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CONSTRUCTIVE APPROXIMATION

Construction of Continuous Functions with Prescribed Local Regularity

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Abstract. In this paper we investigate from both a theoretical and a practical point of view the following problem: Let s be a function from $[0; 1]$ to $[0; 1]$. Under which conditions does there exist a continuous function f from $[0; 1]$ to $\mathbf{R}$ such that the regularity of f at x, measured in terms of Hölder exponent, is exactly $s(x)$, for all $x \in [0; 1]$?

We obtain a necessary and sufficient condition on s and give three constructions of the associated function f. We also examine some extensions regarding, for instance, the box or Tricot dimension or the multifractal spectrum. Finally, we present a result on the "size" of the set of functions with prescribed local regularity.

1. Introduction

Since Riemann [1], a number of authors have been interested in constructing nowhere differentiable continuous functions. Some use geometrical constructions, of which the best-known examples are probably Von Koch's [2], Peano's and Hilbert's [3] curves, while others are based on analytical tools. The very well-known example in this case is the Weierstrass function, which was shown by Weierstrass to be continuous and nowhere differentiable [4]. This result was later greatly enhanced by Hardy [5] who showed that

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n x \pi)$$

has nowhere a finite derivative, provided that

$$0 < b < 1, \qquad a > 1, \qquad ab \geq 1.$$

Hardy also analyzed the Hölder conditions satisfied by $f(x)$. If $ab > 1$, let $\xi < 1$ be defined by $\xi = \log(1/b)/\log a$. Then, for $h \to 0$,

$$|f(x+h) - f(x)| = O(|h|^{\xi}) \qquad \text{for every } x,$$

but

$$|f(x+h) - f(x)| = o(|h|^{\xi}) \qquad \text{for no } x.$$

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Another example of nowhere differentiable functions, which fits well the main ideas of this paper, is the Takagi function [6] defined by

$$T(x) = \sum_{j=0}^{\infty} 2^{-j} \theta^*(2^j x),$$

where $\theta^*(x)$ is the periodic function of period 1 defined on [0; 1] by $\theta^*(x) = 2x$ if $0 \leq x \leq \frac{1}{2}$ and $\theta^*(x) = 2 - 2x$ if $\frac{1}{2} \leq x \leq 1$.

Indeed, we consider in the sequal three different constructions of nowhere differentiable functions: one is based on a generalization of the Weierstrass function, another on an expansion in the Schauder basis, and the last on a generalization of IFS theory. The construction of the Takagi function bears some analogy with that of the Weierstrass function. On the other hand, the restriction of T to [0; 1] has the following expansion in the Schauder basis:

$$T(x) = \sum_{j \geq 0} \sum_{0 \leq k < 2^j} 2^{-j} \theta(2^j x - k),$$

where $\theta(x) = \theta^*(x)$ if $x \in [0; 1]$ and $\theta(x) = 0$ if $x \notin [0; 1]$. Finally, the graph of the restriction of T to [0; 1] is the attractor of the IFS defined by the two functions $w_1(x, y) = (x/2, (2x + y)/2)$ and $w_2(x, y) = (x/2 + \frac{1}{2}, (y - 2x)/2 + 1)$.

Hata [7] considered the following generalization. Let g be the continuous function defined by

$$g(x) = \sum_{n=0}^{\infty} b^n q(a^n x \pi),$$

where q is a continuous function and $0 < b < 1$. He showed in particular that, when $q(x) = \cos(x + \theta)$ ($\theta \in \mathbf{R}$), which leads to

$$g(x) = \sum_{n=0}^{\infty} b^n \cos(a^n x \pi + \theta),$$

then the continuous function g has nowhere a finite or infinite derivative if

$$ab \geq 1 + \pi^2.$$

He also found related results when the function q is almost periodic. His results were later improved by Hu and Lau [8]. Mauldin and Williams [9] also considered a generalization of the Weierstrass function, namely

$$W_\beta(x) = \sum_{-\infty}^{+\infty} \beta^{-\alpha n} (\varphi(\beta^n x + \theta_n) - \varphi(\theta_n)),$$

where $\beta > 1,\ 0 < \alpha < 1$, each θ_n is an arbitrary number, and φ is a function which has period one. They showed that there exists a constant $C > 0$ such that, if β is large enough, then the Hausdorff dimension of the graph of W_β is bounded from below by $2 - \alpha - C/\log \beta$.

Several other techniques are now employed for constructing continuous nowhere differentiable functions. One powerful scheme is to use wavelet decompositions. For instance, Jaffard [10] has given a construction of a function with prescribed multifractal spectrum $(\alpha, f(\alpha))$. Choosing in such a construction $f(\alpha)$ such that $f(\alpha)_{|]-\infty;0]\cup[1;+\infty[} = -\infty$ leads to a nowhere differentiable continuous function.

Another method that has been investigated a lot these past years is based on Iterated Function System (IFS). Although the study of iteration of matrices dates back to Doeblin and Fortet [11] and Dubbins and Freedman [12], it was Hutchinson [13] who really laid the foundations of IFS theory. Subsequently, several authors have explored this path (see for instance [14], [15], [16], [17], and many others). Barnsley [14] showed that, under some conditions, it is possible to construct an IFS whose attractor is the graph of a continuous nowhere differentiable function. More precise results are now known, concerning the almost sure Hölder exponent of such functions [17] or their multifractal spectrum [18], [19].

We will hereafter call α_f the Hölder function of f, which associates, to each point x, the Hölder exponent of the function f at x. The main objective of the present work is to solve the following problem which was raised by J. Lévy Véhel:

Let s be a function from $[0; 1]$ *to* $[0; 1]$. *Under what conditions on s does there exist a continuous function f from* $[0; 1]$ *to* $\mathbf{R}$ *such that* $\alpha_f(x) = s(x)$ *for all x in* $[0; 1]$?

S. Jaffard proposed the Schauder basis construction that we recall in Section 4 (the wavelet basis construction presented in [26] is an adaptation to the case where the Hölder exponents are greater than 1). Y. Meyer realized that this construction allows us to obtain the most general Hölder functions. K. Daoudi and J. Lévy Véhel independantly performed two other constructions that also yield the general result and which are presented in Sections 5 and 6.

The motivation for this investigation stems partly from applications in signal processing. Indeed, in some cases, it is desirable to model highly irregular signals while precisely controlling the irregularity at each point. This happens, for instance, when the significant information lies in the singularities of the signal more than in its amplitude. In such cases, we want to tune the value of $\alpha_f(x)$ *everywhere* and not merely *almost everywhere*. An example in speech modeling is presented in [19] and [20].

Our main result is the following:

Theorem. *Let s be a function from* $[0; 1]$ *to* $[0; 1]$. *Then, the following conditions are equivalent*:

(i) *s is the Hölder function of a continuous function f from* $[0; 1]$ *to* $\mathbf{R}$.
(ii) *There exists a sequence* $(s_n)_{n\geq 1}$ *of continuous functions such that*:

$$s(x) = \liminf_{n\to+\infty} s_n(x), \qquad \forall\, x \in [0; 1].$$

The proof of (i) ⇒ (ii) is easy and is given in Section 3. The proof of (ii) ⇒ (i) requires more work.

For practical purposes, we are interested here in constructive proofs, i.e, we want to derive explicit methods to construct the function f. We present below three such proofs which highlight different aspects of the problem. We also investigate related problems, as for instance the evaluation of the local box dimension of f at each point or the computation of the multifractal spectrum of f. Finally, for practical applications, we want to construct functions f with a prescribed Hölder function that satisfies additional constraints, as for instance interpolating a finite number of points $(x_i, y_i) \in [0; 1] \times \mathbf{R}$, $i = 1, 2, \ldots, N$. This naturally leads to a characterization of the set of functions with a prescribed Hölder function.

The remainder of this paper is organized as follows: in Section 2, we recall some basic definitions about the local regularity of functions, the Hausdorff, Tricot, and box dimension. We also prove a new relation between the local box dimension and the Hölder exponent. In Section 4 we construct functions with prescribed local regularity $s(x)$ at each point using the Schauder basis. In Section 5, we give another solution based on a generalized Weierstrass function. In Section 6, we use IFS to give a solution which constructively allows us to interpolate a given finite set of equispaced points. In Section 7, we propose some desirable extensions that would allow us to measure more finely the local structure of graphs of continuous functions. Section 8 shows some implementation results.

2. Recalls and a Result Relating the Local Box Dimension and the Hölder Exponent

In this section we recall some basic definitions useful for the sequel. The definitions are not given in full generality, but only in the form adapted to our problem.

2.1. *Definition of the Hausdorff Dimension*

Let E be a nonempty set of $\mathbf{R}^2$.

Define

$$|E| := \sup_{x,y}\{|x-y|;\ x, y \in E\}$$

to be the *diameter* of E.

If $E \subset \bigcup_{i\in N} E_i$ with $0 < |E_i| \le \delta$ for each i, then $\{E_i\}_{i\in N}$ is called a (countable) δ-*cover* of E.

For $\delta > 0$ and $r \ge 0$, define

$$\mathcal{H}^r_\delta(E) := \inf\left\{\sum_{i=1}^{+\infty} |E_i|^r / \{E_i\}_{i\in N}\,\delta\text{-cover of } E\right\},$$

$\mathcal{H}^r_\delta(E)$ is a nonincreasing function of δ, and we note

$$\mathcal{H}^r(E) := \lim_{\delta\to 0} \mathcal{H}^r_\delta(E) = \sup_{\delta>0} \mathcal{H}^r_\delta(E)$$

the *Hausdorff r-dimensional outer measure* of E.

The Hausdorff dimension of E is the unique value $\dim_{\mathrm{H}}(E)$ such that [21]

$$\mathcal{H}^r(E) = \begin{cases} +\infty & \text{if } r < \dim_{\mathrm{H}}(E), \\ 0 & \text{if } r > \dim_{\mathrm{H}}(E). \end{cases}$$

2.2. *Definition of the Box Dimension*

For any $\delta > 0$, we consider the set of δ-*mesh squares* in $\mathbf{R}^2$ of the form $[i\delta, (i+1)\delta] \times [j\delta, (j+1)\delta]$ with i, j integers. For any bounded subset F of $\mathbf{R}^2$, we denote by $N_\delta(F)$

the number of δ-mesh squares which intersect F. The box dimension of F is then defined by [22]

$$\dim_B(F) = \lim_{\delta \to 0} \left(\frac{\log N_\delta(F)}{-\log \delta} \right),$$

whenever this limit exists.

When the limit exists, its value is unaffected if we change the definition of $N_\delta(F)$ and take any of the following:

1. the smallest number of squares of size δ that cover F;
2. the smallest number of closed balls of diameter δ that cover F;
3. the smallest number of sets of diameter δ that cover F; and
4. the largest number of disjoint balls of diameter δ with centers in F.

2.3. *Definition of the Tricot (Packing) Dimension*

Let F be a nonempty set of $\mathbf{R}^n$, where $n \geq 1$ is an integer, and

$$\mathcal{P}^r_\delta(F) = \sup \left\{ \sum_{i \in \mathbf{N}} |B_i|^r \right\},$$

where $\{B_i\}_{i\in N}$ is a collection of disjoint balls of radii at most δ whose centers belong to F. Consider

$$\mathcal{P}^r_0(F) = \lim_{\delta \to 0} \mathcal{P}^r_\delta(F),$$

this limit exists since $\mathcal{P}^r_\delta(F)$ decreases with δ.

Define now the r-dimensional Tricot measure [23], [22] $\mathcal{P}^r$ by

$$\mathcal{P}^r(F) = \inf \left\{ \sum_{i=1}^{\infty} \mathcal{P}^r_0(F_i) \ : F \subset \bigcup_{i=1}^{\infty} F_i \right\},$$

then, the Tricot (or packing) dimension $\dim_P$ is defined as follows:

$$\dim_P \ F = \sup\{r \ : \mathcal{P}^r(F) = +\infty\} = \inf\{r \ : \mathcal{P}^r(F) = 0\}.$$

2.4. *Definition of the Hölder Spaces and the Hölder Exponent*

Let I be an interval in $\mathbf{R}$, f a continuous function from I to $\mathbf{R}$, and $\beta \in \bar{\mathbf{R}}^*_+ \backslash \mathbf{N}$.

Definition 1. f is said to belong to the global Hölder space $C^\beta(I)$ iff there exists a positive constant c, such that for every $x_0 \in I$, there exists a polynomial P_{x_0} of degree less than or equal to the integer part of β, such that

$$|f(x) - P_{x_0}(x - x_0)| \leq c|x - x_0|^\beta \qquad \forall x \in I.$$

Definition 2. Let t_0 be in I. Then f is said to belong to the pointwise Hölder space $C^\beta(t_0)$ iff there exists a polynomial P of degree less than or equal to the integer part of

β, and a positive constant c such that, for every t in the neighborhood of t_0, we have

$$|f(t) - P(t-t_0)| \leq c|t-t_0|^{\beta}.$$

Recall that if $\beta \in \mathbf{N}^*$, the space C^{β} must be replaced by the Zygmund β-class [24].

Definition 3. A function f is said to have Hölder exponent β at point t_0 iff:

(i) for every real $\gamma < \beta$

$$\lim_{h\to 0} \frac{|f(t_0+h) - P(h)|}{|h|^{\gamma}} = 0;$$

(ii) if $\beta < +\infty$, for every real $\gamma > \beta$

$$\limsup_{h\to 0} \frac{|f(t_0+h) - P(h)|}{|h|^{\gamma}} = +\infty,$$

where P is a polynomial whose degree is less than or equal to the integer part of β.

When $\beta < +\infty$, this is equivalent to

$$f \in \bigcap_{\varepsilon>0} C^{\beta-\varepsilon}(t_0) \qquad \text{but} \quad f \notin \bigcup_{\varepsilon>0} C^{\beta+\varepsilon}(t_0).$$

It is also equivalent to

$$\beta = \sup\{\theta > 0 \,:\, f \in C^{\theta}(t_0)\}.$$

Notice that $f \in C^{\beta}(I)$ does not imply that $\beta = \inf_{t\in I} \alpha_f(t)$. As an example, consider the continuous function f defined on $\mathbf{R}$ by

$$f(t) = \begin{cases} |t| \sin\left(\dfrac{1}{|t|}\right) & \text{if } t \in \mathbf{R}^*, \\ 0 & \text{if } t = 0, \end{cases}$$

then, $f \in C^{1/2}(\mathbf{R})$, but f is C^{∞} at each point, except at 0 where $\alpha_f(0) = 1$.

2.5. *A Relation Between the Local Box Dimension and the Hölder Exponent*

In [25], the authors investigate the relation between the global upper box dimension of the graph of a function and its global smoothness. They give precise characterizations of the global upper box dimension of the graph of a continuous function in terms of its membership in Besov spaces and variation of Wiener spaces.

In this section, we are rather interested in the relation between the local box dimension of the graph of a function and its membership in pointwise Hölder spaces. We propose a new result that links the local box dimensions of the graph of a continuous function and its Hölder exponents.

Let f be a continuous function from $[0; 1]$ to $\mathbf{R}$. We suppose that $s(x) = \alpha_f(x) \in [0; 1]$ for all $x \in [0; 1]$. Let $x \in]0; 1[$, $\varepsilon > 0$ such that $]x-\varepsilon\,; x+\varepsilon[\subset [0; 1]$ and $\delta \in]0\,; \varepsilon[$.

We cover the plane by a δ-mesh, i.e., a grid of squares of the form $[i\delta\,; (i+1)\delta] \times [j\delta\,; (j+1)\delta]$, with i, j integers.

Let N_δ^ε be the number of squares that intersect *graph* $f_{|]x-\varepsilon;x+\varepsilon[}$. We define, respectively, the upper and lower local box dimension [22] of the graph of f at the point x by

$$\overline{\dim}_B^x \, graph\ f = \lim_{\varepsilon\to 0} \limsup_{\delta\to 0} -\frac{\log N_\delta^\varepsilon}{\log \delta}$$

and

$$\underline{\dim}_B^x \, graph\ f = \lim_{\varepsilon\to 0} \liminf_{\delta\to 0} -\frac{\log N_\delta^\varepsilon}{\log \delta}.$$

When these numbers coincide, we denote by $\dim_B^x \, graph\ f$ the local box dimension of f at x. For $t \in]0;1[$ such that $B(t,\varepsilon) =]t-\varepsilon\,;t+\varepsilon[\subset [0,1]$, define

$$\bar{c}(t,\varepsilon) = \inf\{c \in \mathbf{R}_+^* \,:\, \forall u \in B(t,\varepsilon)\,,\ |f(t)-f(u)| \le c|t-u|^{s(t)}\}$$

and

$$\underline{c}(t,\varepsilon) = \sup\{c \in \mathbf{R}_+^* \,:\, \exists u \in B(t,\varepsilon)\,:\, |f(t)-f(u)| \ge c\varepsilon^{s(t)}\}.$$

Proposition 1. *Let x be a real in* $]0;1[$. *Define the following conditions*:

(c_1) *there exists $\varepsilon' > 0$ such that*

$$\bar{C}(x,\varepsilon) = \sup_{t\in B(x,\varepsilon)} \bar{c}(t,\varepsilon) < +\infty \qquad \textit{for every} \quad \varepsilon < \varepsilon';$$

(c_2) *there exists $\varepsilon' > 0$ such that*

$$\underline{C}(x,\varepsilon) = \inf_{t\in B(x,\varepsilon)} \underline{c}(t,\varepsilon) \ne 0 \qquad \textit{for every} \quad \varepsilon < \varepsilon'.$$

Then, if (c_1) *holds, we have the following inequality*

$$\overline{\dim}_B^x \, graph\ f \le 2 - \min\Big(\liminf_{t\to x} s(t), s(x)\Big)$$

and, if (c_2) *holds, we have*

$$2 - \max\left(\limsup_{t\to x} s(t), s(x)\right) \le \underline{\dim}_B^x \, graph\ f.$$

Proof. Let ε be a real such that $0 < \varepsilon < \varepsilon'$. We denote

$$\begin{aligned} \underline{s}_x^\varepsilon &= \inf\{s(t);\, t \in]x-\varepsilon; x+\varepsilon[\}, \\ \overline{s}_x^\varepsilon &= \sup\{s(t);\, t \in]x-\varepsilon; x+\varepsilon[\}, \\ R_f[t_1;t_2] &= \sup_{t_1<u<v<t_2} |f(u)-f(v)|. \end{aligned}$$

Let m be the least integer greater than or equal to $2\varepsilon/\delta$. Thus, if

$$I_i(\varepsilon,\delta) =]x-\varepsilon+i\delta; x-\varepsilon+(i+1)\delta[,$$

then

$$]x-\varepsilon; x+\varepsilon[\subset \bigcup_{i=0}^{m-1} I_i(\varepsilon,\delta).$$

However, since f is continuous, the number of squares of the δ-mesh that intersect $graph\ f_{|_{I_i(\varepsilon,\delta)}}$ is at least $R_f(I_i(\varepsilon,\delta))/\delta$ and at most $2 + R_f(I_i(\varepsilon,\delta))/\delta$. Summing over all such intervals gives

$$\delta^{-1}\sum_{i=0}^{m-1} R_f(I_i(\varepsilon,\delta)) \le N_\delta^\varepsilon \le 2m + \delta^{-1}\sum_{i=0}^{m-1} R_f(I_i(\varepsilon,\delta)). \tag{1}$$

Let now $u, v \in\,]x-\varepsilon; x+\varepsilon[$, with $u < v$. Then

$$|f(u) - f(v)| \le \bar{c}(u,\varepsilon)|u-v|^{s(u)},$$

thus

$$|f(u) - f(v)| \le \bar{C}(x,\varepsilon)|u-v|^{\underline{s}_x^\varepsilon}.$$

We deduce that

$$R_f[t_1;t_2] \le \bar{C}(x,\varepsilon)|t_1-t_2|^{\underline{s}_x^\varepsilon} \qquad \forall t_1,t_2 \in\,]x-\varepsilon; x+\varepsilon[,$$

but $m \le 1 + 2\varepsilon\delta^{-1}$, and using (1) we get

$$\begin{aligned} N_\delta^\varepsilon &\le (1+2\varepsilon\delta^{-1})(2+\bar{C}(x,\varepsilon)\delta^{-1}\delta^{\underline{s}_x^\varepsilon}) \\ &\le c_1\varepsilon\delta^{\underline{s}_x^\varepsilon - 2}, \end{aligned}$$

where $c_1 > 0$ only depends on x and ε and is finite.

We deduce

$$-\frac{\log N_\delta^\varepsilon}{\log\delta} \le 2 - \underline{s}_x^\varepsilon - h(\delta),$$

where

$$h(\delta) = \frac{\log c_1}{\log\delta} + \frac{\log\varepsilon}{\log\delta}.$$

Since $\lim_{\delta\to 0} h(\delta) = 0$, we obtain

$$\limsup_{\delta\to 0} -\frac{\log N_\delta^\varepsilon}{\log\delta} \le 2 - \underline{s}_x^\varepsilon,$$

which implies

$$\lim_{\varepsilon\to 0}\limsup_{\delta\to 0} -\frac{\log N_\delta^\varepsilon}{\log\delta} \le 2 - \lim_{\varepsilon\to 0}\underline{s}_x^\varepsilon,$$

but

$$\lim_{\varepsilon\to 0}\underline{s}_x^\varepsilon = \min\left(\liminf_{t\to x} s(t), s(x)\right),$$

and finally

$$\overline{\dim}_B^x\, graph\ f \le 2 - \min\left(\liminf_{t\to x} s(t), s(x)\right).$$

Now we establish the other inequality.

For all $v \in\,]x-\varepsilon; x+\varepsilon[$, there exists u such that

$$|f(u) - f(v)| \ge \underline{c}(v,\varepsilon)\varepsilon^{s(v)},$$

thus

$$|f(u) - f(v)| \geq \underline{C}(x,\varepsilon)\varepsilon^{\overline{s}_x^\varepsilon}.$$

We deduce

$$R_f[t_1;t_2] \geq \underline{C}(x,\varepsilon)|t_1 - t_2|^{\overline{s}_x^\varepsilon} \qquad \forall t_1, t_2 \in]x-\varepsilon; x+\varepsilon[,$$

but $m \geq 2\varepsilon\delta^{-1}$, and using (1) we get

$$\begin{aligned} N_\delta^\varepsilon &\geq 2\underline{C}(x,\varepsilon)\varepsilon\delta^{-1}\delta^{-1}\delta^{\overline{s}_x^\varepsilon} \\ &= 2\underline{C}(x,\varepsilon)\varepsilon\delta^{\overline{s}_x^\varepsilon - 2}. \end{aligned}$$

Thus

$$-\frac{\log N_\delta^\varepsilon}{\log \delta} \geq 2 - \overline{s}_x^\varepsilon - h(\delta),$$

where

$$h(\delta) = \frac{\log 2\underline{C}(x,\varepsilon)\varepsilon}{\log \delta}.$$

Since $\lim_{\delta\to 0} h(\delta) = 0$, we get

$$\liminf_{\delta\to 0} -\frac{\log N_\delta^\varepsilon}{\log \delta} \geq 2 - \overline{s}_x^\varepsilon,$$

but

$$\lim_{\varepsilon\to 0} \overline{s}_x^\varepsilon = \max\left(\limsup_{t\to x} s(t), s(x)\right)$$

and finally

$$\underline{\dim}_B^x \, graph \, f \geq 2 - \max\left(\limsup_{t\to x} s(t), s(x)\right). \qquad \blacksquare$$

This result shows in particular that:

Corollary 1. *Whenever s is continuous at point x and conditions* (c_1) *and* (c_2) *hold, the local box dimension of f at x exists and is equal to* $2 - s(x)$.

Note that the converse is not true: the existence of the local box dimension of f at x does not tell anything about the continuity of s at x.

Besides, when s is not continuous at x, $s(x)$ and $\dim_B^x$ *graph* f can greatly differ (take, for instance, $f(x) = \sqrt{|x|}$ at $x = 0$). Another consequence is that we can think of the local box dimension as a more "local" quantity, and of the Hölder exponent as a more "pointwise" quantity: in the case of $f(x) = \sqrt{|x|}$, the local box dimension, which is equal to 1, is dominated by the local behavior of f around 0, as the Hölder exponent, $\frac{1}{2}$, reflects the behavior of f solely at 0.

Let us give an example which shows the necessity of condition (c_1). Consider the continuous function f defined by

$$f(x) = \begin{cases} x^u \cos(x^{-v}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $0 < u < v$. This function does not verify condition (c_1). Now, we can prove that, for $x = 0$, $\alpha_f(x) = u$ and that [24, p. 126]

$$\underline{\dim}_B^x \, graph \; f = \overline{\dim}_B^x \, graph \; f = 2 - \frac{u+1}{v+1}.$$

Hence, when $v < 1/u$, the second inequality in the proposition above does not hold.

2.6. *A Relation Between the Tricot Dimension and the Local Hölder Exponent*

Let f be a continuous function on [0; 1], and define, for $x \in [0; 1]$ and $\varepsilon > 0$

$$V_\varepsilon(x) = \sup\{|f(x') - f(x'')| : |x - x'| \leq \varepsilon, |x - x''| \leq \varepsilon\},$$

$V_\varepsilon(x)$ is called the local ε-oscillation of f at x.

Define now the conditions (p_1) and (p_2) by

$$(p_1) \quad \exists s_1 > 0 / \forall x \in [0; 1], \qquad \exists a_1(x) > 0 / V_\varepsilon \leq a_1(x)\varepsilon^{s_1},$$

$$(p_2) \quad \exists s_2 > 0 / \forall x \in [0; 1], \qquad \exists a_2(x) > 0 / V_\varepsilon \geq a_2(x)\varepsilon^{s_2}.$$

Condition (p_1) implies that

$$\liminf_{\varepsilon \to 0} \frac{\log V_\varepsilon(x)}{\log \varepsilon} \geq s_1,$$

which means that $\alpha_f(x) \geq s_1$ for every $x \in [0; 1]$.

In the same way, condition (p_2) implies that

$$\liminf_{\varepsilon \to 0} \frac{\log V_\varepsilon(x)}{\log \varepsilon} \leq s_2,$$

which means that $\alpha_f(x) \leq s_2$ for every $x \in [0; 1]$.

Then we have the following result, due to Claude Tricot:

Proposition 2. *If condition* (p_1) *holds, then*

$$\dim_P graph \; f \leq \max(1, 2 - s_1).$$

This result remains true when condition (p_1) *holds for every* $x \in [0; 1]$ *except on a set* E *such that* $\dim_P(E) = 0$.

If condition (p_2) *holds, then*

$$\dim_P graph \; f \geq 2 - s_2.$$

This result remains true when condition (p_2) *holds for every* $x \in [0; 1]$ *except on a set of Lebesgue measure zero.*

3. Characterization of the Set of Hölder Functions of Continuous Function

Theorem 1. *Let* f *be a nowhere differentiable continuous function from* $[0; 1]$ *to* $\mathbf{R}$. *Then, there exists a sequence* $\{s_n\}_{n \in \mathbf{N}}$ *of continuous functions such that*

$$\alpha_f(x) = \liminf_{n \to \infty} s_n(x) \qquad \forall x \in [0; 1].$$

Conversely, let s be a function from $[0; 1]$ *to* $[0; 1]$ *such that* $s(x) = \liminf_{n\to\infty} s_n(x)$, *where the s_n's are continuous functions. Then there exists a continuous function f from* $[0; 1]$ *to* **R** *such that*

$$\alpha_f(x) = s(x).$$

The first part of the theorem is easy to prove. Indeed, take

$$s_n(x) = \inf_{2^{-n}\le|h|<2^{-n+1}} \left\{ \frac{\log(|f(x+h) - f(x)| + 2^{-n^2})}{\log|h|} \right\}.$$

Then s_n is continuous for every integer $n \geq 1$, and since

$$\alpha_f(x) = \liminf_{h\to 0} \frac{\log|f(x+h) - f(x)|}{\log|h|},$$

it is easy to see that

$$\alpha_f(x) = \liminf_{n\to\infty} s_n(x) \qquad \forall x \in [0; 1].$$

In the following sections, we will give three constructive proofs of the second part of the theorem. We will denote by $\mathcal{H}$ the set of all functions, defined from [0; 1] to [0; 1], which are the lower limit of a sequence of continuous functions.

4. Construction Using the Schauder Basis

This construction is due to S. Jaffard [26], and is based on the well-known relation between the pointwise regularity of a function and the coefficients of its expansion in the Schauder basis.

4.1. *Recalls on the Schauder Basis*

Consider the function $\theta(x)$ from **R** to **R** defined by

$$\theta(x) = \begin{cases} 1 - |2x - 1| & \text{if } x \in [0; 1], \\ 0 & \text{if } x \notin [0; 1]. \end{cases}$$

It is well known that if f is a continuous function from [0; 1] to **R**, and if $f(0) = f(1) = 0$, then

$$f(x) = \sum_{j\geq 0} \sum_{0\leq k<2^j} c(j,k)\theta_{j,k}(x),$$

where

$$\theta_{j,k}(x) = \theta(2^j x - k)$$

and

$$c(j,k) = f((k+\tfrac{1}{2})2^{-j}) - \tfrac{1}{2}(f(k2^{-j}) + f((k+1)2^{-j})).$$

We have the following results:

Proposition 3. *If $f \in C^s(x_0)$ for some $x_0 \in [0; 1]$ and $s > 0$, then there exists a constant C such that*

$$|c(j,k)| \le C(2^{-j} + |k2^{-j} - x_0|)^s.$$

The proof of this proposition is straightforward.

Proposition 4. *Suppose that there exists a constant C such that for every $x \in [0; 1]$ we have*

$$|f(x+h) - f(x)| \le C\omega(h) \qquad \text{when} \quad h \to 0, \tag{2}$$

where ω is a strictly increasing function from $[0; 1]$ to $\mathbf{R}$, which verifies

$$w(0) = 0 \qquad \text{and} \qquad w(h) = O(|\log h|^{-N}) \qquad \forall N \ge 1.$$

Suppose also that for some $x_0 \in [0; 1]$ and $s > 0$, there exists a constant C such that

$$|c(j,k)| \le C(2^{-j} + |k2^{-j} + |2^{-j} - x_0|)^s.$$

Then

$$f \in C^{s-\varepsilon}(x_0) \qquad \forall \varepsilon > 0.$$

Proof. Let x_1 be a real in the neighborhood of x_0, and let j_0 be the integer such that

$$2^{-j_0} \le |x_1 - x_0| < 2^{-(j_0-1)}.$$

We define the integer j_1 such that $w(2^{-j_1}) = 2^{-sj_0}$. Then

$$|f(x_1) - f(x_0)| \le W + X + Y + Z,$$

where

$$\begin{aligned}
W &= \sum_{0\le j\le j_0} \sum_{0\le k<2^j} |c(j,k)(\theta_{j,k}(x_1) - \theta_{j,k}(x_0))|,\\
X &= \sum_{j>j_0} \sum_{0\le k<2^j} |c(j,k)|\theta_{j,k}(x_0),\\
Y &= \sum_{j_0<j\le j_1} \sum_{0\le k<2^j} |c(j,k)|\theta_{j,k}(x_1),\\
Z &= \left|\sum_{j>j_1} \sum_{0\le k<2^j} c(j,k)\theta_{j,k}(x_1)\right|.
\end{aligned}$$

For $j < j_0 - 1$, $\theta_{j,k}(x_0) \ne 0$ implies $\theta_{j,k}(x_1) \ne 0$. Furthermore, for each j, there exists a unique k such that $\theta_{j,k}(x_0) \ne 0$ or $\theta_{j,k}(x_1) \ne 0$. In this case, we have $|k2^{-j} - x_0| \le 2^{-j}$. Finally, remark that $|\theta_{j,k}(x_1) - \theta_{j,k}(x_0)| \le 2^j|x_1 - x_0|$. Hence, we have

$$W \le \sum_{0\le j\le j_0} 2^{j(1-s)}|x_1 - x_0| \le C|x_1 - x_0|^s.$$

It is easy to prove that $X \le C2^{-j_0 s}$, which leads to

$$X \le C|x_1 - x_0|^s.$$

When $\theta_{j,k}(x_1) \neq 0$, we have $|k2^{-j} - x_1| \leq 2^{-j}$, and if $j > j_0$, this implies that

$$|c(j,k)| \leq C|x_1 - x_0|^s,$$

hence,

$$Y \leq C(j_1 - j_0)|x_1 - x_0|^s.$$

For every integer $N \geq 1$, there exists a constant C_N such that $\omega(2^{-j_1}) \leq C_N {j_1}^{-N}$. Hence,

$$j_1 \leq {C_N}^{1/N} 2^{j_0(s/N)},$$

since $\omega(2^{-j_1}) = 2^{-j_0 s}$. This implies that, for every $\varepsilon > 0$, there exists a constant C_ε such that

$$j_1 - j_0 \leq C_\varepsilon |x_1 - x_0|^{-\varepsilon}.$$

Finding an upper bound for Z requires the following results:

Lemma 1. *Denote*

$$S_q(f)(x) = \sum_{0 \leq j \leq q} \sum_{0 \leq k < 2^j} c(j,k)\theta_{j,k}(x),$$

then $S_q(f)$ is the continuous piecewise affine function which satisfies

$$S_q(f)(k2^{-q}) = f(k2^{-q}) \qquad \forall k = 0, \ldots, 2^q.$$

Corollary 2.

$$||f - S_q(f)||_\infty \leq \omega(2^{-q}).$$

The proofs of the lemma and the corollary are easy.

We remark that $Z \leq ||f - S_{j_1}(f)||_\infty$, and since

$$\omega(2^{-j_1}) = 2^{-j_0 s} \leq |x_1 - x_0|^s,$$

the proof of the proposition is completed. ■

4.2. *Construction of the Desired Function*

The following result will be used in the proof of the theorem.

Lemma 2. *Let $s \in \mathcal{H}$. Then there exists a sequence $\{Q_n\}_{n \geq 1}$ of polynomials such that*

$$\begin{cases} s(t) = \liminf_{n \to +\infty} Q_n(t) & \forall t \in [0;1], \\ ||Q_n'||_\infty \leq n & \forall n \geq 1, \end{cases} \tag{3}$$

where Q_n' is the derivative of Q_n.

Proof. Since $s \in \mathcal{H}^*$, there exists a sequence $\{s_k\}_{k\in\mathbf{N}^*}$ of continuous functions such that

$$s(t) = \liminf_{k\to+\infty} s_k(t) \qquad \forall t \in [0;1].$$

Thus there exists a sequence $\{P_k\}$ of polynomials such that

$$s(t) = \liminf_{k\to+\infty} P_k(t) \qquad \forall t \in [0;1].$$

Let $\{q_k\}_{k\in\mathbf{N}^*}$ be a sequence of integers such that

$$q_1 \geq M_1$$

and

$$q_k \geq \max(M_k, q_{k-1}) \qquad \text{for} \quad k \geq 2,$$

where

$$M_k = ||P_k'||_\infty.$$

Define the sequence $\{Q_j\}_{j\geq 1}$ by

$$Q_j(t) = 0 \qquad \text{if} \quad 1 \leq j < q_1$$

and

$$Q_j(t) = P_k(t) \qquad \text{if} \quad q_k \leq j < q_{k+1} \quad \text{for} \quad k \geq 1.$$

Of course, $s(t) = \liminf_{j\to+\infty} Q_j(t)\, \forall t \in [0;1]$. On the other hand,

$$|Q_j'(t)| = |P_k'(t)| \qquad \text{if} \quad q_k \leq j < q_{k+1},$$

and

$$|P_k'(t)| \leq M_k \leq q_k \qquad \forall t \in [0;1]$$

hence

$$|Q_j{}'(t)| \leq j \qquad \forall j \geq 1 \quad \text{and} \quad \forall t \in [0;1].$$ ■

Proposition 5. *Let $s \in \mathcal{H}$ and let $(Q_n)_{n\geq 1}$ be the associated sequence of polynomials verifying* (3).

Consider the continuous function f defined on $[0;1]$ *by*

$$f(x) = \sum_{j\geq 0} \sum_{0\leq k<2^j} c(j,k)\theta_{j,k}(x),$$

where

$$c(j,k) = \inf(2^{-j/\log j}, 2^{-jQ_j(k2^{-j})}).$$

Then

$$\alpha_f(x) = s(x) \qquad \forall x \in [0;1].$$

Proof. We first prove that $\alpha_f(x_0) \le s(x_0)$ for every $x_0 \in [0; 1]$.

Let $j \ge 1$ be an integer, and let k be the integer such that $x_0 \in [k2^{-j}; (k+1)2^{-j}[$. Hence, $|Q_j(k2^{-j}) - Q_j(x_0)| \le j2^{-j}$. This implies that for every $\varepsilon > 0$ there exists an integer j_0, such that $c(j,k) > 2^{-j(s(x_0)+\varepsilon)}$ for every $j > j_0$. Using Proposition 3, we conclude that $\alpha_f(x_0) \le s(x_0)$.

Let us now show that $\alpha_f(x_0) \ge s(x_0) - \varepsilon$ for every $\varepsilon > 0$. Remark that there exists j_ε such that

$$s(x_0) - \varepsilon < Q_j(k2^{-j})$$

for every $j \ge j_\varepsilon$, and k such that $x_0 \in [k_0 2^{-j}; (k+1)2^{-j}[$. This implies that

$$c(j,k) \le 2^{-j(s(x_0)-\varepsilon)}.$$

Furthermore, since $c(j,k) \le 2^{-j/\log j}$, it is easy to see that condition (2) holds. Hence, we conclude using Proposition 4 that $\alpha_f(x_0) \ge s(x_0) - \varepsilon$. ∎

5. Use of Weierstrass-Type Functions

In this section, we show that a simple generalization of the Weierstrass function allows us to control the regularity at each point. For a related result, see [24, p. 282].

We first recall some properties of the Weierstrass function, which is defined by

$$W(t) = \sum_{k=1}^{+\infty} \lambda^{-ks} \sin(\lambda^k t),$$

where $\lambda > 1$ and $s \in]0; 1[$.

It is well known [27] that $\alpha_W(t) = s$ for all t and that $\dim_B \mathit{graph}\ W = 2 - s$. However the value of $\dim_H \mathit{graph}\, W$ is not yet known. Of course, $\dim_H \mathit{graph}\, W \le \dim_B \mathit{graph}\, W$, and using mass distribution methods depending on estimates for the Lebesgue measure of the set $\{t/(t, W(t)) \in D\}$ where D is a disk, it can be shown [9] that there exists a constant $c > 0$ such that $\dim_H \mathit{graph}\ W \ge s - c/\log\lambda$.

As mentioned in the Introduction, several authors have considered generalizations of the Weierstrass function, by replacing the sinus with other types of function. Here we consider another type of generalization.

Proposition 6. *Let $s(t)$ be a function from $[0; 1]$ to $[a; b] \subset]0; 1[$, which is the lower limit of a sequence of continuous functions. Let a' and b' be two reals such that $0 < a' < a < b < b' < 1$, and consider the sequence $\mathbf{L} = (l_p)_{p\ge1}$ defined by*

$$\begin{cases} l_1 = 1, \\ l_{p+1} = \left[\dfrac{1-a'}{1-b'} l_p\right] + 1, \end{cases} \tag{4}$$

where $[.]$ denotes the integer part. Then:

- *There exists a sequence $\{Q_n\}_{n\ge1}$ of polynomials such that*

$$\begin{cases} s(t) = \liminf\limits_{\substack{n\to+\infty \\ n\in\mathbf{L}}} Q_n(t) \quad \forall t \in [0; 1], \\ ||Q_n'||_\infty \le n \quad \forall n \ge 1, \end{cases} \tag{5}$$

  where Q_n' is the derivative of Q_n.

- *Define*

$$f(t) = \sum_{k \in \mathbf{L}} \lambda^{-kQ_k(t)} \sin(\lambda^k t).$$

Then, provided that λ *is an even integer large enough, we have*

$$\alpha_f(t) = s(t) \qquad \forall t \in [0; 1].$$

Proof. The proof of the first item is similar to that of Lemma 2; the only difference is that we now define the sequence q_k by

$$q_k \geq \max\left(M_k, \frac{1-a'}{1-b'} q_{k-1} + 1\right) \qquad \text{for} \quad k > 1.$$

Now, we give the proof of the second item. Let t be fixed and let ε be a positive real such that $s(t) + \varepsilon < b'$ and $s(t) - \varepsilon > a'$. We begin by proving that $f \in C^{s(t)-\varepsilon}(t)$.

There exists an integer k_0 such that $Q_k(t) > s(t) - \varepsilon$, for every $k > k_0$. Let h be a real such that $0 < |h| < \lambda^{-k_0}$. Then we have

$$\begin{aligned} |f(t+h) - f(t)| &= \left| \sum_{k \in \mathbf{L}} (\lambda^{-kQ_k(t+h)} \sin(\lambda^k(t+h)) - \lambda^{-kQ_k(t)} \sin(\lambda^k t)) \right| \\ &\leq A + A'_{k_0} + A', \end{aligned}$$

where

$$\begin{aligned} A &= \sum_{k=1}^{+\infty} |(\lambda^{-kQ_k(t+h)} - \lambda^{-kQ_k(t)}) \sin(\lambda^k(t+h))|, \\ A'_{k_0} &= \sum_{k=1}^{k_0} \lambda^{-kQ_k(t)} |\sin(\lambda^k(t+h)) - \sin(\lambda^k t)|, \end{aligned}$$

and

$$A' = \sum_{k=k_0+1}^{+\infty} \lambda^{-kQ_k(t)} |\sin(\lambda^k(t+h)) - \sin(\lambda^k t)|.$$

Let us give an upper bound for A. We have

$$A \leq \sum_{k=1}^{+\infty} |\lambda^{-kQ_k(t+h)} - \lambda^{-kQ_k(t)}|$$

but

$$\lambda^{-kQ_k(t+h)} - \lambda^{-kQ_k(t)} = -(\log \lambda) \times [Q_k(t+h) - Q_k(t)] \times (k\lambda^{-k\tau}),$$

where $\tau \in [\min(Q_k(t), Q_k(t+h)); \max(Q_k(t), Q_k(t+h))]$.

Thus

$$A \leq (\log \lambda) \sum_{k=1}^{+\infty} k\lambda^{-k\tau} |Q_k(t+h) - Q_k(t)|.$$

Since

$$|Q_k(t+h) - Q_k(t)| \leq k|h|,$$

we have

$$|A| \leq c_1 |h| \leq c_1 |h|^{s(t)-\varepsilon},$$

with $c_1 = \log\lambda \sum_{k=1}^{+\infty} k^2 \lambda^{-ka}$.

Let us now give an upper bound for A'. For this purpose, we consider the integer N such that

$$\lambda^{-(N+1)} \leq |h| \leq \lambda^{-N}.$$

We have, using the mean value theorem,

$$A' \leq |h| X + 2Y,$$

where

$$X = \sum_{k=1}^{N} \lambda^{-k(s(t)-\varepsilon-1)}$$

and

$$Y = \sum_{k=N+1}^{+\infty} \lambda^{-k(s(t)-\varepsilon)},$$

but

$$\begin{aligned} X &\leq \frac{1}{1-\lambda^{s(t)-1}} |h|^{s(t)-\varepsilon-1}, \\ Y &\leq \frac{1}{1-\lambda^{-s(t)}} |h|^{s(t)-\varepsilon}. \end{aligned}$$

Since $s(t)$ is bounded, there exists a constant $c_2 > 0$ depending only on t and ε such that

$$A' \leq c_2 |h|^{s(t)-\varepsilon}.$$

Finally, it easy to see that there exists a positive constant c_3, which depends only on t and ε such that

$$|A'_{k_0}| \leq c_3 |h| \leq c_3 |h|^{s(t)-\varepsilon}.$$

Hence, if $c = 3\max(c_1, c_2, c_3)$, we have

$$|f(t+h) - f(t)| \leq c|h|^{s(t)-\varepsilon}.$$

Now we will prove that $\alpha_f(t) \leq s(t)$.

There exists an infinite set $\Gamma = \Gamma(t, \varepsilon) \subset \mathbf{L}$ such that $s(t) - \varepsilon < Q_k(t) < s(t) + \varepsilon$, for every $k \in \Gamma$. Let N be an integer in Γ such that $N \gg k_0$. Let $h = [\pi/c(N)]\lambda^{-N}$, where $c(N)$ is chosen in the set $\{\pm 1, \pm 2\}$ so that

$$\left| \sin\left(\lambda^N t + \frac{\pi}{c(N)}\right) - \sin(\lambda^N t) \right| > \tfrac{1}{10}.$$

Hence, if λ is an even integer, we have

$$|f(t+h) - f(t) - \lambda^{-NQ_N(t)}(\sin(\lambda^N(t+h)) - \sin(\lambda^N t))| \leq A + A'_{k_0} + A'' + A''',$$

where

$$A'' = \sum_{\substack{k>k_0\\ k\in \mathbf{L}\backslash\Gamma}} \lambda^{-kQ_k(t)} |\sin(\lambda^k(t+h)) - \sin(\lambda^k t)|,$$
$$A''' = \sum_{\substack{k<N\\ k\in\Gamma}} \lambda^{-kQ_k(t)} |\sin(\lambda^k(t+h)) - \sin(\lambda^k t)|.$$

Since $Q_k(t) \geq s(t) + \varepsilon$ if $k \in \mathbf{L}\backslash\Gamma$ and $k > k_0$, we have

$$A'' \leq \sum_{k\in\mathbf{N}} \lambda^{-k(s(t)+\varepsilon)} |\sin(\lambda^k(t+h)) - \sin(\lambda^k t)|,$$

thus there exists a positive constant c_4 such that

$$A'' \leq c_4 |h|^{s(t)+\varepsilon}.$$

Let N_l be the highest integer in Γ less than N. Then

$$\begin{aligned} A''' &\leq \sum_{k=0}^{N_l} \lambda^{-k(s(t)-\varepsilon)} |\sin(\lambda^k(t+h)) - \sin(\lambda^k t)| \\ &\leq |h| \sum_{k=0}^{N_l} \lambda^{k(1-(s(t)-\varepsilon))} \\ &\leq |h| \frac{\lambda^{N_l(1-(s(t)-\varepsilon))}}{\lambda^{1-(s(t)-\varepsilon)} - 1}. \end{aligned}$$

Using the fact that $1 - s(t) + \varepsilon < 1 - a'$ and $1 - s(t) - \varepsilon > 1 - b'$, we get

$$A''' \leq |h| \frac{\lambda^{N(1-s(t)-\varepsilon)}}{\lambda^{1-(s(t)-\varepsilon)} - 1}.$$

Thus there exists a positive constant c_5 such that

$$A''' \leq c_5 |h|^{s(t)+\varepsilon}.$$

We can choose λ large enough so that the constants c_1, c_3, c_4, and c_5 are less than $\frac{1}{80}$. Hence we end up with

$$|f(t+h) - f(t)| > \tfrac{1}{20} |h|^{s(t)+\varepsilon}. \qquad \blacksquare$$

In the case where s is a continous function, we have the following result:

Proposition 7. *Let s be a continuous function from $[0;1]$ to $[a;b] \subset]0;1[$ such that*

$$s(x) < \alpha_s(x) \qquad \forall x \in [0;1].$$

Assume also that there exsits a constant $M > 0$ such that

$$|s(t) - s(u)| \leq M|t-u|^{\alpha_s(t)} \qquad \forall (t,u) \in [0;1] \times [0;1].$$

Then the function $f(x) = \sum_{k\in\mathbf{N}} \lambda^{-ks(x)} \sin(\lambda^k x)$ is such that

$$2 - \dim_B^x \textit{graph } f = \alpha_f(x) = s(x).$$

Proof. See Appendix.

6. Construction Using an Iterated Function System (IFS)

The third construction of a continuous function with a prescribed Hölder function is based upon a generalization of the notion of IFS. This construction bears some analogy with the first one, but here we directly manipulate the contraction ratios of affine functions instead of working on the coefficients of the expansion in the Schauder basis. To begin with, we recall some basic facts about IFS. More details can be found in [13], [28], [16], [17], [15], and [29] and others.

6.1. *Recalls*

Let K be a compact metric space whose distance is denoted by $d(x, y)$ for $x, y \in K$. Let H be the set of all nonempty closed subsets of K. Then H is a compact metric space with the Hausdorff metric [13]

$$h(A, B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x, y), \sup_{x\in B}\inf_{y\in A} d(x, y)\right\},$$

which is defined whenever A and B are subsets of K.

Let $w_n\colon K \to K$ for $n \in \{1, 2, \ldots, N\}$ be N continuous functions. Then $\{K, w_n\colon n = 1, 2, \ldots, N\}$ is called an iterated function system (IFS). Define $W\colon H \to H$ by

$$W(A) = \bigcup_{n=1}^{N} w_n(A) \qquad \text{for} \quad A \in H.$$

Any set $G \in H$ such that

$$W(G) = G$$

is called an attractor of the IFS $\{K, w_n : n = 1, 2, \ldots, N\}$. An IFS always admits at least one attractor. Indeed, start with any $S \in H$, then the closure of the set of all accumulation points of $\{W^{om}(S)\}_{m=1}^{\infty}$, with $W^{om}(S) = W(W^{o(m-1)}(S))$ is an attractor of the IFS.

If for some $s \in [0, 1[$ and all $n \in \{1, \ldots, N\}$

$$d(w_n(x), w_n(y)) \leq sd(x, y) \qquad \forall (x, y) \in K \times K,$$

then the IFS is termed hyperbolic. In this case W is a contraction mapping, hence it admits a unique fixed point which is the unique attractor of the IFS.

When the attractor G of an IFS is unique, it may be obtained as follows [14]: let $p = (p_1, \ldots, p_N)$ be a probability vector with each $p_n > 0$ and $\sum_n p_n = 1$. Start from the fixed point x_0 of w_1 and define a sequence (x_m) by choosing successively $x_m \in \{w_1(x_{m-1}), \ldots, w_N(x_{m-1})\}$ for $m \in \{1, 2, 3, \ldots\}$, where probability p_n is attached to the event $x_m = w_n(x_{m-1})$. Then the orbit $\{x_m\}_{m\in N}$ is dense in G. The p_n's allow us to generate a unique probability measure μ on K which is stationary for the discrete-time Markov process defined as follows.

The probability of transfer of $x \in K$ to a Borel subset B of K is

$$p(x, B) = \sum_n p_n \delta_{w_n(x)}(B),$$

where

$$\delta_y(B) = \begin{cases} 1 & \text{if } y \in B, \\ 0 & \text{if } y \notin B. \end{cases}$$

We will not develop here this aspect of IFS theory, and will now focus on the use of the IFS for constructing graphs of continuous functions [14].

Given a set of points $\{(x_n, y_n) \in [0; 1] \times [u; v], n = 0, 1, \ldots, N\}$, with $(u, v) \in \mathbf{R}^2$, consider the IFS given by the N contractions w_n $(n = 1, \ldots, N)$ defined on $[0; 1]\times[u; v]$, by

$$w_n(x, y) = (L_n(x); F_n(x, y)),$$

where L_n is a contraction that maps $[0; 1]$ to $[x_{n-1}; x_n]$ and $F_n\colon [0; 1] \times [u; v] \to [u\,; v]$ is a function, contractive with respect to the second variable, such that

$$F_n(x_0, y_0) = y_{n-1}; \qquad F_n(x_N, y_N) = y_n. \tag{6}$$

The attractor of this IFS is the graph of a continuous function f which interpolates the points (x_n, y_n) [14].

If the L_n's are affine, $L_n(x) = a_n x + h_n$, and if, for each $n \in \{1, \ldots, N\}$,

$$t_n d(x, y) \leq d(w_n(x), w_n(y)) \leq s_n d(x, y) \qquad \text{for all} \quad x, y \in K,$$

where $0 < t_n \leq s_n < 1$, then

$$\min(2, l) \leq \dim_H \textit{graph } f \leq u,$$

where l and u are the positive solutions of

$$\sum_{n=1}^{N} t_n^l = 1 \qquad \text{and} \qquad \sum_{n=1}^{N} s_n^u = 1,$$

and where the lower bound holds when

$$t_1 t_N \leq \min(a_1, a_N) \left(\sum_{n=1}^{N} t_n^l \right)^{2/l}.$$

Concerning the box dimension, if each F_n is affine with contraction ratio equal to c_n, and if the interpolation points are equally spaced, then it is a classical result that [22]

$$\dim_B \; graph \; f = 1 + \frac{\log(c_1 + \cdots + c_N)}{\log N}.$$

6.2. *Local Behavior of Self-Affine Functions*

Under some conditions on the F_n's, the function f defined above is nowhere differentiable. But here we want more, namely to control the regularity of f at each point.

In this section we obtain the local Hölder exponent of f at each point $x \in [0; 1]$ in the case where the F_n's are affine functions, and the interpolation points are equally spaced. We also derive the multifractal spectrum of f and recover the classical formula for the box dimension of the graph of f. Related results concerning the almost-sure Hölder exponent of f have already been obtained in [17]. Results concerning the multifractal spectrum were independently obtained in [18] and [19].

It is convenient to rewrite our setting in the following form: Let S_i $(0 \leq i < m)$ be affine transformations represented in matrix notation by

$$S_i \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1/m & 0 \\ a_i & c_i \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} + \begin{pmatrix} i/m \\ b_i \end{pmatrix}.$$

We suppose $0 \le t \le 1$ and $1/m < c_i < 1$. Let f be the function whose graph is the attractor G of the IFS defined by the S_i's (with conditions on a_i and b_i corresponding to (6) to ensure the continuity of f). Our result concerning the local regularity of f is the following one:

Proposition 8. *Let $0.i_1 \dots i_k \dots$ be the terminating base-m expansion of a real $t \in [0; 1)$. Then the Hölder exponent α of f at point t is*

$$\alpha = \min\left(\liminf_{k\to+\infty} \frac{\log(c_{i_1}\dots c_{i_k})}{\log(m^{-k})}, \liminf_{k\to+\infty} \frac{\log(c_{j_1}\dots c_{j_k})}{\log(m^{-k})}, \liminf_{k\to+\infty} \frac{\log(c_{l_1}\dots c_{l_k})}{\log(m^{-k})} \right),$$

where, for any positive integer k, the k-tuples $(j_1, \dots, j_k)$ and $(l_1, \dots, l_k)$ of nonnegative integers strictly smaller than m, are uniquely determined by

$$t_k = m^{-k}[m^k t],$$

$$\textit{if} \quad t_k + m^{-k} < 1, \quad \textit{then} \quad t_k^+ = t_k + m^{-k} = \sum_{p=1}^{k} j_p m^{-p} \quad \textit{else} \quad t_k^+ = t_k,$$

$$\textit{if} \quad t_k - m^{-k} > 0, \quad \textit{then} \quad t_k^- = t_k - m^{-k} = \sum_{p=1}^{k} l_p m^{-p} \quad \textit{else} \quad t_k^- = t_k.$$

Proof. The proof is an adaptation of the classical computation of the box dimension of the graph of self-affine curves [22].

Let k be a positive integer and let $(n_1, \dots, n_k)$ be a k-tuple of integers such that $0 \le n_p < m$ for every $p = 1, \dots, k$. Let $I_{n_1 \dots n_k}$ be the interval of reals in [0; 1) whose base-m expansion begins with $n_1 \dots n_k$. Then $graph\ f_{|_{I_{n_1\dots n_k}}} = S_{n_1} \circ \dots \circ S_{n_k}(G)$, which is a translation of $T_{n_1} \circ \dots \circ T_{n_k}(G)$, where T_i is the linear part of S_i. It is easily seen that the matrix representing $T_{n_1} \circ \dots \circ T_{n_k}$ is

$$\begin{pmatrix} m^{-k} & 0 \\ m^{1-k}a_{n_1} + m^{2-k}c_{n_1}a_{n_2} + \dots + c_{n_1}c_{n_2}\dots c_{n_{k-1}}a_{n_k} & c_{n_1}c_{n_2}\dots c_{n_k} \end{pmatrix}.$$

Note $a = \max|a_i|$, $c = \min(c_i)$, $r = a/[c(1-(mc)^{-1})]$. We have

$$|m^{1-k}a_{n_1} + m^{2-k}c_{n_1}a_{n_2} + \dots + c_{n_1}c_{n_2}\dots c_{n_{k-1}}a_{n_k}| \le r c_{n_1}\dots c_{n_k},$$

so that if s is the height of the rectangle containing G, then $graph\ f_{|_{I_{n_1\dots n_k}}}$ is contained in the rectangle whose height is $(r+s)c_{n_1}\dots c_{n_k}$.

Consider now a real $\beta < \alpha$; there exists a positive integer k_0 such that, for every integer $k > k_0$, we have

$$\beta(i_k) > \beta, \qquad \beta(j_k) > \beta \qquad \text{and} \qquad \beta(l_k) > \beta,$$

where

$$\beta(n_k) = \frac{\log(c_{n_1}\dots c_{n_k})}{\log(m^{-k})}.$$

Let h be a real small enough so that the integer k, defined by $m^{-k-1} \le |h| < m^{-k}$, verifies $k > k_0$. Then either (i), (ii), or (iii) is true:

(i) $(t, t+h) \subset I_{i_1\dots i_k}$;
(ii) $(t, t+h) \subset I_{i_1\dots i_k} \cup I_{j_1\dots j_k}$;
(iii) $(t, t+h) \subset I_{i_1\dots i_k} \cup I_{l_1\dots l_k}$.

Denote $r_1 = r + s$

Case (i).

We have

$$|f(t+h) - f(t)| \leq r_1 c_{i_1} \ldots c_{i_k}.$$

Case (ii).

Since f is continuous, we have

$$|f(t+h) - f(t)| \leq r_1 c_{i_1} \ldots c_{i_k} + r_1 c_{j_1} \ldots c_{j_k}.$$

Case (iii).

Using again the continuity of f, we have

$$|f(t+h) - f(t)| \leq r_1 c_{i_1} \ldots c_{i_k} + r_1 c_{l_1} \ldots c_{l_k}.$$

Hence, we always have

$$|f(t+h) - f(t)| \leq 2r_1 |h|^{\beta}.$$

This implies that $f \in C^{\alpha-\varepsilon}(t)$ for every $\varepsilon > 0$.

On the other hand, consider now a real $\gamma > \alpha$. Assume without loss of generality that

$$\alpha = \liminf_{k \to +\infty} \frac{\log(c_{j_1} \ldots c_{j_k})}{\log(m^{-k})}$$

(the other cases are treated by simply changing j to i or l).

Then there exists a subsequence $\sigma(k)$ such that, for every k, we have

$$\frac{\log(c_{j_1} \ldots c_{j_{\sigma(k)}})}{\log(m^{-\sigma(k)})} < \gamma.$$

If q_1, q_2, and q_3 are three noncollinear points in G, then $S_{j_1} \circ \cdots \circ S_{j_{\sigma(k)}}(G)$ contains the points $(x_n, f(x_n)) = S_{j_1} \circ \cdots \circ S_{j_{\sigma(k)}}(q_n)$ $(n = 1, 2, 3)$. The height $d_{\sigma(k)}$ of the triangle with these vertices is at least $dc_{j_1} \ldots c_{j_{\sigma(k)}}$ where d is the vertical distance from q_2 to $[q_1; q_3]$. Thus, for every k, there exists a real h_k such that $|h_k| < 2m^{-\sigma(k)}$ and

$$|f(t+h_k) - f(t)| \geq \frac{d}{2} c_{j_1} \ldots c_{j_{\sigma(k)}},$$

which implies that

$$|f(t+h_k) - f(t)| \geq \frac{d}{2} |h_k|^{\gamma}.$$

This shows that $f \notin C^{\alpha+\varepsilon}(t)$ for every $\varepsilon > 0$, and the proof is complete. ■

Using this proposition, it is easy to deduce the spectrum $(\alpha, F(\alpha))$ of the singularity of f. The proof is analogous to the one for multinomial measures.

Corollary 3. *With the same notations as above, and assuming that the proportion* $\varphi_i(t)$ *of* $(i-1)$*'s in the base-m expansion of* t *exists for each* i, *we have*

$$\alpha_f(t) = -\sum_{i=0}^{m-1} \varphi_i(t) \log_m c_i; \quad F(\alpha) = -\sum_{i=0}^{m-1} \varphi_i \log_m \varphi_i; \quad \tau(q) = -\log_m \sum_{i=0}^{m-1} c_i{}^q,$$

(*for definition of* F *and* τ, *see, for instance,* [30].)

Remark 1. Using the relation $\dim_B graph\ f = 1 - \tau(1)$ we recover the classical result [22]

$$\dim_B graph\ f = 1 + \log_m \sum_{i=0}^{m-1} c_i .$$

It is now clear that, with this construction, we cannot hope to control the local regularity at each point, since almost all points have the same Hölder exponent (because the almost-sure value of $\varphi_i(t)$ with respect to the Lebesgue measure is $1/m$). We thus need to use some generalization, which will be presented in the next section.

6.3. *Recursive Construction*

We set up here another way to construct fractals recursively, originally due to Andersson [29]. We consider a collection of sets $(F^k)_{k\in\mathbf{N}^*}$, where each F^k is a nonempty finite set of contractions S_i^k in K for $i = 0, \ldots, N_k - 1$, $N_k \geq 1$, being an integer which denotes the cardinal of F^k. We denote by c_i^k the contraction ratio of S_i^k for $i = 0, \ldots, N_k - 1$ and $k \in \mathbf{N}^*$.

For $n \in \mathbf{N}^*$, let $\Im_{N_I}^n$ be the set of sequences of length n, defined as follows

$$\Im_{N_i}^n = \{\sigma = (\sigma_1, \ldots, \sigma_n) \ : \sigma_i \in \{0, \ldots, N_i - 1\},\ i \in \mathbf{N}^*\},$$

and

$$\Im_{N_i}^\infty = \{\sigma = (\sigma_1, \sigma_2, \ldots) \ : \sigma_i \in \{0, \ldots, N_i - 1\},\ i \in \mathbf{N}^*\}.$$

Define the operator W^k: $H \to H$ by

$$W^k(A) = \bigcup_{n=1}^{N_k} S_n^k(A) \qquad \text{for} \quad A \in H,$$

where N_k is the cardinal of F_k. Define the conditions:

$$\text{(c)} \qquad \lim_{n\to\infty} \sup_{(\sigma_1,\ldots,\sigma_n)\in\Im_{N_i}^n} \left\{ \prod_{k=1}^{n} c_{\sigma_k}^k \right\} = 0,$$

$$\text{(c}'\text{)} \qquad \lim_{n\to\infty} \sup_{(\sigma_1,\sigma_2,\ldots)\in\Im_{N_i}^\infty} \left\{ \sum_{j=n}^{\infty} d(S_{\sigma_{j+1}}^{j+1}x, x) \prod_{k=1}^{j} c_{\sigma_k}^k \right\} = 0.$$

The proof of the following proposition can be found in [29].

Proposition 9. *If the conditions* (c) *and* (c′) *hold, then there exists a unique compact G such that*

$$\lim_{k\to\infty} W^k \circ \cdots \circ W^1(A) = G \qquad \textit{for every} \quad A \in H.$$

We call G the attractor of the IFS $(K, \{F^k\}_{k\in\mathbf{N}^*})$.

We will use this generalized result to obtain more flexibility in the construction of our functions.

Let F^k be the set of affine transformations S_i^k $(0 \leq i < m)$ represented in matrix notation by

$$S_i^k \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1/m & 0 \\ a_i^k & c_i^k \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} + \begin{pmatrix} i/m \\ b_i^k \end{pmatrix}.$$

We suppose $0 \leq t \leq 1$ and $1/m < c_i^k < 1$. We also assume that conditions (c) and (c′) hold to ensure that we have a unique and compact attractor. Then if the a_i^k's and the b_i^k's satisfy some relations, analogous to those proposed in Subsection 6.1, one can prove, using the same techniques as in [14], that the attractor of the IFS $(K, \{F^k\}_{k\in N})$ is the graph of a continuous function f. We then have the following result:

Proposition 10. *Let* $0.i_1 \ldots i_k \ldots$ *be the base-m expansion of a real* $t \in [0; 1)$. *Then*

$$\alpha_f(t) = \min\left(\liminf_{k\to+\infty} \frac{\log(c_{i_1}^1 \ldots c_{i_k}^k)}{\log(m^{-k})}, \liminf_{k\to+\infty} \frac{\log(c_{j_1}^1 \ldots c_{j_k}^k)}{\log(m^{-k})}, \liminf_{k\to+\infty} \frac{\log(c_{l_1}^1 \ldots c_{l_k}^k)}{\log(m^{-k})} \right),$$

where, for any integer k, *if we denote* $t_k = m^{-k}[m^k t]$, *the* k*-tuples* $(j_1, \ldots, j_k)$ *and* $(l_1, \ldots, l_k)$ *are given by*

$$t_k^+ = t_k + m^{-k} = \sum_{p=1}^{k} j_p m^{-p},$$

$$t_k^- = t_k - m^{-k} = \sum_{p=1}^{k} l_p m^{-p}.$$

Proof. The proof uses the same techniques as in Proposition 8.

Although this generalization allows more flexibility in the choice of $\alpha_f(t)$, it is still too much constrained. Indeed, it is easy to see that if two reals differ only at a finite number of ranks in their base-*m* expansion, then they will have the same Hölder exponent. Hence we cannot control the regularity independently at each point.

To do so, now let F^k be defined as the set of affine transformations S_i^k $(0 \leq i \leq m^k - 1)$, each S_i^k operating only on $[[i/m]m^{-k+1}; ([i/m]+1)m^{-k+1}]$ and maps to $[im^{-k}; (i+1)m^{-k}]$. Suppose, also, that we want to interpolate the points $(i/m, y_i)$ for $i = 0, \ldots, m$, $m \geq 2$, and $y_i \in \mathbf{R}$. Let the compact K be a rectangle containing the (x_i, y_i)'s and write

$$S_i^k \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1/m & 0 \\ a_i^k & c_i^k \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} + \begin{pmatrix} i/m^k \\ b_i^k \end{pmatrix}.$$

We call $(K, (F^k))$ a *generalized affine IFS*. Define the following conditions, which allow the attractor to be the graph of a continuous function f (for the sake of simplicity we will give conditions when $m = 2$, the general case being handled similarly): start with the graph of any nonaffine continuous function φ and denote

$$\varphi(0) = u, \qquad \varphi(1) = v.$$

Then choose the contractions (or, more precisely, the a_i^k and b_i^k) so that they verify the following conditions:

for $i = 0, 1$:

$$S_i^1(0,u) = \left(\frac{i}{m}, y_i\right); \qquad S_i^1(1,v) = \left(\frac{i+1}{m}, y_{i+1}\right),$$

$$S_0^2(0,y_0) = (0,y_0); \quad S_0^2(\tfrac{1}{2},y_1) = S_1^2(0,y_0); \qquad S_1^2(\tfrac{1}{2},y_1) = (\tfrac{1}{2},y_1),$$
$$S_2^2(1/2,y_1) = (\tfrac{1}{2},y_1); \quad S_2^2(1,y_2) = S_3^2(1/2,y_1); \quad S_3^2(1,y_2) = (1,y_2).$$

for $k > 2$ and for $i = 0, \ldots, 2^k - 1$:

if i is even, then:

if $i < 2^{k-1}$:

$$\begin{aligned} S_i^k \circ S_{i/2}^{k-1} \circ S_{[i/2^2]}^{k-2} \circ \cdots \circ S_{[i/2^{k-2}]}^2(0,y_0) &= S_{i/2}^{k-1} \circ S_{[i/2^2]}^{k-2} \circ \cdots \circ S_{[i/2^{k-2}]}^2(0,y_0),\\ S_i^k \circ S_{i/2}^{k-1} \circ S_{[i/2^2]}^{k-2} \circ \cdots \circ S_{[i/2^{k-2}]}^2(\tfrac{1}{2},y_1) &= S_{i+1}^k \circ S_{[(i+1)/2]}^{k-1} \circ S_{[(i+1)/2^2]}^{k-2}\\ &\quad \circ \cdots \circ S_{[(i+1)/2^{k-2}]}^2(0,y_0), \end{aligned}$$

if $i \geq 2^{k-1}$:

$$\begin{aligned} S_i^k \circ S_{i/2}^{k-1} \circ S_{[i/2^2]}^{k-2} \circ \cdots \circ S_{[i/2^{k-2}]}^2(\tfrac{1}{2},y_1) &= S_{i/2}^{k-1} \circ S_{[i/2^2]}^{k-2} \circ \cdots \circ S_{[i/2^{k-2}]}^2(\tfrac{1}{2},y_1),\\ S_i^k \circ S_{i/2}^{k-1} \circ S_{[i/2^2]}^{k-2} \circ \cdots \circ S_{[i/2^{k-2}]}^2(1,y_2) &= S_{i+1}^k \circ S_{[(i+1)/2]}^{k-1} \circ S_{[(i+1)/2^2]}^{k-2}\\ &\quad \circ \cdots \circ S_{[(i+1)/2^{k-2}]}^2(\tfrac{1}{2},y_1), \end{aligned}$$

if i is odd, then:

if $i < 2^{k-1}$:

$$S_i^k \circ S_{[i/2]}^{k-1} \circ S_{[i/2^2]}^{k-2} \circ \cdots \circ S_{[i/2^{k-2}]}^2(\tfrac{1}{2},y_1) = S_{[i/2]}^{k-1} \circ S_{[i/2^2]}^{k-2} \circ \cdots \circ S_{[i/2^{k-2}]}^2(\tfrac{1}{2},y_1),$$

if $i \geq 2^{k-1}$:

$$S_i^k \circ S_{[i/2]}^{k-1} \circ S_{[i/2^2]}^{k-2} \circ \cdots \circ S_{[i/2^{k-2}]}^2(1,y_2) = S_{[i/2]}^{k-1} \circ S_{[i/2^2]}^{k-2} \circ \cdots \circ S_{[i/2^{k-2}]}^2(1,y_2).$$

Our main result is the following:

Proposition 11. *Suppose that conditions* (c) *and* (c′) *hold. Then the attractor of the IFS defined above is the graph of a continuous function f such that*

$$f\left(\frac{i}{m}\right) = y_i \qquad \forall i = 0, \ldots, m,$$

and

$$\alpha_f(t) = \min(\alpha_1, \alpha_2, \alpha_3),$$

where

$$\begin{cases} \alpha_1 = \liminf\limits_{k\to+\infty} \dfrac{\log(c^k_{m^{k-1}i_1+m^{k-2}i_2+\cdots+mi_{k-1}+i_k} \cdots c^2_{mi_1+i_2} c^1_{i_1})}{\log(m^{-k})},\\ \alpha_2 = \liminf\limits_{k\to+\infty} \dfrac{\log(c^k_{m^{k-1}j_1+m^{k-2}j_2+\cdots+mj_{k-1}+j_k} \cdots c^2_{mj_1+j_2} c^1_{j_1})}{\log(m^{-k})},\\ \alpha_3 = \liminf\limits_{k\to+\infty} \dfrac{\log(c^k_{m^{k-1}l_1+m^{k-2}l_2+\cdots+ml_{k-1}+l_k} \cdots c^2_{ml_1+l_2} c^1_{l_1})}{\log(m^{-k})}, \end{cases} \tag{7}$$

and where the i_p's, j_p's, and l_p's are defined as in Proposition 8.

Proof. Let $I_{n_1 \dots n_k}$ be the interval of reals whose base-m expansion begins with $n_1 \dots n_k$. Define G^k to be the set obtained after k iterations in the process of generation of the attractor G, i.e.,

$$G^k = W^k \circ \cdots \circ W^1(G).$$

Then, it easy to see that

$$G^k{}_{|_{I_{n_1 \dots n_k}}} = S^k_{m^{k-1}n_1+m^{k-2}n_2+\cdots+mn_{k-1}+n_k} \circ \cdots \circ S^2_{mn_1+n_2} \circ S^1_{n_1}(G).$$

Using the same techniques as in the proof of proposition 8, the announced result follows. ■

Remark 2. Given m reals $r_1, \dots, r_m \in]1/m\,;1[$, define, for every integer $k \geq 1$ and for every $i \in \{0, \dots, m^k - 1\}$, the c_i^k's as follows:

$$c_i^k = r_{i+1-m[i/m]}.$$

Then, we recover the original construction considered in Proposition 8.

The following corollary allows us to control the local singularity at each point, while interpolating the points $(i/m, y_i)$ for $i = 0, \dots, m$. We first need to state the following refinement of Lemma 2.

Lemma 3. *Let $s \in \mathcal{H}$. Then there exists a sequence $\{R_n\}_{n\geq 1}$ of piecewise polynomials such that*

$$\begin{cases} s(t) = \liminf_{n\to+\infty} R_n(t) \quad \forall\, t \in [0;1], \\ ||R_n'^{+}||_\infty \leq n;\ ||R_n'^{-}||_\infty \leq n \quad \forall\, n \geq 1, \\ ||R_n||_\infty \geq 1/\log n, \end{cases} \tag{8}$$

where $R_n'^{+}$ and $R_n'^{-}$ are, respectively, the right and left derivative of R_n.

Proof. Let Q_k be defined as in Lemma 2 and define

$$R_k = \max\left(Q_k, \frac{1}{\log k}\right). \tag{9}$$

Corollary 4. *Let $s(t)$ be a function from $[0;1]$ to $[0;1]$, which is the lower limit of a sequence of continuous functions.*

Then there exists a generalized affine IFS whose attractor is the graph of a continuous function f which verifies

$$\alpha_f(t) = s(t).$$

Proof. Because of the continuity constraints, finding the generalized affine IFS amounts to determining the double sequence $(c_i^k)_{i,k}$.

Let $\{R_n\}_{n\geq 1}$ be a sequence of piecewise polynomials that verifies (8) and let $\mathcal{M}$ be the set of m-adic points of $[0;1]$.

Consider now the sequence $\{r_k\}_{k\geq 1}$ of functions from $\mathcal{M}$ to $\mathbf{R}$ defined as follows. For $t \in \mathcal{M}$, $t = \sum_{p=1}^{k_0} i_p m^{-p}$, let

$$r_1(t) = R_1(i_1 m^{-1}),$$

$$r_k(t) = kR_k(t) - (k-1)R_{k-1}\left(\sum_{p=1}^{k-1} i_p m^{-p}\right) \qquad \text{for} \quad k = 2, \ldots, k_0,$$

and

$$r_k(t) = kR_k(t) - (k-1)R_{k-1}(t) \qquad \text{for} \quad k > k_0.$$

Now, for each $k \geq 1$ and $i = 0, \ldots, m^k - 1$, set

$$c_i^k = m^{-r_k(im^{-k})}.$$

Using (9), we verify that conditions (c) and (c′) are fulfilled.

Using Proposition 11, we get

$$\alpha_f(t) = \min\left(\liminf_{k\to+\infty} \frac{\sum_{j=1}^{k} r_j(t_j)}{k}, \liminf_{k\to+\infty} \frac{\sum_{j=1}^{k} r_j(t_j^+)}{k}, \liminf_{k\to+\infty} \frac{\sum_{j=1}^{k} r_j(t_j^-)}{k}\right).$$

Since

$$\frac{\sum_{j=1}^{k} r_j(t_j)}{k} = R_k(t_k)\,; \qquad \frac{\sum_{j=1}^{k} r_j(t_j^+)}{k} = R_k(t_k^+)\,; \qquad \frac{\sum_{j=1}^{k} r_j(t_j^-)}{k} = R_k(t_k^-),$$

$$\alpha_f(t) = \min\left(\liminf_{k\to+\infty} R_k(t_k), \liminf_{k\to+\infty} R_k(t_k^+), \liminf_{k\to+\infty} R_k(t_k^-)\right).$$

Using (8), we have

$$\liminf_{k\to+\infty} R_k(t) = \liminf_{k\to+\infty} R_k(t_k) = \liminf_{k\to+\infty} R_k(t_k^+) = \liminf_{k\to+\infty} R_k(t_k^-).$$

We end up with

$$\alpha_f(t) = s(t). \qquad \blacksquare$$

7. Concluding Remarks

7.1. *Nonuniqueness of f*

It is easy to see that, given a set of points $\{(x_i, y_i)\}_{i=0,\ldots,N}$ where $x_i = i/N$, and a function $s \in \mathcal{H}$, there is an infinite number of continuous functions that interpolate the (x_i, y_i)'s and whose Hölder function is s. Indeed, take the function f constructed in Subsection 6.3 and consider the function g_λ defined by

$$g_\lambda(x) = \frac{f(x) + \lambda P_L(x)}{1 + \lambda},$$

where $P_L(x)$ is the Legendre polynomial defined by

$$P_L(x) = \sum_{i=0}^{N} y_i \frac{\prod_{j\neq i}^{N}(x - x_j)}{\prod_{j\neq i}^{N}(x_i - x_j)},$$

and λ is a real different from -1. Then, since $P_L \in C^\infty(\mathbf{R})$, it is clear that $\alpha_{g_\lambda} = s$ and, of course, the function g_λ interpolates the (x_i, y_i)'s for every $\lambda \in \mathbf{R}\backslash\{-1\}$.

7.2. *Size of E_s*

Let $s \in \mathcal{H}$ and define

$$E_s = \{w \in C^0([0;1])/\alpha_w(x) = s(x)\ \forall x \in [0;1]\}.$$

Proposition 12. *E_s is dense in $C^0([0;1])$ for the uniform convergence norm $||\cdot||_\infty$.*

Proof. Let $\mathbf{P}$ be the set of polynomials defined on $[0;1]$. It is well known that $\mathbf{P}$ is dense in $C^0([0;1])$ for the uniform convergence norm. For $f \in C^0([0;1])$, let $(P_n)_{n\in\mathbf{N}}$ be a sequence such that $P_n \in \mathbf{P}$ for every $n \in \mathbf{N}$ and

$$||P_n - f||_\infty \to 0 \qquad \text{when} \quad n \to \infty.$$

Now let w be a function in E_s, and consider the sequence $(f_n)_{n\in\mathbf{N}^*}$ defined by

$$f_n = P_n + \frac{w}{n} \qquad \text{for every} \quad n \in \mathbf{N}^*.$$

Since $P_n \in C^\infty([0;1])$ for every $n \in \mathbf{N}$ and $w \in E_s$, it is clear that f_n is in E_s for every $n \in \mathbf{N}^*$. We have

$$||f_n - f||_\infty \le ||P_n - f||_\infty + \frac{||w||_\infty}{n}.$$

w is a continuous function on a compact set, and there exists a constant $C > 0$ such that $||w||_\infty \le C$, hence

$$||f_n - f||_\infty \to 0 \qquad \text{when} \quad n \to \infty. \qquad \blacksquare$$

7.3. *More Refined Ways of Characterizing the Local Regularity*

The local regularity of the graphs of the functions constructed with the three methods we have presented above appears, in some cases, strikingly different (see Section 8). Several improvements may be proposed in order to describe these discrepancies:

- A well-known method to measure more precisely the local structure would be to use finer scales of functions, as for instance functions of the form

  $$g(x) = x^\alpha \left(\log\frac{1}{x}\right)^{\beta_1}\left(\log\log\frac{1}{x}\right)^{\beta_2}\dots\left(\log\log\dots\log\frac{1}{x}\right)^{\beta_n},$$

  the Hölder exponent at a point x_0 would then be a vector $(\alpha, \beta_1, \beta_2, \dots, \beta_n)$.

- Another possibility is to characterize algebraic oscillations instead of taking the absolute values, i.e., consider the two limits

$$\limsup_{h\to 0} \frac{g_-(h)}{h^\gamma} \qquad \text{and} \qquad \limsup_{h\to 0} \frac{g_+(h)}{h^\gamma},$$

  where

$$g(x) = f(x_0+h) - f(x_0), \quad g_+(x) = \max(g(x), 0), \quad g_-(x) = \min(g(x), 0).$$

- Finally, especially for practical purposes, the speed of convergence to the local Hölder exponent at x_0 is of crucial importance. For instance, it is easy to show that, for the Schauder-type function considered in Section 4, if we take $s(x) = x$, then, for $x_0 > 0$ and for some sequence $h_n \to 0$, the best possible lower bound is

$$|f(x_0+h_n) - f(x_0)| \geq c_1|h_n|^{x_0 - c_2|h_n|},$$

  where c_1 and c_2 are constants. But for the Weierstrass-like functions of Section 3, and also with $s(x) = x$, the best possible lower bound is

$$|f(x_0+h_n) - f(x_0)| \geq c' h_n^{x_0},$$

  where c' is a constant.

When working with discrete data, this first-order difference in h can make a big difference (see figures in the next section).

8. Examples

The following figures are graphs of continuous functions with prescribed local regularity. We have implemented the constructions described in Sections 4, 5, 6, and for each case, we show an example with $s(t) = t$ and $s(t) = |\sin(5\pi t)|$. In the IFS construction examples, the set of interpolation points is

$$\left\{(0,0)\,;\,(\tfrac{1}{5},1)\,;\,(\tfrac{2}{5},1)\,;\,(\tfrac{3}{5},1)\,;\,(\tfrac{4}{5},1)\,;\,(1,0)\right\}.$$

Appendix

Proof of Proposition 7. Recall that α_s is the Hölder function of s. We begin by proving that $\alpha_f(t) \geq s(t)$. Let t be fixed, ε_2 be a real such that $0 < \varepsilon_2 \ll 1$, and let h be a real such that $0 < |h| < \varepsilon_2$. Then we have

$$\begin{aligned} f(t+h) - f(t) &= \sum_{k=1}^{+\infty} (\lambda^{-ks(t+h)} \sin(\lambda^k(t+h)) - \lambda^{-ks(t)} \sin(\lambda^k t)) \\ &= A + A', \end{aligned}$$

where

$$A = \sum_{k=1}^{+\infty} (\lambda^{-ks(t+h)} - \lambda^{-ks(t)}) \sin(\lambda^k(t+h)),$$

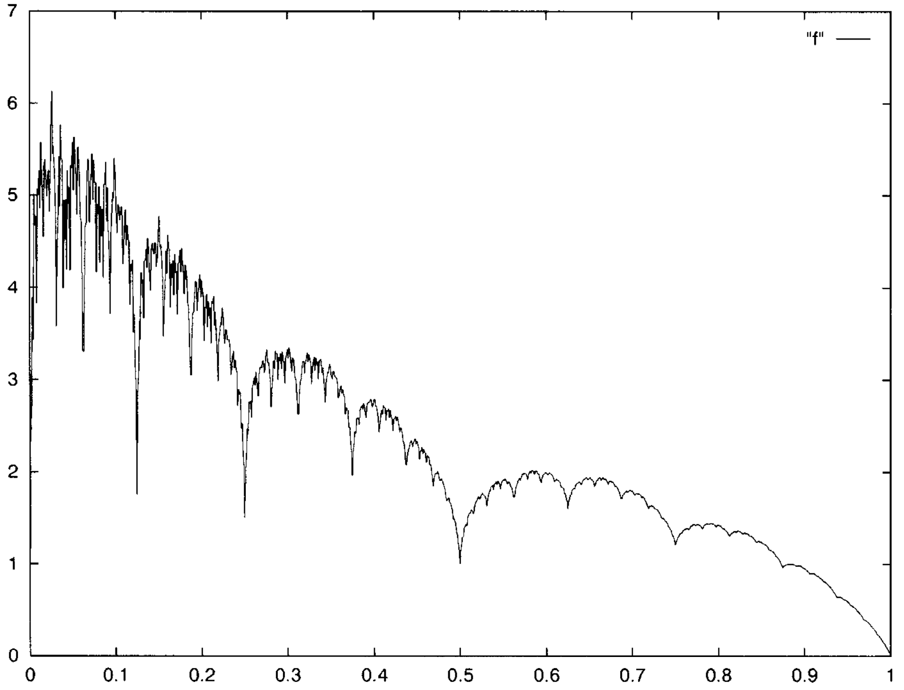

Fig. 1. Construction using the Schauder basis with $s(t) = t$.

and

$$A' = \sum_{k=1}^{+\infty} \lambda^{-ks(t)}(\sin(\lambda^k(t+h)) - \sin(\lambda^k t)).$$

Let us give an upper bound for $|A|$. We have

$$|A| \leq \sum_{k=1}^{+\infty} |\lambda^{-ks(t+h)} - \lambda^{-ks(t)}|,$$

but

$$\lambda^{-ks(t+h)} - \lambda^{-ks(t)} = -(\log\lambda) \times [s(t+h) - s(t)] \times (k\lambda^{-k\tau}),$$

where $\tau \in [\min(s(t), s(t+h)); \max(s(t), s(t+h))]$.

Thus

$$|A| \leq (\log\lambda)|s(t+h) - s(t)| \sum_{k=1}^{+\infty} k\lambda^{-k\tau}.$$

Let $C = \sum_{k=1}^{+\infty} k\lambda^{-k\tau}$ ($0 < C < +\infty$ because this series converges), then since there exists a constant $M > 0$ such that

$$|s(t+h) - s(t)| \leq M|h|^{\alpha_s(t)},$$

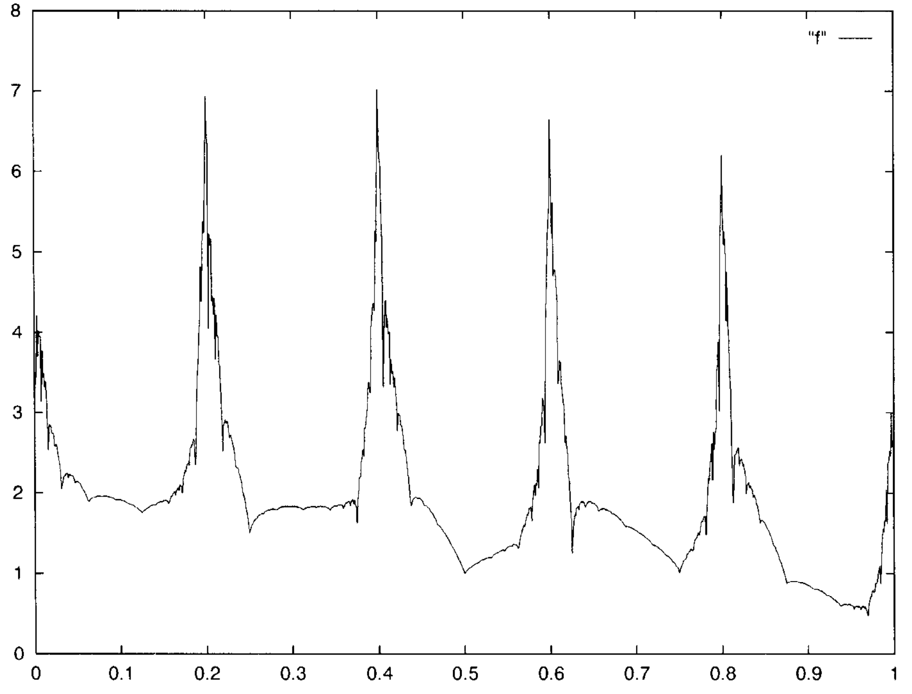

Fig. 2. Construction using the Schauder basis with $s(t) = |\sin(5\pi t)|$.

we have

$$|A| \leq c_1 |h|^{\alpha_s(t)} \leq c_1 |h|^{s(t)},$$

where

$$c_1 = CM \log \lambda.$$

Let us now give an upper bound for $|A'|$. For this purpose, we consider the integer N such that

$$\lambda^{-(N+1)} \leq |h| \leq \lambda^{-N}.$$

We have, using the main value theorem,

$$|A'| \leq |h| X + 2Y,$$

where

$$X = \sum_{k=1}^{N} \lambda^{-k(s(t)-1)}$$

and

$$Y = \sum_{k=N+1}^{+\infty} \lambda^{-ks(t)},$$

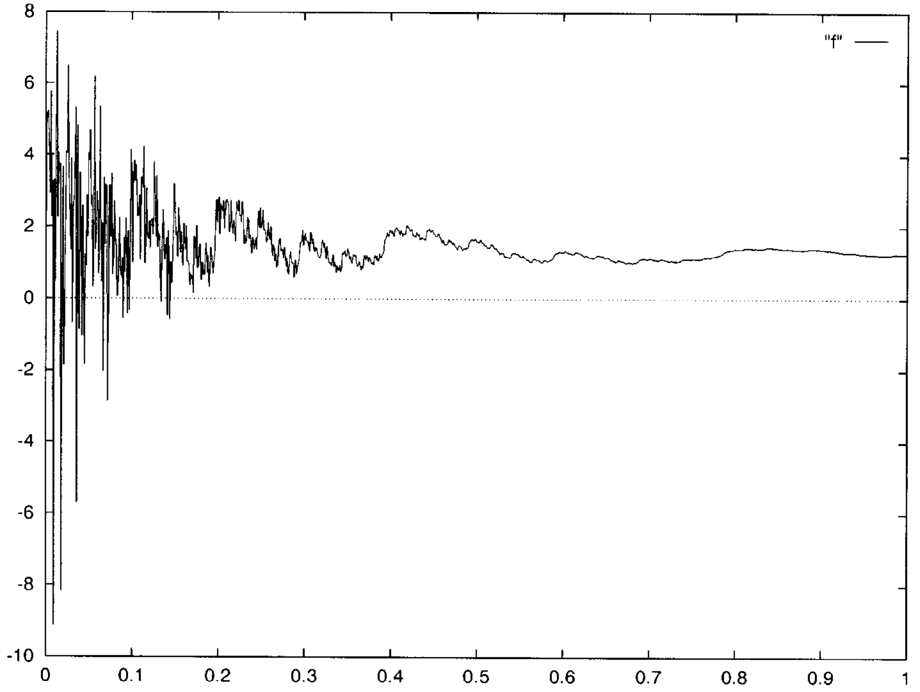

Fig. 3. Construction using the Weierstrass-type function with $s(t) = t$.

but

$$X \leq \frac{1}{1 - \lambda^{s(t)-1}} |h|^{s(t)-1},$$
$$Y \leq \frac{1}{1 - \lambda^{-s(t)}} |h|^{s(t)}.$$

Since $s(t)$ is bounded, there exists a constant $c_2 > 0$ such that

$$|A'| \leq c_2 |h|^{s(t)}.$$

Finally, if $c = c_1 + c_2$, we have

$$|f(t+h) - f(t)| \leq c|h|^{s(t)},$$

which gives

$$(\gamma < s(t)) \quad \Rightarrow \quad \lim_{h \to 0} \frac{|f(t+h) - f(t)|}{|h|^{\gamma}} = 0.$$

Now we will prove that $\alpha_f(t) \leq s(t)$.

Let t be a real in $[0; 1]$ and let δ be a real in $]0; \varepsilon_2[$. Then consider the integer N such

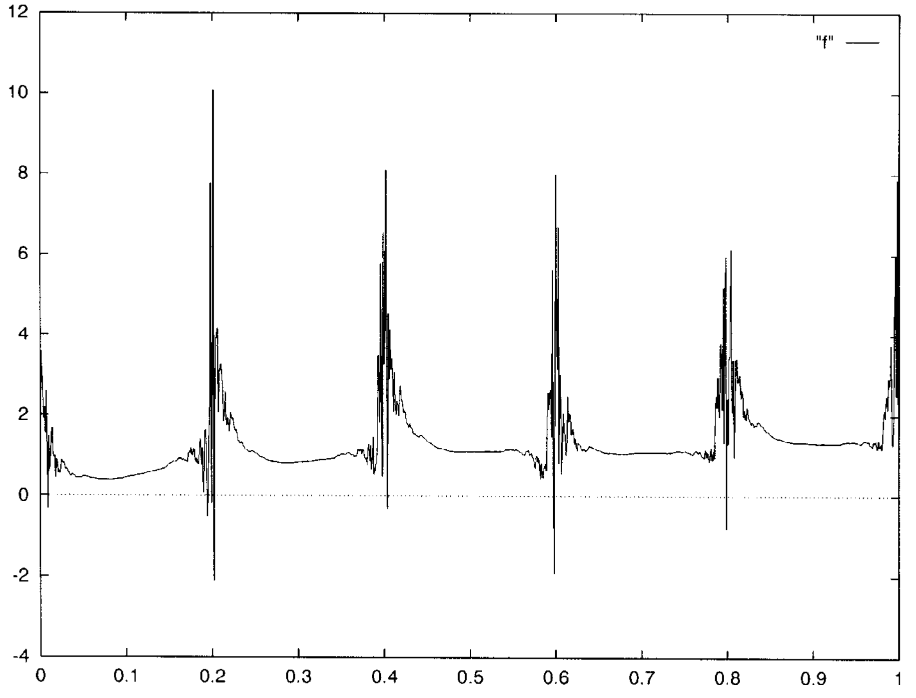

Fig. 4. Construction using the Weierstrass-type function with $s(t) = |\sin(5\pi t)|$.

that $\lambda^{-(N+1)} < \delta \leq \lambda^{-N}$, and let h be a real such that $\lambda^{-(N+1)} < |h| \leq \delta$. We have

$$\begin{aligned} X &= |f(t+h) - f(t) - \lambda^{-Ns(t)}(\sin(\lambda^N(t+h) - \sin(\lambda^N t))| \\ &\leq B + 2\sum_{k=N}^{+\infty} \lambda^{-ks(t)} + |A|, \end{aligned}$$

where $B = \sum_{k=1}^{N-1} \lambda^{-ks(t)} |\sin(\lambda^k(t+h) - \sin(\lambda^k t)|$.

We have

$$B \leq \lambda^{-Ns(t)} \frac{\lambda^{s(t)-1}}{1-\lambda^{(s(t)-1)}}.$$

Since we have seen that

$$|A| \leq c_1 |h|^{s(t)} \leq c_1 \lambda^{-Ns(t)},$$

then

$$X \leq \lambda^{-Ns(t)}(c_1 + c_3),$$

with

$$c_3 = \frac{\lambda^{s(t)-1}}{1-\lambda^{(s(t)-1)}} + 2\frac{\lambda^{-s(t)}}{1-\lambda^{-s(t)}}.$$

Provided that λ is large enough, we may choose c_1 and c_3 such that

$$c_1 \leq \tfrac{1}{40} \qquad \text{and} \qquad c_3 \leq \tfrac{1}{40},$$

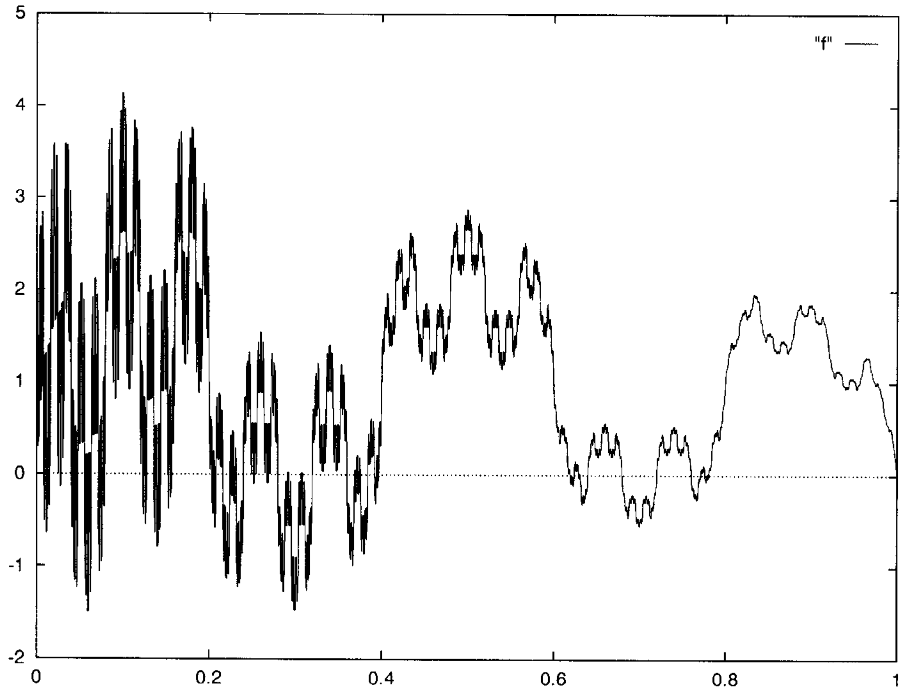

Fig. 5. Construction using generalized affine IFS with $s(t) = t$.

thus

$$X \leq \tfrac{1}{20}\lambda^{-Ns(t)},$$

but

$$X \geq ||f(t+h) - f(t)| - \lambda^{-Ns(t)}|\sin(\lambda^N(t+h)) - \sin(\lambda^N t)||$$

and

$$|f(t+h) - f(t)| \geq \lambda^{-Ns(t)}|\sin(\lambda^N(t+h)) - \sin(\lambda^N t)| - X.$$

There exists a sequence [22] (h_n), with $\lambda^{-(N+1)} < |h_n| \leq \delta \leq \lambda^{-N}$ for every n, such that

$$|\sin(\lambda^N(t+h_n)) - \sin(\lambda^N t)| \geq \tfrac{1}{10} \qquad \forall n,$$

because $1/\lambda \leq |h_n|\lambda^N \leq 1\ \forall n$.

We deduce

$$|f(t+h_n) - f(t)| \geq \tfrac{1}{20}\lambda^{-Ns(t)} \geq \tfrac{1}{20}\delta^{s(t)} \geq \tfrac{1}{20}|h_n|^{s(t)}$$

which gives

$$(\gamma > s(t)) \quad \Rightarrow \quad \limsup_{h\to 0} \frac{|f(t+h) - f(t)|}{|h|^{\gamma}} = +\infty.$$

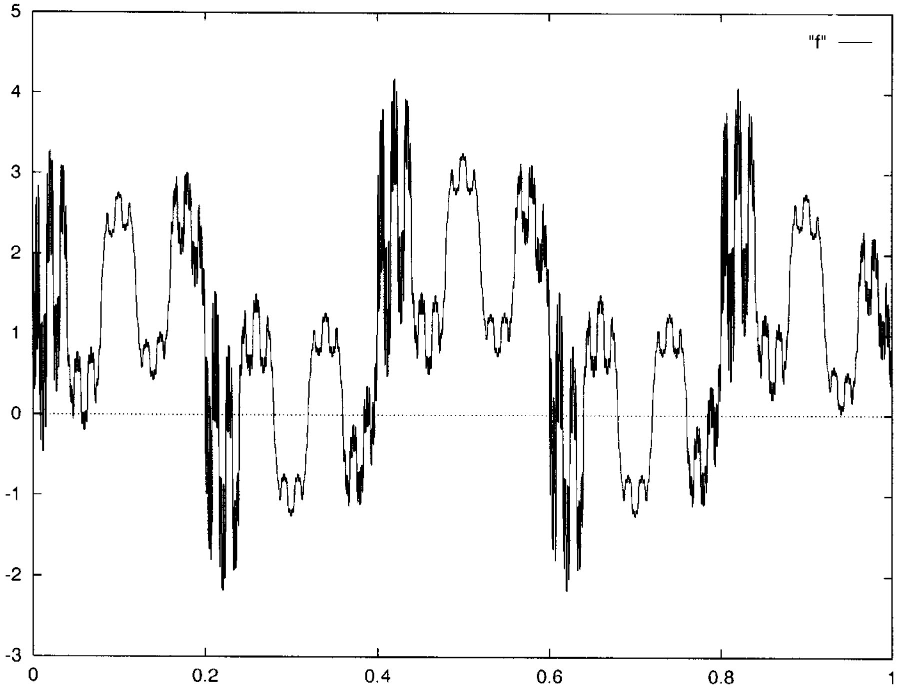

Fig. 6. Construction using generalized affine IFS with $s(t) = |\sin(5\pi t)|$.

Let us now check that f verifies conditions (c_1) and (c_2) of Proposition 1.

Let x be a real in [0; 1] and let ε be a real such that $0 < \varepsilon < \min(\varepsilon_1, \varepsilon_2)$. For every $\delta < \varepsilon$ and $t \in B(x, \delta)$, we have seen that

$$|f(t) - f(u)| \leq \left[c(t) M \log\lambda + \frac{1}{1 - \lambda^{s(t)-1}} + \frac{2}{1 - \lambda^{-s(t)}} \right] |t - u|^{s(t)} \quad \text{for every} \quad u \in B(t, \delta),$$

where

$$c(t) = \sum_{k=1}^{\infty} k\lambda^{-k\tau} \qquad \text{with} \quad \tau \in [\min(s(t), s(u)); \max(s(t), s(u))].$$

This implies that

$$\bar{c}(t, \delta) \leq AM + B \qquad \text{for every} \quad t \in [0; 1] \quad \text{and} \quad \delta < \varepsilon,$$

where

$$A = \log\lambda \sum_{k=1}^{\infty} k\lambda^{-ka} \qquad \text{and} \qquad B = \frac{1}{1 - \lambda^{b-1}} + \frac{2}{1 - \lambda^{-b}}.$$

Hence

$$\bar{C}(x,\delta) < +\infty \qquad \forall \delta < \varepsilon,$$

and condition (c_1) holds.

Condition (c_2) is easy to verify. Indeed, we have seen that there exists a real $u \in B(t,\delta)$ such that

$$|f(t) - f(u)| \geq \tfrac{1}{20}\delta^{s(t)},$$

hence

$$\underline{c}(t,\delta) \geq \tfrac{1}{20} \qquad \forall \delta < \varepsilon,$$

which implies that

$$\underline{C}(x,\varepsilon) \neq 0.$$

Now, since s is continuous, and conditions (c_1) and (c_2) hold, we get, using Proposition 1,

$$2 - \dim_B^x \textit{graph } f = s(x) \qquad \text{for every} \quad x \in [0;1]. \qquad \blacksquare$$

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