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CONSTRUCTIVE APPROXIMATION

Wavelet Expansions and Fractal Dimensions

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Abstract. The bounds of the upper and lower box dimensions of the graph of a function in terms of the coefficients in its wavelet decomposition are given. An example, that the formula for upper box dimension, given in [4], does not hold, is presented.

1. Introduction

This paper consists of two parts. In the first part we propose a method of estimation of the upper and lower box dimensions of graphs of a continuous function in terms of the coefficients in its wavelet expansion. Results of this type for some spline bases have been proved in Z. Ciesielski's papers [1] and [2]. Our proof of the lower bound for the lower box dimension follows the ideas of these two papers.

In the paper by A. Deliu and B. Jawerth [4] the authors have stated a formula for the upper box dimension of the graph of a continuous function (see formula (5) in Section 3). In the second part of this paper we give a counterexample to their statement.

Let us recall the definition of box dimension. For $j \in \mathbf{N}$ and $k \in \mathbf{Z}$, let $I_{j,k} = [k/2^j, (k+1)/2^j]$ be the kth dyadic interval of order j, and for $\underline{k} = (k_1, \ldots, k_d) \in \mathbf{Z}^d$, let $I_{j,\underline{k}} = I_{j,k_1} \times \cdots \times I_{j,k_d}$ denote the $\underline{k}$th dyadic cube of order j. For a compact subset $K \subset \mathbf{R}^d$, let $N(K, j)$ be the number of (d-dimensional) dyadic cubes of order j intersecting K. Then the *upper* and *lower box dimensions* of K $\overline{\dim}_b(K)$ and $\underline{\dim}_b$ are defined by the following formulas:

$$\overline{\dim}_b(K) = \limsup_{j\to\infty} \frac{\log N(K; j)}{j \log 2}, \qquad \underline{\dim}_b(K) = \liminf_{j\to\infty} \frac{\log N(K; j)}{j \log 2}.$$

If $\overline{\dim}_b(K) = \underline{\dim}_b(K)$, then the common value is called the *box dimension* of K and denoted by $\dim_b(K)$. For the equivalent definitions of box dimension, see, for example, [5].

Let us introduce some notation. For a function $h : \mathbf{R}^d \to \mathbf{C}$ denote

$$h_{j,\underline{k}}(\cdot) = 2^{jd/2} h(2^j \cdot - \underline{k}) \qquad \text{for} \quad j \in \mathbf{Z}, \quad \underline{k} \in \mathbf{Z}^d.$$

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For functions $f, g : \mathbf{R}^d \to \mathbf{C}$ we denote $(f, g) = \int_{\mathbf{R}^d} f(\underline{x})\overline{g}(\underline{x})\, d\underline{x}$. $\|\cdot\|$ is the Euclidean norm on $\mathbf{R}^d$; by C we denote a positive constant, which may vary from place to place.

Now we recall briefly the necessary definitions concerning scaling functions and wavelets. Following [6], we introduce r-regular orthonormal scaling functions.

Definition 1.1. Let $\varphi \in L^2(\mathbf{R}^d)$ and

$$V_j = \mathrm{clos}_{L^2(\mathbf{R}^d)}\{\varphi_{j,\underline{k}},\ \underline{k} \in \mathbf{Z}^d\}, \qquad j \in \mathbf{Z}.$$

The function φ is called an orthonormal scaling function if the sequence of spaces $\{V_j,\ j \in \mathbf{Z}\}$ fulfills the following conditions:

(1) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$;
(2) $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R}^d)$;
(3) $\bigcap_{j\in\mathbf{Z}} V_j = \{0\}$; and
(4) the system $\{\varphi_{0,\underline{k}}\}_{\underline{k}\in\mathbf{Z}^d}$ is an orthonormal basis in V_0.

The function φ is called an r-regular scaling function if, in addition, $\varphi \in C^r(\mathbf{R}^d)$ and it satisfies the following decay conditions: for any $p \in \mathbf{N}$ there is a constant $C_{p,r}$ such that

$$|\partial^{\underline{\alpha}}\varphi(\underline{x})| \le \frac{C_{p,r}}{(1 + \|\underline{x}\|)^p} \quad \textit{for} \quad \underline{\alpha} = (\alpha_1, \ldots, \alpha_d), \qquad |\underline{\alpha}| \le r,$$

where $\partial^{\underline{\alpha}} = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_d)^{\alpha_d}$ and $|\underline{\alpha}| = \alpha_1 + \cdots + \alpha_d$.

Let φ be an r-regular scaling function. Then there are $q = 2^d - 1$ orthonormal functions $\psi_1, \ldots, \psi_q \in V_1$ such that the functions $\psi_l(\cdot - \underline{k})$ with $1 \le l \le q$ and $\underline{k} \in \mathbf{Z}^d$ form an orthonormal basis for W_0, where W_0 is the orthogonal complement of V_0 in V_1. Moreover, the functions $\psi_1, \ldots, \psi_q$ can be chosen in such a way that they satisfy the same regularity and decay conditions as φ, i.e., $\psi_l \in C^r(\mathbf{R}^d)$ and for any $p \in \mathbf{N}$ there is a constant $C_{p,r}$, such that

$$|\partial^{\underline{\alpha}}\psi_l(\underline{x})| \le \frac{C_{p,r}}{(1 + \|\underline{x}\|)^p} \qquad \text{for} \quad |\underline{\alpha}| \le r.$$

It follows that the functions $\{2^{dj/2}\psi_l(2^j x - \underline{k}),\ 1 \le l \le q,\ \underline{k} \in \mathbf{Z}^d,\ j \in \mathbf{Z}\}$ form an orthonormal basis for $L^2(\mathbf{R}^d)$. (For details, see, e.g., [6, Chap. 3].)

The functions $\psi_1, \ldots, \psi_q$ are called *wavelets associated with the scaling function* φ.

It follows from the properties of scaling functions that without loss of generality we can assume that

$$(1) \qquad \sum_{\underline{k}\in\mathbf{Z}^d} \varphi(\underline{x} - \underline{k}) = 1 \qquad \text{for all} \quad \underline{x} \in \mathbf{R}^d \qquad \text{and} \qquad \int_{\mathbf{R}^d} \varphi(\underline{x})\, d\underline{x} = 1.$$

2. The Bounds for Upper and Lower Box Dimensions of a Graph

Throughout this section we assume that φ is a 1-regular scaling function on $\mathbf{R}^d$, and that ψ_l, $l = 1, \ldots, 2^d - 1$, are the associated wavelets; moreover, to simplify notation we write $\psi_{l,j,\underline{k}} = (\psi_l)_{j,\underline{k}}$.

For a function $f : \mathbf{R}^d \to \mathbf{R}$ and a parallelepiped $Q = [a_1, b_1] \times \cdots \times [a_d, b_d]$, let

$$\Gamma(f, Q) = \{(\underline{x},\ f(\underline{x})) : \underline{x} \in Q\} \subset \mathbf{R}^{d+1}$$

be the part of the graph of f lying over Q.

2.1. *The Upper Bound for Box Dimensions of a Graph*

It is well known that if a function f satisfies the Hölder condition with exponent α, $0 < \alpha \le 1$, on a d-dimensional parallelepiped, then $\overline{\dim}_b(\Gamma(f, Q)) \le d + 1 - \alpha$ (see, e.g., [5, Chap. 11]). Now we formulate some conditions for the wavelet coefficients of f, which imply the Hölder condition with exponent α for f on Q.

Let $f : \mathbf{R}^d \to \mathbf{R}$ and let $Q \subset \mathbf{R}^d$ be a nondegenerated parallelepiped. Given $0 < \alpha \le 1$ and $0 \le \gamma < 1$ consider the following conditions:

(i) There are a constant $C > 0$ and $n \in \mathbf{N}$ such that for all $\underline{k} \in \mathbf{Z}^d$

$$|(f, \varphi_{0,\underline{k}})| \le C(1 + |\underline{k}|)^n.$$

(ii) There are a constant $C > 0$ and $n \in \mathbf{N}$ such that for all $1 \le l \le 2^d - 1$, $j \ge 0$, and $\underline{k} \in \mathbf{Z}^d$ with $\operatorname{dist}(\underline{k}/2^j, Q) \ge 1/2^{j\gamma}$

$$|(f, \psi_{l,j,\underline{k}})| \le C 2^{-dj/2}(1 + 2^{-j}|\underline{k}|)^n.$$

(iii) There are a constant $C > 0$ and $n \in \mathbf{N}$ such that for all $1 \le l \le 2^d - 1$, $j \ge 0$, and $\underline{k} \in \mathbf{Z}^d$ with $\operatorname{dist}(\underline{k}/2^j, Q) < 1/2^{j\gamma}$

$$|(f, \psi_{l,j,\underline{k}})| \le C 2^{-j(\alpha+d/2)} \left(1 + 2^j \operatorname{dist}\left(\frac{\underline{k}}{2^j}, Q\right)\right)^n.$$

Note that the decay conditions imposed on φ and ψ_l imply that if f is of polynomial growth, i.e., there are $C > 0$ and $m \in \mathbf{N}$ such that

$$|f(\underline{x})| \le C(1 + \|\underline{x}\|)^m \qquad \text{for} \quad \underline{x} \in \mathbf{R}^d,$$

then f satisfies conditions (i) and (ii).

Lemma 2.1. *Let $0 < \alpha < 1$, $0 \le \gamma < 1$, and let $Q \subset \mathbf{R}^d$ be a nondegenerated parallelepiped. Suppose that $f : \mathbf{R}^d \to \mathbf{R}$ satisfies conditions* (i), (ii), *and* (iii). *Then f satisfies the Hölder condition with exponent α on Q.*

Proof. Using the estimate

$$|\psi_l(\underline{x}) - \psi_l(\underline{y})| \le \min\left(|\psi_l(\underline{x})| + |\psi_l(\underline{y})|, \|\underline{x} - \underline{y}\| \sup_{0 \le \theta \le 1} \sum_{i=1}^{d} \left| \frac{\partial \psi_l}{\partial t_i}(\theta \underline{x} + (1-\theta)\underline{y}) \right| \right),$$

the decay conditions imposed on ψ_l, and condition (iii), we check that the function

$$\sum_{l=1}^{2^d-1}\sum_{j\geq 0}\sum_{\operatorname{dist}(\underline{k}/2^j,Q)<2^{-j\gamma}}(f,\psi_{l,j,\underline{k}})\psi_{l,j,\underline{k}}(\cdot)$$

satisfies the Hölder condition with exponent α on Q.

Moreover, the decay properties of φ and ψ_l, and conditions (i) and (ii), imply that

$$\sum_{\underline{k}\in\mathbf{Z}^d}(f,\varphi_{0,\underline{k}})\varphi_{0,\underline{k}}(\cdot)+\sum_{l=1}^{2^d-1}\sum_{j\geq 0}\sum_{\operatorname{dist}(\underline{k}/2^j,Q)\geq 2^{-j\gamma}}(f,\psi_{l,j,\underline{k}})\psi_{l,j,\underline{k}}(\cdot)$$

satisfies the Lipschitz condition on Q.

The details of the calculations are omitted, as they are analogous to the proof of Theorem 6.5 in [6]. ■

Theorem 2.2. *Let* $0<\alpha\leq 1$, $0\leq\gamma<1$, *and let* $Q\subset\mathbf{R}^d$ *be a nondegenerated parallelepiped. Suppose that* $f:\mathbf{R}^d\to\mathbf{R}$ *satisfies conditions* (i), (ii), *and* (iii). *Then* $\overline{\dim}_b(\Gamma(f,Q))\leq d+1-\alpha$.

Proof. Suppose $\alpha<1$. Then it follows from Lemma 2.1 that f satisfies the Hölder condition with exponent α on Q, which gives $\overline{\dim}_b(\Gamma(f,Q))\leq d+1-\alpha$. If $\alpha=1$, then for any $\varepsilon>0$ f satisfies condition (iii) with $\alpha'=1-\varepsilon$, which implies $\overline{\dim}_b(\Gamma(f,Q))\leq d+\varepsilon$ for any $\varepsilon>0$, and finally $\overline{\dim}_b(\Gamma(f,Q))\leq d$. ■

Remark. If the scaling function φ and the associated wavelets ψ_l are of compact support (e.g., if φ is the tensor product of the compactly supported orthonormal scaling functions of one variable), then assumptions (i) and (ii) of Theorem 2.2 can be omitted, and assumption (iii) can be replaced by the following:

(iii′) There is a constant $C>0$ such that for all $j\geq 0$ and $\underline{k}\in\mathbf{Z}^d$ with $\operatorname{supp}\psi_{l,j,\underline{k}}\cap\operatorname{int}Q\neq\emptyset$

$$|(f,\psi_{l,j,\underline{k}})|\leq C2^{-j(\alpha+d/2)}.$$

2.2. *The Lower Bound for Box Dimensions of a Graph*

The next theorem gives an estimate from below for the lower box dimension of $\Gamma(f,Q)$. Recall that $I_{j,\underline{k}}$ denotes the $\underline{k}$th dyadic cube of order j. For a function f denote by $\operatorname{osc}(f,I_{j,\underline{k}})$ the oscillation of f over $I_{j,\underline{k}}$, i.e.,

$$\operatorname{osc}(f,I_{j,\underline{k}})=\sup_{\underline{x}\in I_{j,\underline{k}}}f(\underline{x})-\inf_{\underline{x}\in I_{j,\underline{k}}}f(\underline{x}).$$

Moreover, for the given parallelepiped Q define

$$\operatorname{Osc}_Q(f,j)=\sum_{I_{j,\underline{k}}\subset Q}\operatorname{osc}(f,I_{j,\underline{k}}).$$

Before the formulation of the theorem, let us introduce some notation: for two vectors $\underline{n} = (n_1, \ldots, n_d), \underline{m} = (m_1, \ldots, m_d)$ we write $\underline{n} \wedge \underline{m} = (\min(n_1, m_1), \ldots, \min(n_d, m_d))$, $\underline{n} \vee \underline{m} = (\max(n_1, m_1), \ldots, \max(n_d, m_d))$, and $\underline{n} \leq \underline{m}$ iff $n_i \leq m_i$ for all $i = 1, \ldots, d$.

Theorem 2.3. *Let $Q \subset \mathbf{R}^d$ be a nondegenerated parallelepiped and let $f : \mathbf{R}^d \to \mathbf{R}$ be a function of polynomial growth, continuous on Q, $0 \leq \beta \leq 1$. If*

$$\liminf_{j\to\infty} \left(2^{j(\beta-d/2)} \sum_{l=1}^{q} \sum_{I_{j,\underline{k}} \subset Q} |(f, \psi_{l,j,\underline{k}})| \right) > 0, \tag{2}$$

then

$$\underline{\dim}_b(\Gamma(f, Q)) \geq d + 1 - \beta.$$

For the proof of Theorem 2.3, we need the following lemma (see [6, Prop. 6.7]):

Lemma 2.4. *There exist constants c_1 and c_2 such that for any $j \geq 0$ and for any $g_j(x) = \sum_{l=1}^{q} \sum_{\underline{k}\in\mathbf{Z}^d} a_{l,j,\underline{k}} \psi_{l,j,\underline{k}}(x)$ we have*:

$$c_1 ||g_j||_1 \leq 2^{-jd/2} \sum_{l=1}^{q} \sum_{\underline{k}\in\mathbf{Z}^d} |a_{l,j,\underline{k}}| \leq c_2 ||g_j||_1.$$

Proof of Theorem 2.3. Let T_j be the orthogonal projection onto V_j, i.e.,

$$T_j f(\underline{x}) := \int_{\mathbf{R}^d} K_j(\underline{x}, \underline{y}) f(y)\, dy,$$

where

$$K_j(\underline{x}, \underline{y}) = \sum_{\underline{k}\in\mathbf{Z}^d} \varphi_{j,\underline{k}}(\underline{x}) \overline{\varphi}_{j,\underline{k}}(\underline{y}) \qquad \text{for} \quad \underline{x}, \underline{y} \in \mathbf{R}^d,$$

and put $Q_j = \bigcup_{I_{j,\underline{k}} \subset Q} I_{j,\underline{k}}$. Denote $a_{l,j,\underline{k}} = (f, \psi_{l,j,\underline{k}})$; now we have

$$T_{j+1} f - T_j f = \sum_{l=1}^{q} \sum_{\underline{k}\in\mathbf{Z}^d} a_{l,j,\underline{k}} \psi_{l,j,\underline{k}}.$$

Applying Lemma 2.4 we get

$$\begin{aligned} 2^{-jd/2} \sum_{l=1}^{q} \sum_{I_{j,\underline{k}} \subset Q} |a_{l,j,\underline{k}}| \; &\leq \; C \int_{\mathbf{R}^d} \left| \sum_{l=1}^{q} \sum_{I_{j,\underline{k}} \subset Q} a_{l,j,\underline{k}} \psi_{l,j,\underline{k}}(\underline{x}) \right| d\underline{x} \\ &\leq \; C \left(\int_{Q_j} |T_{j+1} f(\underline{x}) - T_j f(\underline{x})|\, d\underline{x} \right. \end{aligned}$$

$$+ \int_{Q_j} \sum_{l=1}^{q} \sum_{I_{j,\underline{k}} \not\subset Q} |a_{l,j,\underline{k}} \psi_{l,j,\underline{k}}(\underline{x})| \, d\underline{x}$$
$$\left. + \int_{\mathbf{R}^d \setminus Q_j} \sum_{l=1}^{q} \sum_{I_{j,\underline{k}} \subset Q} |a_{l,j,\underline{k}} \psi_{l,j,\underline{k}}(\underline{x})| \, d\underline{x} \right) .$$

We will show that each of the terms appearing on the right-hand side of this inequality can be bounded by $2^{-(d+1)j} N(\Gamma(f, Q), j+1))$.

At first, using (1), we get

$$\begin{aligned}
\int_{Q_j} | f(\underline{x}) - T_j f(\underline{x})| \, d\underline{x} &= \int_{Q_j} \left| f(\underline{x}) - \sum_{\underline{k} \in \mathbf{Z}^d} (f, \varphi_{j,\underline{k}}) \varphi_{j,\underline{k}}(\underline{x}) \right| d\underline{x} \\
&= \int_{Q_j} \left| f(\underline{x}) 2^{-jd/2} \sum_{\underline{k} \in \mathbf{Z}^d} \varphi_{j,\underline{k}}(\underline{x}) - \sum_{\underline{k} \in \mathbf{Z}^d} (f, \varphi_{j,\underline{k}}) \varphi_{j,\underline{k}}(\underline{x}) \right| d\underline{x} \\
&\le \int_{Q_j} \left(\sum_{\underline{k} \in \mathbf{Z}^d} |\varphi_{j,\underline{k}}(\underline{x})| \int_{\mathbf{R}^d} | f(\underline{x}) - f(\underline{s})| |\varphi_{j,\underline{k}}(\underline{s})| d\underline{s} \right) d\underline{x} \\
&= \left(\int_{Q_j} \int_{Q_j} + \int_{Q_j} \int_{\mathbf{R}^d \setminus Q_j} \right) \\
&\quad \times \sum_{\underline{k} \in \mathbf{Z}^d} | f(\underline{x}) - f(\underline{s})| |\varphi_{j,\underline{k}}(\underline{x})| |\varphi_{j,\underline{k}}(\underline{s})| \, d\underline{x} \, d\underline{s}.
\end{aligned}$$

The function f has polynomial growth, so for some $C > 0$ and $\lambda \in \mathbf{N}$

$$| f(\underline{x})| \le C(1 + \|\underline{x}\|)^{\lambda}.$$

Using this property of f and the decay conditions imposed on φ we now get for some $p \in \mathbf{N}$ big enough

$$\begin{aligned}
&\int_{Q_j} \int_{\mathbf{R}^d \setminus Q_j} \sum_{\underline{k} \in \mathbf{Z}^d} | f(\underline{x}) - f(\underline{s})| |\varphi_{j,\underline{k}}(\underline{x})| |\varphi_{j,\underline{k}}(\underline{s})| \, d\underline{x} \, d\underline{s} \\
&\qquad \le C \sum_{I_{j,\underline{n}} \subset Q} \sum_{I_{j,\underline{m}} \not\subset Q} \sum_{\underline{k} \in \mathbf{Z}^d} \int_{I_{j,\underline{n}}} \int_{I_{j,\underline{m}}} (1 + \|\underline{x}\|)^{\lambda} (1 + \|\underline{s}\|)^{\lambda} |\varphi_{j,\underline{k}}(\underline{x})| |\varphi_{j,\underline{k}}(\underline{s})| \, d\underline{x} \, d\underline{s} \\
&\qquad \le \frac{C}{2^{dj}} \sum_{I_{j,\underline{n}} \subset Q} \sum_{I_{j,\underline{m}} \not\subset Q} \sum_{\underline{k} \in \mathbf{Z}^d} \prod_{i=1}^{d} \frac{(1 + 2^{-j} |m_i|)^{\lambda}}{(1 + |n_i - k_i|)^p (1 + |m_i - k_i|)^p} \\
&\qquad \le \frac{C}{2^{dj}} \sum_{I_{j,\underline{n}} \subset Q} \sum_{I_{j,\underline{m}} \not\subset Q} \prod_{i=1}^{d} \frac{(1 + 2^{-j} |m_i|)^{\lambda}}{(1 + |n_i - m_i|)^{p-1}} \le \frac{C}{2^j}.
\end{aligned}$$

On the other hand, we have for $\underline{x} \in I_{j,\underline{n}}$, $\underline{s} \in I_{j,\underline{m}}$

$$| f(\underline{x}) - f(\underline{s})| \le \sum_{\underline{m} \wedge \underline{n} \le \underline{l} \le \underline{m} \vee \underline{n}} \mathrm{osc}(f, I_{j,\underline{l}}).$$

Using this inequality and the decay properties of φ we get

$$
\begin{aligned}
&\int_{Q_j}\int_{Q_j}\sum_{\underline{k}\in\mathbf{Z}^d}|f(\underline{x})-f(\underline{s})||\varphi_{j,\underline{k}}(\underline{x})||\varphi_{j,\underline{k}}(\underline{s})|\,d\underline{x}\,d\underline{s}\\
&\qquad=\sum_{I_{j,\underline{m}},I_{j,\underline{n}}\subset Q}\sum_{\underline{k}\in\mathbf{Z}^d}\int_{I_{j,\underline{m}}}\int_{I_{j,\underline{n}}}|f(\underline{x})-f(\underline{s})||\varphi_{j,\underline{k}}(\underline{x})||\varphi_{j,\underline{k}}(\underline{s})|\,d\underline{x}\,d\underline{s}\\
&\qquad\le\sum_{I_{j,\underline{m}},I_{j,\underline{n}}\subset Q}\sum_{\underline{k}\in\mathbf{Z}^d}\sum_{\underline{m}\wedge\underline{n}\le\underline{l}\le\underline{m}\vee\underline{n}}\operatorname{osc}(f,I_{j,\underline{l}})\int_{I_{j,\underline{m}}}\int_{I_{j,\underline{n}}}|\varphi_{j,\underline{k}}(\underline{x})||\varphi_{j,\underline{k}}(\underline{s})|\,d\underline{x}\,d\underline{s}\\
&\qquad\le\frac{C}{2^{dj}}\sum_{I_{j,\underline{l}}\subset Q}\operatorname{osc}(f,I_{j,\underline{l}})\cdot\left(\sum_{\underline{m}\wedge\underline{n}\le\underline{l}\le\underline{m}\vee\underline{n}}\sum_{\underline{k}\in\mathbf{Z}^d}\prod_{i=1}^{d}\frac{1}{(1+|n_i-k_i|)^p(1+|m_i-k_i|)^p}\right)\\
&\qquad\le\frac{C}{2^{dj}}\sum_{I_{j,\underline{l}}\subset Q}\operatorname{osc}(f,I_{j,\underline{l}})\sum_{\underline{m}\wedge\underline{n}\le\underline{l}\le\underline{m}\vee\underline{n}}\prod_{i=1}^{d}\frac{1}{(1+|n_i-m_i|)^{p-1}}\\
&\qquad\le\frac{C}{2^{dj}}\sum_{I_{j,\underline{l}}\subset Q}\operatorname{osc}(f,I_{j,\underline{l}})=\frac{C}{2^{dj}}\operatorname{Osc}_Q(f,j).
\end{aligned}
$$

The function f is continuous on Q, so

$$\operatorname{Osc}_Q(f,j)\le\frac{1}{2^j}N(\Gamma(f,Q),j).$$

Moreover, note that

$$2^{dj}\le CN(\Gamma(f,Q),j),$$

so we obtain

$$\int_{Q_j}|f(\underline{x})-T_jf(\underline{x})|\,d\underline{x}\le\frac{C}{2^{(d+1)j}}N(\Gamma(f,Q),j).$$

The bounds for the growth of f imply that

$$|a_{l,j,\underline{k}}|\le\frac{C}{2^{dj/2}}(1+2^{-j}|\underline{k}|)^{\lambda}.$$

Therefore, using the decay properties of ψ_l, we get

$$
\begin{aligned}
\sum_{I_{j,\underline{k}}\not\subset Q}\int_{Q_j}|a_{l,j,\underline{k}}\psi_{l,j,\underline{k}}(\underline{x})|\,d\underline{x}&\le\frac{C}{2^{dj}}\sum_{I_{j,\underline{k}}\not\subset Q}\sum_{I_{j,\underline{n}}\subset Q}\prod_{i=1}^{d}\frac{(1+2^{-j}|k_i|)^{\lambda}}{(1+|n_i-k_i|)^p}\\
&\le\frac{C}{2^j}\le\frac{C}{2^{(d+1)j}}N(\Gamma(f,Q),j),
\end{aligned}
$$

and

$$\begin{aligned}\sum_{I_{j,\underline{k}}\subset Q}\int_{\mathbf{R}^d\setminus Q_j}|a_{l,j,\underline{k}}\psi_{l,j,\underline{k}}(x)|\,dx &\le \frac{C}{2^{dj/2}}\sum_{I_{j,\underline{k}}\subset Q}\sum_{I_{j,\underline{n}}\not\subset Q}\int_{I_{j,\underline{n}}}|\psi_{l,j,k}(x)|\,dx\\ &\le \frac{C}{2^{dj}}\sum_{I_{j,\underline{k}}\subset Q}\sum_{I_{j,\underline{n}}\not\subset Q}\prod_{i=1}^{d}\frac{1}{(1+|n_i-k_i|)^p}\\ &\le \frac{C}{2^{(d+1)j}}N(\Gamma(f,Q),j).\end{aligned}$$

The above calculations now give

$$\begin{aligned}2^{-jd/2}\sum_{l=1}^{q}\sum_{I_{j,\underline{k}}\subset Q}|a_{l,j,\underline{k}}| \le C\Bigg(&\int_{Q_j}|T_j f(\underline{x})-f(\underline{x})|\,d\underline{x}+\int_{Q_{j+1}}|T_{j+1}f(\underline{x})-f(\underline{x})|\,d\underline{x}\\ &+\int_{Q_j}\sum_{l=1}^{q}\sum_{I_{j,\underline{k}}\not\subset Q}|a_{l,j,\underline{k}}\psi_{l,j,\underline{k}}(\underline{x})|d\underline{x}\\ &+\int_{\mathbf{R}^d\setminus Q_j}\sum_{l=1}^{q}\sum_{I_{j,\underline{k}}\subset Q}|a_{l,j,\underline{k}}\psi_{l,j,\underline{k}}(\underline{x})|\,d\underline{x}\Bigg)\\ \le C\,&\frac{N(\Gamma(f,Q),j+1)}{2^{(d+1)(j+1)}}.\end{aligned}$$

This inequality and assumption (2) imply that

$$N(\Gamma(f,Q),j)\ge C2^{j(d+1-\beta)}$$

and

$$\liminf_{j\to\infty}\frac{\log N(\Gamma(f,Q),j)}{j\log 2}\ge d+1-\beta,$$

so the proof is complete. ∎

Remark. If the wavelets ψ_l are of compact support, then the assumption on the growth of f in Theorem 2.3 can be replaced by the condition that f is bounded in some neighborhood of Q.

2.3. *Local Upper and Lower Box Dimensions of Graphs*

Let $f:\mathbf{R}^d\to\mathbf{R}$ and $\underline{x}\in\mathbf{R}^d$. The upper and lower box dimensions of the graph of f at point $\underline{x}$ are defined by the formulas

$$\overline{\dim}_b(f;\underline{x})=\limsup_{\operatorname{diam}(Q)\to 0}\overline{\dim}_b(\Gamma(f,Q)),$$

$$\underline{\dim}_b(f;\underline{x})=\liminf_{\operatorname{diam}(Q)\to 0}\underline{\dim}_b(\Gamma(f,Q)),$$

respectively, where lim sup and lim inf are taken over all parallelepipeds with edges parallel to the corresponding coordinate axes and containing a given point $\underline{x}$ in its interior and $\mathrm{diam}(Q)$ is the diameter of Q (see also [3]).

As a corollary of the results of this section we get an estimate of the local upper and lower box dimensions of the graph of a function in terms of the asymptotics of its wavelet coefficients.

Proposition 2.5. *Let $\tau_{j,\underline{k}}$ be the center of the cube $I_{j,\underline{k}}$ and let $f : \mathbf{R}^d \to \mathbf{R}$ be a function of polynomial growth. Let $\alpha, \beta : \mathbf{R}^d \to [0, 1]$.*

If $\underline{x}$ is a point of the lower semicontinuity of $\alpha(\cdot)$ and there are a neighborhood U of $\underline{x}$ and a constant $C > 0$ such that

(3) $$|(f, \psi_{l,j,\underline{k}})| \le C 2^{-dj/2 - j\alpha(\tau_{j,\underline{k}})} \qquad \text{for} \quad l = 1, \dots, 2^d - 1, \quad I_{j,\underline{k}} \subset U,$$

then

$$\overline{\dim}_b(f; \underline{x}) \le d + 1 - \alpha(\underline{x}).$$

If f is continuous in a neighborhood V of $\underline{x}$, $\underline{x}$ is a point of the upper semicontinuity of $\beta(\cdot)$, and there are a neighborhood U of $\underline{x}$ and a constant $C > 0$ such that

(4) $$|(f, \psi_{l,j,\underline{k}})| \ge C 2^{-dj/2 - j\beta(\tau_{j,\underline{k}})} \qquad \text{for} \quad l = 1, \dots, 2^d - 1, \quad I_{j,\underline{k}} \subset U,$$

then

$$\underline{\dim}_b(f; \underline{x}) \ge d + 1 - \beta(\underline{x}).$$

Proof. Let us prove the upper bound for $\overline{\dim}_b(f; \underline{x})$. As $\Gamma(f, Q) \subset \mathbf{R}^{d+1}$, we have $\overline{\dim}_b(f; \underline{x}) \le d + 1$, so the assertion holds if $\alpha(\underline{x}) = 0$.

Assume now that $\underline{x}$ is a point of the lower semicontinuity of $\alpha(\cdot)$ and $\alpha(\underline{x}) > 0$. Let condition (3) be satisfied on a neighborhood U of $\underline{x}$. For $0 < \eta < \alpha(\underline{x})$, let Q_η be a cube containing $\underline{x}$ in its interior such that

$$\alpha(\underline{y}) \ge \alpha(\underline{x}) - \eta \qquad \text{for} \quad \underline{y} \in Q_\eta.$$

Let $\tau_{j,\underline{k}} \in Q_\eta \cap U$. Then

$$|(f, \psi_{l,j,\underline{k}})| \le C 2^{-dj/2 - j\alpha(\tau_{j,\underline{k}})} \le C 2^{-dj/2 - j(\alpha(\underline{x}) - \eta)}.$$

Consequently, if $Q \subset Q_\eta \cap U$, then the assumptions of Theorem 2.2 are satisfied with $\alpha = \alpha(\underline{x}) - \eta$, which gives

$$\overline{\dim}_b(\Gamma(f, Q)) \le d + 1 - \alpha(\underline{x}) + \eta,$$

so applying the definition of $\overline{\dim}_b(f; \underline{x})$ we get

$$\overline{\dim}_b(f; \underline{x}) \le d + 1 - \alpha(\underline{x}).$$

The lower bound of $\underline{\dim}_b(f; \underline{x})$ is proved in an analogous way, with the use of Theorem 2.3. ∎

Corollary 2.6. Let $f : \mathbf{R}^d \to \mathbf{R}$ be a continuous function of polynomial growth. If conditions (3) and (4) are satisfied in some neighborhood of $\underline{x}$ with $\alpha(\cdot) = \beta(\cdot)$ and $\underline{x}$ is a point of continuity of $\alpha(\cdot)$, then

$$\overline{\dim}_b(f; \underline{x}) = \underline{\dim}_b(f; \underline{x}) = d + 1 - \alpha(\underline{x}).$$

If the function $\alpha(\cdot)$ is Riemann integrable on a nondegenerated parallelepiped Q, then the above equality holds almost everywhere (in the Lebesgue sense) on Q.

Remark. If the wavelets under consideration are of compact support, then the assumptions concerning the growth of f can be omitted.

3. Box Dimensions and Smoothness of Function

It would be interesting to have an intrinsic characterization of the class of functions with given box dimensions. An attempt to find such a characterization has been made in [4]. Those authors have stated the following formula for the upper box dimension of a given continuous function $f : [0, 1] \to \mathbf{R}$:

$$\text{(5)} \qquad \overline{\dim}_b(\Gamma_f) = 2 - \gamma \qquad \text{iff} \quad \gamma = \sup\{\alpha \in (0, 1) : \; B^\alpha_{1,\infty} \ni f\},$$

where $B^\alpha_{1,\infty}$ is the space of a function satisfying the Hölder condition on $[0, 1]$ with exponent α in the L^1-norm. ($\Gamma_f := \Gamma(f, [0, 1])$ to simplify the notation.) However, there is a mistake in their proof and the example presented below shows that the above formula for $\overline{\dim}_b(\Gamma_f)$ does not hold.

Following [4], let V^α, $0 < \alpha < 1$, be the space of all measurable functions f on $[0, 1]$, such that

$$\sup_{j \geq 0} \frac{\mathrm{Osc}_{[0,1]}(f, j)}{2^{j(1-\alpha)}} < \infty.$$

Then (see Theorem 3.1 of [4]), for a continuous function $f : [0, 1] \to \mathbf{R}$ and $\gamma \in (0, 1)$, we have

$$\overline{\dim}_b(\Gamma_f) = 2 - \gamma \qquad \text{iff} \quad \gamma = \sup\{\alpha \in (0, 1); \; V^\alpha \ni f\}.$$

Let us recall the definition of the Besov space $B^\alpha_{1,\infty}$:

$$f \in B^\alpha_{1,\infty} \quad \Leftrightarrow \quad f \in L^1[0, 1] \qquad \text{and} \qquad \sup_{0<\delta\leq 1} \frac{\omega_1(f; \delta)}{\delta^\alpha} < \infty,$$

where $\omega_1(f, \delta) = \sup_{0<h<\delta} \int_0^{1-h} |f(x) - f(x + h)|\, dx$ is the modulus of smoothness of f in the L^1-norm.

For a continuous function f denote

$$\begin{aligned} \gamma_V(f) &= \sup\{\alpha \in (0, 1) : \; V^\alpha \ni f\}, \\ \gamma_B(f) &= \sup\{\alpha \in (0, 1) : \; B^\alpha_{1,\infty} \ni f\}. \end{aligned}$$

In terms of $\gamma_V(f)$ and $\gamma_B(f)$, formula (5) means that $\gamma_V(f) = \gamma_B(f)$. Now we give an example of a function f_β with $\gamma_V(f_\beta) < \gamma_B(f_\beta)$.

Let $s^{j,k}$ denote the kth Faber–Schauder function from the jth generation, i.e., let $s^{j,k} : [0,1] \to \mathbf{R}$ be given by the formula

$$s^{j,k}(x) = \max(0,\ 1 - |2^{j+1}x - (2k-1)|), \qquad j = 0, 1, \dots; \quad k = 1, 2, \dots, 2^j.$$

For fixed β, $0 < \beta < 1$, consider the function $f_\beta : [0,1] \to \mathbf{R}$ defined as

$$f_\beta(x) = \sum_{j=0}^{\infty} 2^{-j\beta} \sum_{k=1}^{2^j} s^{2j,2^j k}(x).$$

Note that in the $2j$th generation we have 2^{2j} functions $s^{2j,k}$, but for the construction of f_β we take every 2^jth function from this generation, so only the 2^j functions from the $2j$th generation are used. The function f_β is not only continuous, but it satisfies the Hölder condition with exponent $\beta/2$ in the uniform norm as well.

Let us start with the estimation from below of $\gamma_B(f_\beta)$. For the modulus of smoothness we have

$$\omega_1(g, \delta) \le 2||g||_1, \qquad \omega_1(g, \delta) \le \delta||g'||_1.$$

Applying these inequalities to the functions $s^{j,k}$ we obtain

$$\begin{aligned}\omega_1\left(f_\beta, \frac{1}{2^m}\right) &\le \sum_{j=0}^{\infty} 2^{-j\beta} \sum_{k=1}^{2^j} \omega_1\left(s^{2j,2^j k}, \frac{1}{2^m}\right) \le \sum_{j=0}^{\infty} 2^{j(1-\beta)} \min\left(\frac{1}{2^{2j}}, \frac{1}{2^m}\right) \\ &\le \sum_{0 \le j < m/2} 2^{j(1-\beta-m)} + \sum_{j \ge m/2} 2^{-j(1+\beta)} \le C 2^{-m(\beta+1)/2},\end{aligned}$$

where C depends only on β. This means that $f_\beta \in B^{(\beta+1)/2}_{1,\infty}$, and therefore

$$\gamma_B(f_\beta) \ge \frac{\beta+1}{2}.$$

Now we estimate from above $\gamma_V(f_\beta)$. For $m \ge 0$ and $0 \le l \le 2^m - 1$ denote

$$x^{(1)}_{m,l} = \frac{2^{m+1}(l+1) - 2}{2^{2m+1}}, \qquad x^{(2)}_{m,l} = \frac{2^{m+1}(l+1) - 1}{2^{2m+1}}, \qquad x^{(3)}_{m,l} = \frac{2^{m+1}(l+1)}{2^{2m+1}},$$

and

$$M_{m,l} = \max(|f_\beta(x^{(2)}_{m,l}) - f_\beta(x^{(1)}_{m,l})|,\ |f_\beta(x^{(2)}_{m,l}) - f_\beta(x^{(3)}_{m,l})|).$$

As $x^{(i)}_{m,l} \in I_{m,l}$, $i = 1, 2, 3$, we have $\operatorname{osc}(f_\beta, I_{m,l}) \ge M_{m,l}$. Note that for all $j > m$ and $1 \le k \le 2^{2j}$ we have $s^{2j,k}(x^{(i)}_{m,l}) = 0$, $i = 1, 2, 3$; moreover, the function

$$\sum_{j=0}^{m-1} 2^{-j\beta} \sum_{k=1}^{2^j} s^{2j,2^j k}(x)$$

is linear on $[x_{m,l}^{(1)}, x_{m,l}^{(3)}]$. In addition,

$$s^{2m,2^m k}(x_{m,l}^{(i)}) = 0 \qquad \text{for} \quad 1 \le k \le 2^m, \quad k \ne l+1, \quad i = 1, 2, 3,$$

$$s^{2m,2^m(l+1)}(x_{m,l}^{(2)}) = 1 \qquad \text{and} \qquad s^{2m,2^m(l+1)}(x_{m,l}^{(1)}) = s^{2m,2^m(l+1)}(x_{m,l}^{(3)}) = 0.$$

This implies that $M_{m,l} \ge 2^{-\beta m}$, and consequently

$$\operatorname{osc}(f_\beta, I_{m,l}) \ge 2^{-\beta m},$$

which leads to

$$\operatorname{Osc}_{[0,1]}(f_\beta, m) \ge 2^{m(1-\beta)}.$$

Now, if $f \in V_\alpha$, then

$$\sup_{m\in\mathbf{N}} 2^{m(\alpha-\beta)} \le \sup_{m\in\mathbf{N}} 2^{m(\alpha-1)} \operatorname{Osc}_{[0,1]}(f_\beta, m) < \infty,$$

which means that $\alpha \le \beta$; therefore,

$$\gamma_V(f_\beta) \le \beta.$$

Thus we have obtained

$$\gamma_V(f_\beta) \le \beta < \frac{\beta+1}{2} \le \gamma_B(f_\beta),$$

and therefore formula (5) is not correct.

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