13, 14] and [15].

In addition, there are various analytic and numerical techniques to solve this type of fractional equations, such as fractional Lagrange polynomials [16], fractional-order of Legendre polynomials [17], fractional-order Taylor method [18], homotopy analysis transform method [19], q-fractional

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homotopy analysis transform method [20], Volterra integral equation method [21], homotopy analysis method [22], operational matrix of Legendre polynomials [23], the mixture homotopy analysis technique, Sumudu transform approach and homotopy polynomials [24], and so on.

Several numerical techniques have been presented for solving integer and non-integer order of delay differential equations, such as Legendre wavelet method [25], Chebyshev polynomials [26], Bernoulli wavelet [27], fractional-order Bernoulli wavelet [28], predictor-corrector method [29], Legendre spectral element method [30], finite difference/finite element method [31], etc.

Moreover, some scholars discussed the stability and analysis of the behavior of delay fractional differential equations or a system of delay fractional differential equations. This type of study can be seen in some works like [32, 33].

Delay fractional differential equations describe the effectiveness of some physical objects such as amorphous semiconductors, fractional random walk [34], and so on. Other applications of this kind of equation happen in some fields such as: fluid flow, viscoelasticity, electrical networks, probability and statistics, optics and signal processing, rheology, etc. Interested readers can find more details in Refs. [35–38].

Piecewise constant basis functions, orthogonal polynomials, and sine-cosine functions in the Fourier series are three classes of the orthogonal functions.

Utilizing the derivative or integration operational matrices of orthogonal functions, the dynamical system problems reduce to solving a system of algebraic equations. These operational matrices can be specified uniquely, using the special orthogonal functions [39]. The Fibonacci polynomials are not orthogonal functions. However, these polynomials have the fractional-order integration operational matrix. A Riemann–Liouville integral operational matrix of Fibonacci polynomials is proposed in [40].

Recently, hybrid functions have included a combination of pulse-block functions and some polynomials such as Legendre polynomials, Lagrange polynomials [41], Chebyshev polynomials [42], and Bernoulli polynomials [43], Bernstein polynomials [44], and Taylor series [45] as strong base functions in solving various kind of equations.

In the current study, we present fractional-order Fibonacci polynomials. Then, using these fractional functions, we introduce another new set of fractional functions, that is called fractional-order Fibonacci-hybrid functions.

FFHFs are appropriate for the approximation of smooth and piecewise smooth functions defined on symmetric and asymmetric subintervals. Moreover, the introduced fractional-order Fibonacci-hybrid functions have three free parameters  $(N, M, \alpha)$ , and so, by different choice of them especially  $\alpha$ , we can obtain more accurate solution for problems, while the previous hybrid functions are dependent to  $N$  and  $M$ .

The structure of this work is organized as follows. The next section includes some necessary definitions and mathematical preliminaries required for the next development. Section 3 is devoted to construct the Fibonacci polynomials of fractional-order and fractional-order Fibonacci-hybrid functions. In Sect. 4, we derive the FFHFs operational matrices of fractional integration and delay. In Sect. 5, we present the numerical technique for solving delay fractional differential equations. Section 6 is devoted to the error analysis. In Sect. 7, by considering some example, we propose the numerical results and display the effectiveness of this method. In addition, we employ this method for solving the Hutchinson model.

## 2 Preliminaries

**Definition 1** Suppose that  $f : [a, b] \rightarrow R$  be a function,  $\nu \in R, \nu > 0$  and  $n = [\nu]$ , the fractional-order Riemann–Liouville integral is defined as follows [16]:

$$I^\nu f(x) = \begin{cases} \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t) dt = \frac{1}{\Gamma(\nu)} x^{\nu-1} * f(x) & \nu > 0, \\ f(x) & \nu = 0, \end{cases} \tag{1}$$

where  $*$  denotes the convolution product. In addition, we have the following: [16]

$$(D^\nu I^\nu f)(x) = f(x), \tag{2}$$

$$(I^\nu D^\nu f)(x) = f(x) - \sum_{i=0}^{[\nu]-1} \frac{x^i}{i!} f^{(i)}(0). \tag{3}$$

$D^\nu$  denotes fractional derivative in Caputo sense which is presented in [16].

**Definition 2** (Generalized Taylor’s formula) [45] Let  $D^{k\alpha} f(x) \in C(0, 1]$  for  $k = 0, 1, \dots, n + 1$ . Then, for  $0 < \xi \leq x, \forall x \in (0, 1]$ , we have the following:

$$f(x) = \sum_{k=0}^n \frac{x^{k\alpha}}{\Gamma(k\alpha + 1)} D^{k\alpha} f(0^+) + \frac{x^{(n+1)\alpha}}{\Gamma((n+1)\alpha + 1)} D^{(n+1)\alpha} f(\xi),$$

Moreover, we get

$$|f(x) - \sum_{k=0}^n \frac{x^{k\alpha}}{\Gamma(k\alpha + 1)} D^{k\alpha} f(0^+)| \leq M_\alpha \frac{x^{(n+1)\alpha}}{\Gamma((n+1)\alpha + 1)}, \tag{4}$$

where  $M_\alpha \geq \sup_{\xi \in (0,1]} |D^{(n+1)\alpha} f(\xi)|$ .

### 3 Fractional-order Fibonacci-hybrid functions

Here, we recall the definitions of Fibonacci polynomials, and in continue, we introduce fractional-order Fibonacci polynomials. In the end, we propose the fractional-order Fibonacci-hybrid functions.

#### 3.1 Fibonacci polynomials

**Definition 3** For any  $k \in R^+$ , the recurrent form of  $k$ -Fibonacci sequence is as follows [46]:

$$\tilde{F}_{k,n+1} = k\tilde{F}_{k,n} + \tilde{F}_{k,n-1}, \quad n \geq 1 \tag{5}$$

subject to  $\tilde{F}_{k,0} = 0, \tilde{F}_{k,1} = 1$ .

If  $k$  is a real variable  $x$  in Eq. (5), then  $\tilde{F}_{k,n} = \tilde{F}_{x,n}$ . Therefore, the general formula of Fibonacci polynomials is as follows [46]:

$$\tilde{F}_n(x) = \begin{cases} 1, & n = 0, \\ x, & n = 1, \\ x\tilde{F}_{n-1}(x) + \tilde{F}_{n-2}(x), & n > 1. \end{cases} \tag{6}$$

or the power form representation of Fibonacci polynomials is as follows[46]:

$$\tilde{F}_n(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} x^{n-2i}, \quad n \geq 0, \tag{7}$$

**Remark 1** The matrix form of Fibonacci polynomials is as follows:

$$\tilde{F}(x) = BT(x), \tag{8}$$

in which

$$\tilde{F}(x) = [\tilde{F}_0(x), \tilde{F}_1(x), \tilde{F}_2(x), \dots]^T, \quad T(x) = [1, x, x^2, x^3, \dots]^T,$$

and  $B$  is coefficients matrix which is the lower triangular matrix:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 3 & 0 & 1 & 0 & 0 & \dots \\ 0 & 3 & 0 & 4 & 0 & 1 & 0 & \dots \\ 1 & 0 & 6 & 0 & 5 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{9}$$

In this matrix, the non-zero inputs exactly build the diagonals of the Pascal triangle and the sum of elements in each row is the same as the classical Fibonacci sequence [47].

#### 3.2 Fractional-order Fibonacci polynomials

To construct fractional-order Fibonacci polynomials, we utilize the change of variable  $x$  to  $x^\alpha$ , ( $\alpha > 0$ ), and we represent these fractional-order polynomials with  $\tilde{F}_n^\alpha(x)$ . Using Eq. (6), the analytic form of  $\tilde{F}_n^\alpha(x)$  is as follows:

$$\tilde{F}_n^\alpha(x) = \begin{cases} 1, & n = 0, \\ x^\alpha, & n = 1, \\ x^\alpha \tilde{F}_{n-1}^\alpha(x) + \tilde{F}_{n-2}^\alpha(x), & n > 1. \end{cases} \tag{10}$$

In addition, using Eq. (7), we achieve

$$\tilde{F}_n^\alpha(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} x^{\alpha(n-2i)}, \quad n \geq 0, \tag{11}$$

For example, the fractional-order Fibonacci polynomials for  $n = 3$  are as follows:

$$\tilde{F}_0^\alpha(x) = 1, \quad \tilde{F}_1^\alpha(x) = x^\alpha, \quad \tilde{F}_2^\alpha(x) = x^{2\alpha} + 1, \quad \tilde{F}_3^\alpha(x) = x^{3\alpha} + 2x^\alpha.$$

#### 3.3 Fractional-order Fibonacci-hybrid functions

Now, a new set of functions called fractional-order Fibonacci-hybrid functions (FFHFs) is proposed. FFHFs are made by change of variable  $x$  to  $x^\alpha$ , ( $0 < \alpha \leq 1$ ), on the Fibonacci-hybrid functions. The FFHFs  $f_{nm}^\alpha(x)$  are specified by  $f_{nm}^\alpha(x)$ . Then, using the Fibonacci-hybrid functions and the fractional-order Fibonacci polynomials, we have the following:

$$f_{nm}^\alpha(x) = \begin{cases} \tilde{F}_n(Mx^\alpha - m + 1), & \frac{m-1}{M} \leq x^\alpha < \frac{m}{M}, \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

where  $n = 0, 1, \dots, N$  and  $m = 1, 2, \dots, M$ , and  $n$  is the order of fractional Fibonacci polynomials and the order of block-pulse function is denoted by  $m$ . In addition, we have Fibonacci-hybrid functions, for  $\alpha = 1$ . For example, in  $N = 2, M = 3, \alpha = \frac{1}{2}$ , we obtain the following:

$$\begin{aligned}
 f_{01}^{\frac{1}{2}}(x) &= \begin{cases} 1, & 0 \leq x < \frac{1}{9} \\ 0 & \text{otherwise} \end{cases}, & f_{11}^{\frac{1}{2}}(x) &= \begin{cases} 3\sqrt{x}, & 0 \leq x < \frac{1}{9} \\ 0 & \text{otherwise} \end{cases}, \\
 f_{21}^{\frac{1}{2}}(x) &= \begin{cases} 9x + 1, & 0 \leq x < \frac{1}{9} \\ 0 & \text{otherwise} \end{cases} \\
 f_{02}^{\frac{1}{2}}(x) &= \begin{cases} 1, & \frac{1}{9} \leq x < \frac{4}{9} \\ 0 & \text{otherwise} \end{cases}, & f_{12}^{\frac{1}{2}}(x) &= \begin{cases} 3\sqrt{x} - 1, & \frac{1}{9} \leq x < \frac{4}{9} \\ 0 & \text{otherwise} \end{cases}, \\
 f_{22}^{\frac{1}{2}}(x) &= \begin{cases} 9x - 6\sqrt{x} + 2, & \frac{1}{9} \leq x < \frac{4}{9} \\ 0 & \text{otherwise} \end{cases} \\
 f_{03}^{\frac{1}{2}}(x) &= \begin{cases} 1, & \frac{4}{9} \leq x < 1 \\ 0 & \text{otherwise} \end{cases}, & f_{13}^{\frac{1}{2}}(x) &= \begin{cases} 3\sqrt{x} - 2, & \frac{4}{9} \leq x < 1 \\ 0 & \text{otherwise} \end{cases}, \\
 f_{23}^{\frac{1}{2}}(x) &= \begin{cases} 9x - 12\sqrt{x} + 5, & \frac{4}{9} \leq x < 1 \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

Moreover, Figs. 1 and 2 show the FFHF's with  $N = 2, M = 2$  and  $\alpha = 1, 0.5$

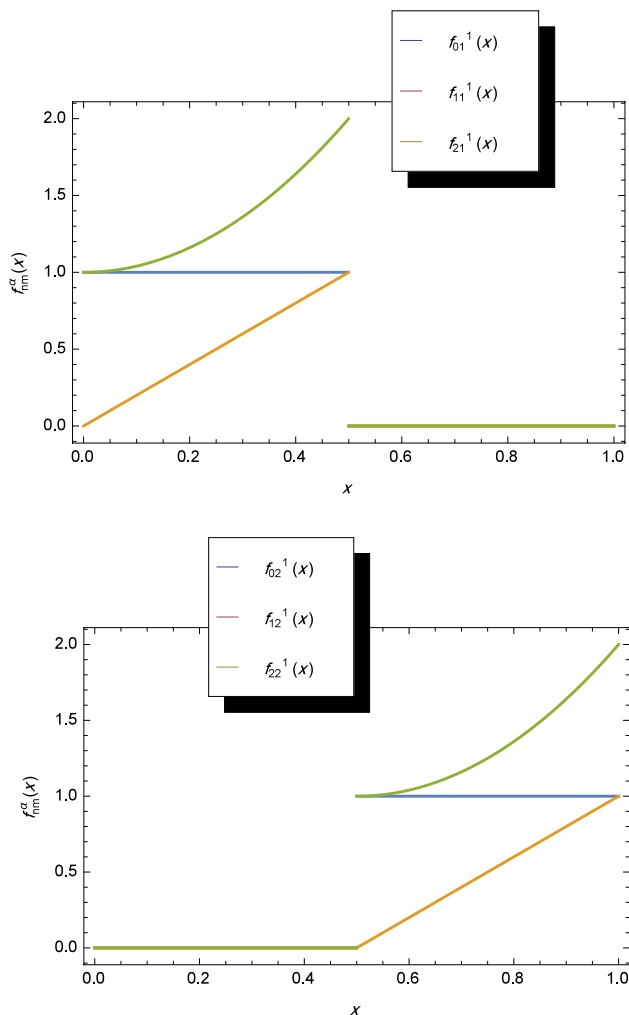


Fig. 1 Plot of the FFHF's with  $N = 2, M = 2$  and  $\alpha = 1$

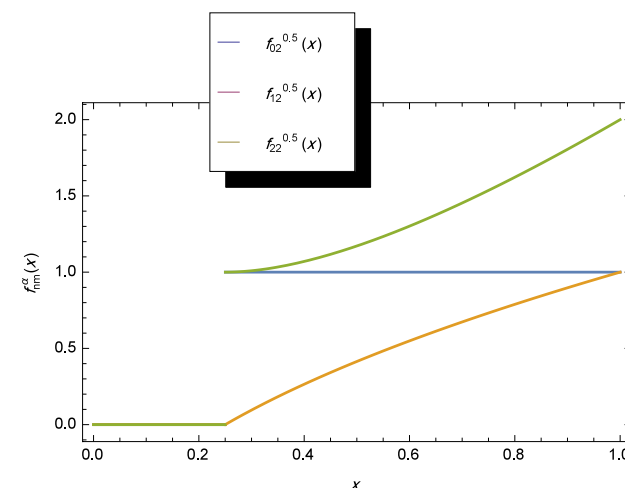
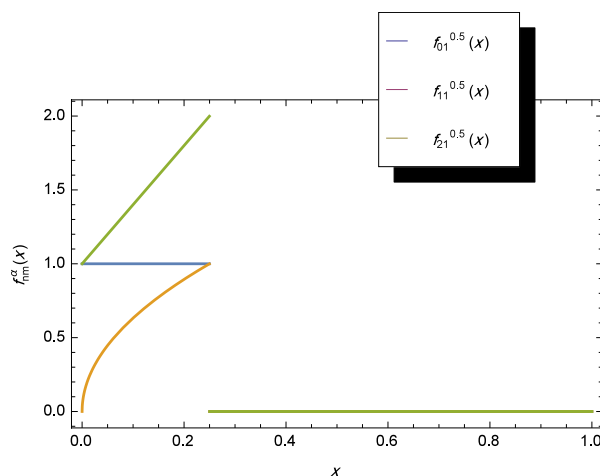


Fig. 2 Plot of the FFHF's with  $N = 2, M = 2$  and  $\alpha = 0.5$

### 3.4 Function approximation

Suppose that  $H = L^2[0, 1]$ , and  $\{f_{01}^\alpha(x), f_{11}^\alpha(x), \dots, f_{NM}^\alpha(x)\} \subset H$  is the set of Fibonacci-hybrid functions of fractional order, and  $Y = span\{f_{01}^\alpha(x), f_{11}^\alpha(x), \dots, f_{NM}^\alpha(x)\}$ . Let  $g$  be an arbitrary element in  $H$ .  $Y$  is a finite-dimensional vector space, so  $\tilde{g} \in Y$  is the unique best approximation out of  $Y$ , which is  $\forall y \in Y, \|g - \tilde{g}\| \leq \|g - y\|$ .

Furthermore, since  $\tilde{g} \in Y$ , there exist coefficients  $c_{01}, c_{11}, \dots, c_{NM}$ , uniquely as follows:

$$g(x) \simeq \tilde{g}(x) = \sum_{n=0}^N \sum_{m=1}^M c_{nm} f_{nm}^\alpha(x) = C^T F^\alpha(x), \tag{13}$$

where

$$\begin{aligned}
 C &= [c_{01}, c_{11}, \dots, c_{N1}, c_{02}, c_{12}, \dots, c_{N2}, \dots, c_{0M}, \dots, c_{NM}]^T \\
 F^\alpha(x) &= [f_{01}^\alpha(x), \dots, f_{N1}^\alpha(x), \dots, f_{0M}^\alpha(x), \dots, f_{NM}^\alpha(x)]^T \\
 &= [f_0^\alpha(x), f_1^\alpha(x), \dots, f_{NM}^\alpha(x)]^T.
 \end{aligned}
 \tag{14}$$

The coefficient vector  $C$  can be obtained as  $C^T = G^T D^{-1}$ , where

$$\begin{aligned}
 D &= \langle F^\alpha, F^\alpha \rangle = \int_0^1 F^\alpha(x) F^{\alpha T}(x) x^{\alpha-1} dx, \\
 G &= [g_{01}, g_{11}, \dots, g_{N1}, g_{02}, g_{12}, \dots, g_{N2}, \dots, g_{0M}, \dots, g_{NM}]^T,
 \end{aligned}
 \tag{15}$$

and

$$g_{nm} = \int_0^1 g(x) f_{nm}^\alpha(x) x^{\alpha-1} dx, \quad n = 0, 1, 2, \dots, N, \quad m = 1, 2, \dots, M.
 \tag{16}$$

### 4 FFHF operational matrix of the Riemann–Liouville integral

Suppose  $F^\alpha(x)$  is FFHFs vector presented in Eq. (7), then

$$I^\nu F^\alpha(x) \simeq P^{(\nu, \alpha)} F^\alpha(x),
 \tag{17}$$

where the fractional Riemann–Liouville integral operational matrix of order  $\nu$  is denoted by  $P^{(\nu, \alpha)}$ . Now, by using Eq. (1), we have the following:

$$I^\nu f_{nm}^\alpha(x) = \frac{1}{\Gamma(\nu)} x^{\nu-1} * f_{nm}^\alpha(x), \quad n = 0, 1, \dots, N \quad m = 1, 2, \dots, M.
 \tag{18}$$

Taking the Laplace transform on Eq. (18), we obtain the following:

$$\begin{aligned}
 \mathcal{L}[I^\nu f_{nm}^\alpha(x)] &= \mathcal{L}\left[\frac{1}{\Gamma(\nu)} x^{\nu-1}\right] \mathcal{L}[f_{nm}^\alpha(x)], \quad n = 0, 1, \dots, N \\
 m &= 1, 2, \dots, M,
 \end{aligned}
 \tag{19}$$

where

$$\mathcal{L}\left[\frac{1}{\Gamma(\nu)} x^{\nu-1}\right] = r^{-\nu}.
 \tag{20}$$

In addition, for  $f_{nm}^\alpha(x)$ , we have the following:

$$\begin{aligned}
 f_{nm}^\alpha(x) &= \mu_{\left(\frac{m-1}{M}\right)^\frac{1}{\alpha}}(x) \tilde{F}_n(Mx^\alpha - m + 1) \\
 &\quad - \mu_{\left(\frac{m}{M}\right)^\frac{1}{\alpha}}(x) \tilde{F}_n(Mx^\alpha - m + 1) \\
 &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (Mx^\alpha - m + 1)^{\alpha(n-2i)} \\
 &\quad \left( \mu_{\left(\frac{m-1}{M}\right)^\frac{1}{\alpha}}(x) - \mu_{\left(\frac{m}{M}\right)^\frac{1}{\alpha}}(x) \right),
 \end{aligned}
 \tag{21}$$

where  $\mu_c(x)$  is unit step function.

By employing the Laplace transform for Eq. (21) and using Eq. (19), we obtain  $\mathcal{L}(I^\nu f_{nm}^\alpha(x)) = \varphi_{nm}^{(\nu, \alpha)}(r)$ . Now, utilizing the inverse Laplace transform of  $\varphi_{nm}^{(\nu, \alpha)}(r)$ , yields

$$I^\nu f_{nm}^\alpha(x) = \tilde{\varphi}_{nm}^{(\nu, \alpha)}(x),
 \tag{22}$$

where  $\tilde{\varphi}_{nm}^{(\nu, \alpha)}(x) = \mathcal{L}^{-1}(\varphi_{nm}^{(\nu, \alpha)}(r))$ . We can expand  $\tilde{\varphi}_{nm}^{(\nu, \alpha)}(x)$  regarding FFHFs as follows:

$$\tilde{\varphi}_{nm}^{(\nu, \alpha)}(x) \simeq \sum_{\tau=0}^N \sum_{\rho=1}^M \tilde{c}_{\tau\rho} f_{\tau\rho}^\alpha(x) = \tilde{C}_{nm}^T F^\alpha(x).
 \tag{23}$$

Hence, we have the following:

$$P^{(\nu, \alpha)} = [\tilde{C}_{nm}], \quad n = 0, 1, \dots, N \quad m = 1, 2, \dots, M.
 \tag{24}$$

### 5 Delay operational matrix of FFHFs

For Taylor polynomials, we have the following [48]:

$$T_N(x - \xi) = \theta(\xi) T_N(x),
 \tag{25}$$

where  $\theta(\xi)$  is the following matrix:

$$\theta(\xi) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -\xi & 1 & \dots & 0 \\ (-\xi)^2 & -2\xi & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-\xi)^N & \binom{N}{N-1}(-\xi)^{N-1} & \dots & 1 \end{bmatrix}.$$

and  $T_N(x) = [1, x, x^2, \dots, x^N]^T$ .

According to Remark 1 and using Eq. (25), we obtain the following:

$$\tilde{F}(x - \xi) = B\theta(\xi) T_N(x) = B\theta(\xi) B^{-1} \tilde{F}(x).$$

Then, we can rewrite

$$\tilde{F}^\alpha(x - \xi) = B\theta(\xi) B^{-1} \tilde{F}^\alpha(x),
 \tag{26}$$

where  $\tilde{F}^\alpha(x) = [\tilde{F}_0^\alpha(x), \tilde{F}_1^\alpha(x), \dots, \tilde{F}_N^\alpha(x)]^T$ .

We get  $\Lambda_\xi = B\theta(\xi) B^{-1}$ , where  $\Lambda_\xi$  is delay operational matrix of fractional Fibonacci polynomials. Moreover, we know

$$\tilde{F}_n(M(x^\alpha - \xi) - m + 1) = \Lambda_{M\xi} \tilde{F}_n(Mx^\alpha - m + 1), \quad n = 0, 1, \dots, N;
 \tag{27}$$

then, we obtain

$$F^\alpha(x - \xi) = \Omega_\xi F^\alpha(x),
 \tag{28}$$

where  $\Omega_\xi$  is delay operational matrix of FFHFs and  $\Omega_\xi = \underbrace{\text{diag}[\Lambda_{M\xi}, \Lambda_{M\xi}, \dots, \Lambda_{M\xi}]}_M$ .

### 6 Numerical method

We focus on the following fractional delay differential equation:

$$\begin{cases} F(x, y(x), y(x - \xi), D^\nu y(x)) = 0, & 0 \leq x \leq 1, \quad l - 1 < \nu \leq l, \quad 0 < \xi < 1, \\ y^{(i)}(0) = \lambda_i, & i = 0, 1, \dots, l - 1, \quad l \in N \\ y(x) = \Phi(x), & x < 0. \end{cases} \tag{29}$$

We can approximate  $D^\nu y(x)$  in this problem with the FFHF's as follows:

$$D^\nu y(x) \simeq C^T F^\alpha(x), \tag{30}$$

then, using Eq. (17) and property of fractional integration in Eq. (3), we achieve the following:

$$y(x) \simeq C^T P^{(\nu, \alpha)} F^\alpha(x) + \sum_{i=0}^{l-1} \frac{x^i}{i!} \lambda_i \simeq C^T P^{(\nu, \alpha)} F^\alpha(x) + E^T F^\alpha(x), \tag{31}$$

where

$$E(x) = \sum_{i=0}^{l-1} \frac{x^i}{i!} \lambda_i, \quad E(x) \simeq E^T F^\alpha(x).$$

In addition, for  $\xi < x \leq 1$  and using Eqs. (28) and (30):

$$\begin{aligned} y(x - \mu) &= C^T P^{(\nu, \alpha)} F^\alpha(x - \mu) + E^T F^\alpha(x - \mu) \\ &= C^T P^{(\nu, \alpha)} \Omega_\xi F^\alpha(x) + E^T \Omega_\xi F^\alpha(x), \end{aligned}$$

and  $\Phi(x - \xi) \simeq A^T F^\alpha(x), 0 \leq x \leq \xi$ , then, we get

$$y(x - \xi) \simeq \begin{cases} A^T F^\alpha(x), & 0 \leq x \leq \xi, \\ C^T P^{(\nu, \alpha)} \Omega_\xi F^\alpha(x) + E^T \Omega_\xi F^\alpha(x), & \xi < x \leq 1 \end{cases} \tag{32}$$

Substituting Eqs. (30)–(32) in Eq. (29), an algebraic equation is derived:

$$\begin{cases} F(x, C^T P^{(\nu, \alpha)} F^\alpha(x) + E^T F^\alpha(x), A^T F^\alpha(x), C^T F^\alpha(x)) = 0, & 0 \leq x \leq \xi, \\ F(x, C^T P^{(\nu, \alpha)} F^\alpha(x) + E^T F^\alpha(x), C^T P^{(\nu, \alpha)} \Omega_\xi F^\alpha(x) + E^T \Omega_\xi F^\alpha(x), C^T F^\alpha(x)) = 0, & \xi < x \leq 1. \end{cases}$$

Therefore, we collocate above equation at  $x_p = \frac{p}{NM}, p = 0, 1, 2, \dots, NM$ . Thus, we get a system of algebraic equations. We solve this system and derive the vector  $C$ , using Newton's iterative method.

### 7 Error analysis

**Theorem 1** Let  $D^{i\alpha} g \in C(0, 1], i = 0, 1, \dots, N, (2N + 1)\alpha \geq 1, (\hat{N} = MN)$  and  $Y_N^\alpha = \text{span}\{F_0^\alpha(x), F_1^\alpha(x), \dots, F_N^\alpha(x)\}$ . If  $g_N(x) = A^T \tilde{F}^\alpha(x)$  is the best approximation of  $g(x)$  out of  $Y_N^\alpha$  on  $[\frac{m-1}{M}, \frac{m}{M}]$ . Then, for the approximate solution  $g_{\hat{N}}(x)$  using FFHF's series on  $[0, 1]$ , we have the following:

$$\|g - g_{\hat{N}}\|_2 \leq \frac{\sup_{x \in [0, 1]} |D^{(N+1)\alpha} g(x)|}{\Gamma((N + 1)\alpha + 1) \sqrt{(2N + 3)\alpha}}. \tag{33}$$

**Proof** We define the following:

$$g_1(x) = \sum_{i=0}^N \frac{x^{i\alpha}}{\Gamma(i\alpha + 1)} D^{i\alpha} f(0^+),$$

and by using Definition 2, we get the following:

$$|g(x) - g_1(x)| \leq \frac{x^{(N+1)\alpha}}{\Gamma((N + 1)\alpha + 1)} \sup_{x \in I_{m, M}} |D^{(N+1)\alpha} g(x)|,$$

where  $I_{m, M} = [\frac{m-1}{M}, \frac{m}{M}]$ . Given that  $g_N(x) = A^T \tilde{F}^\alpha(x)$  is the best approximation of  $g(x)$  out of  $Y_N^\alpha$  on the interval  $I_{m, M}$ , and  $g_1(x) \in Y_N^\alpha$ , then

$$\begin{aligned} \|g - g_N\|_{L^2[0, 1]}^2 &= \|g - C^T F^\alpha\|_{L^2[0, 1]}^2 = \sum_{m=1}^M \|g - A^T \tilde{F}^\alpha\|_{L^2[\frac{m-1}{M}, \frac{m}{M}]}^2 \leq \sum_{m=1}^M \|g - g_1\|_{L^2[\frac{m-1}{M}, \frac{m}{M}]}^2 \\ &\leq \sum_{m=1}^M \int_{I_{m,M}} \left[ \frac{x^{(N+1)\alpha}}{\Gamma((N+1)\alpha + 1)} \sup_{x \in I_{m,M}} |D^{(N+1)\alpha} g(x)| \right]^2 x^{\alpha-1} dx \\ &\leq \int_0^1 \left[ \frac{x^{(N+1)\alpha}}{\Gamma((N+1)\alpha + 1)} \sup_{x \in [0, 1]} |D^{(N+1)\alpha} g(x)| \right]^2 x^{\alpha-1} dx \\ &\leq \frac{1}{\Gamma((N+1)\alpha + 1)^2 (2N+3)\alpha} (\sup_{x \in [0, 1]} |D^{(N+1)\alpha} g(x)|)^2. \end{aligned}$$

**Lemma 1** Assume that  $g \in L^2[0, 1]$  is approximated by fractional-order Fibonacci polynomials as follows:

$$g(x) \simeq g_N(x) = A^T \tilde{F}^\alpha(x);$$

so, we have  $\lim_{N \rightarrow \infty} e_N(g) = 0$ , where

$$e_N(g) = \int_0^1 [g(x) - g_N(x)]^2 dx.$$

**Theorem 2** Let  $H$  is a Hilbert space and  $Y$  is a close subspace of  $H$ , such that  $\dim Y < \infty$  and  $y_1, y_2, \dots, y_n$ , is any basis for  $Y$ . Let  $z$  be an arbitrary element in  $H$ , and  $y^*$  be the unique best approximation to  $z$  out of  $Y$ . Therefore [49]

$$\|z - y^*\|_2^2 = \frac{G(z, y_1, y_2, \dots, y_n)}{G(y_1, y_2, \dots, y_n)},$$

where

$$G(x, y_1, y_2, \dots, y_n) = \begin{pmatrix} \langle x, x \rangle & \langle x, y_1 \rangle & \dots & \langle x, y_n \rangle \\ \langle y_1, x \rangle & \langle y_1, y_1 \rangle & \dots & \langle y_1, y_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle y_n, x \rangle & \langle y_n, y_1 \rangle & \dots & \langle y_n, y_n \rangle \end{pmatrix}.$$

To obtain an upper bound for the error vector of  $P^{(\nu, \alpha)}$ , we consider

$$E_I^{(\nu)} = P^{(\nu, \alpha)} F^\alpha(x) - I^\nu F^\alpha(x), \quad E_I^{(\nu)} = \begin{bmatrix} e_{I0} \\ e_{I1} \\ \vdots \\ e_{INM} \end{bmatrix}, \quad (34)$$

and, from Eqs. (15), (23) and approximated  $\tilde{\varphi}_i^{(\nu, \alpha)}(x)$ , we get

$$\tilde{\varphi}_i^{(\nu, \alpha)}(x) \simeq \sum_{i=0}^{NM} \tilde{c}_i f_i^\alpha(x),$$

□ we obtain  $\tilde{c}_i$  with the best approximation. From Theorem 2, we have the following:

$$\left\| \tilde{\varphi}_i^{(\nu, \alpha)}(x) - \sum_{i=0}^{NM} \tilde{c}_i f_i^\alpha(x) \right\|_2 = \left( \frac{G(\tilde{\varphi}_i^{(\nu, \alpha)}(x), f_0^\alpha, \dots, f_{NM}^\alpha)}{G(f_0^\alpha, \dots, f_{NM}^\alpha)} \right)^{\frac{1}{2}}. \quad (35)$$

Then, from Eqs. (22)–(29), we obtain the following:

$$\begin{aligned} \|e_{li}\|_2 &= \left\| I^\nu f_i^\alpha(x) - \sum_{i=0}^{NM} \tilde{c}_i f_i^\alpha(x) \right\| \\ &\leq \left\| \tilde{\varphi}_i^{(\nu, \alpha)}(x) - \sum_{i=0}^{NM} \tilde{c}_i f_i^\alpha(x) \right\| \\ &\leq \left( \frac{G(\tilde{\varphi}_i^{(\nu, \alpha)}(x), f_0^\alpha, \dots, f_{NM}^\alpha)}{G(f_0^\alpha, \dots, f_{NM}^\alpha)} \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore, considering Theorem 1, we can see that  $E^{(\nu)}$  tends to zero by increasing the number of FFHFs. In addition, notice that we obtain delay operational matrix without any approximation, and then, the vector error of this operational matrix is zero.

For example, considering  $N = 2, M = 3, 5$  and different values of  $\nu, \alpha$ , we obtain the upper bound of the error for this operational matrix. For  $\alpha = \nu = 1$ , we have the following:

$$\begin{aligned} M = 3, \quad E_I^1 &\leq [0, 0, 0, 0.00223, 0, 0, 0.00223]^T, \\ M = 5, \quad E_I^1 &\leq [0, 0, 0, 0, 8.64 \times 10^{-5}, 0, 0, 0, 0, 8.64 \times 10^{-5}]^T. \end{aligned}$$

**Table 1** Absolute errors and CPU time (in seconds) of numerical solution in  $N = 2, M = 2$  and different values of  $\xi$  for Example 1

$x$	$\xi = 0.0001$	$\xi = 0.001$	$\xi = 0.01$
0.1	$3.1919 \times 10^{-15}$	$2.7756 \times 10^{-17}$	$2.4147 \times 10^{-15}$
0.3	$3.7748 \times 10^{-15}$	$2.7756 \times 10^{-17}$	$3.4417 \times 10^{-15}$
0.5	$6.2728 \times 10^{-15}$	$1.6653 \times 10^{-16}$	$3.6915 \times 10^{-15}$
0.7	$2.2204 \times 10^{-16}$	$1.1102 \times 10^{-16}$	0
0.9	$2.2204 \times 10^{-16}$	$1.1102 \times 10^{-16}$	0
CPU	0.016	0.026	0.031

**Table 2** Absolute errors and CPU time of numerical results in  $N = 4, M = 2, \xi = 0.1$  with different values of  $\alpha$  for Example 1

$x$	$\nu = \frac{1}{2}, \alpha = 1$	$\nu = \alpha = \frac{1}{2}$
0.1	$3.37404 \times 10^{-4}$	$1.58642 \times 10^{-5}$
0.3	$1.50058 \times 10^{-4}$	$7.84349 \times 10^{-6}$
0.5	$1.37223 \times 10^{-3}$	$4.44156 \times 10^{-6}$
0.7	$1.09563 \times 10^{-2}$	$1.24645 \times 10^{-5}$
0.9	$9.51393 \times 10^{-3}$	$3.43936 \times 10^{-5}$
CPU	0.016	0.031

**Table 3** Comparison of the numerical solution with analytical solution in Example 2

$x$	Exact solution	Hermit wavelet [50] $\hat{n} = 25$	Bernoulli wavelet [28] $\hat{n} = 14$	Our method $\hat{n} = 7$
0.2	0.8187307	0.8187	0.8187	0.8187307
0.4	0.6703200	0.6703	0.6703	0.6703201
0.6	0.5488116	0.5488	0.5488	0.5488118
0.8	0.4493289	0.4493	0.4493	0.4493292
CPU	–	–	–	0.047

For  $\alpha = 1, \nu = 0.5$ , we have the following:

$$M = 3, \quad E_I^{0.5} \leq [0.011398, 0.001791, 0.013765, 2.57 \times 10^{-4}, 5.05 \times 10^{-4}, 0.0024675]^T,$$

$$M = 5, \quad E_I^{0.5} \leq [0.004029, 1.87 \times 10^{-4}, 0.00441, 4.06 \times 10^{-4}, 0.0042, 3.59 \times 10^{-6}, 4.73 \times 10^{-6}, 1.12 \times 10^{-5}, 2.63 \times 10^{-5}, 1.18 \times 10^{-5}]^T.$$

For  $\alpha = \nu = 0.5$ , we have the following:

$$M = 3, \quad E_I^{0.5} \leq [0, 0, 0, 0.002686, 0, 0, 0.006006]^T,$$

$$M = 5, \quad E_I^{0.5} \leq [0, 0, 0, 0, 1.32 \times 10^{-4}, 0, 0, 0, 0, 3.07 \times 10^{-4}]^T.$$

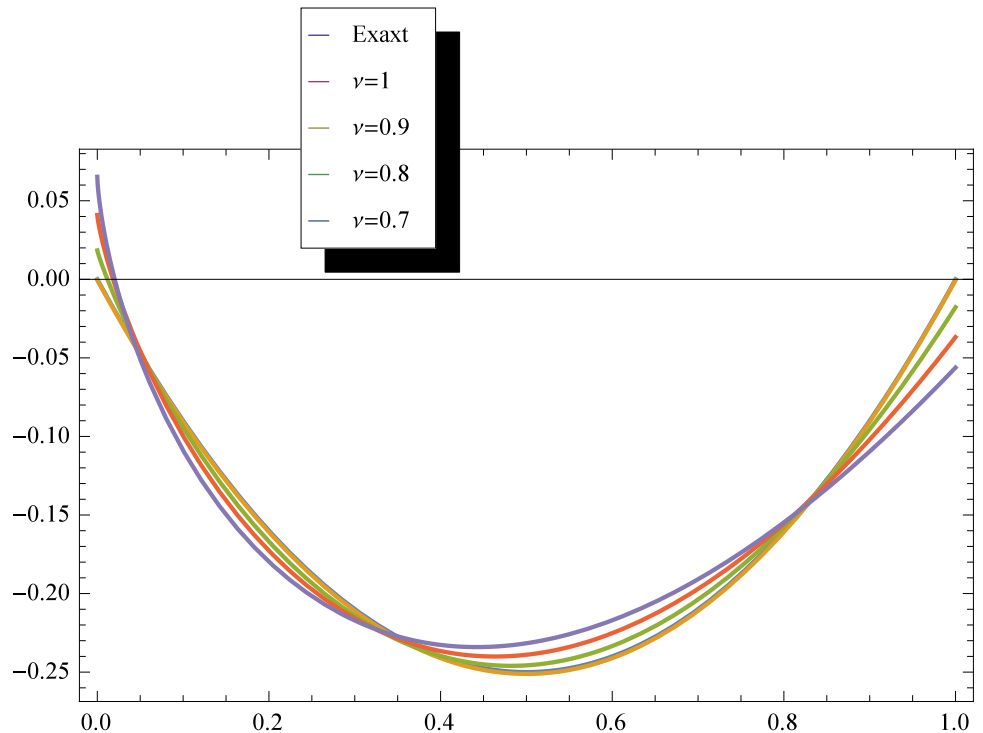
Then, in this example, we can see that, by increasing bases, the vector  $E_I^\nu$  tends to zero.

### 8 Numerical results

Here, we employ the proposed technique for solving the following test examples.

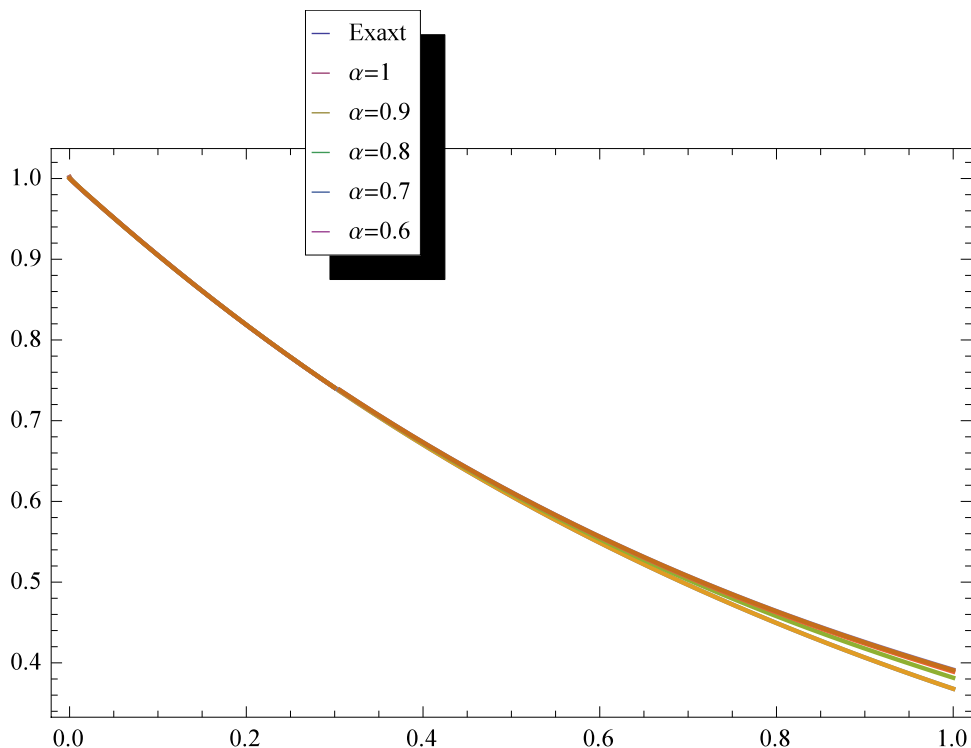
**Example 1** We consider the following fractional delay differential equation [28]:

**Fig. 3** Comparison of our results for  $N = 2, M = 2, \xi = 0.01, \alpha = \nu$  and analytic solution in Example 1





**Fig. 4** Comparison of  $y(x)$  for  $N = 6, M = 1, \nu = 3$  and various of  $\alpha$  and the analytical solution, in Example 2



$$\begin{cases} D^\nu y(x) = y(x - \xi) - y(x) + \frac{2}{\Gamma(3-\nu)}x^{2-\nu} - \frac{1}{\Gamma(2-\nu)}x^{1-\nu} + 2\xi x - \xi^2 - \xi, & 0 \leq x \leq 1, \quad 0 < \nu \leq 1, \\ y(x) = 0, & x \leq 0 \end{cases} \quad (36)$$

For  $\nu = 1, \frac{1}{2}$ , the analytic solution of Eq. (36) is  $y(x) = x^2 - x$ .

We apply this method for  $\nu = \alpha = 1$ , various choices of  $\xi$  and  $N = 2, M = 2$ . Figure 3 displays the numerical results achieved for various values of  $\alpha = \nu$  and the exact solution in  $\xi = 0.01$ .

Table 1 demonstrates the efficiency and accuracy of FFHFs to solve this problem. The numerical results show that the numerical solutions converge to the analytical solution.

Moreover, we solve this problem in  $\nu = \frac{1}{2}$ . In Table 2, our results show the effectiveness of our method in this problem.

**Example 2** Consider the linear fractional delay differential equation as follows [28, 50]:

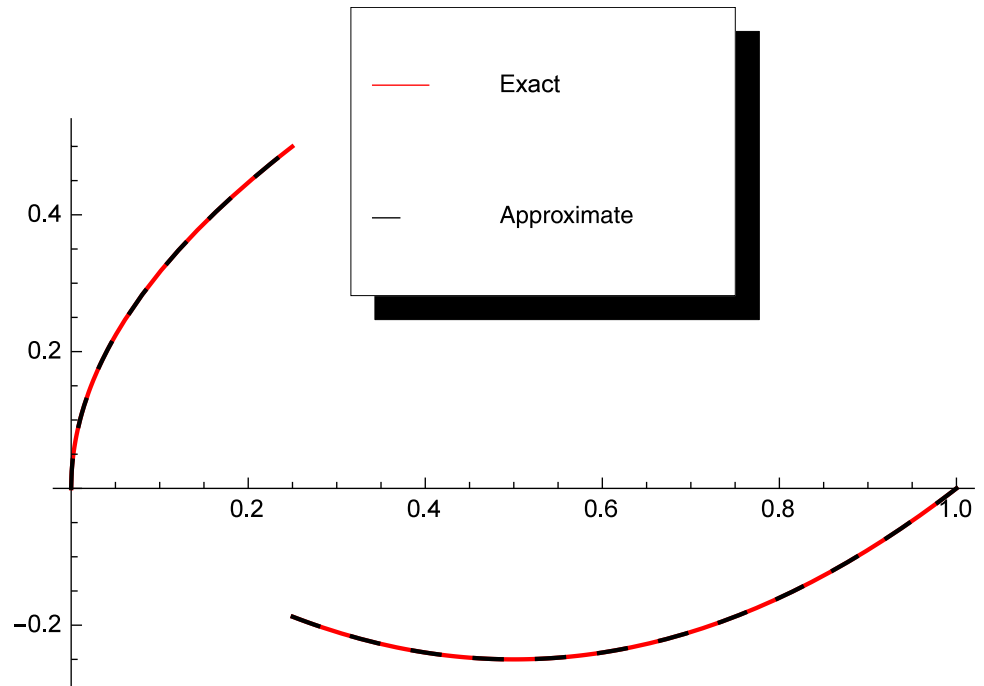
$$\begin{cases} D^\nu y(x) = -y(x) - y(x - 0.3) + e^{-x+0.3}, & 0 \leq x \leq 1, \quad 2 < \nu \leq 3, \\ y(x) = 1, \quad y'(0) = -1, \quad y''(0) = 1, & \\ y(x) = e^{-x}, & x < 0. \end{cases} \quad (37)$$

For  $\nu = 3$ , the analytic solution of Eq. (37) is  $y(x) = e^{-x}$ . We employ the proposed method for  $M = 1, N = 6$  in  $\alpha = 1$ .

Table 3 demonstrates the numerical results achieved for various values of  $x$  using our method in  $M = 1, N = 6, \alpha = 1$  or  $\hat{n} = 7$ , the Hermite wavelet method in  $\hat{n} = 25$  [50], Bernoulli wavelets method with  $k = 2, M = 7$  or  $\hat{n} = 14$  [28], and the exact solution. In addition, Fig. 4 shows the numerical solutions obtained with various values of  $\nu$  and the analytical solution for  $N = 7, M = 1, \alpha = 1$ .

Moreover, maximum absolute error for this problem using our method for  $n = 6$  is  $4.71097 \times 10^{-7}$ , and this value obtained by one-point fifth-order predictor-corrector method and two-point fifth-order predictor-corrector method with  $h = 0.1$  are  $8.692389 \times 10^{-5}$  and  $8.804834 \times 10^{-5}$ , respectively [29].

**Fig. 5** Comparison of  $y(x)$  for  $N = 4, M = 2, \alpha = \frac{1}{2}$  and the analytical solution in Example 3



**Table 4** Absolute errors and CPU time of our method for various values of  $\alpha$  in Example 3

$x$	$\alpha = 1$	$\alpha = \frac{1}{2}$
0.1	$1.31740 \times 10^{-3}$	$7.56818 \times 10^{-6}$
0.3	$7.49422 \times 10^{-2}$	$3.52562 \times 10^{-8}$
0.5	$1.85252 \times 10^{-2}$	$1.27839 \times 10^{-8}$
0.7	$1.64537 \times 10^{-3}$	$7.71325 \times 10^{-8}$
0.9	$1.27992 \times 10^{-2}$	$2.29497 \times 10^{-7}$
CPU	0.063	0.015

**Table 5** The  $\|Res_n\|^2$  and CPU time with various values of  $\nu, \alpha$  for Example 4

		$\ Res_n\ ^2$	CPU
$\nu = \alpha = 1$	$N = 4, M = 1$	$3.3863 \times 10^{-16}$	0.012
	$N = 4, M = 2$	$2.5841 \times 10^{-18}$	0.031
$\nu = 0.5, \alpha = 1$	$N = 4, M = 1$	$1.4814 \times 10^{-12}$	0.178
	$N = 4, M = 2$	$8.6164 \times 10^{-14}$	0.031
$\nu = \alpha = 0.5$	$N = 4, M = 1$	$4.4648 \times 10^{-15}$	0.031
	$N = 4, M = 2$	$2.2085 \times 10^{-17}$	0.078
$\nu = 0.75, \alpha = 1$	$N = 4, M = 1$	$2.9882 \times 10^{-12}$	0.078
	$N = 4, M = 2$	$6.3247 \times 10^{-15}$	0.266
$\alpha = \nu = 0.75$	$N = 4, M = 1$	$2.0648 \times 10^{-15}$	0.063
	$N = 4, M = 2$	$9.6071 \times 10^{-16}$	0.734

**Example 3** We consider the following fractional equation:

$$\begin{cases} D^{\frac{1}{2}}y(x) = y(x - \xi) + f(x), & 0 \leq x \leq 1, \\ y(x) = \sqrt{x}, & -\xi \leq x \leq 0 \end{cases} \quad (38)$$

with

$$f(x) = \begin{cases} \frac{\sqrt{\pi}}{2} + \sqrt{x - \xi}, & 0 \leq x < \frac{1}{4}, \\ \xi - \frac{2\sqrt{x}}{\sqrt{\pi}} - x + \frac{8x^{\frac{3}{2}}}{3\sqrt{\pi}} + (x - \xi)^2, & \frac{1}{4} \leq x \leq 1, \end{cases}$$

and  $\xi = 0.000001$ . The analytic solution of Eq. (38) is as follows:

$$y(x) = \begin{cases} \sqrt{x}, & 0 \leq x < \frac{1}{4}, \\ x^2 - x, & \frac{1}{4} \leq x \leq 1. \end{cases}$$

We employ our method to solve this equation in  $N = 4, M = 2, \alpha = \frac{1}{2}$  and 1. Table 4 compares the numerical results in this method. We can see the effectiveness of presented method to solve this problem with piecewise smooth functions.

In addition, numerical results for  $\alpha = \frac{1}{2}$  and exact solution are shown in Fig. 5.

**Example 4** Consider the Hutchinson model. This model describes the rate of population growth. In each particular time, the rate of population growth is dependent on some unique relationships at that time, and its equation is as follows [51]:

$$\begin{cases} D^{\nu}y(x) = py(x) - \frac{p}{q}y(x)y(x - \xi), & x \in [0, 1], \\ y(x) = 0.5, & x \in [-\xi, 0]. \end{cases} \quad (39)$$

The analytical solution of Eq. (39) is not available [52], and then, our main purpose is to ensure the convergence of the proposed method. For this approach, consider the residual error norm as follows:

$$\begin{aligned} \text{Res}_n(x) &= C^T F^{\alpha}(x) - p(C^T P^{(\nu, \alpha)} F^{\alpha}(x) + 0.5) \\ &\quad + \frac{p}{q}(C^T P^{(\nu, \alpha)} F^{\alpha}(x) + 0.5)(C^T P^{(\nu, \alpha)} \Omega_{\xi} F^{\alpha}(x) + 0.5)^T, \\ \|\text{Res}_n\|^2 &= \int_0^1 \text{Res}_n^2(x) dx. \end{aligned}$$

The calculated residual error and numerical convergence order for the approximate solution of the considered models and CPU time (in seconds) are tabulated in Table 5 for  $\xi = 0.2$ ,  $p = 0.015$ ,  $q = 1$ .

## 9 Conclusion

In the current study, we introduced fractional-order Fibonacci polynomials. Next, we presented new functions called fractional-order Fibonacci-hybrid functions (FFHFs), based on block-pulse functions and fractional-order Fibonacci polynomials. The FFHF operational matrix of fractional-order integration and delay operational matrix of FFHFs have been obtained. These matrices and collocation method are used to numerical solution of linear and nonlinear delay fractional differential equations. Moreover, we employed this method for numerical solution of the Hutchinson model. The results displayed that the present method is more accurate than some existing method.

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## References

- Zhang T, Meng X, Zhang T (2014) SVEIRS: a new epidemic disease model with time delays and impulsive effects. *Abstr Appl Anal* 2014:15

- Sipahi R, Niculescu SI (2009) Deterministic time-delayed traffic flow models: a survey. In: Atay F (ed) *Complex time-delay systems. understanding complex systems*. Springer, Berlin, Heidelberg
- Myshkis AD (1949) General theory of differential equations with a retarded argument. *Uspehi Mat. Nauk* 22 (134), (in Russian). *Amer. Math. soc. transl. no. 55*, 1951, pp 21–57
- Krasovskii NN (1963) *Stability of motion*. Stanford University Press, Stanford
- Bellman R, Cooke KL (1963) *Differential-difference equation*. Academic, New York
- El'sgol'c LE, Norkin SB (1973) *Introduction to the theory of differential equations with deviating argument*, 2th edn. Nauka, Moscow (in Russian), 1971. *Mathematics in science and Eng.*, vol 105. Academic Press, New York
- Hale JK (1977) *Theory of functional differential equations*. Springer, New York
- Khosraviyan-Arab H, Dehghan M, Eslahchi MR (2015) Fractional SturmLiouville boundary value problems in unbounded domains: theory and applications. *J Comput Phys* 299:526–560
- Singh J (2019) A new analysis for fractional rumor spreading dynamical model in a social network with Mittag-Leffler law. *Chaos Interdiscip J Nonlinear Sci* 29(1):013137
- Kumar D, Singh J, Baleanu D (2018) Analysis of regularized long-wave equation associated with a new fractional operator with Mittag-Leffler type kernel. *Phys A Stat Mech Appl* 492:155–167
- Loverro A (2004) *Fractional calculus: history, definitions and applications for the engineer*. Department of Aerospace and Mechanical Engineering, Rapport technique, Univeristy of Notre Dame, Notre Dame, pp 1–28
- Atangana A, Secer A (2013) A note on fractional order derivatives and table of fractional derivatives of some special functions. *Abstract and applied analysis*, vol 2013. Hindawi, Cairo
- Dumitru B, Kai D, Enrico S (2012) *Fractional calculus: models and numerical methods*, vol 3. World Scientific, Singapore
- Kumar D, Singh J, Baleanu D (2018) A new analysis of the Fornberg-Whitham equation pertaining to a fractional derivative with Mittag-Leffler-type kernel. *Eur Phys J Plus* 133(2):70. <https://doi.org/10.1140/epjp/i2018-11934-y>
- Singh J, Kumar D, Baleanu D (2018) On the analysis of fractional diabetes model with exponential law. *Adv Differ Equ* 2018(1):231. <https://doi.org/10.1186/s13662-018-1680-1>
- Sabermahani S, Ordokhani Y, Yousefi SA (2017) Numerical approach based on fractional-order Lagrange polynomials for solving a class of fractional differential equations. *Comput Appl Math* 2017:1–23. <https://doi.org/10.1007/s40314-017-0547-5>
- Kazem S, Abbasbandy S, Kumar S (2013) Fractional-order Legendre functions for solving fractional-order differential equations. *Appl Math Model* 37(7):5498–5510
- Krishnasamy VS, Razzaghi M (2016) The numerical solution of the Bagley-Torvik equation with fractional Taylor method. *J Comput Nonlinear Dyn* 11(5):051010-051010–6
- Kumar D, Singh J, Baleanu D, Rathore S (2018) Analysis of a fractional model of the Ambartsumian equation. *Eur Phys J Plus* 133(7):259. <https://doi.org/10.1140/epjp/i2018-12081-3>
- Singh J, Secer A, Swroop R, Kumar D (2018) A reliable analytical approach for a fractional model of advection-dispersion equation. *Nonlinear Engineering*. <https://doi.org/10.1515/nleng-2018-0027>
- Assari P, Cuomo S (2018) The numerical solution of fractional differential equations using the Volterra integral equation method based on thin plate splines. *Eng Comput*. <https://doi.org/10.1007/s00366-018-0671-x>
- Dehghan M, Manafian J, Saadatmandi A (2010) Solving nonlinear fractional partial differential equations using the homotopy analysis method. *Numer Methods Partial Differ Equ Int J* 26(2):448–479

23. Saadatmandi A, Dehghan M (2010) A new operational matrix for solving fractional-order differential equations. *Comput Math Appl* 59(3):1326–1336
24. Singh J, Kumar D, Baleanu D, Rathore S (2018) An efficient numerical algorithm for the fractional Drinfeld–Sokolov–Wilson equation. *Appl Math Comput* 335:12–24
25. Sadeghi Hafshejani M, Karimi Vanani S, Sedighi Hafshejani J (2011) Numerical solution of delay differential equations using Legendre wavelet method. *World Appl Sci* 13:27–33
26. Sedaghat S, Ordokhani Y, Dehghan M (2012) Numerical solution of the delay differential equations of pantograph type via Chebyshev polynomials. *Commun Nonlinear Sci Numer Simul* 17:4815–4830
27. Rahimkhani P, Ordokhani Y, Babolian E (2017) Numerical solution of fractional pantograph differential equations by using generalized fractional-order Bernoulli wavelet. *Comput Appl Math* 309:493–510
28. Rahimkhani P, Ordokhani Y, Babolian E (2017) A new operational matrix based on Bernoulli wavelets for solving fractional delay differential equations. *Numer Algor*. 74:223–245
29. Seong HY, Majid ZA (2014) Fifth order predictor-corrector methods for solving third order delay differential equations. *AIP Conf Proc* 1635(1):94–98
30. Dehghan M, Abbaszadeh M (2018) A Legendre spectral element method (SEM) based on the modified bases for solving neutral delay distributed-order fractional damped diffusion-wave equation. *Math Methods Appl Sci* 41(9):3476–3494
31. Dehghan M, Abbaszadeh M (2018) An efficient technique based on finite difference/finite element method for solution of two-dimensional space/multi-time fractional Bloch-Torrey equations. *Appl Numer Math* 131:190–206
32. Baleanu D, Magin RL, Bhalekar S, Daftardar-Gejji V (2015) Chaos in the fractional order nonlinear Bloch equation with delay. *Commun Nonlinear Sci Numer Simul* 25(1–3):41–49
33. Moghaddam BP, Yaghoobi S, Machado JT (2016) An extended predictor-corrector algorithm for variable-order fractional delay differential equations. *J Comput Nonlinear Dyn* 11(6):061001. <https://doi.org/10.1115/1.4032574>
34. Ohira T, Milton J (2009) Delayed random walks: Investigating the interplay between delay and noise. *Delay differential equations*. Springer, Boston, pp 1–31
35. Huang C, Guo Z, Yang Z, Chen Y, Wen F (2015) Dynamics of delay differential equations with its applications 2014. Abstract and applied analysis. Hindawi, Cairo
36. an der Heiden U (1979) Delays in physiological systems. *J Math Biol* 8:345–364
37. an der Heiden U, Mackey MC, Walther HO (1981) Complex oscillations in a simple deterministic neuronal network. In: Hoppensteadt F (ed) *Mathematical aspects of physiology*. American Mathematical Society, Providence, pp 355–360
38. Radziunas M (2016) New multi-mode delay differential equation model for lasers with optical feedback. *Opt Quant Electron* 48:470. <https://doi.org/10.1007/s11082-016-0736-2>
39. Mashayekhi S, Razzaghi M (2016) Numerical solution of distributed order fractional differential equations by hybrid functions. *J Comput Phys* 315(15):169–181
40. Youssri YH, Abd-Elhameed WM (2017) Spectral solutions for multi-term fractional initial value problems using a new Fibonacci operational matrix of fractional integration. *Progr Fract Differ Appl* 2(2):141–151
41. Razzaghi M, Marzban HR (2000) A hybrid analysis direct method in the calculus of variations. *Int J Comput Math* 75(3):259–269
42. Tavassoli Kajani M, Hadi Vencheh A (2005) Solving second kind integral equations with Hybrid Chebyshev and Block-Pulse functions. *Appl Math Comput* 163:71–77
43. Behroozifar M (2013) Numerical solution of delay differential equations via operational matrices of hybrid of block-pulse functions and Bernstein polynomials. *Comput Methods Differ Equ* 2(1):78–95
44. Marzban HR, Razzaghi M (2006) Solution of multi-delay systems using hybrid of block-pulse functions and Taylor series. *J Sound Vib* 292(3):954–963
45. Odibat Z, Shawagfeh NT (2007) Generalized Taylor’s formula. *Appl Math Comput* 186(1):286–293
46. Falcon S, Plaza A (2007) The k-Fibonacci sequence and the Pascal 2-triangle. *Chaos Solitons Fract* 33(1):38–49
47. Falcon S, Plaza A (2009) On k-Fibonacci sequences and polynomials and their derivatives. *Chaos Solitons Fract* 39:1005–1019
48. Rabiee K, Ordokhani Y, Babolian E (2017) Fractional-order Bou-baker functions and their applications in solving delay fractional optimal control problems. *J Vib Control* 24:1–14
49. Kreyszig E (1978) *Introductory functional analysis with applications*. Wiley, New York
50. Saeed U, Rehman MU (2014) Hermite wavelet method for fractional delay differential equations. *J Differ Equ* 2014:1–8
51. Fowler AC (2005) Asymptotic methods for delay equations. *J Eng Math* 53(3–4):271–290
52. Moghaddam BP, Mostaghim ZS (2013) A numerical method based on finite difference for solving fractional delay differential equations. *J Taibah Univ Sci* 7(3):120–127

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