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CONSTRUCTIVE APPROXIMATION

Blossoming: A Geometrical Approach

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Abstract. A geometrical approach of a notion of blossom for piecewise smooth Chebyshev functions is developed by considering convenient intersections of osculating flats. A subblossoming principle allows us to obtain all the expected properties and leads to the notion of blossom for splines based on a given piecewise smooth Chebyshev function.

1. Introduction

The now well-know theory of blossoming for polynomial functions and splines, first introduced by L. Ramshaw [34], [35], permits a particularly elegant treatment of the different tools and algorithms found in traditional CAGD (*control points*, *de Casteljau and de Boor algorithms*, *knot insertion*, *subdivision*, *recurrence relations* . . .). Recall that the *blossom* of a polynomial function F of degree less than or equal to k is the unique function f of k variables which is symmetric, affine with respect to each variable and which, restricted to the diagonal of $\mathbf{R}^k$, gives F. Let us mention the following fundamental result: two polynomial functions F_1, F_2 of degree less than or equal to k have a C^s contact ($s \leq k$) at $a \in \mathbf{R}$ iff their blossoms f_1, f_2 coincide on any k-tuple containing at least $(k-s)$ times the point a [34]. This contact theorem is the key tool for defining the blossom of a polynomial spline, which has the same properties as that of a polynomial function, except that it is defined only for particular k-tuples, said to be *admissible* with respect to the corresponding knot vector [27].

Up to now, two main approaches have been developed in order to extend the theory of blossoming beyond the strict framework of polynomial functions or splines. On the one hand, a geometrical one, at the root of which we find a remarkable geometrical property of polynomial blossom. To be more precise, when a polynomial function F of degree k is nondegenerate (i.e., when the affine space spanned by its image is of dimension k), its blossom can be interpreted in geometric terms as follows. Given r distinct real numbers $\tau_1, \ldots, \tau_r$ and r positive integers $\mu_1, \ldots, \mu_r$ whose sum is equal to k, consider the k-tuple $\mathcal{T} = (\tau_1^{\mu_1} \cdots \tau_r^{\mu_r})$, where the notation $\tau_i^{\mu_i}$ means that the point τ_i is repeated

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μ_i times. Then, the value at $\mathcal{T}$ of the blossom f of F satisfies

$$\{f(\mathcal{T})\} = \bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i} F(\tau_i), \tag{1.1}$$

$\mathrm{Osc}_i\, F(t)$ standing for the osculating flat of order i of F at t. A similar interpretation exists for nondegenerate polynomial splines [27].

The possibility of defining a blossom by considering intersections of osculating flats as in (1.1) had first been pointed out by H.-P. Seidel [37] for *geometrically continuous polynomial splines*, then it has been adapted in the case of Q-splines [19], [27]. The same idea has also been used by H. Pottmann [32], [33], [39] (see also M.-L. Mazure and H. Pottmann [30], M.-L. Mazure [24], [26]) in order to develop the blossoming theory for *extended Chebyshev spaces* which, in one variable, appear like the natural generalization of polynomial spaces. Moreover, within this new framework, the blossoming principle provides a characterization of the C^s contact between two functions belonging to *the same* extended Chebyshev space, which can be stated exactly as in the case of polynomials. Consequently, all the tools and results known for parametric polynomial splines do exist for parametric splines based on *a single* given extended Chebyshev space.

On the other hand, an algebraic approach can be derived from the classical formula given by C. de Boor and G. Fix [6] for calculating the coefficients of a polynomial spline of degree k in the B-spline basis. Actually, when the multiplicity at each knot t_i is equal to one, this formula leads to the following expression of the value at $\mathcal{T} = (t_{j+1}, \ldots, t_{j+k})$ of the blossom s of such a spline S:

$$s(\mathcal{T}) = \sum_{i=0}^{k} S^{(i)}(a)\,(-1)^{k-i}\,\Psi_{\mathcal{T}}^{(k-i)}(a), \tag{1.2}$$

where a is an arbitrary point in $]t_j, t_{j+k+1}[$ and $\Psi_{\mathcal{T}}$ stands for the unique polynomial of degree k which vanishes on $\mathcal{T}$ and satisfies $\Psi_{\mathcal{T}}{}^{(k)} \equiv (-1)^k$, i.e., $\Psi_{\mathcal{T}}(t) = (t_{j+1} - t) \ldots (t_{j+k} - t)/k!$.

Recently, P. J. Barry has defined the blossom for a spline each segment of which belongs to an arbitrary extended Chebyshev space, through an extension of the de Boor–Fix formula, the ordinary derivatives involved in (1.2) now being replaced by differential operators related to each section [2]. This is possible as soon as the connections (with respect to these differential operators) are expressed by means of *totally positive* matrices, the underlying reason being that, under a total positivity assumption, the number of zeros of a nonzero function belonging to some related $(k+1)$-dimensional space is bounded by k. P. J. Barry's work is in keeping with the general context of duality between piecewise smooth spaces investigated by M.-L. Mazure and P.-J. Laurent [28], [29] which enables the interpretation of the blossoming principle through the notions of bilinear form and reproducing function.

The approach of blossom that we propose here is a geometrical one: hence, osculating flats will be our basic tools. In particular, we show in Section 2 that it is the relevant geometrical notion to express the (possibly left or right) C^s contact between *geometrically regular functions of order* k, that is to say, functions which are smooth except at a finite number of points, their left and right derivatives up to order k in these points being linked by lower triangular matrices with positive diagonal elements, and for which the k first (left or right) derivatives are everywhere linearly independent. In Section 3, such a function Φ is said to be *a piecewise smooth Chebyshev function of order k* when, whatever the k-tuple $\mathcal{T} = (\tau_1^{\mu_1} \cdots \tau_r^{\mu_r})$ may be, the corresponding intersection $\bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i} \Phi(\tau_i)$ consists of a single point: in a natural way, this point is labeled $\varphi(\mathcal{T})$ and φ is called the blossom of Φ. Section 4 is devoted to the fundamental *subblossoming principle*: for any fixed a, the function $t \mapsto \varphi(at^{k-1})$ is a piecewise smooth Chebyshev function of order $k-1$ with values in $\mathrm{Osc}_{k-1}\Phi(a)$. This is the key tool to prove that the blossom behaves as in the polynomial case, except that the affinity with respect to each variable is now replaced by a pseudo-affinity property. In particular, the possibility of characterizing the (left or right) contact through the blossom enables the definition of blossoms for splines based on a given piecewise smooth Chebyshev function, which is the object of Section 5. Finally, in Section 6, thanks to the result of P. J. Barry on the number of zeros mentioned above [2], we give sufficient conditions to construct piecewise smooth Chebyshev functions. Let us emphasize the fact that all the results obtained in the present paper can be applied in the general case of splines whose sections belong to *different* extended Chebyshev spaces, whereas all the previous papers based on a similar geometrical approach considered only splines built from a single extended Chebyshev space.

2. Splines Based on a Geometrically Regular Function

Let $\mathcal{A}$ be a finite-dimensional affine space, let $(A_0, \ldots, A_p)$ be an affine frame of $\mathcal{A}$, and let I be a real interval with a nonempty interior. Consider a function $\Phi : I \to \mathcal{A}$. Then, it can be expressed in a unique way as follows:

$$\Phi(t) = \sum_{i=0}^{p} \Phi_i(t)\, A_i, \qquad \text{with} \qquad \sum_{i=0}^{p} \Phi_i(t) = 1, \qquad t \in I. \tag{2.1}$$

2.1. *Nondegenerate Functions*

Definition 2.1. The *order of* Φ is defined as the dimension of the affine space $\mathrm{aff}\,\mathrm{Im}(\Phi)$ spanned by the image of Φ. The space $\mathcal{E} := \mathrm{span}(\Phi_0, \ldots, \Phi_p) = \mathrm{span}(\mathbf{1}, \Phi_1, \ldots, \Phi_p)$ will be called *the space associated with* Φ.

We can easily verify that $\mathcal{E}$ depends neither on the affine space $\mathcal{A}$ containing $\mathrm{Im}(\Phi)$, nor on the chosen frame $(A_0, \ldots, A_p)$ in $\mathcal{A}$.

Theorem 2.2. *The function Φ is of order k iff its associated space is of dimension $k+1$.*

Proof. We can prove that $k+1$ real valued functions $G_0, \ldots, G_k$ defined on I are linearly independent iff there exist $x_0, \ldots, x_k \in I$ such that the determinant $\det(G_i(x_j))_{0\le i,j\le k}$ is nonzero.

Consequently, the space $\mathcal{E}$ is of dimension greater than or equal to $\ell+1$ iff there exist $t_0, \ldots, t_\ell \in I$ such that the rank of the $(p+1, \ell+1)$ matrix $(\Phi_i(t_j))_{i=0,\ldots,p,\, j=0,\ldots,\ell}$ is equal to $\ell + 1$. This condition means that the $\ell + 1$ points $\Phi(t_0), \ldots, \Phi(t_\ell)$ are affinely independent. Hence, the condition $\dim \mathcal{E} \geq \ell + 1$ is satisfied iff $\dim(\operatorname{aff} \operatorname{Im}(\Phi)) \geq \ell$. ■

As a direct consequence, Theorem 2.2 leads to the following result:

Corollary 2.3. *Consider the function Φ defined in* (2.1) *and let $\mathcal{E}$ denote its associated space. Then, the following three statements are equivalent*:

(i) Φ *is of order* p;
(ii) $A_0, \ldots, A_p$ *belong to* $\operatorname{aff} \operatorname{Im}(\Phi)$;
(iii) $(\Phi_0, \ldots, \Phi_p)$ *is a basis of* $\mathcal{E}$.

Definition 2.4. Let $\mathcal{E}$ be a $(k + 1)$-dimensional space of real valued functions defined on I. Then, a function F defined on I, with values in a finite-dimensional affine space $\mathcal{C}$ will be called an *$\mathcal{E}$-function* if its affine coordinates in any affine frame of $\mathcal{C}$ belong to $\mathcal{E}$ (in other words, if its associated space is a subspace of $\mathcal{E}$, which implies in particular that the order of F is less than or equal to k). An $\mathcal{E}$-function F will be said to be *nondegenerate* if it is of order k (i.e., if its associated space is $\mathcal{E}$).

Theorem 2.5. *Let $\mathcal{E}$ be a $(k+1)$-dimensional space of real valued functions defined on I and let Φ be a nondegenerate $\mathcal{E}$-function. Then, a function $F : I \to \mathcal{C}$ is an $\mathcal{E}$-function iff there exists an affine map $h : \operatorname{aff} \operatorname{Im}(\Phi) \to \mathcal{C}$ such that $F = h \circ \Phi$. An $\mathcal{E}$-function F defined by $F = h \circ \Phi$ is nondegenerate iff h is one-to-one.*

Proof. Let $(P_0, \ldots, P_k)$ be an affine frame of $\operatorname{aff} \operatorname{Im}(\Phi)$, so that we can write

$$\Phi(t) = \sum_{j=0}^{k} B_j(t)\ P_j, \qquad \sum_{j=0}^{k} B_j(t) = 1, \qquad t \in I. \tag{2.2}$$

It follows from Corollary 2.3 that $(B_0, \ldots, B_k)$ is a basis of $\mathcal{E}$. Given an affine frame $(C_0, \ldots, C_\ell)$ of $\mathcal{C}$ and a function $F : I \to \mathcal{C}$, we can write

$$F(t) = \sum_{i=0}^{\ell} F_i(t)\ C_i, \qquad \sum_{i=0}^{\ell} F_i(t) = 1 \qquad \text{for all} \quad t \in I.$$

If F is an $\mathcal{E}$-function, each F_i belongs to $\mathcal{E}$. Then $F_i = \sum_{j=0}^{k} a_{ji} B_j$, $i = 0, \ldots, \ell$. Hence,

$$F(t) = \sum_{j=0}^{k} B_j(t) \sum_{i=0}^{\ell} a_{ji} C_i. \tag{2.3}$$

On the other hand, the equality $\sum_{i=0}^{\ell} F_i = \mathbf{1}$ implies that

$$\sum_{j=0}^{k} \sum_{i=0}^{\ell} a_{ji} B_j = \mathbf{1}. \tag{2.4}$$

Since $(B_0, \ldots, B_k)$ is a basis of $\mathcal{E}$, comparing (2.4) and $\sum_{j=0}^{k} B_j = \mathbf{1}$ proves that, for $j = 0, \ldots, k$, $\sum_{i=0}^{\ell} a_{ji} = 1$. Consequently, setting $h(P_j) := \sum_{i=0}^{\ell} a_{ji} C_i$ for $j = 0, \ldots, k$, provides an affine map $h : \mathrm{aff\,Im}(\Phi) \to \mathcal{C}$ such that $F = h \circ \Phi$. Clearly, $\mathrm{aff\,Im}(F) = h(\mathrm{aff\,Im}(\Phi))$. Hence F is of order k iff h is one-to-one.

The converse part is obvious. ■

2.2. *Geometrically Regular Functions*

Consider a function $\Phi : I \to \mathcal{A}$. Let us recall that, if Φ is C^k on I, its *osculating flat of order i* ($0 \le i \le k$) at a point $a \in I$ is the affine flat going through $\Phi(a)$ and the direction of which is the linear space spanned by $\Phi'(a), \ldots, \Phi^{(i)}(a)$. It will be denoted by $\mathrm{Osc}_i\, \Phi(a)$. In particular, $\mathrm{Osc}_0\, \Phi(a) = \{\Phi(a)\}$. Let us observe that

$$\mathrm{Osc}_k\, \Phi(a) \subset \mathrm{aff\,Im}(\Phi). \tag{2.5}$$

Consequently, if Φ is of order k and if the k derivatives $\Phi'(a), \ldots, \Phi^{(k)}(a)$ are linearly independent, then $\mathrm{Osc}_k\, \Phi(a) = \mathrm{aff\,Im}(\Phi)$.

More generally, suppose now that a is an interior point of I, and consider the two intervals $I^- := \{x \in I \mid x \le a\}$ and $I^+ := \{x \in I \mid x \ge a\}$. Suppose that Φ is continuous on I and C^k on I^- and I^+ separately. Then, for $0 \le i \le k$, it is possible to define similarly $\mathrm{Osc}_i^-\, \Phi(a)$ from the left derivatives $\Phi'(a^-), \ldots, \Phi^{(i)}(a^-)$ of Φ at a and $\mathrm{Osc}_i^+\, \Phi(a)$ from its right derivatives $\Phi'(a^+), \ldots, \Phi^{(i)}(a^+)$. When $\mathrm{Osc}_i^-\, \Phi(a) = \mathrm{Osc}_i^+\, \Phi(a)$, this affine flat will be simply denoted by $\mathrm{Osc}_i\, \Phi(a)$.

Suppose that the k left (or right) derivatives of Φ at a are linearly independent. Then, the existence of the k osculating flats $\mathrm{Osc}_i\, \Phi(a)$, $i = 1, \ldots, k$, is guaranteed iff there is a (unique) regular lower triangular matrix M of order k such that

$$D_k\Phi(a^+) = M \cdot D_k\Phi(a^-), \tag{2.6}$$

where, for $\varepsilon = -$ or $\varepsilon = +$, $D_k\Phi(a^\varepsilon)$ is defined by

$$D_k\Phi(a^\varepsilon) := (\Phi'(a^\varepsilon), \ldots, \Phi^{(k)}(a^\varepsilon))^T. \tag{2.7}$$

However, this is not a sufficient condition for Φ to provide a "smooth" curve. Indeed, if $\Phi'(a^+) = -\Phi'(a^-)$, the tangent line $\mathrm{Osc}_1\, \Phi(a)$ does exist, and yet, the curve defined by Φ has a cusp at the point $\Phi(a)$. In case the k derivatives at a point t are linearly independent, a rough localization of the curve near a point t is obtained by means of the Frénet frame of order k (i.e., the orthonormal system obtained from $(\Phi'(t), \ldots, \Phi^{(k)}(t))$ by the Gram–Schmidt process). So, if we want Φ to provide a "nice" curve, we have (at least) to require the Frénet frames at a^- and a^+ to be identical. As a matter of fact, this occurs iff relation (2.6) holds with the additional assumption that the diagonal elements of M are positive. This will give sense to the definition hereunder.

Throughout this paper, we shall consider a fixed sequence $t_1 < t_2 < \cdots < t_n$ ($n \ge 0$) of interior points of I and the corresponding sequence of consecutive intervals

$$I_0 := \{x \in I \mid x \le t_1\}, \qquad I_n := \{x \in I \mid x \ge t_n\},$$

$$I_i := [t_i, t_{i+1}], \qquad i = 1, \ldots, n-1, \tag{2.8}$$

if $n \geq 1$, with the convention $I_0 := I$ if $n = 0$. Let us set $I_* := I \backslash \{t_1, \ldots, t_n\}$. In all the formulas to come, given $t \in I$, the notation t^ε can be replaced by t when $t \in I_*$, while it is to be read either as t^+ or t^- when t is one of the t_i's. But, of course, in case I has a left endpoint t_0, t_0^ε will stand only for t_0^+, with a similar convention for a possible right endpoint.

Definition 2.6. Suppose that $\Phi : I \to \mathcal{A}$ is continuous on I and C^k on each interval I_j, $j = 0, \ldots, n$. Then Φ is said to be *geometrically k-regular* if the following two properties are satisfied:

(i) for all $t^\varepsilon \in I$, the k vectors $\Phi'(t^\varepsilon), \ldots, \Phi^{(k)}(t^\varepsilon)$ are linearly independent; and
(ii) for all $\ell = 1, \ldots, n$ there exists a lower triangular matrix $M_\ell = (m_{ij}^\ell)_{1 \leq i,j \leq k}$ with positive diagonal elements, such that

$$D_k \Phi(t_\ell^+) = M_\ell \cdot D_k \Phi(t_\ell^-). \tag{2.9}$$

Accordingly, if Φ is geometrically k-regular, for all $i = 1, \ldots, k$ and all $t \in I$, $\mathrm{Osc}_i\, \Phi(t)$ exists and is of dimension i. It results from (2.5) that the order of a geometrically k-regular function is greater than or equal to k. If Φ is of order k and is geometrically k-regular, it will simply be said to be *a geometrically regular function of order k*. As an example, a continuous function $\Phi : I \to \mathcal{A}$, assumed to be piecewise C^1 (i.e., C^1 everywhere except at the t_i's), is a geometrically regular function of order 1 iff it is strictly monotone on I, with values in an affine line.

Remark 2.7. (i) If $\Phi : I \longrightarrow \mathcal{A}$ is a geometrically regular function of order k, any basis $(U_0, \ldots, U_k)$ of its associated space $\mathcal{E}$ satisfies

$$\det({U_i}^{(j)}(t^\varepsilon))_{0 \leq i,j \leq k} \neq 0 \qquad \text{for all} \quad t^\varepsilon \in I. \tag{2.10}$$

Moreover, if the connections for Φ are expressed by (2.9), any $\mathcal{E}$-function $F = h \circ \Phi$ (in particular, any $F \in \mathcal{E}$) also satisfies

$$D_k F({t_\ell}^+) = M_\ell \cdot D_k F({t_\ell}^-), \qquad \ell = 1, \ldots, n. \tag{2.11}$$

Conversely, let $\mathcal{E}$ be a $(k+1)$-dimensional subspace of piecewise C^k functions F satisfying (2.11) where the M_ℓ's are lower triangular matrices with positive diagonal elements, and for which (2.10) holds for a given basis $(U_0, \ldots, U_k)$. Then, any nondegenerate $\mathcal{E}$-function Φ is a geometrically regular function of order k.

(ii) Let Φ be a geometrically regular function of order k and let $\mathcal{E}$ be its associated space. From $F = h \circ \Phi$, it results that the order of an $\mathcal{E}$-function F is equal to the dimension of $\mathrm{Osc}_k\, F(a)$, where a is a given point in I. Accordingly, given a subinterval $J \subset I$ supposed to have a nonempty interior, F and its restriction $F|_J$ have the same order. For instance, as soon as an $\mathcal{E}$-function vanishes on J, it vanishes on the whole interval I.

Suppose that $\Phi : I \to \mathcal{A}$ is a geometrically regular function of order k satisfying (2.9). Let us choose, once and for all, a basis in the direction Δ of $\mathrm{aff\,Im}(\Phi)$ and denote by $\langle \cdot, \cdot \rangle$ the inner product in Δ for which this basis is an orthonormal basis, and by "det" the determinant with respect to this basis. Then, given $k-1$ vectors $W_1, \ldots, W_{k-1} \in \Delta$, $W_1 \wedge \cdots \wedge W_{k-1}$ will stand for the only element of Δ satisfying

$$\det(W_1, \ldots, W_{k-1}, X) = \langle W_1 \wedge \cdots \wedge W_{k-1}, X\rangle \qquad \text{for all} \quad X \in \Delta.$$

So that, for all $t^\varepsilon \in I$, $\Phi'(t^\varepsilon) \wedge \cdots \wedge \Phi^{(k-1)}(t^\varepsilon)$ provides the orthogonal direction to the osculating hyperplane $\mathrm{Osc}_{k-1}\,\Phi(t)$. This direction is also given by the vector $\Phi^\sharp(t^\varepsilon) \in \Delta$ defined by

$$\Phi^\sharp(t^\varepsilon) := \frac{\Phi'(t^\varepsilon) \wedge \cdots \wedge \Phi^{(k-1)}(t^\varepsilon)}{\det(\Phi'(t^\varepsilon), \ldots, \Phi^{(k)}(t^\varepsilon))}, \tag{2.12}$$

which is characterized by the following k relations:

$$\langle \Phi^\sharp(t^\varepsilon), \Phi^{(i)}(t^\varepsilon)\rangle = 0, \quad i = 1, \ldots, k-1, \qquad \langle \Phi^\sharp(t^\varepsilon)), \Phi^{(k)}(t^\varepsilon))\rangle = 1. \tag{2.13}$$

Let us observe that relations (2.9) imply that

$$\Phi^\sharp(t_\ell^+) = \frac{1}{m_{kk}^\ell}\,\Phi^\sharp(t_\ell^-), \qquad \ell = 1, \ldots, n. \tag{2.14}$$

In other words, $\Phi^\sharp$ can be considered as a function defined on the set $I_* \cup \{t_\ell^-, t_\ell^+,\ \ell = 1, \ldots, n\}$: it will be called *the normal function* of Φ. On the other hand, the very definition of $\Phi^\sharp(t^\varepsilon)$, $t^\varepsilon \in I$, provides the following equivalence:

$$P \in \mathrm{Osc}_{k-1}\,\Phi(t) \quad \Leftrightarrow \quad \langle P - \Phi(t), \Phi^\sharp(t^\varepsilon)\rangle = 0. \tag{2.15}$$

As soon as Φ is assumed to be C^{2k-1} on each I_j, its normal function $\Phi^\sharp$ is C^{k-1} on each I_j. Then, by differentiating relations (2.13) on each subinterval, a recursive argument proves that, for all $t^\varepsilon \in I$, for $1 \le i \le k$ and $0 \le j \le k-1$,

$$\langle \Phi^{\sharp(j)}(t^\varepsilon), \Phi^{(i)}(t^\varepsilon)\rangle = \begin{cases} 0 & \text{if } \ i + j \le k-1, \\ (-1)^j & \text{if } \ i + j = k. \end{cases} \tag{2.16}$$

This immediately implies that, for all $t^\varepsilon \in I$, the k vectors $\Phi^\sharp(t^\varepsilon), \ldots, \Phi^{\sharp(k-1)}(t^\varepsilon)$ are linearly independent. Consequently, the two linear spaces $\mathrm{span}(\Phi'(t^\varepsilon), \ldots, \Phi^{(k-i)}(t^\varepsilon))^\perp$ (where $V^\perp$ denotes the subspace orthogonal to V) and $\mathrm{span}(\Phi^\sharp(t^\varepsilon), \ldots, \Phi^{\sharp(i-1)}(t^\varepsilon))$ are both of dimension i. Therefore, relations (2.16) eventually lead to the following equalities:

$$\mathrm{span}(\Phi'(t^\varepsilon), \ldots, \Phi^{(k-i)}(t^\varepsilon))^\perp = \mathrm{span}(\Phi^\sharp(t^\varepsilon), \ldots, \Phi^{\sharp(i-1)}(t^\varepsilon)), \quad 0 \le i \le k. \tag{2.17}$$

Thus, a point P belongs to $\mathrm{Osc}_{k-i}\,\Phi(t)$ iff it satisfies

$$\langle P - \Phi(t), \Phi^{\sharp(s)}(t^\varepsilon)\rangle = 0, \qquad s = 0, \ldots, i-1. \tag{2.18}$$

As an immediate consequence of (2.17), for all $\ell = 1, \ldots, n$ and all $i = 0, \ldots, k-1$, the two spaces spanned, respectively, by $(\Phi^\sharp(t_\ell{}^-), \ldots, \Phi^{\sharp(i-1)}(t_\ell{}^-))$ and

$(\Phi^\sharp(t_\ell{}^+), \ldots, \Phi^{\sharp(i-1)}(t_\ell{}^+))$, are identical. Accordingly, there exist n regular lower triangular matrices $M_1^\sharp, \ldots, M_n^\sharp$, such that

$$(2.19) \qquad (\Phi^\sharp(t_\ell^+), \ldots, \Phi^{\sharp(k-1)}(t_\ell^+))^T = M_\ell^\sharp \cdot (\Phi^\sharp(t_\ell^-), \ldots, \Phi^{\sharp(k-1)}(t_\ell^-))^T, \quad \ell = 1, \ldots, n.$$

Actually, using (2.9) and (2.16), we can verify that the diagonal of matrix $M_\ell^\sharp$ is equal to

$$(2.20) \qquad \left(\frac{1}{m_{kk}^\ell}, \frac{1}{m_{k-1,k-1}^\ell}, \ldots, \frac{1}{m_{11}^\ell}\right).$$

2.3. *$\mathcal{E}$-Splines*

Throughout this subsection, we shall deal with a given geometrically regular function of order k, $\Phi : I \to \mathcal{A}$. We shall denote by $M_1, \ldots, M_n$ the corresponding connection matrices, and by $\mathcal{E}$ the space associated to Φ.

Inside any tuple, we shall use a multiplicative notation, τ^μ meaning that the point τ is repeated μ times. Moreover, associated with an arbitrary p-tuple $\mathcal{T} \in I^p$, we consider the p-tuple $\mathcal{T}^{\text{ord}}$ composed of the same elements as $\mathcal{T}$ but arranged in ascending order. Using the multiplicative notation introduced above, this p-tuple $\mathcal{T}^{\text{ord}}$ will be written $\mathcal{T}^{\text{ord}} = (\tau_1^{\mu_1} \ldots \tau_r^{\mu_r})$, with positive integers μ_i and $\tau_i < \tau_{i+1}$.

2.3.1. *Osculating Flats and Contact.*

For $s \le k$, two $\mathcal{E}$-functions F_1, F_2 will be said *to have a contact of order s at $a \in I$* if

$$(2.21) \qquad F_1^{(i)}(a^\varepsilon) = F_2^{(i)}(a^\varepsilon), \qquad i = 0, \ldots, s,$$

for $\varepsilon = +$ or $\varepsilon = -$ such that $a^\varepsilon \in I$. Observe that, since F_1 and F_2 both satisfy (2.11), if a is any interior point of I, (2.21) holds for $\varepsilon = +$ iff it holds for $\varepsilon = -$.

Theorem 2.8. *Two $\mathcal{E}$-functions $F_1 = h_1 \circ \Phi$ and $F_2 = h_2 \circ \Phi$ have a contact of order $s \le k$ at $a \in I$ iff $h_1(P) = h_2(P)$ for all $P \in \mathrm{Osc}_s\,\Phi(a)$.*

Proof. Let us denote by $\bar{h}_1$ and $\bar{h}_2$ the linear maps associated with h_1 and h_2, respectively. Since $\mathrm{Osc}_s\,\Phi(a)$ is the affine flat going through $\Phi(a)$ and the direction of which is spanned by the linearly independent vectors $\Phi'(a^\varepsilon), \ldots, \Phi^{(s)}(a^\varepsilon)$, h_1 and h_2 are equal on $\mathrm{Osc}_s\,\Phi(a)$ iff

$$h_1(\Phi(a)) = h_2(\Phi(a)), \qquad \bar{h}_1(\Phi^{(i)}(a^\varepsilon)) = \bar{h}_2(\Phi^{(i)}(a^\varepsilon)), \qquad i = 1, \ldots, s.$$

Now, $h_j(\Phi(a)) = F_j(a)$ and $\bar{h}_j(\Phi^{(i)}(a^\varepsilon)) = F_j{}^{(i)}(a^\varepsilon)$ for $j = 1, 2$, $i = 1, \ldots, s$, which concludes the proof. ∎

Theorem 2.9. *If two nondegenerate $\mathcal{E}$-functions F_1 and F_2 have a contact of order $s \le k$ at $a \in I$, then, for any p-tuple $\mathcal{T} \in I^p$ $(p \le k)$ containing (a^{k-s}) (i.e., in which* *the point a is repeated at least $k-s$ times), assuming that $\mathcal{T}^{\mathrm{ord}} = ({\tau_1}^{\mu_1}\dots{\tau_r}^{\mu_r})$, we have*

$$\bigcap_{i=1}^{r}\mathrm{Osc}_{k-\mu_i}\,F_1(\tau_i) = \bigcap_{i=1}^{r}\mathrm{Osc}_{k-\mu_i}\,F_2(\tau_i)\;. \tag{2.22}$$

Proof. Since F_1 and F_2 are nondegenerate, we can define any $\mathcal{E}$-function from F_1 instead of Φ. In particular, $F_2 = h\circ F_1$, where h denotes a one-to-one affine map defined on aff $\mathrm{Im}(F_1)$. Since h is one-to-one, we have:

$$\bigcap_{i=1}^{r} h(\mathrm{Osc}_{k-\mu_i}\,F_1(\tau_i)) = h\left(\bigcap_{i=1}^{r}\mathrm{Osc}_{k-\mu_i}\,F_1(\tau_i)\right). \tag{2.23}$$

Clearly, for all $t\in I$ and all $j\le k$, $\mathrm{Osc}_j\,F_2(t) = h(\mathrm{Osc}_j\,F_1(t))$. Hence, (2.23) can be replaced by

$$\bigcap_{i=1}^{r}\mathrm{Osc}_{k-\mu_i}\,F_2(\tau_i) = h\left(\bigcap_{i=1}^{r}\mathrm{Osc}_{k-\mu_i}\,F_1(\tau_i)\right). \tag{2.24}$$

On the other hand, Theorem 2.8 ensures that

$$h(P) = P \qquad \text{for all} \quad P\in\mathrm{Osc}_s\,F_1(a). \tag{2.25}$$

Since a appears at least $k-s$ times in $\mathcal{T}$, without any loss of generality, we can suppose that $a=\tau_1$, so that $\mu_1\ge k-s$ or, as well, $k-\mu_1\le s$. Consequently, any P in $\bigcap_{i=1}^{r}\mathrm{Osc}_{k-\mu_i}\,F_1(\tau_i)$ belongs to $\mathrm{Osc}_s\,F_1(a)$, hence, by (2.25), P is invariant under h. Consequently, (2.24) proves that

$$\bigcap_{i=1}^{r}\mathrm{Osc}_{k-\mu_i}\,F_1(\tau_i) \subset \bigcap_{i=1}^{r}\mathrm{Osc}_{k-\mu_i}\,F_2(\tau_i).$$

This finally leads to equality (2.22) by exchanging the rôles of F_1 and F_2. ■

2.3.2. *Admissible Tuples.*

Given n fixed integers $m_1,\dots,m_n$, such that $0\le m_i\le k$ for $i=1,\dots,n$, we define the corresponding knot vector by

$$T := (t_1^{m_1}\dots t_n^{m_n}). \tag{2.26}$$

Definition 2.10. Let $\mathcal{T}$ be an element of I^p, $p\le k+1$, with $\mathcal{T}^{\mathrm{ord}} = (\tau_1^{\mu_1}\dots\tau_r^{\mu_r})$. Then, $\mathcal{T}$ will said to be *admissible* with respect to the knot vector T if every t_i $(1\le i\le n)$ belonging to $\mathrm{ri}[\tau_1,\tau_r]$ is repeated at least m_i times in $\mathcal{T}$.

The notation $\mathrm{ri}[\alpha,\beta]$ stands for the relative interior of interval $[\alpha,\beta]$, i.e., $]\alpha,\beta[$ when $\alpha<\beta$ and $\{\alpha\}$ when $\alpha=\beta$. Therefore, for $p\le k+1$ and $1\le i\le n$, the p-tuple (t_i^p) is admissible iff $p\ge m_i$. In particular, since the multiplicity at each knot t_i is supposed to be less than or equal to k, the k-tuple (t^k) is admissible whatever the point $t\in I$ may be.

Definition 2.11. If $\mathcal{T}$ is an admissible p-tuple, $p \leq k$, its *domain* is defined as

$$\mathcal{D}(T) := \{t \in I / (t, \mathcal{T}) \text{ is admissible}\}. \tag{2.27}$$

Theorem 2.12. *Let $\mathcal{T}$ be an admissible p-tuple, $p \leq k$. Then, $\mathcal{D}(T)$ is a union of consecutive intervals I_i, i.e.,*

$$\mathcal{D}(T) = \bigcup_{i \in \mathcal{J}(T)} I_i, \tag{2.28}$$

where $\mathcal{J}(T)$ is a nonempty subset of consecutive integers.

Proof. For simplicity, we shall assume that $m_i > 0$ for $i = 1, \dots, n$. If not, we can get rid of all the t_i's which do not really appear in the knot vector T, and simultaneously join the corresponding consecutive intervals into a single one. Let $\mathcal{N}(T)$ denote the set of all integers i, $1 \leq i \leq n$, such that t_i appears at least m_i times in $\mathcal{T}$. Two possibilities have to be examined.

(1) $\mathcal{N}(T) \neq \emptyset$.
In that case, clearly

$$\mathcal{J}(T) = \{\mathrm{Min}\, \mathcal{N}(T) - 1, \dots, \mathrm{Max}\, \mathcal{N}(T)\}. \tag{2.29}$$

For example, if $\mathcal{T} = (t_i^p)$, with $1 \leq i \leq n$, we have $\mathcal{N}(T) = \{i\}$ since the admissibility of $\mathcal{T}$ implies $p \geq m_i$. Thus, $\mathcal{J}(T) = \{i-1, i\}$, so that $\mathcal{D}(T) = I_{i-1} \cup I_i$.

(2) $\mathcal{N}(T) = \emptyset$.
In that case, on account of the admissibility of $\mathcal{T}$, we can verify that there exists a unique integer $\ell \in \{0, \dots, n\}$ such that $\tau_1, \dots, \tau_r \in I_\ell$ and that $\mathcal{D}(T) = I_\ell$, or, equivalently, that $\mathcal{J}(T) = \{\ell\}$. ∎

2.3.3. *Splines and Osculating Flats.*

Given the sequence $(m_1, \dots, m_n)$ of integers introduced in the previous subsection, for $\ell = 1, \dots, n$, $\widehat{M}_\ell$ will stand for the $(k - m_\ell, k - m_\ell)$ lower triangular matrix obtained by suppressing the m_ℓ last rows and columns of M_ℓ.

Definition 2.13. A continuous function $S : I \to \mathcal{A}$ is said to be *an $\mathcal{E}$-spline* (with respect to the knot vector T) if the following two properties are satisfied:

(i) there exist $n+1$ $\mathcal{E}$-functions $F_j : I \to \mathcal{A}$, $j = 0, \dots, n$, such that

$$S(t) = F_j(t) \qquad \text{for all} \quad t \in I_j \quad \text{and all} \quad j = 0, \dots, n; \tag{2.30}$$

(ii) $D_{k-m_\ell} S(t_\ell{}^+) = \widehat{M}_\ell \cdot D_{k-m_\ell} S(t_\ell{}^-)$ for all $\ell = 1, \dots, n$.

Moreover, the $\mathcal{E}$-spline S will be said to be *nondegenerate* if each F_j is a nondegenerate $\mathcal{E}$-function.

Clearly, $S : I \to \mathcal{A}$ is an $\mathcal{E}$-spline iff it is an $\mathcal{S}$-function, where $\mathcal{S}$ denotes the space of all real valued $\mathcal{E}$-splines. Due to the regularity of Φ, this space $\mathcal{S}$ is a $(k+m+1)$-dimensional space, where $m := \sum_{\ell=1}^n m_\ell$. On account of Remark 2.7(ii), an $\mathcal{E}$-spline S

given by (2.30) satisfies

$$\operatorname{aff}\operatorname{Im}(S|_{I_j}) = \operatorname{aff}\operatorname{Im}(F_j), \qquad j = 0, \dots, n. \tag{2.31}$$

Let us observe in particular that any nondegenerate $\mathcal{S}$-function (i.e., any $\mathcal{E}$-spline such that $\dim(\operatorname{aff}\operatorname{Im}(\Phi)) = k + m + 1$) is a nondegenerate $\mathcal{E}$-spline.

Lemma 2.14. *Consider a nondegenerate $\mathcal{E}$-spline S satisfying* (2.30), *and a p-tuple $\mathcal{T}$ ($p \le k$) supposed to contain $(t_\ell^{m_\ell})$, for a given integer $\ell \in \{1, \dots, n\}$. Then, if $\mathcal{T}^{\mathrm{ord}} = (\tau_1^{\mu_1} \dots \tau_r^{\mu_r})$, we have*

$$\bigcap_{i=1}^{r} \operatorname{Osc}_{k-\mu_i} F_\ell(\tau_i) = \bigcap_{i=1}^{r} \operatorname{Osc}_{k-\mu_i} F_{\ell-1}(\tau_i). \tag{2.32}$$

Proof. According to Theorem 2.9, since $F_{\ell-1}$ and F_ℓ are nondegenerate, it is sufficient to prove that they have a contact of order $k - m_\ell$ at t_ℓ.

Since S is an $\mathcal{E}$-spline, it satisfies condition (ii) of Definition 2.14, which can also be written

$$D_{k-m_\ell} F_\ell(t_\ell^+) = \widehat{M}_\ell \cdot D_{k-m_\ell} F_{\ell-1}(t_\ell^-), \tag{2.33}$$

due to the fact that $F_{\ell-1}$ and F_ℓ coincide with S on $I_{\ell-1}$ and I_ℓ, respectively. On the other hand, any $\mathcal{E}$-function being an $\mathcal{E}$-spline, we have

$$D_{k-m_\ell} F_{\ell-1}(t_\ell^+) = \widehat{M}_\ell \cdot D_{k-m_\ell} F_{\ell-1}(t_\ell^-). \tag{2.34}$$

Comparing (2.33) and (2.34), we obtain

$$D_{k-m_\ell} F_\ell(t_\ell^+) = D_{k-m_\ell} F_{\ell-1}(t_\ell^+), \tag{2.35}$$

and of course a similar equality for t_ℓ^-. As we additionally have $F_\ell(t_\ell) = F_{\ell-1}(t_\ell) = S(t_\ell)$, equality (2.35) means that $F_{\ell-1}$ and F_ℓ have a $k - m_\ell$ contact at t_ℓ. ■

Lemma 2.15. *Let S be a nondegenerate $\mathcal{E}$-spline and let $\mathcal{T} \in I^p$ be an admissible p-tuple ($p \le k$) such that $\mathcal{T}^{\mathrm{ord}} = (\tau_1^{\mu_1} \dots \tau_r^{\mu_r})$. Then, the affine flat $\bigcap_{i=1}^r \operatorname{Osc}_{k-\mu_i} F_\ell(\tau_i)$ does not depend on $\ell \in \mathcal{J}(T)$.*

Proof. As soon as $\mathcal{J}(T)$ contains two consecutive integers $\ell - 1$ and ℓ, $1 \le \ell \le n$, the point t_ℓ appears necessarily at least m_ℓ times in $\mathcal{T}$. Consequently, equality (2.32) is valid. This yields the desired result. ■

For $\ell \in \{1, \dots, n\}$ and $i \le k$, $\operatorname{Osc}_i^+ S(t_\ell) = \operatorname{Osc}_i F_\ell(t_\ell)$ and $\operatorname{Osc}_i^- S(t_\ell) = \operatorname{Osc}_i F_{\ell-1}(t_\ell)$. If $i \le k - m_\ell$, as an obvious application of (2.32) we have $\operatorname{Osc}_i F_\ell(t_\ell) = \operatorname{Osc}_i F_{\ell-1}(t_\ell)$, i.e., $\operatorname{Osc}_i^+ S(t_\ell) = \operatorname{Osc}_i^- S(t_\ell)$. In other words, $\operatorname{Osc}_i S(t_\ell)$ is well defined for all $i \le k - m_\ell$. On the contrary, for $i > k - m_\ell$, we can deal only with $\operatorname{Osc}_i^+ S(t_\ell)$ and $\operatorname{Osc}_i^- S(t_\ell)$. On the other hand, $\operatorname{Osc}_i S(t)$ is well defined for any $t \in I_*$ and any $i \le k$.

Theorem 2.16. *With the same assumptions as in Lemma* 2.15*, let us set* $D := \mathcal{D}(T)$ *and denote by* S_D *the restriction of* S *to* D*. Then, for all* $\ell \in \mathcal{J}(T)$*,*

$$\bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i}\, F_\ell(\tau_i) = \bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i}\, S_D(\tau_i). \tag{2.36}$$

Proof. (1) On account of the admissibility of $\mathcal{T}$, it may be the case that $\mathrm{Osc}_{k-\mu_i}\, S(\tau_i)$ is not defined only if $r \geq 2$, more precisely in the following two situations:

Either $i = 1$, $\tau_1 = t_\ell$, with $1 \leq \ell \leq n$ and $\mu_1 < m_\ell$, in which case τ_1 is the left endpoint of D. Then, $\mathrm{Osc}_{k-\mu_1}\, S_D(\tau_1)$ stands for $\mathrm{Osc}_{k-\mu_1}\, F_\ell(t_\ell)$.

Or $i = r$, $\tau_r = t_\ell$, with $1 \leq \ell \leq n$ and $\mu_r < m_\ell$, in which case τ_r is the right endpoint of D. Then, $\mathrm{Osc}_{k-\mu_r}\, S_D(\tau_r)$ stands for $\mathrm{Osc}_{k-\mu_r}\, F_{\ell-1}(t_\ell)$.

(2) Taking Lemma 2.15 into account, it is sufficient to prove the existence of an integer $\ell \in \mathcal{J}(T)$ such that

$$\bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i}\, F_\ell(\tau_i) = \bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i}\, S_D(\tau_i). \tag{2.37}$$

The proof will be done by induction on r. Observe first that, whenever $\tau_1, \ldots, \tau_r$ are all located in a union of consecutive subintervals $I_j, \ldots, I_{j+r}$ such that $m_{j+1} = \cdots = m_{j+r-1} = 0$, any integer $\ell \in \{j, \ldots, j+r-1\}$ belongs to $\mathcal{J}(T)$, and, since $F_j = \cdots = F_{j+r-1}$, we have additionally $\mathrm{Osc}_{k-\mu_i}\, S_D(\tau_i) = \mathrm{Osc}_{k-\mu_i}\, F_\ell(\tau_i)$ for all $i = 1, \ldots, r$. Therefore, (2.37) is trivially satisfied by any such integer ℓ. In particular, on account of the admissibility of $\mathcal{T}$, this occurs as soon as $r \leq 2$.

So, assume that $r \geq 3$ and that the result has already been proved for $r - 1$. Then $\mathcal{T}' := (\tau_1{}^{\mu_1} \ldots \tau_{r-1}{}^{\mu_{r-1}})$ is also admissible.

According to the observation above, we can also suppose that there exists at least one knot of nonzero multiplicity in $]\tau_1, \tau_r[$. Let t_ℓ be the greatest one. The multiplicity of all possible knots located between t_ℓ and τ_r being equal to 0, we have

$$\mathrm{Osc}_{k-\mu_r}\, S_D(\tau_r) = \mathrm{Osc}_{k-\mu_r}\, F_\ell(\tau_r). \tag{2.38}$$

On the other hand, the admissibility of $\mathcal{T}$ implies that we have either $\tau_{r-1} > t_\ell$ or $\tau_{r-1} = t_\ell$ with $\mu_{r-1} \geq m_\ell$. In both cases, we can derive that ℓ belongs to $\mathcal{J}(T')$. Hence, the recursive hypothesis applied to $\mathcal{T}'$ leads to

$$\bigcap_{i=1}^{r-1} \mathrm{Osc}_{k-\mu_i}\, F_\ell(\tau_i) = \bigcap_{i=1}^{r-1} \mathrm{Osc}_{k-\mu_i}\, S_{D'}(\tau_i), \tag{2.39}$$

where $D' := \mathcal{D}(T')$. Moreover, for $i = 1, \ldots, r-1$, τ_i is not the right endpoint of D'. Thus,

$$\mathrm{Osc}_{k-\mu_i}\, S_{D'}(\tau_i) = \mathrm{Osc}_{k-\mu_i}\, S_D(\tau_i) \qquad \text{for} \quad i = 1, \ldots, r-1. \tag{2.40}$$

Finally, we can write

$$\bigcap_{i=1}^{r-1} \mathrm{Osc}_{k-\mu_i}\, F_\ell(\tau_i) = \bigcap_{i=1}^{r-1} \mathrm{Osc}_{k-\mu_i}\, S_D(\tau_i), \tag{2.41}$$

which, together with (2.38), proves that the integer ℓ satisfies (2.37). ∎

3. Blossom

For some particular geometrically regular functions, it will be possible to define a notion of blossom by means of intersections of convenient osculating flats.

3.1. *Piecewise Smooth Chebyshev Functions and Blossoming*

Definition and Theorem 3.1. *A geometrically regular function of order k, $\Phi : I \to \mathcal{A}$, will be said to be a* piecewise smooth Chebyshev function of order k *on I if, for all distinct points $\tau_1, \dots, \tau_r \in I$ and all positive integers $\mu_1, \dots, \mu_r$ whose sum is equal to k, the affine flat $\bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i} \Phi(\tau_i)$ consists of a single point. If so, for all k-tuple $\mathcal{T} \in I^k$, such that $\mathcal{T}^{\mathrm{ord}} = (\tau_1^{\mu_1} \dots \tau_r^{\mu_r})$, we shall set*

$$\{\varphi(\mathcal{T})\} := \bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i} \Phi(\tau_i). \tag{3.1}$$

The function $\varphi : I^k \to \mathcal{A}$ so defined will be called the blossom of Φ. *It is a symmetric function and it satisfies*

$$\varphi(t^k) = \Phi(t) \qquad \text{for all} \quad t \in I\ . \tag{3.2}$$

Proof. The symmetry of φ is evident. On the other hand, if $t \in I$, by (3.1), $\{\varphi(t^k)\} := \mathrm{Osc}_0 \Phi(t)$, whence (3.2). ∎

Suppose that Φ is a piecewise smooth Chebyshev function of order k on I, the connections being still given by (2.9). Then, if a, b are two distinct points of I, for all $i = 0, \dots, k$, the value of the blossom φ at the k-tuple $(a^{k-i} b^i)$ is given by

$$\{\varphi(a^{k-i} b^i)\} := \mathrm{Osc}_i \Phi(a) \ \cap\ \mathrm{Osc}_{k-i} \Phi(b). \tag{3.3}$$

So, for $a^\varepsilon, b^{\varepsilon'} \in I$, the linear system

$$\Phi(a) + \sum_{s=1}^{i} \lambda_s \Phi^{(s)}(a^\varepsilon) = \Phi(b) + \sum_{s=1}^{k-i} \nu_s \Phi^{(s)}(b^{\varepsilon'})$$

has a unique solution, which implies the linear independence of the k vectors $\Phi'(a^\varepsilon), \dots, \Phi^{(i)}(a^\varepsilon), \Phi'(b^{\varepsilon'}), \dots, \Phi^{(k-i)}(b^{\varepsilon'})$.

Let us consider the function $N : I \to \mathbf{R}$ defined for $t \in I$ by

$$N(t) := \det(\Phi'(a^\varepsilon), \dots, \Phi^{(k-1)}(a^\varepsilon), \Phi(t) - \Phi(a)). \tag{3.4}$$

Clearly, function N belongs to the space $\mathcal{E}$ associated with Φ, hence it satisfies (2.11). Furthermore, for all $t^{\varepsilon'} \in I$,

$$N'(t^{\varepsilon'}) = \det(\Phi'(a^\varepsilon), \dots, \Phi^{(k-1)}(a^\varepsilon), \Phi'(t^{\varepsilon'})).$$

Thus, from the two properties $N'(t^{\varepsilon'}) \neq 0$ for $t \neq a$, and $N'(t_\ell{}^+) = m_{11}^{\ell} N'(t_\ell{}^-)$, $\ell = 1, \dots, n$, we can derive that N is strictly monotone on I. Accordingly, the obvious equality $N(a) = 0$ implies that $N(t) \neq 0$ for $t \in I \backslash a$. Equivalently, this means that

$$\Phi(t) \notin \mathrm{Osc}_{k-1} \Phi(a) \qquad \text{for} \quad t \neq a. \tag{3.5}$$

In particular, it results from (3.5) that Φ is one-to-one on I.

Definition 3.2. Let Φ be a given piecewise smooth Chebyshev function of order k, and let $\mathcal{E}$ be its associated space. For any affine map $h : \mathrm{aff}\,\mathrm{Im}(\Phi) \to \mathcal{C}$, the blossom of the $\mathcal{E}$-function $F := h \circ \Phi$ will be defined by

$$f := h \circ \varphi. \tag{3.6}$$

Theorem 3.3. *Let Φ be a given piecewise smooth Chebyshev function of order k, let $\mathcal{E}$ be its associated space, and let F be an $\mathcal{E}$-function. Then, F is a piecewise smooth Chebyshev function of order k iff it is nondegenerate. If so, the blossom f of F satisfies*

$$\{f(\tau_1^{\mu_1} \dots \tau_r^{\mu_r})\} = \bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i} F(\tau_i), \tag{3.7}$$

for all distinct points $\tau_1, \dots, \tau_r \in I$ and all positive integers $\mu_1, \dots, \mu_r$ whose sum is equal to k.

Proof. On account of Remark 2.7, F is a geometrically regular function of order k iff it is nondegenerate. Now, assume that $F = h \circ \Phi$ is nondegenerate, i.e., by Theorem 2.5, that h is one-to-one. Then, for all distinct points $\tau_1, \dots, , \tau_r \in I$ and all positive integers $\mu_1, \dots, \mu_r$ whose sum is equal to k,

$$h\left(\bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i} \Phi(\tau_i)\right) = \bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i} F(\tau_i). \tag{3.8}$$

This equality shows that the affine flat appearing in the right-hand side of (3.8) consists of the single point $h(\varphi(\tau_1^{\mu_1} \dots \tau_r^{\mu_r}))$. Hence, F is a Chebyshev function and, using definition (3.6) yields (3.7). ■

In particular, the blossom of an $\mathcal{E}$-function depends only on $\mathcal{E}$, not on the particular function Φ which defines $\mathcal{E}$.

3.2. *A Characterization of Piecewise Smooth Chebyshev Functions*

Theorem 3.4. *Let Φ be a geometrically regular function of order k, supposed to be C^{2k-1} on each interval I_j, $j = 0, \dots, n$, and let $\Phi^\sharp$ be the normal function of Φ. Then, Φ is a piecewise smooth Chebyshev function of order k iff, for all distinct points $\tau_1, \dots, \tau_r \in I$, all $\varepsilon_i = +$ or $-$ such that $\tau_i^{\varepsilon_i} \in I$, and all positive integers $\mu_1, \dots, \mu_r$ whose sum is equal to k, the k vectors $\Phi^\sharp(\tau_1)$, $\Phi^{\sharp\prime}(\tau_1^{\varepsilon_1}) \dots$, $\Phi^{\sharp(\mu_1-1)}(\tau_1^{\varepsilon_1})$, $\Phi^\sharp({\tau_2}^{\varepsilon_2}), \dots, \Phi^\sharp(\tau_r^{\varepsilon_r}), \dots, \Phi^{\sharp(\mu_r-1)}(\tau_r^{\varepsilon_r})$, are linearly independent.*

Proof. Let us fix r distinct points $\tau_1, \dots, \tau_r \in I$, and r positive integers $\mu_1, \dots, \mu_r$

such that $\sum_{i=1}^{r} \mu_i = k$. Using (2.18), for $X \in \mathcal{A}$, we can write

$$X \in \bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i}\, \Phi(\tau_i) \quad \Leftrightarrow \quad \langle\, X - \Phi(\tau_i)\,,\, \Phi^{\sharp(j)}(\tau_i^{\varepsilon_i})\,\rangle = 0, \qquad 1 \le i \le r, \quad 0 \le j \le \mu_i - 1. \tag{3.9}$$

The right-hand side of (3.9) can be regarded as a linear system of $k = \sum_{i=1}^{r} \mu_i$ equations in k unknowns. This system has a unique solution iff the k vectors $\Phi^{\sharp(j)}(\tau_i^{\varepsilon_i})$, $1 \le i \le r$, $0 \le j \le \mu_i - 1$, are linearly independent.

Hence, Φ is a piecewise smooth Chebyshev function of order k iff this holds for any choice of distinct points $\tau_1, \ldots, \tau_r \in I$, and of positive integers $\mu_1, \ldots, \mu_r$ whose sum is equal to k. ■

Corollary 3.5. *Let Φ be a piecewise smooth Chebyshev function of order k, supposed to be C^{2k-1} on each interval I_j. Then, for all distinct points $\tau_1, \ldots, \tau_r \in I$ and all positive integers $\mu_1, \ldots, \mu_r$ such that $\sum_{i=1}^{r} \mu_i \le k$,*

$$\dim\left(\bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i}\, \Phi(\tau_i)\right) = k - \sum_{i=1}^{r} \mu_i. \tag{3.10}$$

Proof. Given any r distinct points $\tau_1, \ldots, \tau_r \in I$ (with $\tau_i^{\varepsilon_i} \in I$) and any positive integers $\mu_1, \ldots, \mu_r$ such that $\sum_{i=1}^{r} \mu_i \le k$, the equivalence (3.9) is still valid. By the previous theorem, $\Phi^{\sharp(j)}(\tau_i^{\varepsilon_i})$, $1 \le i \le r$, $0 \le j \le \mu_i - 1$, are linearly independent, so that the solutions of the linear system involved in (3.9) now form a $(k - \sum_{i=1}^{r} \mu_i)$-dimensional affine flat. ■

In particular, let us fix $k-1$ points $x_1, \ldots, x_{k-1} \in I$ and suppose that $(x_1, \ldots, x_{k-1})^{\mathrm{ord}} = (\tau_1^{\mu_1} \ldots \tau_r^{\mu_r})$. Then, according to Corollary 3.5, the affine space

$$\mathcal{D} = \bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i}\, \Phi(\tau_i), \tag{3.11}$$

is an affine line. Now, it follows from the definition of the blossom that, while t varies on I, the point $\varphi(x_1, \ldots, x_{k-1}, t)$ moves along the affine line $\mathcal{D}$. As a matter of fact, it will be pointed out in the following section that function φ satisfies a pseudo-affinity property with respect to each variable in the sense that $\varphi(x_1, \ldots, x_{k-1}, \cdot)$ is always a strictly monotone function.

4. The Subblossoming Principle

This section is devoted to the subblossoming principle, that is to say, to the possibility of constructing piecewise smooth Chebyshev functions of lesser orders from a given one by fixing some of the variables in its blossom.

Let $\Phi : I \to \mathcal{A}$ be a piecewise smooth Chebyshev function of order k. Again, we denote by $M_1, \ldots, M_n$ the connection matrices and by $\mathcal{E}$ the space associated with Φ.

For simplicity, we shall assume that Φ is infinitely many times differentiable on each interval I_j, although the results can be adapted in cases of lower order of differentiability (see [26]). So, from now on, the expression "piecewise smooth" (with respect to the t_i's) is always to be interpreted as "piecewise C^∞".

4.1. *Constructing Subblossoms*

Theorem 4.1. *Given $a \in I$, the function $\widetilde{\Phi} : I \to \mathrm{Osc}_{k-1}\Phi(a)$, defined for all $t \in I$ by*

$$\widetilde{\Phi}(t) = \varphi(at^{k-1}), \tag{4.1}$$

is a piecewise smooth Chebyshev function of order $(k-1)$ on I, the blossom of which is given by

$$\widetilde{\varphi}(t_1, \ldots, t_{k-1}) = \varphi(a, t_1, \ldots, t_{k-1}) \qquad \text{for all} \quad t_1, \ldots, t_{k-1} \in I. \tag{4.2}$$

Proof. The proof includes several steps.

(1) Let us first show that $\widetilde{\Phi}$ is a geometrically regular function of order $k-1$.

By (4.1), we have $\widetilde{\Phi}(a) = \Phi(a)$ and, for each $t \in I\backslash\{a\}$,

$$\{\widetilde{\Phi}(t)\} = \mathrm{Osc}_{k-1}\,\Phi(a) \cap \mathrm{Osc}_1\,\Phi(t). \tag{4.3}$$

Since the values of $\widetilde{\Phi}$ all belong to $\mathrm{Osc}_{k-1}\,\Phi(a)$, it is in fact sufficient to prove that $\widetilde{\Phi}$ is $(k-1)$-regular.

- Suppose first that $a \in I_*$.

(i) Let us show that $\widetilde{\Phi}$ is C^∞ on each interval I_j.

For all $t \in I$, there exist real numbers $\mu(t^\varepsilon), \lambda_1(t), \ldots, \lambda_{k-1}(t)$ such that

$$\widetilde{\Phi}(t) = \Phi(t) + \mu(t^\varepsilon)\Phi'(t^\varepsilon) = \Phi(a) + \sum_{s=1}^{k-1} \lambda_s(t)\Phi^{(s)}(a), \tag{4.4}$$

and, on account of (3.5), $\mu(t^\varepsilon) \neq 0$ for $t \neq a$. Observe that

$$\mu(t_\ell^+) = \frac{\mu(t_\ell^-)}{m_{11}^\ell}, \qquad \ell = 1, \ldots, n. \tag{4.5}$$

In order to prove that $\widetilde{\Phi}$ is C^∞ on each I_j, it is sufficient to prove that μ is a C^∞ function on each I_j. For this purpose, let us consider the function $N : I \to \mathbf{R}$ introduced in (3.4). Here,

$$N(t) := \det(\Phi'(a), \ldots, \Phi^{(k-1)}(a), \Phi(t) - \Phi(a)), \tag{4.6}$$

and, for all $t^\varepsilon \in I$,

$$N^{(i)}(t^\varepsilon) = \det(\Phi'(a), \ldots, \Phi^{(k-1)}(a), \Phi^{(i)}(t^\varepsilon)). \tag{4.7}$$

Indeed, when $t \in I_*$, we can get rid of ε everywhere. In particular, for $t = a$, (4.6) and (4.7) lead to

$$\begin{aligned} N(a) &= N'(a) = \cdots = N^{(k-1)}(a) = 0, \\ \Delta &:= N^{(k)}(a) = \det(\Phi'(a), \ldots, \Phi^{(k)}(a)) \neq 0. \end{aligned} \tag{4.8}$$

Moreover, it results from Section 3 that $N'(t^\varepsilon) \neq 0$ for all $t \in I\backslash\{a\}$.

On the other hand, from (4.4) we can deduce that

$$\Phi(t) - \Phi(a) = -\mu(t^\varepsilon)\Phi'(t^\varepsilon) + \sum_{s=1}^{k-1} \lambda_s(t)\Phi^{(s)}(a),$$

which proves that

$$\mu(t^\varepsilon) = \begin{cases} 0 & \text{if } t = a, \\ -\dfrac{N(t)}{N'(t^\varepsilon)} & \text{if } t \neq a. \end{cases} \tag{4.9}$$

Given $j \in \{0, \dots, n\}$, if $a \notin I_j$, μ is clearly C^∞ on I_j. Actually, the lemma which follows will prove that μ is C^∞ on I_j even if $a \in I_j$ and that, additionally,

$$\mu'(a) = -1/k. \tag{4.10}$$

Lemma 4.2. *Let J be a real interval containing a. Suppose that $f : J \to \mathbf{R}$ is C^∞ on J and satisfies $f(a) = f'(a) = \cdots = f^{(k-1)}(a) = 0$, $f^{(k)}(a) \neq 0$, and $f'(t) \neq 0$ for all $t \in J \backslash \{a\}$. Then the function g defined on J by*

$$g(t) = \frac{f(t)}{f'(t)} \quad \text{if} \quad t \neq 0, \quad g(a) = 0, \tag{4.11}$$

is C^∞ on J and $g'(a) = 1/k$.

Proof. The assumption $f(a) = f'(a) = \cdots = f^{(k-1)}(a) = 0$ implies the existence of a function f_1 which is C^∞ on J and which satisfies (see [26])

$$f(t) = (t-a)^k\ f_1(t) \qquad \text{for all} \quad t \in J.$$

Consequently, $f'(t) = (t-a)^{k-1}[kf_1(t) + (t-a)\,f_1{}'(t)]$, and the assumption $f'(t) \neq 0$ for all $t \in J\backslash\{a\}$ implies that $kf_1(t) + (t-a)\,f_1{}'(t) \neq 0$ for $t \neq a$. Therefore, we have

$$g(t) = (t-a)\ \frac{f_1(t)}{kf_1(t) + (t-a)\,f_1{}'(t)} \qquad \text{for all} \quad t \in J\backslash\{a\}.$$

As a matter of fact, since $g(a) = 0$ and $f_1(a) = f^{(k)}(a)/k! \neq 0$, this expression is still valid for $t = a$, which proves that g is C^∞ on the whole interval J. Moreover,

$$g'(a) = \lim_{t\to a} \frac{f_1(t)}{kf_1(t) + (t-a)\,f_1{}'(t)} = \frac{1}{k}. \qquad \blacksquare$$

(ii) Let us study the connections at $t_1, \dots, t_n$. According to part (i), we can differentiate (4.4) up to order i ($i \leq k-1$) on each I_ℓ, which gives

$$\widetilde{\Phi}^{(i)}(t^\varepsilon) = \mu(t^\varepsilon)\Phi^{(i+1)}(t^\varepsilon) + (1 + i\mu'(t^\varepsilon))\Phi^{(i)}(t^\varepsilon) + G_i(t^\varepsilon), \tag{4.12}$$

where $G_i(t^\varepsilon)$ is a linear combination of $\Phi'(t^\varepsilon), \dots, \Phi^{(i-1)}(t^\varepsilon)$. In particular, since $\mu(a) = 0$, for $t = a$, (4.12) reduces to

$$\widetilde{\Phi}^{(i)}(a) = (1 + i\mu'(a))\Phi^{(i)}(a) + G_i(a). \tag{4.13}$$

Observe that the coefficient $(1 + i\mu'(a))$ does not vanish for $i \le k - 1$, due to equality (4.10). So, we eventually obtain

$$\mathrm{Osc}_i\, \widetilde{\Phi}(a) = \mathrm{Osc}_i\, \Phi(a), \qquad i = 0, \dots, k - 1. \tag{4.14}$$

On the contrary, for $t \neq a$, relations (4.12), together with the left equality in (4.4), can be summarized as follows:

$$(\widetilde{\Phi}(t) - \Phi(t), \widetilde{\Phi}'(t^\varepsilon), \dots, \widetilde{\Phi}^{(k-1)}(t^\varepsilon))^T = R(t^\varepsilon) \cdot D_k\Phi(t^\varepsilon), \tag{4.15}$$

where $R(t^\varepsilon)$ stands for a lower triangular matrix of order k, all the diagonal elements of which are equal to $\mu(t^\varepsilon)$. For $t \neq a$, on account of (3.5) and (4.4), $\mu(t^\varepsilon) \neq 0$, and (4.15) proves the linear independence of the k vectors involved in its left-hand side. In particular, for all $i \le k-1$, $\mathrm{Osc}_i^\varepsilon\, \widetilde{\Phi}(t)$ is of dimension i. Moreover, relation (4.12) clearly implies that for all $t \neq a$,

$$\mathrm{Osc}_i^\varepsilon\, \widetilde{\Phi}(t) \subset \mathrm{Osc}_{i+1}\Phi(t) \cap \mathrm{Osc}_{k-1}\, \Phi(a), \qquad i = 0, \dots, k - 1. \tag{4.16}$$

By Corollary 3.5, the right-hand side of (4.16) has dimension i, i.e., the same dimension as $\mathrm{Osc}_i^\varepsilon\, \widetilde{\Phi}(t)$. Thus, we obtain the equality

$$\mathrm{Osc}_i^\varepsilon\, \widetilde{\Phi}(t) = \mathrm{Osc}_{i+1}\, \Phi(t) \cap \mathrm{Osc}_{k-1}\, \Phi(a) \qquad \text{for all} \quad t \neq a. \tag{4.17}$$

Consequently, for $\ell = 1, \dots, n$, (4.17) proves that $\mathrm{Osc}_i^+\, \widetilde{\Phi}(t_\ell) = \mathrm{Osc}_i^-\, \widetilde{\Phi}(t_\ell)$ for all $i = 0, \dots, k - 1$. This proves the existence of n regular lower triangular matrices $\widetilde{M}_\ell$ of order $k - 1$ such that

$$D_{k-1}\widetilde{\Phi}(t_\ell{}^+) = \widetilde{M}_\ell \cdot D_{k-1}\widetilde{\Phi}(t_\ell{}^-). \tag{4.18}$$

Moreover, on account of (4.15) and (4.5), the diagonal elements of $\widetilde{M}_\ell$ are

$$\widetilde{m}_{ii}^\ell = \frac{\mu(t_\ell{}^+)}{\mu(t_\ell{}^-)}\, m_{i+1,i+1}^\ell = \frac{m_{i+1,i+1}^\ell}{m_{11}^\ell}, \qquad i = 1, \dots, k - 1, \tag{4.19}$$

hence they are positive.

- Suppose now that $a = t_{\ell_0}$, $\ell_0 \in \{1, \dots, n\}$.
  For $t \neq a$, we can write

$$\widetilde{\Phi}(t) = \Phi(t) + \mu(t^\varepsilon)\Phi'(t^\varepsilon) = \Phi(a) + \sum_{s=1}^{k-1} \lambda_s(t)\Phi^{(s)}(a^+), \tag{4.20}$$

where $\mu(t^\varepsilon)$ is defined as in (4.9), but now with

$$N(t) := \det(\Phi'(a^+), \dots, \Phi^{(k-1)}(a^+), \Phi(t) - \Phi(a)).$$

Again, it appears clearly that μ is C^∞ on each I_j, hence so is $\widetilde{\Phi}$. Moreover, formulas (4.12) and (4.17) are still valid for $t \neq a$. In particular, the connections at the points t_ℓ, $\ell \neq \ell_0$, are still given by (4.18) and (4.19). On the other hand, (4.13) must now be replaced by

$$(4.21)\quad \widetilde{\Phi}^{(i)}(a^+) = (1 + i\mu'(a^+))\Phi^{(i)}(a^+) + G_i(a^+), \qquad i = 1, \ldots, k-1.$$

Here, $\mu'(a^+) = -1/k$. Therefore, relations (4.21) can be summarized by

$$(4.22)\qquad D_{k-1}\widetilde{\Phi}(a^+) = R_+ \cdot D_{k-1}\Phi(a^+),$$

where R_+ is a lower triangular matrix of order $k-1$, with $(1-i/k)_{i=1\ldots,k-1}$ as its diagonal elements. Starting from the left derivatives of Φ at a instead of its right derivatives in (4.20) would symmetrically give

$$(4.23)\qquad D_{k-1}\widetilde{\Phi}(a^-) = R_- \cdot D_{k-1}\Phi(a^-),$$

where the lower triangular matrix R_- has the same diagonal elements as R_+. Taking into account the equality $D_k\Phi(a^+) = M_{\ell_0} \cdot D_k\Phi(a^-)$, (4.22) and (4.23) eventually lead to the following relation:

$$D_{k-1}\widetilde{\Phi}({t_{\ell_0}}^+) = \widetilde{M}_{\ell_0} \,.\, D_{k-1}\widetilde{\Phi}({t_{\ell_0}}^-),$$

with $\widetilde{M}_{\ell_0} := R_+ \cdot \overline{M}_{\ell_0} \cdot R_-^{-1}$, the matrix $\overline{M}_{\ell_0}$ being obtained by suppressing the last row and column of M_{ℓ_0}. Hence, the diagonal elements of $\widetilde{M}_{\ell_0}$ are $(m_{11}^{\ell_0}, \ldots, m_{k-1,k-1}^{\ell_0})$. Again, they are positive.

(2) In any case, we have proved that $\widetilde{\Phi}$ is a regular function of order $k-1$ and that its osculating flats at a point t are obtained by (4.14) or (4.17) depending on whether $t = a$ or not. The proof of Theorem 4.1 will be carried out by verifying that

$$(4.24)\qquad \bigcap_{i=1}^{r} \mathrm{Osc}_{k-1-\mu_i}\,\widetilde{\Phi}(\tau_i) = \{\varphi(a{\tau_1}^{\mu_1}\ldots{\tau_r}^{\mu_r})\},$$

for any distinct $\tau_1, \ldots, \tau_r \in I$ and any positive integers $\mu_1, \ldots, \mu_r$ such that $\sum_{i=1}^r \mu_i = k-1$. Suppose first that $a \notin \{\tau_1, \ldots, \tau_r\}$. Then, equality (4.17) implies

$$\bigcap_{i=1}^{r} \mathrm{Osc}_{k-1-\mu_i}\,\widetilde{\Phi}(\tau_i) = \bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i}\,\Phi(\tau_i) \;\cap\; \mathrm{Osc}_{k-1}\,\Phi(a)\;,$$

from which (4.24) results by definition (3.1).

Suppose now that $a \in \{\tau_1, \ldots, \tau_r\}$, for instance $a = \tau_r$. Then, we can derive from (4.17) and (4.14) that

$$\begin{aligned}\bigcap_{i=1}^{r} \mathrm{Osc}_{k-1-\mu_i}\,\widetilde{\Phi}(\tau_i) &= \bigcap_{i=1}^{r-1} \big(\mathrm{Osc}_{k-\mu_i}\,\Phi(\tau_i) \cap \mathrm{Osc}_{k-1}\,\Phi(a)\big) \;\cap\; \mathrm{Osc}_{k-1-\mu_r}\,\Phi(a)\\ &= \bigcap_{i=1}^{r-1} \mathrm{Osc}_{k-\mu_i}\,\Phi(\tau_i) \;\cap\; \mathrm{Osc}_{k-(\mu_r+1)}\,\Phi(a)\\ &= \{\varphi(\tau_1^{\mu_1}\ldots\tau_{r-1}^{\mu_{r-1}}a^{\mu_r+1}\},\end{aligned}$$

which, in this case, is the exact equality (4.24). ■

4.2. *A de Casteljau-Type Algorithm*

The construction of *subblossoms* can be iterated: this eventually leads to the pseudo-affinity property of the blossom.

Corollary 4.3. *Let $x_1, \ldots, x_{k-1}$ be any fixed points in I. Then, given $a, b \in I$, with $a < b$, there exists a strictly increasing continuous function $\alpha : I \to \mathbf{R}$ such that, for all $\mathcal{E}$-functions F and for all $t \in I$,*

$$f(x_1, \ldots, x_{k-1}, t) = [1-\alpha(t)]\, f(x_1, \ldots, x_{k-1}, a) + \alpha(t)\, f(x_1, \ldots, x_{k-1}, b), \tag{4.25}$$

with $\alpha(a) = 0$ and $\alpha(b) = 1$. Moreover, this function α is C^∞ on each interval I_j.

Proof. Step by step, it follows from Theorem 4.1 that the function $\widehat{\Phi}$ defined on I by

$$\widehat{\Phi}(t) = \varphi(x_1, \ldots, x_{k-1}, t), \qquad t \in I, \tag{4.26}$$

is a piecewise smooth Chebyshev function of order 1 on I with values in the affine line $\mathcal{D} = \bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i} \Phi(\tau_i)$, where $(x_1, \ldots, x_{k-1})^{\mathrm{ord}} = (\tau_1^{\mu_1} \ldots \tau_r^{\mu_r})$. As already noticed in Section 3, such a function is one-to-one on I, hence strictly monotone. Thus, it can be written as

$$\varphi(x_1, \ldots, x_{k-1}, t) = [1-\alpha(t)]\varphi(x_1, \ldots, x_{k-1}, a) + \alpha(t)\varphi(x_1, \ldots, x_{k-1}, b), \tag{4.27}$$

with the required properties for α. Given any $\mathcal{E}$-function F, equality (4.25) is obtained by taking the image of (4.27) under the affine map h which satisfies $F = h \circ \Phi$. ■

Setting $\mathcal{T} := (x_1, \ldots, x_{k-1})$, Corollary 4.3 may be summarized by the following diagram:

$$\begin{array}{ccc} f(\mathcal{T}, a) & & f(\mathcal{T}, b) \\ & \searrow \quad \swarrow & \\ & f(\mathcal{T}, t) & \end{array} \tag{4.28}$$

in which the two arrows stand for the affine combination involved in (4.25). Then, starting from the $(n+1)$ points $f(a^{k-i}b^i)$, $i = 0, \ldots, k$, for a given $t \in I$, we can compute $F(t)$ in k steps as follows:

first iteration: compute $f(a^{k-1-i}b^i t)$, $i = 0, \ldots, k-1$;
second iteration: compute $f(a^{k-2-i}b^i t^2)$, $i = 0, \ldots, k-2$;

and so forth up to:

next to last iteration: compute $f(at^{k-1})$, $f(bt^{k-1})$;
last iteration: compute $f(t^k) = F(t)$.

Each computation is obtained by means of an affine combination the coefficients of which do not depend on F and are positive as soon as t belongs to $]a, b[$.

This algorithm will be called *the Chebyshev–de Casteljau algorithm with respect to* (a, b).

Definition and Theorem 4.4. *Let a, b be two distinct points of I. Then, for any $\mathcal{E}$-function F, the $k+1$ points $P_i := f(a^{k-i}b^i)$, $i = 0, \ldots, k$, are called the* Chebyshev–Bézier points of F with respect to (a, b). *They satisfy*

$$\text{aff}\,\text{Im}(F) = \text{aff}(P_0, \ldots, P_k) \tag{4.29}$$

(*in particular, F is nondegenerate iff its Chebyshev–Bézier points are affinely independent*), *and, for all $i = 0, \ldots, k$,*

$$\text{Osc}_i\; F(a) = \text{aff}(P_0, \ldots, P_i), \qquad \text{Osc}_i\; F(b) = \text{aff}(P_{k-i}, \ldots, P_k). \tag{4.30}$$

Proof. Let us denote by $\Pi_0, \ldots, \Pi_k$, the Chebyshev–Bézier points of Φ with respect to (a, b), i.e., $\Pi_i = \varphi(a^{k-i}b^i)$, $i = 0, \ldots, k$. By the Chebyshev–de Casteljau algorithm, each $\Phi(t)$, $t \in I$, can be obtained by means of an affine combination of $\Pi_0, \ldots, \Pi_k$. Hence, $\text{Im}(\Phi) \subset \text{aff}(\Pi_0, \ldots, \Pi_k)$. Since $\text{aff}\,\text{Im}(\Phi)$ is of dimension k, it follows that:

$$\text{aff}\,\text{Im}(\Phi) = \text{aff}\,(\Pi_0, \ldots, \Pi_k), \tag{4.31}$$

and that $\Pi_0, \ldots, \Pi_k$ are affinely independent.

As a matter of fact, $\Pi_0, \ldots, \Pi_k$ are defined by

$$\{\Pi_i\} = \text{Osc}_i\; \Phi(a) \cap \text{Osc}_{k-i}\; \Phi(b), \qquad 0 \le i \le k. \tag{4.32}$$

So that, in particular, the $i+1$ points $\Pi_0, \ldots, \Pi_i$ belong to $\text{Osc}_i\; \Phi(a)$. Their affine independence and the fact that $\text{Osc}_i\; \Phi(a)$ is of dimension i proves that

$$\text{Osc}_i\; \Phi(a) = \text{aff}(\Pi_0, \ldots, \Pi_i). \tag{4.33}$$

Now, if $F = h \circ \Phi$, for all $i = 0, \ldots, k$, $P_i = h(\Pi_i)$. Formula (4.29) and the first part of (4.30) are obtained by taking the images of (4.31) and (4.33) under h. The second equality in (4.30) can be obtained by exchanging a and b. ■

Corollary 4.5. *Let $\tau_1, \ldots, \tau_r$ be any distinct points of I and let $\mu_1, \ldots, \mu_r$ be any positive integers such that $\mu := \sum_{i=1}^r \mu_i \le k$. Then, for any nondegenerate $\mathcal{E}$-function F,*

$$\bigcap_{i=1}^{r} \text{Osc}_{k-\mu_i}\; F(\tau_i) = \text{aff}\{f(\tau_1^{\mu_1} \ldots \tau_r^{\mu_r} t_1 \ldots t_{k-\mu}) \mid t_1, \ldots, t_{k-\mu} \in W\}, \tag{4.34}$$

where W denotes any subset of I containing at least two distinct points.

Proof. Applying Theorem 3.3 and Corollary 3.5 to function F proves that the affine flat $\bigcap_{i=1}^r \text{Osc}_{k-\mu_i} F(\tau_i)$ is of dimension $k - \mu$. Furthermore, each point

$$f(\tau_1^{\mu_1} \ldots \tau_r^{\mu_r} t_1 \ldots t_{k-\mu})$$

clearly belongs to $\bigcap_{i=1}^r \text{Osc}_{k-\mu_i} F(\tau_i)$.

On the other hand, iterating Theorem 4.1 shows that the function Ψ defined by

$$\Psi(t) = f(\tau_1^{\mu_1} \ldots \tau_r^{\mu_r} t^{k-\mu}),$$

is a piecewise smooth Chebyshev function of order $k-\mu$ on I, the blossom of which is given by

$$\psi(t_1,\ldots,t_{k-\mu}) = f(\tau_1^{\mu_1}\ldots\tau_r^{\mu_r}t_1\ldots t_{k-\mu}) \qquad \text{for all} \quad (t_1,\ldots,t_{k-\mu})\in I^{k-\mu}.$$

Hence, by Theorem 4.4, the $k-\mu+1$ Chebyshev–Bézier points of Ψ with respect to any two distinct elements of W are affinely independent; they belong to $\bigcap_{i=1}^{r}\mathrm{Osc}_{k-\mu_i}F(\tau_i)$, which proves equality (4.34). ■

Theorem 4.6. *Two $\mathcal{E}$-functions F_1 and F_2 have a contact of order $s\le k$ at a point $a\in I$ iff $f_1(\mathcal{T})=f_2(\mathcal{T})$ for all $\mathcal{T}\in I^k$ containing (a^{k-s}).*

Proof. As a particular case of Corollary 4.5, we obtain

$$\mathrm{Osc}_s\,\Phi(a) = \mathrm{aff}\{\varphi(a^{k-s}t_1\ldots t_s)\mid t_1,\ldots,t_s\in I\}. \tag{4.35}$$

Hence, if $F_1=h_1\circ\Phi$, $F_2=h_2\circ\Phi$, Theorem 2.8 ensures that F_1 and F_2 have a contact of order s at a iff $h_1(\varphi(a^{k-s}t_1\ldots t_s)) = h_2(\varphi(a^{k-s}t_1\ldots t_s))$ for all $t_1,\ldots,t_s\in I$, i.e., by (3.6), iff $f_1(\mathcal{T})=f_2(\mathcal{T})$ for all $\mathcal{T}\in I^k$ containing (a^{k-s}). ■

According to Theorem 4.4, the Chebyshev–Bézier points $(\Pi_0,\ldots,\Pi_k)$ of Φ with respect to (a,b) form an affine frame of $\mathrm{aff}\,\mathrm{Im}(\Phi)$. So, we can express Φ as follows:

$$\Phi(t)=\sum_{i=0}^{k}\mathcal{B}_i(t)\Pi_i, \qquad \sum_{i=0}^{k}\mathcal{B}_i(t)=1, \qquad t\in I. \tag{4.36}$$

Theorem and Definition 4.7. *The functions $(\mathcal{B}_0,\ldots,\mathcal{B}_k)$ form a basis of $\mathcal{E}$, called the* Chebyshev–Bernstein basis *of $\mathcal{E}$ with respect to (a,b). They satisfy*

$$0<\mathcal{B}_i(t)<1 \quad \textit{for all } t\in\,]a,b[\textit{ and for all } i=0,\ldots,k, \tag{4.37}$$

$$\mathcal{B}_i{}^{(j)}(a^{\varepsilon})=0 \quad \textit{for } j<i\,, \quad \mathcal{B}_i{}^{(j)}(b^{\varepsilon'})=0 \quad \textit{for } j<k-i, \tag{4.38}$$

$$\mathcal{B}_i^{(i)}(a^{\varepsilon}) = \frac{\det[\Phi'(a^{\varepsilon}),\ldots,\Phi^{(i)}(a^{\varepsilon}),\Phi'(b^{\varepsilon'}),\ldots,\Phi^{(k-i)}(b^{\varepsilon'})]}{\det[\Phi'(a^{\varepsilon}),\ldots,\Phi^{(i-1)}(a^{\varepsilon}),\Phi(b)-\Phi(a),\Phi'(b^{\varepsilon'}),\ldots,\Phi^{(k-i)}(b^{\varepsilon'})]}. \tag{4.39}$$

Proof. The fact that $(\mathcal{B}_0,\ldots,\mathcal{B}_k)$ is a basis of $\mathcal{E}$ results from Corollary 2.3. When $t\in\,]a,b[$, as a straightforward consequence of the positivity of all the coefficients of the Chebyshev–de Casteljau algorithm, we can derive that $\Phi(t)$ is a strictly convex combination of the points $\Pi_0,\ldots,\Pi_k$, which gives (4.37). Finally, formulas (4.38) and (4.39) can be deduced from (4.30) by differentiating (4.36). ■

Of course, taking the image of both sides of equality (4.36) under any affine map h defined on $\mathrm{aff}\,\mathrm{Im}(\Phi)$ proves that any $\mathcal{E}$-function F (for instance, any $F\in\mathcal{E}$) can be written by means of its Chebyshev–Bézier points $P_0,\ldots,P_k$ as

$$F(t)=\sum_{i=0}^{k}\mathcal{B}_i(t)P_i. \tag{4.40}$$

Thus, given two distinct points $a, b \in I$, the corresponding Chebyshev–Bernstein basis depends only on the space $\mathcal{E}$. Furthermore, by applying equality (4.40) to the Chebyshev–Bernstein function $\mathcal{B}_i$ $(0 \le i \le k)$, we can conclude that its Chebyshev–Bézier points with respect to (a, b) are all equal to 0, except the one of index i which is equal to 1, i.e., its blossom b_i satisfies

$$b_i(a^{k-j}b^j) = \delta_{ij}, \qquad j = 0, \ldots, k. \tag{4.41}$$

Finally, formulas (4.38) and (4.39) lead to the following expressions (see [26]):

(4.42)

$$\begin{aligned}\mathcal{B}_i(t) \;=\; &\frac{\det[\Phi'(a^\varepsilon), \ldots, \Phi^{(i)}(a^\varepsilon), \Phi'(b^{\varepsilon'}), \ldots, \Phi^{(k-i)}(b^{\varepsilon'})]}{\det[\Phi'(a^\varepsilon), \ldots, \Phi^{(i-1)}(a^\varepsilon), \Phi(b)-\Phi(a), \Phi'(b^{\varepsilon'}), \ldots, \Phi^{(k-i)}(b^{\varepsilon'})]}\\ &\times\frac{\begin{array}{c}\det[\Phi(b)-\Phi(a), \Phi'(a^\varepsilon), \ldots, \Phi^{(i-1)}(a^\varepsilon),\\ \Phi(t)-\Phi(a), \Phi'(b^{\varepsilon'}), \ldots, \Phi^{(k-i-1)}(b^{\varepsilon'})]\end{array}}{\det[\Phi(b)-\Phi(a), \Phi'(a^\varepsilon), \ldots, \Phi^{(i)}(a^\varepsilon), \Phi'(b^{\varepsilon'}), \ldots, \Phi^{(k-i-1)}(b^{\varepsilon'})]}\end{aligned}$$

for $1 \le i \le k-1$, whereas

$$\mathcal{B}_k(t) = \frac{\det[\Phi'(a^\varepsilon), \ldots, \Phi^{(k-1)}(a^\varepsilon), \Phi(t)-\Phi(a)]}{\det[\Phi'(a^\varepsilon), \ldots, \Phi^{(k-1)}(a^\varepsilon), \Phi(b)-\Phi(a)]}, \tag{4.43}$$

the function $\mathcal{B}_0$ being obtained symmetrically by exchanging a and b.

5. Splines Based on a Piecewise Smooth Chebyshev Function

As in the previous section, we suppose that $\Phi : I \to \mathcal{A}$ is a piecewise smooth Chebyshev function of order k, $\mathcal{E}$ denoting its associated space. We shall now deal with the corresponding $\mathcal{E}$-splines associated with the knot vector T defined in (2.26).

5.1. *The Blossom of a Spline*

Theorem and Definition 5.1. *Consider a nondegenerate $\mathcal{E}$-spline S given by* (2.30) *and an admissible k-tuple $\mathcal{T}$ such that $\mathcal{T}^{\text{ord}} = (\tau_1^{\mu_1} \ldots \tau_r^{\mu_r})$. Then, if $D := \mathcal{D}(T)$, the affine flat $\bigcap_{i=1}^r \mathrm{Osc}_{k-\mu_i} S_D(\tau_i)$ consists of a single point. When setting*

$$\{s(\mathcal{T})\} := \bigcap_{i=1}^r \mathrm{Osc}_{k-\mu_i} S_D(\tau_i), \tag{5.1}$$

we define a symmetric function s on the set of all admissible k-tuples, called the blossom of S. *It satisfies*

$$s(\mathcal{T}) = f_\ell(\mathcal{T}) \qquad \textit{for all} \quad \ell \in \mathcal{J}(T). \tag{5.2}$$

Proof. For any $\ell = 0, \ldots, n$, F_ℓ is a nondegenerate $\mathcal{E}$-function. Thus, by Theorems 2.16 and 3.3,

$$\bigcap_{i=1}^r \mathrm{Osc}_{k-\mu_i} S_D(\tau_i) = \{f_\ell(\mathcal{T})\} \qquad \text{for all} \quad \ell \in \mathcal{J}(T). \qquad \blacksquare$$

Corollary 5.2. *If S is a nondegenerate $\mathcal{E}$-spline and $\mathcal{T} \in I^{\mu}$ an admissible μ-tuple ($\mu \le k$), with $\mathcal{T}^{\mathrm{ord}} = (\tau_1^{\mu_1} \dots \tau_r^{\mu_r})$ and $D = \mathcal{D}(T)$, then*

$$\bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i}\, S_D(\tau_i) = \mathrm{aff}\{s(\mathcal{T}, t_1, \dots, t_{k-\mu}) \mid t_1, \dots, t_{k-\mu} \in W\}, \tag{5.3}$$

where W denotes any subset of D containing at least two distinct points.

Proof. For a given $\ell \in \mathcal{J}(T)$, we can apply Corollary 4.5 to the nondegenerate $\mathcal{E}$-function F_ℓ. More precisely, by taking Theorem 2.16 into account, (4.34) gives

$$\bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i}\, S_D(\tau_i) = \mathrm{aff}\{ f_\ell(\mathcal{T}, t_1, \dots, t_{k-\mu}) \mid t_1, \dots, t_{k-\mu} \in W\}. \tag{5.4}$$

Now, for any $t_1, \dots, t_{k-\mu} \in W$, ℓ also belongs to $\mathcal{J}(\mathcal{T}, t_1, \dots, t_{k-\mu})$. Hence, according to (5.2),

$$f_\ell(\mathcal{T}, t_1, \dots, t_{k-\mu}) = s(\mathcal{T}, t_1, \dots, t_{k-\mu}),$$

which proves (5.3). ■

Let us select a particular nondegenerate $\mathcal{S}$-function Σ (i.e., an $\mathcal{E}$-spline Σ such that $\dim(\mathrm{aff}\,\mathrm{Im}(\Sigma)) = k + m + 1$); such a spline Σ plays the same rôle as the universal spline introduced by H.-P. Seidel [37]. Since it is a nondegenerate $\mathcal{E}$-spline, its blossom σ can be defined as in (5.1). Now, any $\mathcal{E}$-spline S can be written as $S = h \circ \Sigma$, h being an affine map defined on the affine space $\mathrm{aff}\,\mathrm{Im}(\Sigma)$. If the spline S is nondegenerate, for any admissible k-tuple $\mathcal{T}$, (5.1) leads to

$$\{s(\mathcal{T})\} = \bigcap_{i=1}^{r} h(\mathrm{Osc}_{k-\mu_i}\, \Sigma_D(\tau_i)) \supset h\left(\bigcap_{i=1}^{r} \mathrm{Osc}_{k-\mu_i}\, \Sigma_D(\tau_i)\right) = \{h(\sigma(\mathcal{T}))\},$$

which means that $s = h \circ \sigma$. More generally, the blossom s of any (possibly degenerate) $\mathcal{E}$-spline $S = h \circ \Sigma$ will be defined as

$$s := h \circ \sigma, \tag{5.5}$$

If Φ_ℓ is the nondegenerate $\mathcal{E}$-function which coincides with Σ on I_ℓ, by (5.2), for all admissible k-tuples $\mathcal{T}$ and all $\ell \in \mathcal{J}(T)$, $\sigma(\mathcal{T}) = \varphi_\ell(\mathcal{T})$. Thus, we also have $h \circ \sigma(\mathcal{T}) = h \circ \varphi_\ell(\mathcal{T})$, which means that

$$s(\mathcal{T}) = f_\ell(\mathcal{T}) \qquad \text{for all} \quad \ell \in \mathcal{J}(T),$$

where F_ℓ is the $\mathcal{E}$-function which coincides with S on I_ℓ. This shows that equality (5.2) is still valid even if the spline S is degenerate. As a matter of fact, due to the contact theorem 4.6, equality (5.2) could directly have been taken as the definition of the blossom of any $\mathcal{E}$-spline S, whether this spline is nondegenerate or degenerate. However, although different piecewise smooth Chebyshev functions Φ may lead to the same space $\mathcal{S}$ of splines, Definition 5.1 proves that they all provide the same notion of blossom for $\mathcal{S}$.

5.2. *A de Boor-Type Algorithm*

We shall rewrite the knot vector $T=(t_1^{m_1},\dots,t_n^{m_n})$ as follows:

(5.6) $$T=(x_1,\dots,x_m).$$

In particular, if $m=\sum_{i=1}^n m_i>0$, x_1 is the first t_ℓ of nonzero multiplicity, and x_m the last one. Let us choose $2(k+1)$ additional points $x_{-k},\dots,x_0\in I_0$ and $x_{m+1},\dots,x_{m+k+1}\in I_n$ with $x_{-k}\le x_{-k+1}\le\cdots\le x_0<t_1$ and $t_n<x_{m+1}\le\cdots\le x_{m+k}\le x_{m+k+1}$, so as to obtain

(5.7) $$T'=(x_{-k},x_{-k+1},\dots,x_{m+k},x_{m+k+1}).$$

Consider the following k-tuples composed of k consecutive points of the knot vector T':

(5.8) $$X_j=(x_{j+1},\dots,x_{j+k}),\qquad j=-k,\dots,m.$$

These $(m+k+1)$ k-tuples are clearly admissible. This remark gives sense to the definition hereunder.

Definition 5.3. If s denotes the blossom of an $\mathcal{E}$-spline S, the $k+m+1$ points

(5.9) $$Q_j:=s(X_j),\qquad j=-k,\dots,m,$$

are called the `poles` of S with respect to the knot vector T'.

The following algorithm, which we shall refer to as *the Chebyshev–de Boor algorithm with respect to the knot vector* T', will allow us to compute each point $S(t)$, $t\in I$, in k steps from the poles of S.

Lemma 5.4. *Given $i=0,\dots,n$, let us set $j_i:=\sum_{\ell=1}^i m_\ell$ (so that, $j_0=0$ and $j_n=m$). Then, for $j=-k,\dots,m$, $i\in\mathcal{J}(X_j)$ iff $j_i-k\le j\le j_i$.*

Proof. Observe that for $0<i<n$, j_i is the unique integer s such that $I_i\subset[x_s,x_{s+1}]$, this inclusion being strict iff one (at least) of the two multiplicities m_i and m_{i+1} is equal to 0.

We can verify that for any $i=0,\dots,n$, and any $j=-k,\dots,m$,

(5.10) $$i\in\mathcal{J}(X_j)\quad\Leftrightarrow\quad[x_{j_i},x_{j_i+1}]\subset\mathcal{D}(X_j).$$

On the other hand, we have clearly

(5.11) $$[x_j,x_{j+k+1}]\subset\mathcal{D}(X_j),\qquad j=-k,\dots,m,$$

and actually, $\mathcal{D}(X_j)=[x_j,x_{j+k+1}]$ for $0<j<m-k$. Since $\mathcal{D}(X_j)$ is a union of consecutive subintervals, as soon as it contains one of the points $x_{-k},\dots,x_0$, it also contains I_0. Thus, for $j\le 0$, $I_0\subset\mathcal{D}(X_j)$. Similarly, for $j\ge m-k$, $I_n\subset\mathcal{D}(X_j)$.

Due to (5.10) and (5.11), the condition $[x_{j_i},x_{j_i+1}]\subset[x_j,x_{j+k+1}]$ is sufficient to ensure that $i\in\mathcal{J}(X_j)$. In other words,

$$j_i-k\le j\le j_i\quad\Rightarrow\quad i\in\mathcal{J}(X_j).$$

In order to prove the converse property, just observe that x_{j_i} is the right endpoint of $\mathcal{D}(X_{j_i-k-1})$ (for $j_i > 0$, hence $i > 0$) and that x_{j_i+1} is the left endpoint of $\mathcal{D}(X_{j_i+1})$ (for $j_i < m$, hence $i < n$). ■

Equality (5.2) being valid for any $\mathcal{E}$-spline S defined by (2.30), the previous lemma shows that, among the $k+m+1$ poles $Q_j = s(X_j)$, $j = -k, \dots, m$, exactly $k+1$ ones can be labeled by means of the blossom f_i, namely

$$Q_j = f_i(X_j), \qquad j = j_i - k, \dots, j_i. \tag{5.12}$$

For a given $i \in \{0, \dots, n\}$ and a given $\nu \in \{0, \dots, k\}$, we now introduce the following $k - \nu + 1$ points depending on $t \in I$:

$$Q_j^\nu(t) := f_i(x_{j+1}, \dots, x_{j+k-\nu}, t^\nu), \qquad j = j_i - k + \nu, \dots, j_i. \tag{5.13}$$

In particular $Q_j^0(t) = Q_j$ for all $j = j_i - k, \dots, j_i$, and $Q_{j_i}^k(t) = F_i(t)$, which gives $Q_{j_i}^k(t) = S(t)$ when $t \in I_i$.

For $\nu \geq 1$, Corollary 4.3 may be applied to function $f_i(x_{j+1}, \dots, x_{j+k-\nu}, t^{\nu-1}, \cdot)$. Now, for $j = j_i - k + \nu, \dots, j_i$,

$$x_j \leq x_{j_i} < x_{j_i+1} \leq x_{j+k-\nu+1}. \tag{5.14}$$

Therefore, since $Q_j^\nu(t) = f_i(x_{j+1}, \dots, x_{j+k-\nu}, t^{\nu-1}, t)$, there exists a real number $\alpha_j^\nu(t)$, independent of S, such that

$$\begin{aligned} Q_j^\nu(t) \;=\; & [1 - \alpha_j^\nu(t)]\, f_i(x_{j+1}, \dots, x_{j+k-\nu}, t^{\nu-1}, x_j) \\ & + \alpha_j^\nu(t)\, f_i(x_{j+1}, \dots, x_{j+k-\nu}, t^{\nu-1}, x_{j+k-\nu+1}), \end{aligned}$$

that is to say,

$$Q_j^\nu(t) = [1 - \alpha_j^\nu(t)] Q_{j-1}^{\nu-1}(t) + \alpha_j^\nu(t) Q_j^{\nu-1}(t). \tag{5.15}$$

Finally, at the last step of the algorithm described in (5.15), for all $t \in I$, we obtain $Q_{j_i}^k(t) = F_i(t)$ as an affine combination of the $k+1$ poles Q_j, $j = j_i - k, \dots, j_i$. More precisely, we can state the following result:

Theorem 5.5. *Let Q_j, $j = -k, \dots, m$, be the poles of a given $\mathcal{E}$-spline S. Then, for all $i = 0, \dots, n$, and for all $t \in [x_{j_i}, x_{j_i+1}]$ (resp., $t \in]x_{j_i}, x_{j_i+1}[$), $S(t)$ is a convex (resp., strictly convex) combination of $Q_{j_i-k}, \dots, Q_{j_i}$.*

Proof. Taking (5.14) into account, Corollary 4.3 proves that when $t \in]x_{j_i}, x_{j_i+1}[$, all the real numbers $\alpha_j^\nu(t)$, $\nu = 0, \dots, k$, $j = j_i - k + \nu, \dots, j_i$, involved in the Chebyshev–de Boor algorithm belong to $]0, 1[$. Thus, for all $t \in]x_{j_i}, x_{j_i+1}[$, $F_i(t)$ is a strictly convex combination of the poles $Q_{j_i-k}, \dots, Q_{j_i}$. Now, $S(t) = F_i(t)$ for all $t \in [x_{j_i}, x_{j_i+1}]$. Indeed, as soon as there exists $\ell \in \{1, \dots, n\}$ such that $t_\ell \in]x_{j_i}, x_{j_i+1}[$, the corresponding multiplicity m_ℓ is equal to zero, which implies $F_{\ell-1} = F_\ell = F_i$. ■

Corollary 5.6. *The spline S is nondegenerate iff for all $i = 0, \ldots, n$, its $k+1$ consecutive poles $(Q_{j_i-k}, \ldots, Q_{j_i})$ are affinely independent, whereas it is a nondegenerate $\mathcal{S}$-function iff all its poles are affinely independent.*

Proof. Let us first observe that $[x_0, t_1] \subset [x_0, x_1]$. According to Remark 2.7(ii), we know that $\operatorname{aff}\operatorname{Im}(S_{|I_0}) = \operatorname{aff}\operatorname{Im}(S_{|[x_0,t_1]})$, hence $\operatorname{aff}\operatorname{Im}(S_{|I_0}) \subset \operatorname{aff}\operatorname{Im}(S_{|[x_0,x_1]})$. Similarly, we have $\operatorname{aff}\operatorname{Im}(S_{|I_n}) \subset \operatorname{aff}\operatorname{Im}(S_{|[x_m,x_{m+1}]})$. Moreover, for $0 < i < n$, $I_i \subset [x_{j_i}, x_{j_i+1}]$. Thus, the previous theorem implies that

$$\operatorname{aff}\operatorname{Im}(S_{|I_i}) \subset \operatorname{aff}(Q_{j_i-k}, \ldots, Q_{j_i}) \qquad \text{for all} \quad i = 0, \ldots, n. \tag{5.16}$$

Applying (5.16) to the nondegenerate $\mathcal{S}$-function Σ previously selected, we can deduce in particular that $\operatorname{aff}\operatorname{Im}(\Sigma) \subset \operatorname{aff}\{\sigma(X_j) \mid j = -k, \ldots, m\}$, where σ is the blossom of Σ. Since $\operatorname{aff}\operatorname{Im}(\Sigma)$ is of dimension $k+m+1$, we have

$$\operatorname{aff}\operatorname{Im}(\Sigma) = \operatorname{aff}\{\sigma(X_j) \mid j = -k, \ldots, m)\}, \tag{5.17}$$

which proves the linear independence of the $k+m+1$ poles $\sigma(X_j)$, $j = -k, \ldots, m$.

On the other hand, we know that Σ is also a nondegenerate $\mathcal{E}$-spline, so that, for $i = 0, \ldots, n$, $\dim(\operatorname{aff}\operatorname{Im}(\Sigma|_{I_i})) = k$. Hence, (5.16) leads to

$$\operatorname{aff}\operatorname{Im}(\Sigma|_{I_i}) = \operatorname{aff}\{\sigma(X_j) \mid j = j_i - k, \ldots, j_i\}. \tag{5.18}$$

Now, for any spline $S = h \circ \Sigma$, taking the images of (5.19) and (5.20) under the affine map h gives

$$\operatorname{aff}\operatorname{Im}(S) = \operatorname{aff}(Q_{-k}, \ldots, Q_m), \tag{5.19}$$

$$\operatorname{aff}\operatorname{Im}(S_{|I_i}) = \operatorname{aff}(Q_{j_i-k}, \ldots, Q_{j_i}) \qquad \text{for all} \quad i = 0, \ldots, n, \tag{5.20}$$

which completes the proof. ∎

The linear independence of the poles of Σ allows us to write in a unique way

$$\Sigma(t) = \sum_{j=-k}^{m} \mathcal{N}_j(t)\sigma(X_j), \qquad \sum_{j=-k}^{m} \mathcal{N}_j(t) = 1. \tag{5.21}$$

Definition and Theorem 5.7. *The $k+m+1$ functions $\mathcal{N}_j$, $j = -k, \ldots, m$, are called* the Chebyshev B-splines: *they form a basis of $\mathcal{S}$, called the* Chebyshev B-basis. *For $j = -k, \ldots, m$, $\mathcal{N}_j$ is the element of $\mathcal{S}$ the blossom n_j of which satisfies*

$$n_j(X_i) = \delta_{ij}, \qquad i = -k, \ldots, m. \tag{5.22}$$

Moreover, the support of $\mathcal{N}_j$ is given by

$$\operatorname{Supp} \mathcal{N}_j = \mathcal{D}(X_j). \tag{5.23}$$

Proof. The fact that the Chebyshev B-splines form a basis of the space $\mathcal{S}$ is a direct consequence of Corollary 2.3.

When applying Theorem 5.5 to Σ (the poles of which are affinely independent), we obtain

$$0 < \mathcal{N}_j(t) < 1 \qquad \text{for all} \quad t \in \,]x_{j_i}, x_{j_i+1}[\quad \text{and all} \quad j = j_i - k, \ldots, j_i. \tag{5.24}$$

On the other hand, according to Remark 2.7(ii), we have

$$[x_0, t_1] \subset \operatorname{Supp} \mathcal{N}_j \quad \Leftrightarrow \quad I_0 \subset \operatorname{Supp} \mathcal{N}_j, \tag{5.25}$$

and a similar property for $[t_n, x_{m+1}]$. Consequently, on account of (5.25), for a given $j \in \{-k, \ldots, m\}$, relations (5.24) prove that

$$i \in \mathcal{J}(X_j) \quad \Rightarrow \quad I_i \subset \operatorname{Supp}(\mathcal{N}_j). \tag{5.26}$$

Now, for a given integer i, $0 \le i \le n$, comparing (5.20) and (5.21) shows that

$$\mathcal{N}_j(t) = 0 \qquad \text{for all} \quad t \in I_i \quad \text{and all} \quad j \notin \{j_i - k, \ldots, j_i\}. \tag{5.27}$$

Therefore, by (5.26) and (5.27), the support of $\mathcal{N}_j$ is the union of all the intervals I_i, $i \in \mathcal{J}(X_j)$, i.e., (5.23).

Taking the image of equality (5.21) under affine maps proves that any $\mathcal{E}$-spline S (in particular, any $S \in \mathcal{S}$) can be written as

$$S(t) \equiv \sum_{j=-k}^{m} \mathcal{N}_j(t) s(X_j). \tag{5.28}$$

Applied to the Chebyshev B-spline $\mathcal{N}_j$, (5.28) proves (5.22). ■

6. How to Build Piecewise Smooth Chebyshev Functions

Let Φ be a geometrically regular function of order k, $\mathcal{E}$ its associated space, and $\Phi^\sharp$ its normal function. Given a basis $(\bar{D}_1, \ldots, \bar{D}_k)$ in the direction Δ of the affine space spanned by the image of Φ, let us write

$$\Phi^\sharp(t^\varepsilon) = \sum_{i=1}^{k} \Phi_i^\sharp(t^\varepsilon) \bar{D}_i \qquad \text{for all} \quad t^\varepsilon \in I. \tag{6.1}$$

Although $\Phi^\sharp(t^\varepsilon)$ essentially depends on the inner product which has been chosen in Δ, the space $\mathcal{E}^\sharp$ spanned by its coordinates functions $(\Phi_1^\sharp, \ldots, \Phi_k^\sharp)$ depends only on the space $\mathcal{E}$. On the other hand, $\mathcal{E}^\sharp$ is also independent of the regular function defining $\mathcal{E}$. It is a k-dimensional space which will be equally called *the normal space* of Φ or of $\mathcal{E}$.

Any element $U^\sharp \in \mathcal{E}^\sharp$ can be considered as a real valued function defined on $I_* \cup \{t_\ell^-, t_\ell^+, \ell = 1, \ldots, n\}$ and its restriction to each I_j is C^∞ on I_j. Moreover, due to (2.19), $U^\sharp$ satisfies

$$(U^\sharp(t_\ell^+), \ldots, U^{\sharp(k-1)}(t_\ell^+))^T = M_\ell^\sharp \cdot (U^\sharp(t_\ell^-), \ldots, U^{\sharp(k-1)}(t_\ell^-))^T, \quad \ell = 1 \ldots, n. \tag{6.2}$$

Since matrix $M_\ell^\sharp$ is lower triangular and regular, t_ℓ^+ is a zero of order $i \le k$ of $U^\sharp$ iff t_ℓ^- is. In such a case, t_ℓ will simply said to be a *zero of order* i of $U^\sharp$. Furthermore, the linear independence of the k vectors $\Phi^\sharp(t^\varepsilon), \dots, \Phi^{\sharp(k-1)}(t^\varepsilon)$ for all $t \in I$ implies that, if $U^\sharp$ is nonzero, each zero of $U^\sharp$ in I is of order less than or equal to $k-1$. These remarks give sense to considering the upper bound of the numbers of zeros on I (counted with multiplicities) of all nonzero elements of $\mathcal{E}^\sharp$, this number being possibly infinite. It will be denoted by $Z_I(\mathcal{E}^\sharp)$.

Actually, Theorem 3.4 states that Φ is a piecewise smooth Chebyshev function iff, for all distinct points $\tau_1, \dots, \tau_r \in I$ and all positive integers $\mu_1, \dots, \mu_r$ summing to k,

$$
\begin{vmatrix}
\Phi_1^\sharp(\tau_1^{\varepsilon_1}) & \dots & \Phi_1^{\sharp(\mu_1-1)}(\tau_1^{\varepsilon_1}) & \dots & \Phi_1^\sharp(\tau_r^{\varepsilon_r}) & \dots & \Phi_1^{\sharp(\mu_r-1)}(\tau_r^{\varepsilon_r}) \\
\Phi_2^\sharp(\tau_1^{\varepsilon_1}) & \dots & \Phi_2^{\sharp(\mu_1-1)}(\tau_1^{\varepsilon_1}) & \dots & \Phi_2^\sharp(\tau_r^{\varepsilon_r}) & \dots & \Phi_2^{\sharp(\mu_r-1)}(\tau_r^{\varepsilon_r}) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\Phi_k^\sharp(\tau_1^{\varepsilon_1}) & \dots & \Phi_k^{\sharp(\mu_1-1)}(\tau_1^{\varepsilon_1}) & \dots & \Phi_k^\sharp(\tau_r^{\varepsilon_r}) & \dots & \Phi_k^{\sharp(\mu_r-1)}(\tau_r^{\varepsilon_r})
\end{vmatrix} \neq 0, \tag{6.3}
$$

whatever the ε_i's may be, provided that $\tau_i^{\varepsilon_i} \in I$. Clearly, this result can also be stated as follows:

Theorem 6.1. *The geometrically regular function Φ of order k is a piecewise smooth Chebyshev function of order k iff its normal space $\mathcal{E}^\sharp$ satisfies $Z_I(\mathcal{E}^\sharp) \le k-1$.*

6.1. *Chebyshev Spaces*

In this subsection, we shall give a compact presentation of the necessary tools on Chebyshev spaces. For the proofs and more details, see, for instance, [15], [25], and [36].

Definition 6.2. Given a real interval J, a k-dimensional space $\mathcal{U}$ contained in $C^\infty(J)$ is said to be an *extended Chebyshev space* (EC space) on J if any nonzero element of $\mathcal{U}$ has at most $k-1$ zeros (counted with multiplicities) in J (i.e., if $Z_J(\mathcal{U}) \le k-1$). It is said to be a *complete extended Chebyshev space* (ECC space) on J if there exists a nested sequence

$$\mathcal{U}_1 \subset \mathcal{U}_2 \subset \cdots \subset \mathcal{U}_{k-1} \subset \mathcal{U}_k = \mathcal{U}, \tag{6.4}$$

where, for $i = 1, \dots, k, \mathcal{U}_i$ is an i-dimensional EC space on J.

Theorem 6.3. *A given k-dimensional subspace $\mathcal{U}$ of $C^\infty(J)$ is an ECC space on J iff there exist k positive functions $w_1, \dots, w_k \in C^\infty(J)$ (called* weight functions associated with *$\mathcal{U}$) such that $\mathcal{U} = \operatorname{Ker} D \circ L_k$, where D stands for the ordinary differentiation and, for $L_1, \dots, L_k$, for the differential operators defined on $C^\infty(J)$ by*

$$L_1 U := \frac{1}{w_1} U, \qquad L_i U := \frac{1}{w_i}(L_{i-1}U)', \qquad i = 2, \dots, k. \tag{6.5}$$

Proof. Any k nonvanishing functions $w_1, \dots, w_k \in C^\infty(J)$ generate a nested sequence similar to (6.4) by means of the corresponding operators, namely

$$\mathcal{U}_i := \operatorname{Ker} L_{i+1}, \quad i = 1, \dots, k-1, \qquad \mathcal{U} := \operatorname{Ker} DL_k. \tag{6.6}$$

We can prove that each space involved in (6.6) is in fact an EC space. For the converse part we refer to [15] and [25]. ■

Let us observe that different systems of weight functions may lead to the same ECC space. From the previous theorem, we can easily deduce the following result:

Corollary 6.4. *Let $\mathcal{U}$ be a $(k+1)$-dimensional subspace of $C^\infty(I)$ containing the constant functions. Then, $D\mathcal{U}$ is an ECC space with associated weight functions $w_1, \dots, w_k$, iff $\mathcal{U}$ is an ECC space with associated weight functions $\mathbf{1}, w_1, \dots, w_k$.*

Remark 6.5. Using Definition 6.2, it is straighforward to check that, when $D\mathcal{U}$ is an EC space, $\mathcal{U}$ is an EC space containing the constant functions. Contrary to the case of ECC spaces, the converse property does not hold, except if the interval I is supposed to be closed and bounded. This can be obtained through the crucial result stated hereunder.

Theorem 6.6 ([32] and [25]). *Over a closed bounded interval $I = [a, b]$, an EC space is an ECC space.*

Suppose $\mathcal{U}$ is a k-dimensional EC space on J, and choose a basis $(U_1, \dots, U_k)$ of $\mathcal{U}$. Then, for any $t \in J$, the linear system

(6.7) $$\sum_{j=1}^{k} U_j{}^{(i)}(t) U_j^*(t) = \delta_{k-1,i}, \qquad i = 0, \dots, k-1,$$

has a unique solution. This provides k functions $U_1^*, \dots, U_k^*$ which are C^∞ on J. The space $\mathcal{U}^*$ spanned by $(U_1^*, \dots, U_k^*)$ depends only on $\mathcal{U}$, not on the basis $(U_1, \dots, U_k)$: $\mathcal{U}^*$ is called *the dual space of* $\mathcal{U}$ while $(U_1^*, \dots, U_k^*)$ is called *the dual basis* of $(U_1, \dots, U_k)$.

Theorem 6.7. *If $\mathcal{U}$ is a k-dimensional ECC space on J, with weight functions $(w_1, \dots, w_k)$, then its dual space $\mathcal{U}^*$ is the ECC space associated with the weight functions*

(6.8) $$\widehat{w}_1 := \frac{1}{\prod_{i=1}^{k} w_i}, \qquad \widehat{w}_2 := w_k, \dots, \qquad \widehat{w}_k := w_2.$$

Moreover, a given basis $(U_1, \dots, U_k)$ of $\mathcal{U}$ and its dual basis $(U_1^*, \dots, U_k^*)$ satisfy:

(6.9) $$\mathcal{L}(U_1, \dots, U_k)(t)^T \cdot \widehat{\mathcal{L}}(U_1^*, \dots, U_k^*)(t) = \mathcal{R} \qquad \textit{for all} \quad t \in J.$$

Here, $\mathcal{L}(U_1, \dots, U_k)(t)$ and $\widehat{\mathcal{L}}(U_1^*, \dots, U_k^*)(t)$ are the (k, k) matrices defined by

(6.10) $$\begin{aligned} \mathcal{L}(U_1, \dots, U_k)(t)_{i,j} &:= L_j U_i(t), \\ \widehat{\mathcal{L}}(U_1^*, \dots, U_k^*)(t)_{i,j} &:= \widehat{L}_j U_i^*(t), \qquad i, j = 1, \dots, k, \end{aligned}$$

$\widehat{L}_1, \dots, \widehat{L}_k$ standing for the differential operators defined from the weight functions $\widehat{w}_1, \dots, \widehat{w}_k$, similarly to (6.5)*, and $\mathcal{R}$ standing for the antidiagonal matrix such that $\mathcal{R}_{k+1-j,j} = (-1)^{j-1}$, $j = 1, \dots, k$.*

Proof. Setting $\vec{U} := (U_1, \ldots, U_k)^T$ and $\vec{U}^* := (U_1^*, \ldots, U_k^*)^T$, the dual basis satisfies

$$\vec{U}^*(t) = \frac{\vec{U}(t) \wedge \cdots \wedge \vec{U}^{(k-2)}(t)}{\det(\vec{U}(t), \ldots, \vec{U}^{(k-1)}(t))} \qquad \text{for all} \quad t \in J. \tag{6.11}$$

Since there exist real numbers $a_{i\ell}$ such that $L_i\vec{U} = 1/w_1 \ldots w_i\, \vec{U}^{(i-1)} + \sum_{\ell=0}^{i-2} a_{i\ell}\, \vec{U}^{(\ell)}$ for $1 \leq i \leq k$, we can check that (6.11) leads to

$$L_1\vec{U}(t) \wedge \cdots \wedge L_{k-1}\vec{U}(t) = \delta(t)\widehat{L}_1\vec{U}^*(t), \tag{6.12}$$

where $\delta(t) := \det(L_1\vec{U}(t), \ldots, L_k\vec{U}(t))$. Now, from (6.5) and $\mathcal{U} = \operatorname{Ker} DL_k$, we can derive that

$$DL_i\vec{U} = w_{i+1}L_{i+1}\vec{U}, \quad i = 1, \ldots, k-1, \qquad DL_k\vec{U} = 0. \tag{6.13}$$

Relations (6.13) imply in particular that $D\delta(t) = 0$ for all $t \in J$, hence δ is a constant function on J. A simple recursive argument starting from (6.12) and based on (6.13) proves that

$$\begin{aligned} \delta\widehat{L}_i\vec{U}^*(t) \ &= \ L_1\vec{U}(t) \wedge \cdots \wedge L_{k-i}\vec{U}(t) \\ &\quad \wedge L_{k-i+2}\vec{U}(t) \wedge \cdots \wedge L_k\vec{U}(t), \qquad i = 1, \ldots, k. \end{aligned} \tag{6.14}$$

On the other hand, it is straightforward to verify that, for all $t \in J$,

$$\begin{aligned} &\langle\, L_j\vec{U}(t), L_1\vec{U}(t) \wedge \cdots \wedge L_{k-i}\vec{U}(t) \wedge L_{k-i+2}\vec{U}(t) \wedge \cdots \wedge L_k\vec{U}(t)\rangle \\ &\qquad = \begin{cases} 0 & \text{if } \ j \neq k-i+1\ , \\ (-1)^{i-1}\,\delta & \text{if } \ j = k-i+1\ . \end{cases} \end{aligned}$$

Taking this latter equality into account, (6.14) eventually gives (6.9). Moreover, due to (6.13), (6.14) also implies $D\widehat{L}_k\vec{U}^* = 0$, i.e., $\mathcal{U}^* = \operatorname{Ker} D\widehat{L}_k$, which carries out the proof. ■

Corollary 6.8. *If $\mathcal{U}$ is a k-dimensional EC space on J, its dual space $\mathcal{U}^*$ is also a k-dimensional EC space on J and $\mathcal{U}^{**} = \mathcal{U}$.*

Proof. Being an EC space on J is clearly equivalent to being an EC (hence, due to Theorem 6.6, an ECC) on any closed bounded interval contained in J. Therefore, Corollary 6.8 is a direct consequence of Theorem 6.7. ■

6.2. *A Sufficient Condition for Piecewise Smooth Chebyshev Functions*

Denote by $\mathcal{E}_i$ (resp., $\mathcal{E}_i^\sharp$) the space obtained by restricting the elements of $\mathcal{E}$ (resp., $\mathcal{E}^\sharp$) to I_i, $i = 0, \ldots, n$, so that $\mathcal{E}_i$ and $\mathcal{E}_i^\sharp$ are subspaces of $C^\infty(I_i)$ (of dimension $k+1$ and k, respectively).

Let us first give a necessary condition:

Theorem 6.9. *If Φ is a piecewise smooth Chebyshev function of order k, then, for all $i = 0, \dots, n$, $D\mathcal{E}_i$ is a k-dimensional EC space on I_i.*

Proof. Condition $Z_I(\mathcal{E}^\sharp) \le k-1$ clearly implies that, for all $i = 0, \dots, n$, $Z_{I_i}(\mathcal{E}_i^\sharp) \le k-1$, which means that $\mathcal{E}_i^\sharp$ is a k-dimensional EC-space on I_i. Moreover, comparing (2.13) and (6.7) shows that $D\mathcal{E}_i$ is the dual space of $\mathcal{E}_i^\sharp$. Thus, by Corollary 6.8, $D\mathcal{E}_i$ is also an EC space on I_i. ∎

Remark 6.10. (i) Suppose for a while that $n = 0$, so that $\mathcal{E}$ and $\mathcal{E}^\sharp$ are contained in $C^\infty(I)$. Then, the converse property is also true. Indeed, if $D\mathcal{E}$ is a k-dimensional EC space on I, by Corollary 6.8, so is its dual space $\mathcal{E}^\sharp$. Thus, $Z_I(\mathcal{E}^\sharp) \le k-1$.

(ii) On the contrary, when $n > 0$, the necessary condition stated in Theorem 6.9 is no longer sufficient, as proved by considering the C^∞ space $\mathcal{E}$ spanned by the four functions $(1, x, \cos x, \sin x)$ on $I =]-2\pi, 2\pi[$. Let $\mathcal{E}_0$ and $\mathcal{E}_1$ stand for the restrictions of $\mathcal{E}$ to $I_0 =]-2\pi, 0]$ and $I_1 = [0, 2\pi[$, respectively. Here, the connection matrix M_1 at $t_1 = 0$ is the identity matrix $\mathcal{I}_3$. The space $D\mathcal{E}$ is spanned by the three functions $(\mathbf{1}, \cos, \sin)$ defined on I and we can verify that, for $i = 0, 1$, $D\mathcal{E}_i$ is a three-dimensional EC space on I_i. Moreover, in that case, we have $\mathcal{E}^\sharp = D\mathcal{E}$. Hence, the condition $Z(\mathcal{E}) \le 2$ does not hold: for example, function sin vanishes at three distinct points of I, namely $-\pi, 0, \pi$.

(iii) Suppose that Φ is a piecewise smooth Chebyshev function of order k on I. Then, through Theorem 6.6, the necessary condition stated in Theorem 6.9 proves that, for a given $i = 1, \dots, n-1$, $D\mathcal{E}_i$ is a k-dimensional ECC space on I_i. Thus, thanks to Theorem 6.3 and Corollary 6.4, we can find positive weight functions $w_1^i, \dots, w_k^i \in C^\infty(I_i)$ such that $\mathcal{E}_i$ is the ECC space associated with $(\mathbf{1}, w_1^i, \dots, w_k^i)$. As for $\mathcal{E}_0$ and $\mathcal{E}_n$, without any additional assumption on the two end subintervals, we can only say that both are $(k+1)$-dimensional EC spaces on I_0 and I_n, respectively, condition $Z_{I_i}(D\mathcal{E}_i) \le k-1$ clearly implying $Z_{I_i}(\mathcal{E}_i) \le k$.

We are now searching for conditions sufficient to ensure that $Z(\mathcal{E}^\sharp) \le k-1$. Suppose that, for $i = 0, \dots, n$, $\mathcal{E}_i$ is an ECC space on I_i, with $(\mathbf{1}, w_1^i, \dots, w_k^i)$ as weight functions, and denote by L_j^i, $j = 1, \dots, k$, the differential operators defined on $C^\infty(I_i)$ by

$$(6.15) \qquad L_1^i U = \frac{1}{w_1^i} U, \qquad L_j^i U = \frac{1}{w_j^i}(L_{j-1}^i U)', \qquad j = 2, \dots, k.$$

Without any loss of generality, we can assume that

$$(6.16) \qquad w_j^{\ell-1}(t_\ell) = w_j^\ell(t_\ell) \qquad j = 0, \dots, n, \quad \ell = 1, \dots, q.$$

Instead of expressing the connections by means of the ordinary derivatives as in (2.9), we can now use the previous operators. For $F \in C^\infty(I_i)$ and $t \in I_i$, let us set

$$(6.17) \qquad \Lambda_k^i F(t^\varepsilon) := \left(L_1^i F'(t^\varepsilon), \dots, L_k^i F'(t^\varepsilon)\right)^T.$$

It is straightforward to verify that

$$(6.18) \qquad \Lambda_k^i F(t^\varepsilon) = C_k^i(t^\varepsilon) \cdot D_k F(t^\varepsilon),$$

where $C_k^i(t^\varepsilon)$ is a regular lower triangular matrix with diagonal elements

$$\left(\frac{1}{w_1^i(t)}, \frac{1}{w_1^i(t)w_2^i(t)}, \ldots, \frac{1}{w_1^i(t)\ldots w_k^i(t)}\right).$$

Thus, the space $\mathcal{E}$ can now be described as the space of all continuous functions $F : I \to \mathbf{R}$ such that $F|_{I_i} \in \mathcal{E}_i$, $i = 0, \ldots, n$, and

$$\Lambda_k^\ell F(t_\ell^+) = N_\ell \cdot \Lambda_k^{\ell-1} F(t_\ell^-), \qquad \ell = 1, \ldots, n, \tag{6.19}$$

where N_ℓ is defined by

$$N_\ell := C_k^\ell(t_\ell^+) \cdot M_\ell \cdot C_k^{\ell-1}(t_\ell^-)^{-1}. \tag{6.20}$$

The following theorem is a straightforward extension of a fundamental result due to P. J. Barry [2]:

Theorem 6.11. *Suppose that, for $\ell = 1, \ldots, n$, N_ℓ is* totally positive (*i.e., each minor of N_ℓ is nonnegative*). *Then, $Z_I(\mathcal{E}^\sharp) \le k-1$ (i.e., Φ is a piecewise smooth Chebyshev function of order k).*

Proof. Since $\mathcal{E}_i^\sharp$ is the dual space of $D\mathcal{E}_i$, it follows from Corollary 6.4 and Theorem 6.7 that it is the k-dimensional ECC space associated with the weight functions

$$\widehat{w}_1^i := \frac{1}{\prod_{j=1}^k w_j^i}, \qquad \widehat{w}_j^i := w_{k+2-j}^i, \qquad j = 2, \ldots, k. \tag{6.21}$$

Let us denote by $\widehat{L}_1^i, \ldots, \widehat{L}_k^i$, the corresponding differential operators on $C^\infty(I_i)$. Applying formula (6.9) to each ECC space $D\mathcal{E}_i$, we can prove that the connections in the space $\mathcal{E}^\sharp$ are the following ones:

(6.22)

$$\left(L_1^\ell U^\sharp(t_\ell^+), \ldots, L_k^\ell U^\sharp(t_\ell^+)\right)^T = \widehat{N}_\ell \cdot \left(\widehat{L}_1^{\ell-1} U^\sharp(t_\ell^-), \ldots, \widehat{L}_k^{\ell-1} U^\sharp(t_\ell^-)\right)^T, \quad \ell = 1, \ldots, n,$$

where

$$\widehat{N}_\ell := \mathcal{R}^T \cdot N_\ell^{-T} \cdot \mathcal{R}. \tag{6.23}$$

It follows from [2, Theorem 5] that N_ℓ is totally positive iff $\widehat{N}_\ell$ is. Although matrices $\widehat{N}_\ell$ are not exactly of the same type as the connection matrices used by P. J. Barry, the argument he gives in the proof of [2, Theorem 8] can easily be adapted. So, it allows us to conclude that, as soon as each $\widehat{N}_\ell$ is totally positive, any nonzero element of $\mathcal{E}^\sharp$ has at most $k-1$ zeros in I. ∎

However, the sufficient condition stated in the previous theorem is not necessary as pointed out in the following example. Let $\mathcal{E}$ denote the four-dimensional space spanned by functions $(1, t, \cosh t, \sinh t)$. It is an ECC space on $I = \mathbf{R}$. Then, $D\mathcal{E}$ is a three-dimensional ECC on I, which implies that $\mathcal{E}^\sharp$ satisfies the required condition $Z_I(\mathcal{E}^\sharp) \le 2$. On the other hand, the ECC space $\mathcal{E}$ can be defined from two different systems of weight functions $(\mathbf{1}, w_1^i, w_2^i, w_3^i)$, $i = 0, 1$, namely,

$$w_1^0(t) = 1, \qquad w_2^0(t) = \cosh t, \qquad w_3^0(t) = \frac{1}{\cosh^2 t},$$
$$w_1^1(t) = \cosh t, \qquad w_2^1(t) = \frac{1}{\cosh^2 t}, \qquad w_3^1(t) = \cosh t.$$

The corresponding matrices C_3^0 and C_3^1 introduced in (6.18) are the following ones:

(6.24)
$$C_3^0(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\cosh t & 0 \\ 0 & -\sinh t & \cosh t \end{pmatrix}, \qquad C_3^1(t) = \begin{pmatrix} 1/\cosh t & 0 & 0 \\ -\sinh t & \cosh t & 0 \\ -1 & 0 & 1 \end{pmatrix}, \qquad t \in \mathbf{R}.$$

Now, let us denote by $\mathcal{E}_i$ the restriction of the space $\mathcal{E}$ to each interval I_i, $i = 0, 1$, with $I_0 :=]-\infty, 0]$ and $I_1 := [0, +\infty[$. Using (6.20) and (6.24), the space $\mathcal{E}$ can also be described as the space of all continuous functions $F : I \longrightarrow \mathbf{R}$ such that $F|_{I_i} \in \mathcal{E}_i$, $i = 0, 1$, and which satisfy the connection condition $\Lambda_3^1(0^+) = N \cdot \Lambda_3^0(0^-)$, where N is the following nontotally positive matrix:

$$N := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Open Question. It may be possible to prove that, if the condition $Z_I(\mathcal{E}^\sharp) \le k - 1$ is satisfied, then, in each interval, there exists a convenient choice of the weight functions w_j^i, $j = 1, \dots, k$, ensuring that the corresponding connection matrices are totally positive.

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