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CONSTRUCTIVE APPROXIMATION

A Discrepancy Theorem on Quasiconformal Curves

V. V. Andrievskii and H.-P. Blatt

Abstract. In [7], Blatt and Mhaskar estimated the Erdős-Turán type discrepancy of a signed Borel measure σ on a sufficiently smooth Jordan curve or arc L in terms of the logarithmic potential of σ on a curve enclosing L. We extend this result to a measure σ on an arbitrary quasiconformal curve. As applications, estimates for the distribution of simple zeros of monic polynomials, Fekete points, extreme points of polynomials of best uniform approximation are obtained.

1. Introduction

Let $L \subset \mathbf{C}$ be a bounded Jordan curve or a Jordan arc. We define the discrepancy of a signed (Borel) measure σ given on L by the quantity

$$D[\sigma] := \sup|\sigma(J)|,$$

where the supremum is taken over all subarcs $J \subset L$.

Estimates of this discrepancy $D[\sigma]$ in terms of the logarithmic potential

$$U(\sigma, z) := \int \log \frac{1}{|z-\zeta|} d\sigma(\zeta)$$

attracted the interest of several authors ([5]–[8], [10], [14], [18], [20]).

A major role in these articles is played by the mapping Φ which maps the unbounded component Ω of $\bar{\mathbf{C}} \setminus L$ conformally and univalently onto the exterior $\Delta := \bar{\mathbf{C}} \setminus \bar{D}$ of the unit disk $D := \{z\colon |z| < 1\}$, where $\bar{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ is the extented complex plane and Φ is normalized by the conditions

$$\Phi(\infty) = \infty, \qquad \Phi'(\infty) := \lim_{z\to\infty} \frac{\Phi(z)}{z} > 0.$$

For $\delta > 0$, let us define the level curve

$$L_\delta := \{z \in \Omega\colon |\Phi(z)| = 1 + \delta\}$$

and the bound

$$\varepsilon(\delta) := \|U(\sigma, \cdot)\|_{L_\delta},$$

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where the symbol $\|\cdot\|_A$ always denotes the supremum norm on the subset $A \subset \mathbf{C}$. Then the most recent discrepancy result in terms of $\varepsilon(\delta)$ is the following:

Theorem (Blatt and Mhaskar [7]). *Let L be a Jordan curve or a Jordan arc of class C^{1+}, $\sigma := \sigma^+ - \sigma^-$ be a signed measure on L with positive part σ^+, negative part σ^-, and $\sigma^+(L) = \sigma^-(L) = 1$. Moreover, let $M > 0$, $0 < \gamma \le 1$, be constants such that for all subarcs J of L,*

$$\sigma^+(J) \le M\left(\int_J ds\right)^{\gamma}.$$

Then there exists a constant $c > 0$ depending only on L, M, and γ such that

$$D[\sigma] \le c\varepsilon(\delta)\log\left(\frac{1}{\varepsilon(\delta)}\right)$$

for all δ with $\delta \le 1 + \varepsilon(\delta)^{1+1/\gamma}$ and $\varepsilon(\delta) < 1/e$.

In this article, applying the techniques of the theory of quasiconformal mappings (see [1] and [13]), we extend the statement of the theorem above and some results of [5], [7], and [20] (without essential change of their form) to the case when L is a quasiconformal curve or arc.

2. Formulation of the Main Results

In the sequel, for a bounded Jordan curve L, we denote by $\operatorname{int} L$ the bounded component and by $\operatorname{ext} L$ the unbounded component of $\bar{\mathbf{C}}\setminus L$. For any subset $A \subset \bar{\mathbf{C}}$, let $\bar{A}$ be the closure of A in $\bar{\mathbf{C}}$. The logarithmic capacity of L can be defined by

$$\operatorname{cap} L := \frac{1}{\Phi'(\infty)}.$$

By μ_L we denote the equilibrium measure of L [21]. This notion has a simple interpretation using the conformal mappings Φ and $\Psi := \Phi^{-1}$. Namely, if L is a curve, then Φ can be extended to a homeomorphism $\Phi\colon \bar{\Omega} \to \bar{\Delta}$ and, for any subarc $J \subset L$,

$$\mu_L(J) = \frac{1}{2\pi}\,\text{length}\,\Phi(J).$$

If L is an arc, then Ψ can be extended continuously to a function $\Psi\colon \bar{\Delta} \to \bar{\mathbf{C}}$ and, for any subarc $J \subset L$, there exist two preimages, i.e., arcs $J_1', J_2' \subset \partial\Delta$ such that $\Psi(J_1') = \Psi(J_2') = J$ and $J_1' \cap J_2'$ consists of at most two points. In this case

$$\mu_L(J) = \frac{1}{2\pi}(\text{length}\,J_1' + \text{length}\,J_2').$$

Throughout this article we suppose that L is a quasiconformal curve or arc (see [1] and [13]).

We recall that, by definition, a K-quasiconformal ($K \ge 1$) or, briefly, a quasiconformal curve is the image of the unit circle under some K-quasiconformal mapping $F\colon \bar{\mathbf{C}} \to \bar{\mathbf{C}}$.

Any subarc of a K-quasiconformal curve is called a K-quasiconformal or briefly a quasiconformal arc.

There exists a geometric characterization of quasiconformal curves [13, p. 100] and arcs [17]. For example, for curves it can be formulated as follows: L is a quasiconformal curve if and only if there exists a constant $c > 0$, depending only on L, such that for $z_1, z_2 \in L$

$$\min\{\operatorname{diam} L', \operatorname{diam} L''\} \leq c\,|z_1 - z_2|,$$

where L' and L'' denote the two arcs of which $L \backslash \{z_1, z_2\}$ consists.

Using this criterion, we can easily verify that convex curves, curves of bounded variation without cusps, and rectifiable Jordan curves which have locally the same order of arc length and chord length are quasiconformal. On the other hand, it is of interest to know, that a quasiconformal curve can be everywhere nonrectifiable [4, p. 42].

Theorem 1. *Let L be a K-quasiconformal curve, let $\sigma := \sigma^+ - \sigma^-$ be a signed measure on L with positive part σ^+, negative part σ^-, and $\sigma^+(L) = \sigma^-(L) = 1$. Moreover, let c and β be positive constants such that, for all subarcs $J \subset L$,*

$$\sigma^+(J) \leq c\mu_L(J)^\beta. \tag{2.1}$$

Then there exists a constant $c_1 > 0$, depending only on K, c, and β such that, for $0 < \delta < 1/e$,

$$D[\sigma] \leq c_1 \left(\varepsilon(\delta) \log \left(\frac{1}{\delta} \right) + \delta^{1/(2K^2)} + \delta^{\beta/2} \right). \tag{2.2}$$

In applications the following consequence of Theorem 1 is especially useful.

Theorem 2. *Let L be a quasiconformal curve or arc, and let p_n be a monic polynomial of degree n with simple zeros $z_j \in L$, $j = 1, \ldots, n$, such that*

$$\|p_n\|_L \leq A_n (\operatorname{cap} L)^n,$$

$$|p_n'(z_j)| \geq B_n^{-1} (\operatorname{cap} L)^n,$$

$$C_n := \max(A_n, B_n, n) \leq e^{n/e}.$$

Let ν_n denote the measure which associates the mass $1/n$ with each of the zeros z_j. Then

$$D[\mu_L - \nu_n] \leq c_2 \frac{\log C_n}{n} \log \frac{n}{\log C_n}, \tag{2.3}$$

where $c_2 > 0$ is a constant depending only on L.

Theorems 1 and 2 are extensions of results in [5], [7], and [20] for sufficiently smooth L $(\in C^{1+})$ to the case of an arbitrary quasiconformal curve or arc.

Next, we can apply the technique of [7] to investigate the distribution of Fekete points and extreme points in best polynomial approximation. We restrict ourselves to the formulation of the corresponding two assertions. We will not dwell on their proofs, since they are completely analogous to [7, Theorems 3.1 and 3.2].

Theorem 3. *Let L be a quasiconformal curve or arc and, for any integer $n \geq 2$, let ν_n denote the unit measure associated with an nth Fekete point set of L. Then*

$$D[\mu_L - \nu_n] \leq c_3 \frac{(\log n)^2}{n},$$

where the constant $c_3 > 0$ depends only on L.

Theorem 4. *Suppose that L is a quasiconformal curve or arc and let*

$$E := \overline{\mathbf{C}} \backslash \Omega = \begin{cases} \overline{L} & \text{if } L \text{ is an arc}, \\ \operatorname{int} L & \text{if } L \text{ is a curve}. \end{cases}$$

Let f be a function continuous on E and analytic at the interior points of E. If ν_{n+2} denotes the unit measure that associates the mass $1/(n+2)$ with each point of some $(n+2)$th Fekete point set of the extreme points

$$\{z \in L\colon |f(z) - p_n(z)| = \|f - p_n\|_L\},$$

where p_n is the best Chebyshev approximation to f with respect to the class of polynomials of degree at most n, then there exist infinitely many integers n satisfying

$$D[\mu_L - \nu_{n+2}] \leq \frac{c_4 (\log n)^2}{n^{1/2}}$$

with some constant $c_4 > 0$ depending only on L.

The following fact completes the assertion of Theorem 3. For the case of convex curves it was proved in [11] and [12].

Theorem 5. *Let the points $z_k = \Psi(e^{i\theta_k}), \theta_1 < \theta_2 < \cdots < \theta_n < \theta_{n+1} := \theta_1 + 2\pi$, form a Fekete set on the K-quasiconformal curve L. Then*

$$\frac{c_5}{n} \leq \theta_{j+1} - \theta_j \leq \frac{c_6}{n}, \qquad j = 1, \ldots, n, \tag{2.4}$$

with some positive constants c_5 and c_6 depending only on K.

Finally, we would like to discuss the requirement of quasiconformality in our results. More precisely, we claim that in the case of curves and arcs with cusps, in general, a universal estimate for the discrepancy in terms of (2.3) with some constant c_2 does not exist. We will try to explain this effect by an example to Theorem 2 for domains of the type

$$E = E(f) := \{z = x + iy\colon 0 < x < 1, 0 < y < f(x)\},$$

where f is a positive monotonically increasing function on [0,1] with $f(0) = 0$ and such that there exists an inverse function $g := f^{-1}$ (with the same properties).

In the sequel, we denote by $c, c_1, \ldots$ positive constants (in general, different each time) depending only on the curve or arc L.

For $a > 0$ and $b > 0$, we use the expression $a \preccurlyeq b$ (order inequality) if $a \leq cb$. The expression $a \asymp b$ means that $a \preccurlyeq b$ and $b \preccurlyeq a$ hold simultaneously.

For $n \geq 2$, we consider the Jordan arc

$$S = S(n) := L \setminus (0, g(n^{-2})),$$

where $L := \partial E$.

Let $S_{2/n}$ denote the corresponding level curve of the conformal mapping $\bar{\mathbf{C}} \setminus S \to \Delta$ with standard normalization at ∞. Since for any $z \in S_{2/n}$ and for $n \in \mathbf{N}$ large enough

$$d(z, S) \geqslant \frac{\operatorname{diam} S}{4} \frac{(2/n)^2}{1 + 2/n} > n^{-2}$$

(see [19, p. 181]), we conclude that

$$E \subset \operatorname{int} S_{2/n},$$

and consequently

$$\operatorname{cap} S \leq \operatorname{cap} L \leq \left(1 + \frac{2}{n}\right) \operatorname{cap} S.$$

Further, let $z_1, \ldots, z_n$ be the points of an nth Fekete point set of S.

According to [15], the Fekete monic polynomial

$$q_n(z) := \prod_{j=1}^{n} (z - z_j)$$

satisfies

$$|q_n'| \geq (\operatorname{cap} S)^n \succcurlyeq (\operatorname{cap} L)^n,$$
$$\|q_n\|_L \preccurlyeq \|q_n\|_S \preccurlyeq n^2 (\operatorname{cap} S)^n \leq n^2 (\operatorname{cap} L)^n.$$

Hence, the right-hand side of estimate (2.3) has the order

$$\frac{\log C_n}{n} \log \frac{n}{\log C_n} \asymp \frac{(\log n)^2}{n}.$$

On the other hand, it is obvious that

$$D[\mu_L - \nu_n] \succcurlyeq g(n^{-2}).$$

Since g can tend arbitrarily slowly to zero as $x \to 0$, it is impossible to obtain either an estimate of the same type as in Theorem 2 or any other estimate with a universal right-hand side.

3. Some Relevant Facts from The Theory of Quasiconformal Mappings

We begin with some simple general facts from geometric function theory.

Let $d(A, B)$ denote the distance between $A \subset \mathbf{C}$ and $B \subset \mathbf{C}$.

Lemma 1. *Let f be a conformal mapping of a region $G_1 \subset \mathbf{C}$ onto a region $G_2 \subset \mathbf{C}$. Then, for each $z \in G_1$,*

$$\frac{1}{4} \frac{d(f(z), \partial G_2)}{d(z, \partial G_1)} \leq |f'(z)| \leq 4 \frac{d(f(z), \partial G_2)}{d(z, \partial G_1)}. \tag{3.1}$$

Moreover, if $|\xi - z| \leq d(z, \partial G_1)/2$, then

$$\frac{1}{16} \frac{d(f(z), \partial G_2)}{d(z, \partial G_1)} |\xi - z| \leq |f(\xi) - f(z)| \leq 16 \frac{d(f(z), \partial G_2)}{d(z, \partial G_1)} |\xi - z|. \tag{3.2}$$

Proof. Set $d_1 := d(z, \partial G_1)$, $d_2 := d(f(z), \partial G_2)$ and consider the function

$$g(\zeta) := f(z + d_1\zeta), \qquad \zeta \in D.$$

An argument of [16, p. 22] shows that

$$|f'(z)|\, d_1 = |g'(0)| \le 4\, d_2,$$

which yields the right-hand side of (3.1). The left-hand side of (3.1) follows from the right-hand side written for the inverse function f^{-1}.

Further, consider the function

$$h(\zeta) := \frac{f(z + d_1\zeta) - f(z)}{f'(z) d_1}, \quad \zeta \in D.$$

An elementary argument involving [16, Theorem 1.6, p. 21] shows that, for $|\zeta| < 1/2$,

$$\frac{|\zeta|}{4} \le |h(\zeta)| \le 4|\zeta|.$$

Therefore, setting $\zeta := (\xi - z)/d_1$, we obtain

$$|f(\xi) - f(z)| \le 4\frac{|\xi - z|}{d_1} 4d_2 = 16\frac{d_2}{d_1}|\xi - z|,$$

$$|f(\xi) - f(z)| \ge \frac{|\xi - z|}{4d_1}\frac{d_2}{4} = \frac{1}{16}\frac{d_2}{d_1}|\xi - z|.$$

This completes the proof of the lemma. ■

Let $G \subset \bar{\mathbf{C}}$ be a Jordan domain, $z \in G$, and let $l \subset \partial G$ be an arc. We denote by $\omega(z, G, l)$ the harmonic measure of l at the point z with respect to G (see [21]).

The next lemma is an immediate consequence of the Poisson and Schwarz representation formulas.

Lemma 2. *Let z_1 and z_2 be the endpoints of an arc $l \subset \partial D$. Then for any $z \in D$ the inequalities*

$$\omega(z, D, l) \le c_1 \frac{1 - |z|}{d(z, l)}, \tag{3.3}$$

$$|\operatorname{grad} \omega(z, D, l)| \le c_2 / \min_{j=1,2} |z - z_j| \tag{3.4}$$

hold with some universal constants $c_1, c_2 \ge 1$.

It is well known that the harmonic measure is a conformal invariant. In the next lemma we claim that it is also a quasiconformal "quasi-invariant."

Lemma 3. *Let $G_j \subset \mathbf{C}$, $j = 1, 2$, be arbitrary Jordan domains, and let F be a K-quasiconformal ($K \ge 1$) mapping of G_1 onto G_2 extended continuously to the boundary ∂G_1. Then, for each arc $l \subset \partial G_1$ and each point $z \in G_1$,*

$$\omega(z, G_1, l) \le 8\pi\, \omega(F(z), G_2, F(l))^{1/K}.$$

Proof. According to the previous remark concerning the conformal invariance of the harmonic measure we have only to consider the case $G_1 = G_2 = D$ and $z = F(z) = 0$.

Notice that by a theorem of Mori [9, p. 66], for any $z_1, z_2 \in \bar{D}$,

$$|z_1 - z_2| \le 16|F(z_1) - F(z_2)|^{1/K}. \tag{3.5}$$

Let z_1 and z_2 be the endpoints of the arc $l \subset \partial D$. The interesting case is

$$\omega(0, D, F(l)) < (8\pi)^{-K}.$$

Hence by virtue of (3.5),

$$\omega(0, D, l) < \frac{\pi}{2}|z_1 - z_2| \le 8\pi|F(z_1) - F(z_2)|^{1/K} \le 8\pi\omega(0, D, F(l))^{1/K}.$$

■

The following result is useful in the study of metric properties of the conformal mappings Φ and Ψ.

Lemma 4 ([2, Lemma 1]). *Let $w = F(\zeta)$ be a K-quasiconformal mapping of the complex plane onto itself with $F(\infty) = \infty$, $\zeta_j \in \mathbf{C}$, $w_j := F(\zeta_j)$, $j = 1, 2, 3$, and $|w_1 - w_2| \le c_1|w_1 - w_3|$. Then $|\zeta_1 - \zeta_2| \le c_2|\zeta_1 - \zeta_3|$ and, in addition,*

$$\left|\frac{\zeta_1 - \zeta_3}{\zeta_1 - \zeta_2}\right| \le c_3 \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^K,$$

where $c_i = c_i(c_1, K)$, $i = 2, 3$.

Corollary. *Since F^{-1} is also a K-quasiconformal mapping it follows from the hypothesis $|\zeta_1 - \zeta_2| \le c_1|\zeta_1 - \zeta_3|$ that $|w_1 - w_2| \le c_2|w_1 - w_3|$ and*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right| \le c_3 \left|\frac{\zeta_1 - \zeta_3}{\zeta_1 - \zeta_2}\right|^K.$$

In the following, we will often use the fact that a conformal mapping $\Phi\colon \Omega \to \Delta$ can be extended to a K^2-quasiconformal mapping $\Phi\colon \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ [1, Chapter IV] if L is a K-quasiconformal curve. The inverse mapping $\Psi := \Phi^{-1}$ will also be K^2-quasiconformal in $\bar{\mathbf{C}}$.

Applying Lemma 4 and its corollary to the function $F := \Phi$ it is easily verified that, for any points $\zeta_j \in L$, $w_j := \Phi(\zeta_j)$, $j = 1, 2$, the double inequality

$$c_1|\zeta_1 - \zeta_2|^{K^2} \le |w_1 - w_2| \le c_2|\zeta_1 - \zeta_2|^{1/K^2}$$

holds with some constants $c_j = c_j(G)$, $j = 1, 2$. Hence, for any arc $l \subset L$,

$$c_3(\operatorname{diam} l)^{K^2} \le \mu_L(l) \le c_4(\operatorname{diam} l)^{1/K^2}. \tag{3.6}$$

Next, we note that the same inequality is true for the case of a quasiconformal arc. To be more precise, let L be a K-quasiconformal arc. Denote by z_1 and z_2 its endpoints and, for $j = 1, 2$, set

$$w_j := \Phi(z_j),$$

$$\Delta_1 := \{w\colon |w| > 1; \arg w_1 < \arg w < \arg w_2\},$$

$$\Delta_2 := \Delta\backslash\overline{\Delta_1}, \qquad \Omega_j := \Psi(\Delta_j).$$

A routine category argument shows that $\partial\Delta_1$ and $\partial\Delta_2$ are quasiconformal curves. Moreover, it was proved in [3, Lemma 1] that $\partial\Omega_1 = \partial\Omega_2$ is quasiconformal too. Therefore, the restriction Φ_j of the function Φ to the region Ω_j can be extended to a K_1-quasiconformal mapping $\Phi_j\colon \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ with a suitable constant $K_1 = K_1(L) > 1$ (see [1, Chapter IV]). Using Lemma 4 and its corollary for the function $F := \Phi_j$, we find that, for each pair of points ζ_1 and $\zeta_2 \in L$,

$$c_1|\zeta_1 - \zeta_2|^{K_1} \le |\Phi_j(\zeta_1) - \Phi_j(\zeta_2)| \le c_2|\zeta_1 - \zeta_2|^{1/K_1},$$

where $c_i = c_i(L),\ i = 1, 2$. At last, recalling the definition of the equilibrium measure, we may conclude that, for any subarc $l \subset L$, the inequalities

$$c_3(\operatorname{diam} l)^{K_1} \le \mu_L(l) \le c_4(\operatorname{diam} l)^{1/K_1} \tag{3.7}$$

are satisfied with some $c_i = c_i(L),\ i = 3, 4$.

4. Some Auxiliary Results

Before proving our theorems, we will discuss a special construction which will be the base of the proof of Theorem 1.

Suppose that L is a K-quasiconformal curve and let $\delta > 0$ be sufficiently small. Set $z_0 := \Psi(0),\ a := \delta + \delta^{K^2}$. Denote by φ_a a conformal mapping of $\operatorname{int} L_a$ onto the unit disk D with the normalization $\varphi_a(z_0) = 0$.

Consider the function $g := \varphi_a \circ \Psi$ that maps the disk $\{w\colon |w| < 1 + a\}$ K^2-quasiconformally on D with $g(0) = 0$.

We note that g is conformal in the annulus $\{w\colon 1 < |w| < 1 + a\}$ and, moreover, g can be extended to a K^2-quasiconformal mapping $g\colon \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ if, for $|w| > 1 + a$, we set

$$g(w) := \left[\overline{g\left(\frac{(1+a)^2}{\bar{w}}\right)}\right]^{-1}.$$

The last fact makes it possible to use Lemma 4 and its corollary with $F := g$.

Set $u := g^{-1}$,

$$\Gamma = \Gamma(\delta) := \varphi_a(L_\delta) = \{z \in D\colon |u(z)| = 1 + \delta\}.$$

Lemma 5. *Let $z, \zeta \in \Gamma$ be arbitrary points and let $\Gamma(z, \zeta)$ be the shortest component of $\Gamma\backslash\{z, \zeta\}$. Then the inequality*

$$\operatorname{length} \Gamma(z, \zeta) \le c|z - \zeta|$$

holds with some constant $c = c(K) > 0$.

Proof. Set $w := u(z)$, $\tau := u(\zeta)$,

$$\Gamma'(w, \tau) := u(\Gamma(z, \zeta)), \qquad E := \{t \in D\colon 1 < |u(t)| < 1 + a\}.$$

By virtue of Lemma 4 and its corollary we have $d(z, \partial E) \asymp 1 - |z|$.

Suppose first that $|w - \tau| \le \delta^{K^2}/32$. According to (3.2),

$$\frac{1}{16}\frac{d(z, \partial E)}{\delta^{K^2}}|\tau - w| \le |z - \zeta| \le \frac{d(z, \partial E)}{2}.$$

Since by the same reasoning

$$|z - \xi| \le \frac{d(z, \partial E)}{2} \qquad \text{for any} \quad \xi \in \Gamma(z, \zeta),$$

we obtain by Lemma 1 that, for any $t \in \Gamma'(w, \tau)$,

$$|g'(t)| \asymp |g'(w)| \asymp \frac{1 - |z|}{\delta^{K^2}}.$$

Therefore,

$$\text{length}\,\Gamma(z, \zeta) = \int_{\Gamma'(w,\tau)} |g'(t)||dt| \asymp \frac{1 - |z|}{\delta^{K^2}}|w - \tau| \preccurlyeq |z - \zeta|.$$

Now, let $|w - \tau| > \delta^{K^2}/32$. We divide the arc $\Gamma'(w, \tau)$ by points $t_1 := w, \dots, t_{m+1} := \tau$ in such a way that

$$\frac{\delta^{K^2}}{64} \le |t_k - t_{k+1}| \le \frac{\delta^{K^2}}{32}, \qquad k = 1, 2, \dots, m.$$

Set

$$s_k := t_k \frac{1 + a}{1 + \delta}, \qquad \xi_k := g(s_k), \qquad \eta_k := g(t_k).$$

Using the previous result and Lemma 4, we find that

$$\text{length}\,\Gamma(\eta_k, \eta_{k+1}) \preccurlyeq |\eta_k - \eta_{k+1}| \asymp |\eta_k - \xi_k| \asymp |\xi_{k+1} - \xi_k|.$$

Hence,

$$\begin{aligned} \text{length}\,\Gamma(z, \zeta) &= \sum_{k=1}^{m} \text{length}\,\Gamma(\eta_k, \eta_{k+1}) \\ &\preccurlyeq \sum_{k=1}^{m} |\xi_k - \xi_{k+1}| \preccurlyeq |\xi_1 - \xi_{m+1}| \asymp |z - \zeta|. \end{aligned}$$

■

Now, let $l \subset L$ be an arbitrary arc,

$$\gamma := \Phi(l) = \{e^{i\theta} \colon \theta_1 \le \theta \le \theta_2\}.$$

For simplicity, we always assume that

$$\theta_2 - \theta_1 \leqslant \frac{3}{\pi} \qquad \text{and} \qquad 0 < \delta < \min\{\theta_2 - \theta_1, 1/2\}. \tag{4.1}$$

For $r > 0$, set

$$\begin{aligned} \gamma_r &:= \{(1 + r)e^{i\theta} \colon \theta_1 \leqslant \theta \leqslant \theta_2\}, \qquad l_r := \Psi(\gamma_r); \\ h_{l,a}(z) &:= \omega(z, \text{int}\, L_a, l_a), \qquad z \in \text{int}\, L_a. \end{aligned}$$

Lemma 6. *For an arbitrary signed measure σ on L and sufficiently small δ (satisfying* (4.1)), *the inequality*

$$\left|\int_L h_{l,a}(z)\,d\sigma(z)\right| \leqslant c\,\varepsilon\left(\frac{\delta}{2}\right)\log\left(\frac{1}{\delta}\right) \tag{4.2}$$

holds with some constant $c = c(K) > 0$.

Proof. Defining

$$\omega_l(z) := \omega(z, \Omega, l), \qquad z \in \Omega,$$

and applying the Green formula, we may write, for $z \in L$,

$$\begin{aligned} h_{l,a}(z) \;=\; \omega_l(\infty) + \frac{1}{2\pi}\int_{L_\delta}\Bigg[&\left(\frac{\partial\omega_l(\zeta)}{\partial n} - \frac{\partial h_{l,a}(\zeta)}{\partial n}\right)\log|\zeta - z| \\ &+ (h_{l,a}(\zeta) - \omega_l(\zeta))\frac{\partial}{\partial n}\log|\zeta - z|\Bigg]|d\zeta|, \end{aligned}$$

where $\partial/\partial n$ is the operator of differentiation with respect to the outward normal to the curve L_δ at the point ζ.

Integrating the last relation we get

$$\begin{aligned} \int_L h_{l,a}(z)\,d\sigma(z) \;=\; &\frac{1}{2\pi}\int_{L_\delta} U(\sigma,\zeta)\left(\frac{\partial h_{l,a}(\zeta)}{\partial n} - \frac{\partial\omega_l(\zeta)}{\partial n}\right)|d\zeta| \\ &+ \frac{1}{2\pi}\int_{L_\delta}(\omega_l(\zeta) - h_{l,a}(\zeta))\frac{\partial}{\partial n}U(\sigma,\zeta)|d\zeta|. \end{aligned} \tag{4.3}$$

This formula will be the base of our next reasoning.

First, we will try to estimate the first integral on the right-hand side of (4.3) from above. A routine category argument shows that for our purpose it suffices to establish the inequalities

$$\int_{L_\delta}\left|\frac{\partial}{\partial n}h_{l,a}(\zeta)\right||d\zeta| \;\preccurlyeq\; \log\left(\frac{1}{\delta}\right), \tag{4.4}$$

$$\int_{L_\delta}\left|\frac{\partial}{\partial n}\omega_l(\zeta)\right||d\zeta| \;\preccurlyeq\; \log\left(\frac{1}{\delta}\right). \tag{4.5}$$

The estimate (4.5) is a simple consequence of Lemma 2. In fact, since the harmonic measure is a conformal invariant we obtain

$$\begin{aligned} \int_{L_\delta}\left|\frac{\partial}{\partial n}\omega_l(\zeta)\right||d\zeta| \;&\le\; \int_{|w|=1+\delta}|\operatorname{grad}\omega(w,\Delta,\gamma)||dw| \\ &\preccurlyeq\; \sum_{j=1}^{2}\int_{|w|=1+\delta}\frac{|dw|}{|w - w_j|} \preccurlyeq \log\left(\frac{1}{\delta}\right), \end{aligned}$$

where w_1 and w_2 are the endpoints of the arc $\gamma := \Phi(l)$.

In order to prove (4.4) we consider the conformal mapping φ_a defined as above. Let τ_1 and τ_2 be endpoints of the arc $\varphi_a(l_a)$. Using (3.4), we find that, for $\tau \in D$,

$$|\operatorname{grad}\omega(\tau, D, \varphi_a(l_a))| \preccurlyeq \frac{1}{|\tau - \tau_1|} + \frac{1}{|\tau - \tau_2|}.$$

Denote by $\mu_j(\delta)$, where $j = 1, 2$ and $0 < \delta < 2$, the length of the portion of Γ lying in the open disk with center at τ_j and radius δ. According to Lemma 5 we have $\mu_j(\delta) \preccurlyeq \delta$.

Further, recalling the definition of the function g from the proof of Lemma 5 and applying to it Lemma 4, we obtain

$$d(\Gamma, \partial D) \succcurlyeq \delta^{K^4}.$$

Therefore, integration by parts yields

$$\begin{aligned}\int_{L_\delta} \left| \frac{\partial}{\partial n} h_{l,a}(\zeta) \right| |d\zeta| &\leqslant \int_\Gamma |\operatorname{grad} \omega(\tau, D, \varphi_a(l_a))||d\tau| \\ &\preccurlyeq \sum_{j=1}^{2} \int_{c\delta^{K^4}}^{2} \frac{d\mu_j(x)}{x} = \sum_{j=1}^{2} \left(\frac{\mu_j(2)}{2} + \int_{c\delta^{K^4}}^{2} \frac{\mu_j(x)}{x^2} dx \right) \preccurlyeq \log\left(\frac{1}{\delta}\right),\end{aligned}$$

which is the assertion of (4.4).

In order to estimate the second integral on the right-hand side of (4.3) we set

$$\begin{aligned}\tilde{U}(w) &:= U(\sigma, \Psi(w)), && |w| > 1;\\ \tilde{h}(w) &:= h_{l,a}(\Psi(w)), && |w| > 1;\\ \tilde{\omega}(w) &:= \omega_l(\Psi(w)), && |w| > 1;\\ \tilde{\chi}(w) &:= \begin{cases} 1 & \text{if } w \in \gamma_\delta, \\ 0 & \text{if } w \notin \gamma_\delta. \end{cases}\end{aligned}$$

For $|w| = 1 + \delta$, by Schwarz's formula,

$$|\operatorname{grad} \tilde{U}(w)| \le \frac{1}{\pi} \int_{|\tau - w| = \delta/2} \frac{|\tilde{U}(\tau)|}{|\tau - w|^2} |d\tau| \le \frac{4\,\varepsilon(\delta/2)}{\delta}.$$

Since

$$\int_{L_\delta} |\omega_l(\zeta) - h_{l,a}(\zeta)| \left| \frac{\partial}{\partial n} U(\sigma, \zeta) \right| |d\zeta| \le \int_{|w|=1+\delta} |\tilde{\omega}(w) - \tilde{h}(w)|| \operatorname{grad} \tilde{U}(w)||dw|,$$

it is sufficient, in order to establish an appropriate estimate, to show that

$$\int_{|w|=1+\delta} |\tilde{\omega}(w) - \tilde{\chi}(w)||dw| \preccurlyeq \delta \log \frac{1}{\delta} \tag{4.6}$$

and

$$\int_{|w|=1+\delta} |\tilde{h}(w) - \tilde{\chi}(w)||dw| \preccurlyeq \delta \log \frac{1}{\delta}. \tag{4.7}$$

Now, it follows from (3.3), rewritten for the exterior of the unit disk, that

$$|\tilde{\omega}(w) - \tilde{\chi}(w)| \preccurlyeq \delta \left(\frac{1}{|w - w_1|} + \frac{1}{|w - w_2|} \right), \tag{4.8}$$

where $|w| = 1 + \delta$, and w_1 and w_2 are the endpoints of the arc γ.

Thus, (4.6) is a simple consequence of (4.8).

In order to establish (4.7) we suppose that $|w| = 1 + \delta$, and let t_1 and t_2 be the endpoints of the arc γ_a. Applying Lemma 3 to the restriction of the mapping Φ on int L_a and Lemma 2, we have

$$|\tilde{h}(w) - \tilde{\chi}(w)| \preccurlyeq \left(\frac{\delta^{K^2}}{\min_{j=1,2} |w - t_j|} \right)^{K^{-2}} \le \delta(|w - t_1|^{-K^{-2}} + |w - t_2|^{-K^{-2}}).$$

By integrating the last inequality we get (4.7).

Combining (4.3)–(4.7), we obtain (4.2). ■

5. Proof of the Main Results

Proof of Theorem 1. To see that $D[\sigma]$ is appropriately estimated, we have only to consider an arbitrary arc $J \subset L$ and, for sufficiently small δ, say $\delta < (\pi/4)^2$, to establish the inequality

$$\sigma(J) \geq -c_1\left[\varepsilon\left(\frac{\delta}{2}\right)\log\left(\frac{1}{\delta}\right) + \delta^{1/(2K^2)} + \delta^{\beta/2}\right]. \tag{5.1}$$

Moreover, if

$$J' := \Phi(J) = \{e^{i\theta}\colon\ a \leqslant \theta \leqslant b\},$$

we only need to study the case $b - a \leq \pi$.

Set

$$\gamma := \{e^{i\theta} : a - \delta^{1/2} \leq \theta \leq b + \delta^{1/2}\}, \qquad l := \Psi(\gamma);$$
$$\gamma_1 := \{e^{i\theta} : a - 2\delta^{1/2} \leq \theta \leq b + 2\delta^{1/2}\}, \qquad l_1 := \Psi(\gamma_1).$$

With this choice we get from Lemma 6 that the function $h_{l,a}(z)$ satisfies

$$\int_L h_{l,a}(z)\, d\sigma(z) \geqslant c_2\varepsilon\left(\frac{\delta}{2}\right)\log\delta. \tag{5.2}$$

On the other hand,

$$\begin{aligned}\int_L h_{l,a}(z)\, d\sigma(z) \ &\leq\ \sigma(J) + \int_J [1 - h_{l,a}(z)]\, d\sigma^-(z) + \sigma^+(l_1\backslash J)\\ &\quad + \int_{L\backslash l_1} h_{l,a}(z)\, d\sigma^+(z).\end{aligned} \tag{5.3}$$

Let us estimate each of the integrals on the right-hand side of (5.3) from above.

Using (3.3) and Lemma 3, we find

$$\begin{aligned}1 - h_{l,a}(z) \ &\preccurlyeq\ \delta^{1/(2K^2)}, \qquad z \in J,\\ h_{l,a}(z) \ &\preccurlyeq\ \delta^{1/(2K^2)}, \qquad z \in L\backslash l_1.\end{aligned}$$

Therefore,

$$\int_J [1 - h_{l,a}(z)]\, d\sigma^-(z) + \int_{L\backslash l_1} h_{l,a}(z)\, d\sigma^+(z) \preccurlyeq \delta^{1/(2K^2)}. \tag{5.4}$$

Furthermore, by our assumption (2.1),

$$\sigma^+(l_1\backslash J) \preccurlyeq \delta^{\beta/2},$$

which, in view of (5.2)–(5.4), yields (5.1). ■

Proof of Theorem 2. The idea of the following discussion is originated in [5] and [7]. We give a sketch of the proof to show how the arguments of Blatt and Mhaskar can be modified to obtain the result.

First, suppose that L is a curve and consider the signed measure $\sigma := \mu_L - \nu_n$. Its positive part μ_L satisfies (2.1) with $c = \beta = 1$.

Set

$$\delta = \delta(n) := \left(\frac{\log C_n}{n}\right)^{2K^2}.$$

The same arguments as in the proof of [7, Theorem 2.2] (see also [5]) lead to

$$\varepsilon(\delta) \preccurlyeq \frac{\log C_n}{n}.$$

Hence, inequality (2.3) follows immediately from estimate (2.2).

Now, suppose that L is an arc. Consider a K-quasiconformal curve $\Gamma \supset L$ and define a signed measure σ on Γ as follows:

$$\sigma(J) := \mu_L(J \cap L) - \nu_n(J) \qquad \text{for all} \quad J \subset \Gamma.$$

According to (3.6) and (3.7), the positive part of this measure satisfies inequality (2.1) with some constants c and β depending on L (not only on K!).

Set

$$\delta = \delta(n) := \left(\frac{\log C_n}{n}\right)^{2(K^2+1/\beta)}.$$

Then, as above,

$$\varepsilon(\delta) := \|U(\sigma, \cdot)\|_{\Gamma_\delta} \le \|U(\sigma, \cdot)\|_{L_\delta} \preccurlyeq \frac{\log C_n}{n},$$

from which (2.3) directly follows.

Hence, our proof is complete. ■

6. Proof of Theorem 5

We apply the technique of [11] and some constructions from approximation theory in the complex plane, which can be found in [9].

In what follows we will often use Lemma 4 and its corollary (sometimes without special reference to them) for suitable, each time naturally defined, triplets of points in $\overline{\Omega}$ and $\overline{\Delta}$.

To see that $\theta_{j+1} - \theta_j$ is appropriately bounded from below, we consider the fundamental polynomials

$$q_j(z) = q_j(z_j, z) := \prod_{k \ne j} \frac{z - z_k}{z_j - z_k},$$

associated with the system of points $\{z_j\}_1^n$.

Since $\|q_j\|_L \le 1$, by the Bernstein–Walsh theorem we obtain, for ζ such that $|\zeta - z_j| \le d(z_j, L_{1/n}) =: d_j$, the estimate

$$|q_j(\zeta)| \le e.$$

Consequently, if $|\zeta - z_j| \le d_j/2$, then

$$|q_j'(\zeta)| \le \frac{1}{2\pi} \int_{|\zeta - \xi| = d_j/2} \frac{|q_j(\xi)|}{|\xi - \zeta|^2} |d\xi| \le \frac{2e}{d_j}.$$

We claim that

$$|z_{j+1} - z_j| \succcurlyeq d_j \asymp |z_j - \tilde{z}_j|, \tag{6.1}$$

where $\tilde{z}_j := \Psi[(1 + 1/n)\Phi(z_j)]$.

Indeed, we have only to verify the inequality if $|z_{j+1} - z_j| \leq \frac{1}{2} d_j$. In this case, we have

$$1 = |q_j(z_{j+1}) - q_j(z_j)| \leq \int_{[z_j, z_{j+1}]} |q_j'(z)|\,|dz| \preccurlyeq \frac{|z_{j+1} - z_j|}{d_j}.$$

Then, by (6.1) via Lemma 4 we get the left-hand side of (2.4).

In order to prove the right-hand inequality in (2.4), we use the Jackson-type kernel

$$J_{k,m}(t) := \frac{1}{\gamma_{k,m}} \left(\frac{\sin(mt)/(2)}{\sin(t)/(2)} \right)^{2k},$$

where $k, m \in \mathbf{N}$ and

$$\gamma_{k,m} := \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\sin(mt)/(2)}{\sin(t)/(2)} \right)^{2k} dt.$$

In what follows k will be a fixed number (large enough).

The following properties of $J_{k,m}$ are well known (see [9, Chapter II]):

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} J_{k,m}(t)\, dt = 1; \tag{6.2}$$

$$J_{k,m}(t) = \sum_{j=-k(m-1)}^{k(m-1)} a_{|j|} e^{ijt},$$

where the a_j are real numbers;

$$\int_{-\pi}^{\pi} J_{k,m}(t)|t|^c\, dt \preccurlyeq m^{-c}, \qquad 0 \leq c \leq 2k - 4. \tag{6.3}$$

For $\zeta \in \Omega$ and t real, we set $w := \Phi(\zeta)$ and $\zeta_t := \Psi(we^{-it})$.

Since, for $z \in L$,

$$\frac{1}{\zeta_t - z} = \frac{1}{\Psi(we^{-it}) - z} = \sum_{j=1}^{\infty} \frac{\Pi_j(z)}{w^j} e^{ijt},$$

where $\Pi_j(z),\ j \in \mathbf{N}$, denote the appropriate generalized Faber polynomials, the function

$$\pi_{k,m}(z) = \pi_{k,m}(\zeta, z) := \frac{1}{2\pi} \int_{-\pi}^{\pi} J_{k,m}(t) \frac{dt}{\zeta_t - z} = \sum_{j=1}^{k(m-1)} \frac{a_j}{w^j} \Pi_j(z)$$

is a polynomial in z of degree at most $k(m - 1)$.

Lemma 7. *There exist constants $c > 1$ and $k_0 \in \mathbf{N}$ depending only on K and such that, for $m > 2c,\ k \geq k_0,\ 2 \geq |w| \geq (1 - c/m)^{-1}$, and $z \in L$,*

$$\frac{1}{2|\zeta - z|} \leq |\pi_{k,m}(z)| \leq \frac{3}{2|\zeta - z|}. \tag{6.4}$$

Proof. Let $z \in L$ be arbitrary. From (6.2) we see that

$$\begin{aligned}\left|\frac{1}{\zeta - z} - \pi_{k,m}(z)\right| \\ &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} J_{k,m}(t)\left(\frac{1}{\zeta - z} - \frac{1}{\zeta_t - z}\right)dt\right| \le \frac{1}{2\pi|\zeta - z|}\int_{-\pi}^{\pi} J_{k,m}(t)\left|\frac{\zeta_t - \zeta}{\zeta_t - z}\right| dt.\end{aligned}$$

Set $\xi_t := \Psi((w)/(|w|)e^{-it})$.

Lemma 4 implies the following relations:

$$\begin{aligned}|\zeta_t - z| &\succcurlyeq |\zeta_t - \xi_t|, \\ \left|\frac{\zeta_t - \zeta}{\zeta_t - \xi_t}\right| &\preccurlyeq \left|\frac{wt}{|w| - 1}\right|^{K^{-2}} + \left|\frac{wt}{|w| - 1}\right|^{K^2}.\end{aligned}$$

By (6.3), for $k \ge [K^2 + 2] =: k_0$, we have

$$\begin{aligned}\left|\frac{1}{\zeta - z} - \pi_{k,m}(z)\right| &\le \frac{c_1}{|\zeta - z|}\int_{-\pi}^{\pi} J_{k,m}(t)\left(\left|\frac{wt}{|w| - 1}\right|^{K^{-2}} + \left|\frac{wt}{|w| - 1}\right|^{K^2}\right)dt \\ &\le \frac{c_2}{|\zeta - z|}\left(\left|\frac{w}{(|w| - 1)m}\right|^{K^{-2}} + \left|\frac{w}{(|w| - 1)m}\right|^{K^2}\right),\end{aligned}$$

from which (6.4) immediately follows, if we choose $c := 1 + (4c_2)^{K^2}$. ∎

To continue the proof of Theorem 5 let ζ be chosen such that

$$|w| = \left(1 - \frac{c}{m}\right)^{-1},$$

where c is a constant as in Lemma 7. Further, let $s \in \mathbf{N}$, $s \ge 2K^2$, be fixed. Define the function

$$p_{s,k,m}(z) = p_{s,k,m}(\zeta, z) := \pi_{k,m}(\zeta, z)^s(\zeta - \zeta_L)^s,$$

where $\zeta_L := \Psi(w/|w|)$. Then $p_{s,k,m}(z)$ is a polynomial in z of degree at most $sk(m-1)$, and, by Lemma 7,

$$|p_{s,k,m}(\zeta_L)| \ge 2^{-s}. \tag{6.5}$$

In addition, by Lemma 4, for any $z \in L$ and $t := \Phi(z)$, we have

$$|p_{s,k,m}(z)| \preccurlyeq \left|\frac{\zeta - \zeta_L}{\zeta - z}\right|^s \preccurlyeq (m|w - t|)^{-s/K^2} \preccurlyeq (m|w - t|)^{-2}. \tag{6.6}$$

Now, we are ready to establish the right-hand side of (2.4).

Set $\delta = \delta_j := \frac{1}{2}(\theta_{j+1} - \theta_j)$, $\theta_0 = \theta_{0,j} := \frac{1}{2}(\theta_{j+1} + \theta_j)$. We rename the points $\{\theta_j\}_1^n$ by $\{\theta_j'\}_1^\nu$ and $\{\theta_j''\}_1^{n-\nu}$ in such a way that

$$\begin{aligned}\theta_0 < \theta_1' < \cdots < \theta_\nu' \le \pi + \theta_0, \\ \theta_0 - \pi < \theta_{n-\nu}'' < \cdots < \theta_1'' < \theta_0.\end{aligned}$$

By the left-hand side inequality in (2.4), we obtain

$$\theta_j' - \theta_0 \ \geq\ \frac{c_1 j}{n} + \delta, \qquad j = 1, \ldots, \nu,$$
$$\theta_0 - \theta_j'' \ \geq\ \frac{c_1 j}{n} + \delta, \qquad j = 1, \ldots, n - \nu.$$

Set $w := (1 + c_2/m)e^{i\theta_0},\ \zeta := \Psi(w),\ m := [(n-1)/sk]$, where the constants $c_2 > 0$, s, and $k \in \mathbf{N}$ are chosen in such a way that, for the polynomial $p(z) := p_{s,k,m}(\zeta, z)$ (of degree at most $n - 1$), the relations (6.5) and (6.6) are fulfilled.

Since by Lagrange's interpolation formula

$$p(z) = \sum_{j=1}^{\nu} p(\zeta_j')q(z_j', z) + \sum_{j=1}^{n-\nu} p(z_j'')q(z_j'', z),$$

where $z_j' := \Psi(e^{i\theta_j'})$ and $z_j'' := \Psi(e^{i\theta_j''})$, the quantity δ can easily be estimated from above by the following reasoning:

$$\begin{aligned} 2^{-s} \leq |p(\Psi(e^{i\theta_0}))| \ &\leq\ \sum_{j=1}^{\nu} |p(z_j')| + \sum_{j=1}^{n-\nu} |p(z_j'')| \\ &\preccurlyeq\ n^{-2}\left(\sum_{j=1}^{\nu}\left(\delta + \frac{c_1 j}{n}\right)^{-2} + \sum_{j=1}^{n-\nu}\left(\delta + \frac{c_1 j}{n}\right)^{-2}\right) \\ &\leq\ 2\sum_{j=1}^{\infty}(\delta n + c_1 j)^{-2} \\ &\leq\ 2\int_0^{\infty}\frac{dx}{(\delta n + c_1 x)^2} = \frac{2}{c_1 \delta n}. \end{aligned}$$

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V. V. Andrievskii
Katholische Universität Eichstätt
Ostenstrasse 26–28
D-85071 Eichstätt
Germany

H.-P. Blatt
Katholische Universität Eichstätt
Ostenstrasse 26–28
D-85071 Eichstätt
Germany