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CONSTRUCTIVE APPROXIMATION

Biorthogonal Wavelet Expansions

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Abstract. This paper is concerned with developing conditions on a given finite collection of compactly supported algebraically linearly independent refinable functions that insure the existence of biorthogonal systems of refinable functions with similar properties. In particular, we address the close connection of this issue with stationary subdivision schemes.

1. Introduction

During the past few years the construction of multivariate wavelets has received considerable attention. It is quite apparent that multivariate wavelets with good localization properties in frequency and spatial domains, which constitute an orthonormal basis of $L_2(\mathbb{R}^s)$, are hard to realize. On the other hand, it turns out that in many applications orthogonality is not really important whereas locality, in particular, compact support is very desirable. In this regard, the concept of *biorthogonality* seems to offer more flexibility in practical realizations while still preserving many of the advantages of orthonormality. So far, this concept has been carefully studied in the univariate case (see, e.g., [CDF]). In the multivariate case concrete results have been obtained only for certain special bivariate examples [CS], [CD], and since the start of this paper multivariate studies have appeared in [LC] and [KV].

The point of view taken in this paper is, to avoid trying to relax assumptions on the initial system used for the construction of a biorthogonal system. Instead, we will focus on locality of the initial system, that is, we will insist on local support, finitely supported masks, and linear independence. Then we try to construct a biorthogonal system with the same properties. The above-mentioned results, even for the univariate case, do not seem to answer this question. So far they still require assumptions on both systems. Although one may never be able to answer this question in great generality, the objective of this paper is at least to contribute to the understanding of this issue.

In Section 2 we describe the concept of multiresolution based on finitely many generating refinable functions. The main objective is then to formulate algebraic conditions to be satisfied by the biorthogonal systems.

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The possibility of realizing such conditions will be seen to be closely related to the concept of stationary subdivision. Section 3 is devoted to the discussion of several convergence concepts of subdivision schemes which will be discussed and interrelated. Our findings will also extend previous results from [CDM] and should be of independent interest.

2. Finitely Generated Shift-Invariant Spaces

We will consider here sequences of nested closed subspaces of $L_2(\mathbb{R}^s)$ which are generated by certain dilates and integer shifts of finitely many scalar-valued functions $h_1, \ldots, h_N \in L_2(\mathbb{R}^s)$. By $\mathbf{h} = (h_1, \ldots, h_N)^T$ we will denote the corresponding vector-valued mapping from $\mathbb{R}^s$ into $\mathbb{R}^N$ which is assumed to be in $L_2^N(\mathbb{R}^s) := L_2(\mathbb{R}^s) \times \cdots \times L_2(\mathbb{R}^s)$, N times. Generally, we use boldface letters whenever we are dealing with objects that are associated with N-tuples. Also, we use $L_p^N(\mathbb{R}^s)$, $\ell_p^N(\mathbb{Z}^s)$, $1 \leq p \leq \infty$, for $L_p(\mathbb{R}^s) \times \cdots \times L_p(\mathbb{R}^s)$, $\ell_p(\mathbb{Z}^s) \times \cdots \times \ell_p(\mathbb{Z}^s)$, N times, respectively. For $\mathbf{h} = (h_1, \ldots, h_N)^T \in L_p^N(\mathbb{R}^s)$ we set $\|\mathbf{h}\|_{L_p^N(\mathbb{R}^s)}^p := \sum_{i=1}^N \|h_i\|_{L_p(\mathbb{R}^s)}^p$ and likewise the norm of $\mathbf{c} = (c_1, \ldots, c_N)^T \in \ell_p^N(\mathbb{Z}^s)$ is defined by $\|\mathbf{c}\|_{\ell_p^N(\mathbb{Z}^s)}^p := \sum_{i=1}^N \|c_i\|_{\ell_p(\mathbb{Z}^s)}^p$. Of course, as usual for $c = \{c_\alpha\}_{\alpha \in \mathbb{Z}^s}$ we use $\|c\|_{\ell_p(\mathbb{Z}^s)}^p := \sum_{\alpha \in \mathbb{Z}^s} |c_\alpha|^p$ and $f \in L_p(\mathbb{R}^s)$ has the norm $\|f\|_{L_p(\mathbb{R}^s)} := (\int_{\mathbb{R}^s} |f(x)|^p \, dx)^{1/p}$. Also, we use $|\cdot|_p$ for the ℓ_p-norm on $\mathbb{R}^s$. Therefore, if we write $\mathbf{c} \in \ell_p^N(\mathbb{Z}^s)$ as $\mathbf{c} = \{\mathbf{c}_\alpha\}_{\alpha \in \mathbb{Z}^s}$ where $\mathbf{c}_\alpha \in \mathbb{R}^N$, then we also have $\|\mathbf{c}\|_{\ell_p^N(\mathbb{Z}^s)}^p := \sum_{\alpha \in \mathbb{Z}^s} |\mathbf{c}_\alpha|_p^p$ and similarly $\|\mathbf{f}\|_{L_p^N(\mathbb{R}^s)}^p := \int_{\mathbb{R}^s} |\mathbf{f}(x)|_p^p \, dx$. Finally, we use $\ell_1^{N \times N}(\mathbb{R}^s)$ for all bi-infinite sequences $\mathbf{C} = \{\mathbf{C}_\alpha\}_{\alpha \in \mathbb{Z}^s}$ where each $\mathbf{C}_\alpha$, $\alpha \in \mathbb{Z}^s$, is an $N \times N$ matrix and for some norm $\|\cdot\|$ on such matrices we demand that $\sum_{\alpha \in \mathbb{Z}^s} \|\mathbf{C}_\alpha\| < \infty$.

2.1. *Expanding Scaling Matrices*

Dilates of such mappings $\mathbf{h}$: $\mathbb{R}^s \to \mathbb{R}^N$ can be formed with the aid of *expanding* scaling $s \times s$ matrices M. Here M is called expanding if it has integer entries and all its eigenvalues are greater than one. Perhaps the easiest example is $M = 2I$ where I denotes the identity matrix.

As is well known the order of $\mathbb{Z}^s / M\mathbb{Z}^s$ equals $m := |\det M|$. The following example shows that for any s and any $m \geq 2$ one can find expanding matrices M such that the order of $\mathbb{Z}^s / M\mathbb{Z}^s$ equals m. In fact, suppose A is any unimodular integer matrix, i.e., $|\det A| = 1$, then

$$M = A^{-1} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ m & 0 & 0 & \cdots & 0 \end{pmatrix} A \tag{2.1}$$

has eigenvalues $r\omega_j$, $j = 1, \ldots, s$, where $r^s = m$ and ω_j are the sth order primitive roots of unity. Thus M is expanding for $m \geq 2$ and, moreover, $M^s = mI$.

Given a function f and a fixed expanding matrix M, we will be concerned with its dilates

$$sc^k f = sc^k_M f := f(M^k\cdot).$$

Although we will be mainly interested in mappings $\mathbf{h}$ whose components all have compact support, it is convenient to work with the space $\mathcal{L}_p^N := \mathcal{L}_p \times \cdots \times \mathcal{L}_p$, N-times, where for $1 \leq p \leq \infty$

$$\mathcal{L}_p := \{g \in L_p(\mathbb{R}^s)\colon\ u\Sigma|g| \in L_p([0,1]^s)\},$$

where $u = \{u_\alpha\}_{\alpha\in\mathbb{Z}^s}$, $u_\alpha := 1$, $\alpha \in \mathbb{Z}^s$, and we write, in general,

$$c\Sigma g := \sum_{\alpha\in\mathbb{Z}^s} c_\alpha g(\cdot - \alpha), \qquad c = \{c_\alpha\}_{\alpha\in\mathbb{Z}^s}.$$

We extend this last bit of notation to vector-valued functions $\mathbf{g}$ and sequences $\mathbf{c}$ in $L_p^N(\mathbb{R}^s)$, $\ell_p^N(\mathbb{Z}^s)$, respectively, by setting

$$\mathbf{c}\Sigma\mathbf{g} := \sum_{i=1}^{N} c_i \Sigma g_i,$$

where $c_i, g_i, i = 1, \ldots, N$, denote the ith component of $\mathbf{c}$, $\mathbf{g}$ in $\ell_p(\mathbb{Z}^s)$, $L_p(\mathbb{R}^s)$, respectively. In the same fashion we can even treat the case when $\mathbf{C} = \{\mathbf{C}_\alpha\}_{\alpha\in\mathbb{Z}^s}$ and $\mathbf{C}_\alpha$ is a matrix-valued element of $\ell_1^{N\times N}(\mathbb{Z}^s)$, by letting

$$\mathbf{C}\Sigma\mathbf{h} := \sum_{\alpha\in\mathbb{Z}^s} \mathbf{C}_\alpha \mathbf{h}(\cdot - \alpha),$$

where we require that

$$\sum_{\alpha\in\mathbb{Z}^s} \|\mathbf{C}_\alpha\| < \infty$$

for some norm $\|\cdot\|$ on $N \times N$ matrices. Note that $\mathbf{c}\Sigma\mathbf{g}$ is a scalar, while $\mathbf{C}\Sigma\mathbf{h}$ is a vector with N components.

We always use the letter e for a typical representer of an equivalence class in $\mathbb{Z}^s/M\mathbb{Z}^s$ and E for a set of m distinct such representers for $\mathbb{Z}^s/M\mathbb{Z}^s$. That is,

$$\mathbb{Z}^s = \bigcup_{e\in E}(e + M\mathbb{Z}^s),$$

and, moreover, the sublattices $e+M\mathbb{Z}^s$, $e \in E$ form a partition of $\mathbb{Z}^s$ into disjoint subsets. One possible choice of E is given in [DM2] by the formula

$$E := \mathbb{Z}^s \cap M[0,1)^s.$$

In this case, we will frequently use the notation

$$E_* := E\backslash\{0\}.$$

One easily concludes that for $\mathbf{h} \in \mathcal{L}_2^N$ the spaces

$$S(\mathbf{h}) := \{\mathbf{c}\Sigma\mathbf{h}\colon\ \mathbf{c} \in \ell_2^N(\mathbb{Z}^s)\}, \qquad S^k(\mathbf{h}) := sc^k_M S(\mathbf{h}) = \{sc^k_M\mathbf{f}\colon\ \mathbf{f} \in S(\mathbf{h})\},$$

are closed subspaces of $L_2(\mathbb{R}^s)$, provided $\mathbf{h}$ is *stable*, that is, there is a positive constant d such that

$$\|\mathbf{c}\|_{\ell_2^N(\mathbb{Z}^s)} \leq d\|\mathbf{c}\Sigma\mathbf{h}\|_{L_2^N(\mathbb{R}^s)}.$$

2.2. *Refinement Relation*

To insure that the spaces $S^k(\mathbf{h})$ are nested we require that $\mathbf{h}$ be *refinable* or, more appropriately, $\mathbf{A}$-refinable by which we mean that there exists some *mask* $\mathbf{A} = \{\mathbf{A}_\alpha\}_{\alpha\in\mathbb{Z}^s} \in \ell_1^{N\times N}(\mathbb{Z}^s)$, i.e., each $\mathbf{A}_\alpha$ is an $N \times N$ matrix for $\alpha \in \mathbb{Z}^s$, such that

$$\sum_{\alpha\in\mathbb{Z}^s} \|\mathbf{A}_\alpha\| < \infty$$

and

$$\mathbf{h} = sc(\mathbf{A}\Sigma\mathbf{h}). \tag{2.2}$$

As a matter of notation we use $A_{i,j}$, $1 \le i,\ j \le N$, for the bi-infinite vector $\{(\mathbf{A}_\alpha)_{i,j}\}_{\alpha\in\mathbb{Z}^s}$ where $(\mathbf{A}_\alpha)_{i,j}$ stands for the $(i,\ j)$ entry of the $N \times N$ matrix $\mathbf{A}_\alpha$, $\alpha \in \mathbb{Z}^s$.

Also for $\mathbf{c} \in \ell_1^N(\mathbb{Z}^s)$ the *symbol* of $\mathbf{c}$ is given by

$$\mathbf{c}(z) := \sum_{\alpha\in\mathbb{Z}^s} \mathbf{c}_\alpha z^\alpha,$$

where $z^\alpha = z_1^{\alpha_1}\cdots z_s^{\alpha_s}$, $z = (z_1,\dots z_s)$, $\alpha = (\alpha_1,\dots,\alpha_s)$. Similarly, for $\mathbf{R} \in \ell_1^{N\times N}(\mathbb{Z}^s)$, we use

$$\mathbf{R}(z) = \sum_{\alpha\in\mathbb{Z}^s} \mathbf{R}_\alpha z^\alpha$$

for its corresponding symbol, and the splitting of $\mathbb{Z}^s$, induced by M, gives the *subsymbols*

$$\mathbf{R}_e(z) := \sum_{\alpha\in\mathbb{Z}^s} \mathbf{R}_{e+M\alpha} z^\alpha, \qquad e \in \mathbb{Z}^s.$$

Introducing the Fourier transform of $f \in L_1(\mathbb{R}^s)$ by

$$\hat f(y) := \int_{\mathbb{R}^s} f(x)e^{-ix\cdot y}\,dx,$$

one readily verifies that (2.2) is equivalent to

$$\hat{\mathbf{h}}(y) = m^{-1}\mathbf{A}(e^{-iM^{-T}y})\hat{\mathbf{h}}(M^{-T}y), \tag{2.3}$$

where

$$m := |\det M|.$$

Moreover, for $\mathbf{c} \in \ell_1^N(\mathbb{Z}^s)$, we can assemble the subsymbols

$$\mathbf{c}_e(z) = \sum_{\alpha\in\mathbb{Z}^s} \mathbf{c}_{e+M\alpha} z^\alpha$$

to recapture the symbol via the formula

$$\mathbf{c}(z) = \sum_{e\in E} z^e \mathbf{c}_e(z^M), \tag{2.4}$$

where $z^M := (z^{M^1},\dots,z^{M^s})^T$ and M^j is the jth column vector of the matrix M.

Before starting our analysis of the general setting described so far, we pause to comment briefly on some recent related developments. For any N, examples of *univariate* refinable functions are encountered in connection with cardinal splines where N corresponds to the multiplicity of the integer knots [P2]. Multiresolution induced by several generators is also studied in [H] and corresponding approximation properties are investigated in [P1]. The construction of multiwavelets is discussed in [G] and [GL]. Recently, certain refinable functions for $N > 1$ have been generated with the aid of techniques from the theory of iterated function systems [DGH]. In particular, they may be suitable for tensor product finite-element applications [SS]. All these results are univariate. For $N = 1$, the cube spline provides an important example of a refinable function for $M = 2I$ as a scaling matrix in *any dimension* [DM1]. Also, for $N = 1$ and arbitrary scaling matrices M refinable functions were studied in [GM], which generalized the univariate Haar bases. Specifically, they showed that there exists, for a given expanding matrix M, a finite set $\Gamma \subset \mathbb{Z}^s$, $\#\Gamma = m$, as well as bounded domain $\Omega \subset \mathbb{R}^s$ such that

$$\chi_\Omega = \sum_{\alpha \in \Gamma^s} \chi_\Omega(M \cdot -\alpha),$$

where χ_Ω is the indicator function of Ω. Extensions of this result to the case $N > 1$ appear as special cases of observations made in [MX]. Convolutions of generalized Haar refinable functions again yield refinable functions but with higher regularity [S]. More quantitative smoothness results (for the scalar case) were obtained in [DDL] taking convolutions of indicator functions relative to different fundamental domains associated with M. Finally, we mention, from a different point of view, that the tuple $\mathbf{h}$ as the generator of a vector field was used in [U] to construct compactly supported divergence-free wavelets.

2.3. *Stability and Linear Independence*

Recalling, as before, that $\|c\|_{\ell_p(\mathbb{Z}^s)} := (\sum_{\alpha\in\mathbb{Z}^s} |c_\alpha|^p)^{1/p}$ we can endow $\ell_p^N(\mathbb{Z}^s)$ with the norm $\|\mathbf{c}\|_{\ell_p^N(\mathbb{Z}^s)} := (\sum_{j=1}^N \|c_j\|^p_{\ell_p(\mathbb{Z}^s)})^{1/p}$, where $\mathbf{c} = (c_1, \ldots, c_N), c_j \in \ell_p(\mathbb{Z}^s)$, $j = 1, \ldots, N$. Analogously, we define a norm for $L_p^N(\mathbb{R}^s)$ which will again be denoted by $\|\cdot\|_{L_p^N(\mathbb{R}^s)}$. One can show as in the scalar case (see [JM1]) that

$$\|\mathbf{c}\Sigma\mathbf{h}\|_{L_p^N(\mathbb{R}^s)} \le \|\mathbf{h}\|_{\mathcal{L}_p^N} \|\mathbf{c}\|_{\ell_p^N(\mathbb{Z}^s)}, \tag{2.5}$$

where $\|g\|_{\mathcal{L}_p} := \|u\Sigma|g|\|_{L_p([0,1]^s)}$ and $\|\mathbf{h}\|_{\mathcal{L}_p^N} := (\sum_{j=1}^N \|h_j\|^p_{\mathcal{L}_p})^{1/p}$. The function $\mathbf{h}$ is called ℓ_p-stable if

$$\|\mathbf{c}\|_{\ell_p^N(\mathbb{Z}^s)} \le d\ \|\mathbf{c}\Sigma\mathbf{h}\|_{L_p^N(\mathbb{R}^s)} \tag{2.6}$$

for some constant d independent of $\mathbf{c}$. It is well known [JM2] that $\mathbf{h}$ is ℓ_p-stable if and only if the sequences $\{\hat{h}_j(y + 2\pi\alpha)\}_{\alpha\in\mathbb{Z}^s}$, $j = 1, \ldots, N$, are linearly independent for every $y \in \mathbb{R}^s$. Thus ℓ_p-stability for some p, $1 \le p \le \infty$, implies ℓ_q-stability for any $1 \le q \le \infty$.

When $\mathbf{h}$ has compact support $\mathbf{c}\Sigma\mathbf{h}$ is defined for any vector-valued sequence $\mathbf{c}$. The integer shifts of $\mathbf{h}$ are called (algebraically) linearly independent if the mapping

$$\mathbf{c} \mapsto \mathbf{c}\Sigma\mathbf{h}$$

is injective on the space of *all* sequences $\mathbf{c}$ which is equivalent to the fact that the sequences $\{\hat{h}_j(z + 2\pi\alpha)\}_{\alpha\in\mathbb{Z}^s}$, $j = 1, \ldots, N$, are now linearly independent for all $z \in \mathbb{C}^s$ [JM2]. Here $\hat{\mathbf{h}}$, of course, denotes the Fourier–Laplace transform of $\mathbf{h}$.

2.4. *Properties of the Mask*

In the following $\mathbf{1}$ stands for the vector in $\mathbb{R}^s$ all of whose coordinates are one.

Proposition 2.1. *Let* $\mathbf{A}$ *be in* $\ell_1^{N\times N}(\mathbb{Z}^s)$ *and let* $\mathbf{h} \in \mathcal{L}_2^N$ *be a stable solution of the refinement equation* (2.2). *Then there exists a unique* $\mathbf{y} \in \mathbb{C}^N\backslash\{0\}$ *of unit length satisfying*

$$\mathbf{A}_e(\mathbf{1})^T\mathbf{y} = \mathbf{y}, \qquad e \in E := \mathbb{Z}^s/M\mathbb{Z}^s. \tag{2.7}$$

Proof. Let ρ denote the spectral radius of the matrix $m^{-1}\mathbf{A}(\mathbf{1})$, and let $\mathbf{y} \in \mathbb{C}^N\backslash\{0\}$ be an eigenvector of $(m^{-1}\mathbf{A}(\mathbf{1}))^T$ such that

$$(m^{-1}\mathbf{A}(\mathbf{1}))^T\mathbf{y} = \lambda\mathbf{y} \tag{2.8}$$

with $|\lambda| = \rho$. We claim that $\rho = 1$ and that for $\mathbf{y}$ satisfying (2.8), the function

$$g := \mathbf{y}^T\mathbf{h} := \sum_{j=1}^{N} y_j h_j \tag{2.9}$$

has the property that

$$\hat{g}(2\pi\alpha) = 0, \qquad \alpha \in \mathbb{Z}^s\backslash\{0\}. \tag{2.10}$$

We begin the proof of these assertions by proving that $\rho \geq 1$ and that (2.10) holds. For this purpose we follow the proof of Theorem 2.1 of [M1].

As a consequence of the refinement equation (2.3) we have for any positive integer k that

$$\hat{\mathbf{h}}(y) = \left\{\prod_{j=1}^{k}(m^{-1}\mathbf{A}(e^{-i(M^{-T})^j y}))\right\}\hat{\mathbf{h}}((M^{-T})^k y). \tag{2.11}$$

To use this formula, we keep in mind that because M is expanding one has

$$\lim_{j\to\infty}(M^{-T})^j = 0. \tag{2.12}$$

Now, suppose to the contrary that $\rho < 1$. Then for every $\mu \in (\rho, 1)$ there is a norm $\|\cdot\|$ on $\mathbb{C}^N$ such that

$$\|m^{-1}\mathbf{A}(\mathbf{1})\mathbf{x}\| \leq \mu\|\mathbf{x}\|, \qquad \mathbf{x} \in \mathbf{C}^N,$$

(see [SB, p. 384]). Since $\mathbf{A} \in \ell_1^{N\times N}(\mathbb{Z}^s)$ and $\mathbf{h} \in \mathcal{L}_2^N$ both the symbol of $\mathbf{A}$ and the Fourier transform of $\mathbf{h}$ are continuous. Thus, by (2.12), for every $y \in \mathbb{R}^s$ and $\varepsilon > 0$ there is a positive integer l such that for $j \geq l$ both

$$\|m^{-1}\mathbf{A}(e^{-i(M^{-T})^j y})\mathbf{x}\| \leq (\mu + \varepsilon)\|\mathbf{x}\|, \qquad \mathbf{x} \in \mathbb{C}^N,$$

and

$$\|\hat{\mathbf{h}}((M^{-T})^j y)\| \leq \varepsilon + \|\hat{\mathbf{h}}(0)\|.$$

Therefore, we infer from (2.11) with k replaced by $k + l$ that

$$\|\hat{\mathbf{h}}(y)\| \leq \kappa(\mu + \varepsilon)^k,$$

where

$$\kappa := \left\{ \prod_{j=1}^{l} \|m^{-1}\mathbf{A}(e^{-i(M^{-T})^j y})\| \right\} (\varepsilon + \|\hat{\mathbf{h}}(0)\|).$$

Hence, $\hat{\mathbf{h}}(y) = 0$, $y \in \mathbb{R}^s$, which is an obvious contradiction. Consequently, we have established that

$$\rho \geq 1. \tag{2.13}$$

Next we let $\mathbf{y} \in \mathbb{C}^N \backslash \{\mathbf{0}\}$ be any vector satisfying (2.8) with $|\lambda| = \rho$. Returning to (2.11) and evaluating $\hat{\mathbf{h}}$ at $2\pi(M^T)^k\alpha$ where $\alpha \in \mathbb{Z}^s$, we conclude that

$$\hat{\mathbf{h}}(2\pi(M^T)^k\alpha) = (m^{-1}\mathbf{A}(\mathbf{1}))^k\hat{\mathbf{h}}(2\pi\alpha).$$

For $\alpha \neq 0$, the Riemann–Lebesgue lemma implies that the left-hand side of this equation tends to zero as $k \to \infty$. Since the inner product of the right-hand side of this equation with $\mathbf{y}$ has an absolute value which tends to infinity unless $\hat{g}(2\pi\alpha) = \mathbf{y}^T\hat{\mathbf{h}}(2\pi\alpha) = 0$ we have proved (2.10) as well. Now let ζ be any smooth one-periodic function so that its Fourier series converges absolutely. Thus

$$\begin{aligned} \int_{[0,1]^s} (u\Sigma g)(x)\zeta(x)\,dx &= \sum_{\alpha\in\mathbb{Z}^s} \hat{\zeta}(\alpha) \int_{[0,1]^s} (u\Sigma g)(x)e^{i2\pi\alpha\cdot x}\,dx \\ &= \sum_{\alpha\in\mathbb{Z}^s} \hat{\zeta}(\alpha)(u\Sigma g)\hat{}(-\alpha). \end{aligned} \tag{2.14}$$

Now $\mathbf{h} \in \mathcal{L}_2^N$ implies $g \in \mathcal{L}_2$ and

$$\begin{aligned} \int_{\mathbb{R}^s} |g(x)|\,dx &= \int_{[0,1]^s} \sum_{\alpha\in\mathbb{Z}^s} |g(x-\alpha)|\,dx \\ &\leq \left(\int_{[0,1]^s} \left(\sum_{\alpha\in\mathbb{Z}^s} |g(x-\alpha)| \right)^2 dx \right)^{1/2} < \infty, \end{aligned}$$

which shows that $g \in L_1(\mathbb{R}^s)$. Employing the Lebesgue Dominated Convergence Theorem, one easily confirms that $(u\Sigma g)\hat{}(\alpha) = \hat{g}(2\pi\alpha)$. Hence we infer from (2.14) and (2.10) that

$$\int_{[0,1]^s} (u\Sigma g)(x)\zeta(x)\,dx = \hat{g}(0)\hat{\zeta}(0),$$

which, in turn, proves that

$$\hat{g}(0) = u\Sigma g = u\Sigma\mathbf{y}^T\mathbf{h}.$$

Thus, by the stability of $\mathbf{h}$, we conclude that $\hat{g}(0) \neq 0$. This conclusion about g holds for *any* vector $\mathbf{y}$ satisfying (2.8). If there were two such vectors we could form a nontrivial linear combination of them so as to choose a $\mathbf{y}$ satisfying (2.8) with the additional property that $\mathbf{y}^T\hat{\mathbf{h}}(0) = 0$. Hence the corresponding function g has the property that $\hat{g}(0) = 0$. This contradicts our observation above and so establishes that $\mathbf{y}$ satisfying (2.8) is unique (up to normalization). To prove (2.7) we observe from the Poisson summation formula that for $x \in \mathbb{R}^s$

$$\text{(2.15)} \qquad \begin{aligned} \hat{g}(0) &= (u\Sigma g)(x) = (u\Sigma \mathbf{y}^T\mathbf{h})(x) \\ &= \sum_{\beta\in\mathbb{Z}^s} \mathbf{y}^T\mathbf{h}(x-\beta) = \sum_{\beta\in\mathbb{Z}^s}\sum_{e\in E} \mathbf{y}^T\mathbf{h}(x-M\beta-e). \end{aligned}$$

Similarly, from the refinement equation (2.2) we have for all $x \in \mathbb{R}^s$

$$\begin{aligned} \hat{g}(0) &= (u\Sigma g)(x) = (u\Sigma \mathbf{y}^T sc(\mathbf{A}\Sigma\mathbf{h}))(x) \\ &= \sum_{\beta\in\mathbb{Z}^s}\left(\mathbf{y}^T \sum_{\alpha\in\mathbb{Z}^s} \mathbf{A}_\alpha \mathbf{h}(Mx-M\beta-\alpha)\right) \\ &= \sum_{\beta\in\mathbb{Z}^s}\sum_{e\in E}\sum_{\alpha\in\mathbb{Z}^s} \mathbf{y}^T\mathbf{A}_{e+M\alpha}\mathbf{h}(Mx-M\beta-e-M\alpha) \\ &= \sum_{\beta\in\mathbb{Z}^s}\sum_{e\in E}(\mathbf{A}_e^T(\mathbf{1})\mathbf{y})^T\mathbf{h}(Mx-M\beta-e). \end{aligned}$$

Replacing x by $M^{-1}x$ in this equation, comparing it to (2.15), and using the ℓ_∞-stability of $\mathbf{h}$, we conclude that

$$\text{(2.16)} \qquad \mathbf{A}_e(\mathbf{1})^T\mathbf{y} = \mathbf{y}, \qquad e \in E := \mathbb{Z}^s/M\mathbb{Z}^s.$$

Since

$$\text{(2.17)} \qquad \mathbf{y} = (m^{-1}\mathbf{A}(\mathbf{1}))^T\mathbf{y},$$

in view of (2.8), we also observe that when $\mathbf{h}$ is stable

$$\text{(2.18)} \qquad \rho = 1,$$

and the only eigenvalue of $m^{-1}\mathbf{A}(\mathbf{1})$ on the unit circle $|\lambda| = 1$ is $\rho = 1$. ■

Let us look at some examples of the previous result. The first one is quite special and is univariate. It comes from the simple but important idea of “filling” in a function iteratively by a *stationary subdivision scheme*. In the present context we consider the possibility of Hermite interpolation of function and derivative data. Thus we begin with a vector $\mathbf{v} = (\mathbf{v}_j)_{j\in\mathbb{Z}}$ of data where each $\mathbf{v}_j = (v_j^0, v_j^1)^T$, $j \in \mathbb{Z}$, is to represent the value of a function and its derivative at the integers $j \in \mathbb{Z}$. At the first step of the filling-in process we create new values to be associated with our function at $j/2$, $j \in \mathbb{Z}$, which we call $\mathbf{v}^1 = (\mathbf{v}_j^1)_{j\in\mathbb{Z}}$, in the following way. To insure that our scheme is interpolatory we set $\mathbf{v}_{2j}^1 := \mathbf{v}_j^0 := \mathbf{v}_j$, $j \in \mathbb{Z}$. The remaining values $\mathbf{v}_{2j+1}^1$ are obtained by Hermite interpolation. To this end, we choose an integer N and consider the data $\mathbf{v}_l^0$, $l = j-N+1, \ldots, j+N$. These $4N$ scalar data we interpolate by a polynomial P of degree $4N-1$ which is determined by

$$\text{(2.19)} \qquad \begin{aligned} P(j+r) &= v_{r+j}^0, \qquad r = -N+1, \ldots, N, \\ P'(j+r) &= v_{r+j}^1, \qquad r = -N+1, \ldots, N. \end{aligned}$$

Then we evaluate P and its derivative at the points $(j+1)/2$ and set these values equal to $\mathbf{v}_{2j+1}^1$. That is, we set

$$\mathbf{v}_{2j+1}^1 := \left(P\left(\frac{j+1}{2}\right), P'\left(\frac{j+1}{2}\right)\right)^T.$$

This scheme can be expressed in other terms. For this purpose, we let $\ell_r^0(x)$, $\ell_r^1(x)$, $r = -N+1, \ldots, N$, be the fundamental Hermite polynomials associated with the interpolation scheme (2.19) when $j = 0$. In other words, for $j, r = -N+1, \ldots, N$, we demand that

$$\begin{aligned} \ell_r^0(j) &= \delta_{r,j}, & (\ell_r^0)'(j) &= 0, \\ \ell_r^1(j) &= 0, & (\ell_r^1)'(j) &= \delta_{r,j}. \end{aligned}$$

Thus we have

$$P(x) = \sum_{r=-N+1}^{N} (v_{r+j}^0 \ell_r^0(x-j) + v_{r+j}^1 \ell_r^1(x-j)), \qquad x \in \mathbb{R},$$

and so

$$\mathbf{v}_{2j+1}^1 = \sum_{r=-N+1}^{N} \begin{pmatrix} \ell_r^0(\frac{1}{2}) & \ell_r^1(\frac{1}{2}) \\ (\ell_r^0)'(\frac{1}{2}) & (\ell_r^1)'(\frac{1}{2}) \end{pmatrix} \mathbf{v}_{r+j}^0.$$

We introduce the mask $\mathbf{A} = \{\mathbf{A}_r\}_{r \in \mathbb{Z}^s}$ of 2×2 matrices by setting for r satisfying $-N+1 \le r \le N$

$$\begin{aligned} \mathbf{A}_{2r}^T &:= \mathbf{I}\delta_{0,r}, \\ \mathbf{A}_{1-2r}^T &:= \begin{pmatrix} \ell_r^0(\frac{1}{2}) & \ell_r^1(\frac{1}{2}) \\ (\ell_r^0)'(\frac{1}{2}) & (\ell_r^1)'(\frac{1}{2}) \end{pmatrix}, \end{aligned}$$

and otherwise we set $\mathbf{A}_r^T = \mathbf{0}$. Therefore, the iteration takes the form

$$\mathbf{v}_l^1 = \sum_{j \in \mathbb{Z}} \mathbf{A}_{l-2j}^T \mathbf{v}_j^T.$$

Note that in this case we can take $E = \{0, 1\}$ as representers of $\mathbb{Z}/2\mathbb{Z}$. Thus, we see that

$$\mathbf{A}_0^T(\mathbf{1}) = \mathbf{I},$$

while

$$\mathbf{A}_1^T(\mathbf{1}) = \begin{pmatrix} \sum_{r=-N+1}^{N} \ell_r^0(\frac{1}{2}) & \sum_{r=-N+1}^{N} \ell_r^1(\frac{1}{2}) \\ \sum_{r=-N+1}^{N} (\ell_r^0)'(\frac{1}{2}) & \sum_{r=-N+1}^{N} (\ell_r^1)'(\frac{1}{2}) \end{pmatrix}.$$

Since

$$1 = \sum_{r=-N+1}^{N} \ell_r^0(x),$$

we infer that $\mathbf{y} = (1, 0)^T$ is the common eigenvector of the matrices $\mathbf{A}_e^T(\mathbf{1})$, $e \in \{0, 1\}$. In the special case when $N = 1$ we have

$$\mathbf{A}_0^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{A}_1^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{8} \\ -\frac{3}{2} & -\frac{1}{4} \end{pmatrix}, \qquad \mathbf{A}_{-1}^T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{8} \\ \frac{3}{2} & -\frac{1}{4} \end{pmatrix}.$$

Therefore

$$\mathbf{A}^T(z) = z^{-1} \begin{pmatrix} \frac{1}{2} + z + \frac{1}{2}z^2 & -\frac{1}{8} + \frac{1}{8}z^2 \\ \frac{3}{2} - \frac{3}{2}z^2 & -\frac{1}{4} + z - \frac{1}{4}z^2 \end{pmatrix}, \tag{2.20}$$

and so

$$\mathbf{A}_1^T(\mathbf{1}) = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

Material which relates to this case appears in [M].

Our next example is more general in spirit. We let $\mathbf{h}$ be a refinable vector and let g be a refinable scalar on $\mathbb{R}^s$. That is,

$$\mathbf{h} = sc(\mathbf{A}\Sigma\mathbf{h}), \qquad \mathbf{A} = (\mathbf{A}_\alpha)_{\alpha\in\mathbb{Z}^s},$$

and

$$g = sc(b\Sigma g), \qquad b = (b_\alpha)_{\alpha\in\mathbb{Z}^s}.$$

Define

$$\mathbf{G} := \mathbf{h} * g,$$

that is, $\mathbf{G} = (G_1, \ldots, G_N)^T$ where $G_i = h_i * g$ and $\mathbf{h} = (h_1, \ldots, h_N)^T$. Hence

$$\hat{\mathbf{h}}(y) = m^{-1}\mathbf{A}(e^{-iM^{-T}y})\hat{\mathbf{h}}(M^{-T}y) \tag{2.21}$$

and

$$\hat{g}(y) = m^{-1}b(e^{-iM^{-T}y})\hat{g}(M^{-T}y), \tag{2.22}$$

so that

$$\hat{\mathbf{G}}(y) = m^{-1}\mathbf{c}(e^{-iM^{-T}y})\hat{\mathbf{G}}(M^{-T}y),$$

where $\mathbf{c} = (\mathbf{c}_\alpha)_{\alpha\in\mathbb{Z}^s}$ and $\mathbf{c}_\alpha := m^{-1}\sum_{\beta\in\mathbb{Z}^s}\mathbf{A}_\beta b_{\alpha-\beta}$. That is, $\mathbf{G}$ is refinable. In fact, $\mathbf{G} = sc(\mathbf{c}\Sigma\mathbf{G})$.

The next example extends the important notion of cube spline and is based on some material in [CDM]. To this end, suppose that $\Omega \subset \mathbb{R}^s$ is a bounded measurable set of measure 1 satisfying

$$M\Omega = \bigcup_{e\in E_\Omega}^{m} (e + \Omega) \tag{2.23}$$

for some set E_Ω of representers of $\mathbb{Z}^s/M\mathbb{Z}^s$. As pointed out in [GM], for a given M, one can always construct a set Ω satisfying the self-similarity relation (2.23) by means of an iteration. The integer translates of Ω form a tiling of $\mathbb{R}^s$ if and only if Ω has measure one, as we will assume now. Therefore the sets $\Omega + \alpha$, $\alpha \in \mathbb{Z}^s$ are disjoint and their union is all of $\mathbb{R}^s$. Let $\mathbf{h}$ be a nontrivial continuous refinable manifold on Ω. That is, there exist $N \times N$ matrices $\mathbf{B}_e$, $e \in E_\Omega$, such that

$$\mathbf{h}(M^{-1}(x + e)) = \mathbf{B}_e^T\mathbf{h}(x), \qquad x \in \Omega, \tag{2.24}$$

(see [CDM]). For $d \le s$ and a given $d \times s$ matrix X with integer entries define the vector-valued distribution $\mathbf{F}(\cdot|X)$ by

$$\mathbf{F}(\cdot|X)f := \int_\Omega f(Xt)\mathbf{h}(t)\,dt. \tag{2.25}$$

We observe that if there exists another expanding $d \times d$ integer matrix $\tilde{M}$ such that

$$XM^{-1} = \tilde{M}^{-1}X, \tag{2.26}$$

then $\mathbf{F}$ is refinable relative to $\tilde{M}$. To prove this observation, we note that

$$\begin{aligned}\hat{\mathbf{F}}(y) &= \int_{\mathbb{R}^d} \mathbf{F}(x|X)e^{-iy\cdot x}\,dx \\ &= \int_{\Omega} e^{-iy\cdot Xt}\mathbf{h}(t)\,dt = \hat{\mathbf{h}}(X^T y), \qquad y \in \mathbb{R}^d,\end{aligned} \tag{2.27}$$

where it is to be understood that $\mathbf{h}$ has been extended to be zero outside Ω. Denoting this extension also by $\mathbf{h}$ we observe that (2.24) is equivalent to the refinement equation

$$\mathbf{h}(x) = \sum_{e\in E_\Omega} \mathbf{B}_e^T \mathbf{h}(Mx - e), \qquad x \in \mathbb{R}^s. \tag{2.28}$$

In fact, the reasoning is quite straightforward. By (2.23), $x \notin \Omega$ implies that $Mx - e \notin \Omega$ for all $e \in E_\Omega$ so that (2.28) is trivially satisfied for $x \notin \Omega$. On the other hand, again by (2.23), $x \in \Omega$ if and only if $x \in M^{-1}(\Omega + e')$ for some unique $e' \in E_\Omega$, which means that $Mx - e' \in \Omega$ holds for that particular $e' \in E_\Omega$. Noting that (2.28) therefore reduces to

$$\mathbf{h}(x) = \mathbf{B}_{e'}^T \mathbf{h}(Mx - e'),$$

which, in turn, is equivalent to (2.24) confirming (2.28). Now, from (2.28) we have

$$\hat{\mathbf{h}}(t) = m^{-1} \sum_{e\in E_\Omega} \mathbf{B}_e^T e^{-iM^{-T}t\cdot e}\hat{\mathbf{h}}(M^{-T}t), \qquad t \in \mathbb{R}^s.$$

Set

$$\mathbf{B}(t) := \frac{\tilde{m}}{m} \sum_{e\in E_\Omega} \mathbf{B}_e^T e^{-it\cdot e},$$

where $\tilde{m} = |\det \tilde{M}|$. Then

$$\hat{\mathbf{h}}(t) = \tilde{m}^{-1}\mathbf{B}(M^{-T}t)\hat{\mathbf{h}}(M^{-T}t).$$

But according to (2.27) and (2.26) we have

$$\begin{aligned}\hat{\mathbf{F}}(y|X) &= \tilde{m}^{-1}\mathbf{B}(M^{-T}X^T y)\hat{\mathbf{h}}(M^{-T}X^T y) \\ &= \tilde{m}^{-1}\mathbf{B}(X^T\tilde{M}^{-T}y)\hat{\mathbf{h}}(X^{-T}\tilde{M}^{-T}y) \\ &= \tilde{m}^{-1}\mathbf{B}(X^T\tilde{M}^{-T}y)\hat{\mathbf{F}}(\tilde{M}^{-T}y|X),\end{aligned}$$

which confirms the asserted refinability of $\mathbf{F}(\cdot|X)$.

As a simple example take $N = 1$, $M = 2I$, where I denotes the identity on $\mathbb{R}^s$, and $h = \chi_{[0,1]^s}$. Then $\mathbf{B}_e^T = \mathbf{I}$,

$$\mathbf{B}(t) = 2^{d-s} \sum_{e\in\{0,1\}^s} e^{-it\cdot e} = 2^{d-s}(1 + e^{-it_1})\cdots(1 + e^{-it_s}).$$

Thus $\mathbf{F}(\cdot|X)$ is in this case the cube spline [CDM].

Aside from the fact that the $N \times N$ matrix subsymbols $\mathbf{A}_e(z), e \in E$, evaluated at $\mathbf{1}$ share the same left eigenvector for the common eigenvalue 1, stability and linear independence imply further noteworthy properties of $\mathbf{A}$ which can be expressed in terms of $\mathbf{A}_e(z)$. First, the arguments given in [M1] for the case of $M = 2I$ carry over to the general case of expanding scaling matrices M without any change to establish the following results.

Proposition 2.2. *Suppose* $\mathbf{h} \in \mathcal{L}_2^N$ *is* $\mathbf{A}$*-refinable and stable. Then the* $N \times mN$ *block matrix* $\mathbf{A}^0(z)$ *which consists of* m, $N \times N$ *blocks formed from the* m, $N \times N$ *matrices* $\mathbf{A}_e(z)$, $e \in E$, *has full rank* N *for all* $z \in T^s := \{z \in \mathbb{C}^s\colon |z_i| = 1,\ i = 1, \dots, s\}$.

Proposition 2.3. *Suppose* $\mathbf{h} \in L_2^N(\mathbb{R}^s)$ *has compact support, is* $\mathbf{A}$*-refinable, and has linearly independent integer translates. Then* $\mathbf{A}^0(z)$ *has full rank for all* $z \in (\mathbb{C}\backslash\{0\})^s$.

Likewise, the arguments in [M1] show that, when $\mathbf{h} \in \mathcal{L}_2^N$ has compact support, is stable, and $\mathbf{A}$-refinable, then the entries of $\mathbf{A}_\alpha$ decay exponentially as $|\alpha|$ tends to infinity. More can be said when the translates of $\mathbf{h}$ are linearly independent. To explain this, let, for $\mathbf{v} \in L_p^{N\times l}(\mathbb{R}^s)$, $\mathbf{w} \in L_q^{N\times l}(\mathbb{R}^s)$, $1/p + 1/q = 1$,

$$\langle \mathbf{v}, \mathbf{w}\rangle := \int_{\mathbb{R}^s} \mathbf{v}(x)\mathbf{w}(x)^*\, dx,$$

where we write for any $N \times n$ matrix $\mathbf{C}$

$$\mathbf{C}^* := \overline{\mathbf{C}}^T.$$

Any function $\mathbf{g} \in \mathcal{L}_2^N$ which satisfies

$$\langle \mathbf{g}, \mathbf{h}(\cdot - \alpha)\rangle = \delta_{0,\alpha}\mathbf{I},$$

is called *dual* to $\mathbf{h}$. The following observations extend corresponding known facts for the case $N = 1$ (see, e.g., [CDP]):

Proposition 2.4. *Suppose* $\mathbf{g}$ *and* $\mathbf{h}$ *are dual. Then the following holds*:

(i) *if* $\mathbf{h}$ *has compact support, then its integer translates are lineraly independent*;
(ii) *if both* $\mathbf{g}$ *and* $\mathbf{h}$ *have compact support and* $\mathbf{h}$ *is* $\mathbf{A}$*-refinable, then* $\mathbf{A}$ *has finite support*; *and*
(iii) *if* $\mathbf{h} \in \mathcal{L}_2^N$ *has compact support, linearly independent integer translates, and is* $\mathbf{A}$*-refinable with finitely supported mask* $\mathbf{A}$, *then there exists a finitely supported mask* $\mathbf{D}$ *such that the vector field*

$$\mathbf{g} := sc(\mathbf{D}\Sigma\mathbf{h}) \tag{2.29}$$

is dual to $\mathbf{h}$.

Proof. (i) and (ii) are obvious. As for (iii), let

$$\mathbf{H}_\beta := \langle \mathbf{h}, \mathbf{h}(\cdot + \beta)\rangle, \qquad \mathbf{B}_\beta := \sum_{\mu\in\mathbb{Z}^s} (\mathbf{A}_\mu \mathbf{H}_{\mu-\beta})^*.$$

Suppose that for some $z \in (\mathbb{C}\backslash\{0\})^s$ and some $\mathbf{y} \in \mathbb{C}^N$ one has $\mathbf{B}_e(z)\mathbf{y} = 0$, for all $e \in E$. Since, by (2.2)

$$\begin{aligned}\Big\langle \mathbf{h}(M \cdot - e), \sum_{\alpha\in\mathbb{Z}^s} \overline{z}^{-\alpha}\mathbf{h}(\cdot - \alpha)\Big\rangle &= m^{-1} \sum_{\alpha\in\mathbb{Z}^s} z^{-\alpha} \sum_{\mu\in\mathbb{Z}^s} \mathbf{H}_{e-\mu}\mathbf{A}^*_{\mu-M\alpha}\\ &= m^{-1} \sum_{\alpha\in\mathbb{Z}^s} z^{-\alpha}\mathbf{B}_{e-M\alpha}\\ &= m^{-1}\mathbf{B}_e(z),\end{aligned}$$

one would get

$$\langle \mathbf{h}(M\cdot-\beta), \sum_{\alpha\in\mathbb{Z}^s} \overline{z}^{-\alpha}\mathbf{h}(\cdot-\alpha)\rangle = 0, \qquad \beta\in\mathbb{Z}^s,$$

contradicting linear independence. Thus we have proved:

Remark 2.1. The $(N \times mN)$-matrix $\mathbf{B}^0(z)$ whose first (block) row consists of the blocks $\mathbf{B}_e(z)$, $e \in E$, has full rank for all $z \in (\mathbb{C}\backslash\{0\})^s$.

By the compact support of $\mathbf{h}$ and since, by assumption, $\mathbf{A}$ is finitely supported, all entries of $\mathbf{B}^0(z)$ are Laurent polynomials. Hence the Hilbert Nullstellensatz yields the following result, see [M2].

Remark 2.2. There exists a finitely supported mask $\mathbf{D}$ which is dual to $\mathbf{B}$, i.e., the $(N \times mN)$-matrix $\mathbf{D}^0(z)$ which consist of the row of blocks $\mathbf{D}_e(z)$, $e \in E$, satisfies

$$\mathbf{D}^0(z)\mathbf{B}^0(z^{-1})^T = m\mathbf{I}.$$

We will prove this result for the convenience of the reader. But first we show how it yields (iii) of Proposition 2.4.

To this end, suppose $\mathbf{g}$ has the form (2.29). By the refinement equation (2.2) for $\mathbf{h}$, one obtains

$$\begin{aligned}(2.30)\qquad \langle \mathbf{g}, \mathbf{h}(\cdot-\alpha)\rangle &= \sum_{\beta,\mu\in\mathbb{Z}^s} \langle \mathbf{D}_\beta \mathbf{h}(M\cdot-\beta), \mathbf{A}_{\mu-M\alpha}\mathbf{h}(M\cdot-\mu)\rangle \\ &= m^{-1}\sum_{\beta,\mu\in\mathbb{Z}^s} \mathbf{D}_\beta \mathbf{H}_{\beta-\mu}\mathbf{A}^*_{\mu-M\alpha} = m^{-1}\sum_{\beta\in\mathbb{Z}^s}\mathbf{D}_\beta\mathbf{B}_{\beta-M\alpha}.\end{aligned}$$

Thus

$$\langle \mathbf{g}, \mathbf{h}(\cdot-\alpha)\rangle = \delta_{0,\alpha}\mathbf{I}$$

if and only if

$$m\mathbf{I} = \sum_{e\in E}\mathbf{D}_e(z^{-M})\mathbf{B}_e(z^M).$$

By Remark 2.2 this latter equation has a solution $\mathbf{D}$. This completes the proof of (iii). ■

The claim made in Remark 2.2 follows from the following result (see [M2, Theorem 2.3]).

Proposition 2.5. *Let* $\mathbf{A}(z)$, $z \in (\mathbb{C}\backslash\{0\})^s$, *be a* $(k \times \ell)$ *matrix of Laurent polynomials such that* $k \le \ell$ *and*

$$\operatorname{rank}\mathbf{A}(z) = k, \quad \forall z \in (\mathbb{C}\backslash\{0\})^s .$$

Then there exists an $\ell \times k$ *matrix* $\mathbf{C}(z)$ *of Laurent polynomials such that*

$$\mathbf{A}(z)\mathbf{C}(z) = \mathbf{I}, \quad z \in (\mathbb{C}\backslash\{0\})^s.$$

In the above equation $\mathbf{I}$ stands for the $(k \times k)$ identity matrix. Before we prove this proposition, as a means to illustrate the result, we note some special cases. The first case to consider is $k = 1$. In this case, Proposition 2.5 asserts that whenever $a_1(z), \dots, a_\ell(z)$ are Laurent polynomials with no common zeros there are Laurent polynomials $c_1(z), \dots, c_\ell(z)$ such that

$$a_1(z)c_1(z) + \cdots + c_\ell(z)a_\ell(z) = 1, \qquad z \in (\mathbb{C}\backslash\{0\})^s.$$

This is essentially Hilbert's Nullstellensatz, see [W]. As the Nullstellensatz is usually stated for polynomials we show how to reduce the current situation to that case. For this purpose, we write the Laurent polynomial $a_j(z)$ in the form

$$a_j(z) = \sum_{\alpha \in \mathbb{Z}^s} a_\alpha^j \prod_{\alpha_i > 0} z_i^{\alpha_i} \cdot \prod_{\alpha_i < 0} z_i^{\alpha_i}, \qquad z = (z_1, \dots, z_s),$$

for some constants a_α^j and define polynomials on $\mathbb{C}^{2s}$ by

$$b_j(z, \zeta) = \sum_{\alpha \in \mathbb{Z}^s} a_\alpha^j \prod_{\alpha_i > 0} z_i^{\alpha_i} \prod_{\alpha_i < 0} \zeta_i^{-\alpha_i}, \qquad j = 1, \dots, \ell,$$

so that

$$a_j(z) = b_j(z, z^{-1}), \qquad z \in (\mathbb{C}\backslash\{0\})^s.$$

Also, set

$$u_j(z, \zeta) = 1 - z_j\zeta_j, \qquad j = 1, \dots, \ell,$$

and note that if (z, ζ) is a common zero of $u_1, \dots, u_\ell,\ b_1, \dots, b_\ell$, then $z \in (\mathbb{C}\backslash\{0\})^s$ and z is a common zero of $a_1, \dots, a_\ell$ and $u_1, \dots, u_\ell, b_1, \dots, b_\ell$ have no common zeros on $\mathbb{C}^{2s}$. Hence by the Nullenstellensatz there are polynomials $d_1(z, \zeta), \dots, d_\ell(z, \zeta), c_1(z, \zeta), \dots, c_\ell(z, \zeta)$ such that for all $z, \zeta \in \mathbb{C}^s$

$$1 = \sum_{i=1}^{\ell} (1 - z_i\zeta_i) d_i(z, \zeta) + \sum_{i=1}^{\ell} c_i(z, \zeta) b_i(z, \zeta).$$

Now, for any $z \in (\mathbb{C}\backslash\{0\})^s$, choose $\zeta = z^{-1}$ above and obtain

$$1 = \sum_{i=1}^{\ell} c_i(z, z^{-1}) a_i(z)$$

which proves Proposition 2.5 in this case.

The next case which is elementary is $k = \ell$. For this choice of k it is easy to see that our hypothesis implies that

$$\det \mathbf{A}(z) = \rho z^\alpha, \qquad z \in (\mathbb{C}\backslash\{0\})^s$$

for some $\rho \in \mathbb{C}\backslash\{0\}$ and $\alpha \in \mathbb{Z}^s$ and so Cramer's rule proves the result.

There remains the principal case when $1 < k < \ell$. In this case, we consider for every $1 \le j_1 < \cdots < j_k \le \ell$ the Laurent polynomial

$$\mathbf{A}(z) \begin{pmatrix} 1, \dots, k \\ j_1, \dots, j_k \end{pmatrix}$$

which is the minor of $A(z)$ corresponding to columns $j_1, \ldots, j_k$. According to our hypothesis these Laurent polynomials have no common zeros. Hence by the Nullenstellensatz there are Laurent polynomials $q_{j_1,\ldots,j_k}(z)$ such that

$$\sum_{1\le j_1<\cdots<j_k\le \ell} q_{j_1,\ldots,j_k}(z)\mathbf{A}(z)\begin{pmatrix}1,\ldots,k\\ j_1,\ldots,j_k\end{pmatrix}=1, \qquad z\in(\mathbb{C}\backslash\{0\})^s.$$

To make use of this result, we recall that for any $(r\times r)$ matrix B, the matrix adj B stands for the matrix $B^{-1}\det B$. Hence the elements of adj B are by Cramer's rule polynomials in the elements of B. Next, for every $1\le j_1<\cdots<j_k\le\ell$ we introduce the $\ell\times k$ matrix $\mathbf{D}_{j_1,\ldots,j_k}$ defined as

$$(\mathbf{D}_{j_1,\ldots,j_k})_{\mu\nu}=\delta_{\mu j_\nu}, \qquad \mu=1,\ldots,\ell, \quad \nu=1,\ldots,k,$$

and define the $\ell\times k$ matrix of Laurent polynomials

$$\mathbf{C}(z)=\sum_{1\le j_1<\cdots<j_k\le\ell} q_{j_1,\ldots j_k}(z)\mathbf{D}_{j_1,\ldots,j_k}(z)\,\mathrm{adj}(\mathbf{A}(z)\mathbf{D}_{j_1,\ldots,j_k}(z)).$$

Then for every $z\in(\mathbb{C}\backslash\{0\})^s$

(2.31) $$\mathbf{A}(z)\mathbf{C}(z)=\left(\sum_{1\le j_1<\cdots<j_k\le\ell} q_{j_1,\ldots,j_k}(z)\mathbf{A}(z)\begin{pmatrix}1,\ldots,k\\ j_1,\ldots,j_k\end{pmatrix}\right)\mathbf{I}=\mathbf{I}.$$

2.5. *Multiresolution*

With the prerequisites from Section 2.4 at hand, one may employ the arguments from [M1] to establish the following fact.

Theorem 2.1. *Any* $\mathbf{h}\in\mathcal{L}_2^N$ *which is stable and refinable admits multiresolution, i.e.,*

(i)

$$S^k(\mathbf{h})\subset S^{k+1}(\mathbf{h}), \qquad k\in\mathbb{Z}.$$

(ii)

$$\overline{\bigcup_{k\in\mathbb{Z}} S^k(\mathbf{h})}=L_2(\mathbb{R}^s), \qquad \bigcap_{k\in\mathbb{Z}} S^k(\mathbf{h})=\{0\}.$$

This result could certainly be established under weaker assumptions. But, as mentioned before, we are mainly interested in compactly supported generators $\mathbf{h}$.

2.6. *Discrete Biorthogonality Relations*

Our main goal is to construct for a multiresolution of the above type a Riesz basis consisting of biorthogonal wavelets of compact support.

We begin by collecting a few auxiliary facts. Recall from [DM2] that for $\alpha\in\mathbb{Z}^s$

(2.32) $$\zeta_\alpha:=e^{2\pi i M^{-T}\alpha}, \qquad \hat{\zeta}_\alpha:=e^{2\pi i M^{-1}\alpha},$$

solve the systems

(2.33) $$z^M=1, \qquad z^{M^T}=1,$$

respectively. Let E, $\hat{E}$ denote any set of representers of $\mathbb{Z}^s/M^T\mathbb{Z}^s$, $\mathbb{Z}^s/M\mathbb{Z}^s$, respectively. The following relations have been shown in [CL].

Lemma 2.1. *One has*

$$\sum_{e\in E}\zeta_{e'}^{e}=m\delta_{e',0},\qquad e'\in\hat{E},$$

and, consequently,

$$\sum_{e\in\hat{E}}\zeta_{e}^{e'}\zeta_{e}^{-e''}=m\delta_{e',e''},\qquad e',e''\in E.$$

This leads to the following inversion of the splitting formula (2.4). Here and in the sequel we will always tacitly assume that Laurent series under consideration are well defined. For instance, when $\mathbf{c}$ belongs to $\ell_1(\mathbb{Z}^s)$ the variable z can be restricted to the torus T^s while for finitely supported $\mathbf{c}$ the expressions make sense for all $z\in(\mathbb{C}\backslash\{0\})^s$.

Lemma 2.2. *Defining* $\zeta z:=(\zeta_1 z_1,\ldots,\zeta_s z_s)^T$, *one has for any* $c\in\ell_1(\mathbb{Z}^s)$

$$z^{-e}\mathbf{c}_e(z^M)=m^{-1}\sum_{e'\in\hat{E}}\zeta_{e'}^{-e}\mathbf{c}(\zeta_{e'}z),\qquad e\in E. \tag{2.34}$$

Proof. Substituting (2.4) into the right-hand side of (2.34), yields, in view of (2.33),

$$m^{-1}\sum_{e'\in\hat{E}}\zeta_{e'}^{-e}\left(\sum_{e''\in E}\zeta_{e'}^{e''}z^{e''}\mathbf{c}_{e''}(z^M)\right)=m^{-1}\sum_{e''\in E}z^{e''}\left(\sum_{e'\in\hat{E}}\zeta_{e'}^{-e}\zeta_{e'}^{e''}\right)\mathbf{c}_{e''}(z^M).$$

The assertion now follows from Lemma 2.1. ∎

We will make frequent use of the following relations.

Lemma 2.3. *The following relations are equivalent. Let* $a,b,c\in\ell_1(\mathbb{Z}^s)$:

(i)

$$c_\alpha=\sum_{\beta\in\mathbb{Z}^s}a_{\beta+M\alpha}b_\beta,\qquad \alpha\in\mathbb{Z}^s.$$

(ii)

$$c(z)=\sum_{e\in E}a_e(z)b_e(z^{-1}),\qquad z\in T^s.$$

(iii)

$$c(z^M)=m^{-1}\sum_{e\in\hat{E}}a(\zeta_e z)b(\zeta_{-e}z^{-1}),\qquad z\in T^s.$$

Proof. Multiplying both sides of (i) by $z^{M\alpha}$ and summing over $\alpha\in\mathbb{Z}^s$, gives

$$c(z^M)=\sum_{\alpha\in\mathbb{Z}^s}\sum_{e\in E}\sum_{\beta\in\mathbb{Z}^s}a_{e+M(\beta+\alpha)}z^{M(\alpha+\beta)}b_{e+M\beta}z^{-M\beta},$$

which proves the equivalence of (i) and (ii). Now Lemma 2.2 provides

$$\sum_{e\in E} a_e(z^M)b_e(z^{-M}) \;=\; m^{-2}\sum_{e\in E}\left(\sum_{e'\in\hat E} z^e\zeta_{e'}^{-e}a(\zeta_{e'}z)\right)\left(\sum_{e''\in\hat E} z^{-e}\zeta_{-e''}^{-e}b(\zeta_{-e''}z^{-1})\right)$$
$$= m^{-2}\sum_{e',e''\in\hat E}\left(\sum_{e\in E}\zeta_{e'}^{-e}\zeta_{-e''}^{-e}\right)a(\zeta_{e'}z)b(\zeta_{-e''}z^{-1}).$$

By Lemma 2.1, this completes the proof. ■

In the following, let $\mathbf{h}_0, \mathbf{g}_0 \in \mathcal{L}_2^N$ be $\mathbf{A}^0, \mathbf{B}^0$-refinable functions, respectively. Suppose additional functions $\mathbf{h}_e, \mathbf{g}_e,\ e \in E$, and matrices $\mathbf{A}^e, \mathbf{B}^e$ are given such that

(2.35) $$\mathbf{h}_e \;=\; sc(\mathbf{A}^e\Sigma\mathbf{h}_0), \qquad e \in E,$$

(2.36) $$\mathbf{g}_e \;=\; sc(\mathbf{B}^e\Sigma\mathbf{g}_0), \qquad e \in E\ .$$

Proposition 2.6. *Suppose the functions* $\mathbf{h}_e, \mathbf{g}_e,\ e \in E$, *defined in* (2.35), (2.36), *form a* biorthogonal system, *i.e.*,

(2.37) $$\langle \mathbf{h}_e, \mathbf{g}_{e'}(\cdot - \alpha)\rangle = \delta_{e,e'}\delta_{0,\alpha}\mathbf{I}, \qquad e, e' \in E, \quad \alpha \in \mathbb{Z}^s.$$

Then one has the following discrete biorthogonality relations.

(2.38) $$\sum_{e\in E}\mathbf{A}_e^{e'}(z)(\mathbf{B}_e^{e''})^*(z^{-1}) = m\delta_{e',e''}\mathbf{I}, \qquad e', e'' \in E,$$

or, equivalently,

(2.39) $$\sum_{e\in\hat E}\mathbf{A}^{e'}(\zeta_e z)(\mathbf{B}^{e''})^*(\zeta_{-e}z^{-1}) = m^2\delta_{e',e''}\mathbf{I}, \qquad e', e'' \in E.$$

Proof. As in (2.30) one obtains

(2.40) $$\delta_{e',e''}\delta_{0,\alpha}\mathbf{I} = \langle \mathbf{h}_{e'}, \mathbf{g}_{e''}(\cdot - \alpha)\rangle = m^{-1}\sum_{\beta,\mu\in\mathbb{Z}^s}\mathbf{A}_\beta^{e'}\delta_{0,\mu-\beta}\mathbf{I}(\mathbf{B}_{\mu-M\alpha}^{e''})^*.$$

The assertion follows now from Lemma 2.3. ■

Defining the $(mN \times mN)$-matrices

(2.41) $$\mathbf{A}(z) := (\mathbf{A}_{e'}^{e}(z))_{e,e'\in E}, \qquad \mathbf{B}(z) := (\mathbf{B}_{e'}^{e}(z^{-1}))_{e,e'\in E},$$

condition (2.38) is equivalent to

(2.42) $$\mathbf{A}(z)\mathbf{B}^*(z) = m\mathbf{I}.$$

Proposition 2.4 suggests considering compactly supported functions $\mathbf{h} \in L_2^N(\mathbb{R}^s)$ with linearly independent integer translates. Such functions will be called *admissible*.

Proposition 2.7. *Let* $\mathbf{h}_0$ *be admissible and* $\mathbf{A}^0$*-refinable with finitely supported mask* $\mathbf{A}^0$. *Then there exist additional finitely supported masks* $\mathbf{A}^e,\ e \in E_* := E\backslash\{0\}$, *and* $\mathbf{B}^e,\ e \in E$, *such that the discrete biorthogonality conditions* (2.38) *or* (2.39) *hold.*

Proof. By Proposition 2.3 the matrix $\mathbf{A}^0(z)$ has full rank for all $z \in (\mathbb{C}\backslash\{0\})^s$. Hence, by the Quillen–Suslin theorem it can be extended to a $(mN \times mN)$ matrix $\mathbf{A}(z)$ whose entries are Laurent polynomials and whose determinant equals one for all $z \in (\mathbb{C}\backslash\{0\})^s$. Setting

$$\mathbf{B}(z)^* := m\mathbf{A}(z)^{-1}, \tag{2.43}$$

and defining the masks $\mathbf{B}^e$, $e \in E$, by (2.41), completes the proof. ■

The collection of masks $\mathbf{A}^e, \mathbf{B}^e$, $e \in E$, satisfying (2.38), will be called a *discrete biorthogonal system* (DBS). In particular, $\mathbf{A}^0$ and $\mathbf{B}^0$ are also said to be *dual* to each other. Some properties of $\mathbf{A}^0$ carry over to $\mathbf{B}^0$ when the $\mathbf{A}^e, \mathbf{B}^e$ form a DBS.

Proposition 2.8. *Let* $\mathbf{h}$ *be an admissible* $\mathbf{A}^0$*-refinable function. Suppose the* $\mathbf{A}^e, \mathbf{B}^e$, $e \in E$, *form a DBS and let* $\mathbf{y}$ *denote again the common left eigenvector of the* $\mathbf{A}^0_e(1)$, $e \in E$, *with eigenvalue* 1 (2.16). *Then*

$$\mathbf{A}^0(\zeta_e)^T\mathbf{y} = m\delta_{0,e}\mathbf{y}, \qquad e \in \hat{E}, \tag{2.44}$$

and

$$\overline{\mathbf{B}^e(\mathbf{1})}\mathbf{y} = m\delta_{0,e}\mathbf{y}, \qquad e \in E. \tag{2.45}$$

Proof. By (2.4), one has

$$\mathbf{y}^T\mathbf{A}^0(\zeta_e) = \sum_{e'\in E}\zeta_e^{e'}\mathbf{y}^T\mathbf{A}^0_{e'}(\mathbf{1}) = \left(\sum_{e'\in E}\zeta_e^{e'}\right)\mathbf{y}^T = m\delta_{0,e}\mathbf{y}^T, \qquad e \in \hat{E},$$

where we have used (2.16) and Lemma 2.1. This proves (2.44). Now (2.39) and (2.44) give

$$\begin{aligned} m^2\,\delta_{0,e}\mathbf{y}^T &= \sum_{e'\in\hat{E}}\mathbf{y}^T\mathbf{A}^0(\zeta_{e'})(\mathbf{B}^e)^*(\zeta_{-e'}) \\ &= m\mathbf{y}^T(\mathbf{B}^e)^*(\mathbf{1}), \end{aligned}$$

which completes the proof. ■

It is worthwhile recording the scalar case, $N = 1$, of the above result.

Corollary 2.1. *Let* h *be a* a^0*-refinable admissible function in* $L_2(\mathbb{R}^s)$. *Suppose that the* a^e, b^e, $e \in E$, *form a DBS. Then the following relations hold*:

(i)

$$a^0_e(1) = 1, \qquad e \in E,$$

so that, in particular,

$$a^0(1) = m.$$

(ii)

$$a^0(\zeta_e) = m\delta_{0,e}, \qquad e \in \hat{E}.$$

(iii)

$$b^e(1) = m\delta_{0,e}, \qquad e \in E.$$

Thus, given any a^0-refinable admissible h_0, we can always find a DBS of masks a^e, $e \in E_*$, b^e, $e \in E$, satisfying certain necessary conditions to permit the existence of an associated biorthogonal system of refinable functions. The question arises whether for a given h_0, a^0 one can always find a dual g_0, b^0 such that g_0 is also admissible. The following characterization of duality, which is well known in the scalar case $N = 1$, will be helpful.

Lemma 2.4. $\mathbf{h}, \mathbf{g} \in \mathcal{L}_2^N$ *are dual if and only if*

$$[\hat{\mathbf{h}}, \hat{\mathbf{g}}] := \sum_{\alpha \in \mathbb{Z}^s} \hat{\mathbf{h}}(\cdot + 2\pi\alpha)\hat{\mathbf{g}}(\cdot + 2\pi\alpha)^* = \mathbf{I}. \tag{2.46}$$

Proof. As in [JM1] we can show that the left-hand side of (2.46) is well defined. Moreover, as in the scalar case, one has

$$\begin{aligned} \langle \mathbf{h}, \mathbf{g}(\cdot + \alpha)\rangle &= (2\pi)^{-s} \int_{\mathbb{R}^s} \hat{\mathbf{h}}(\omega)\hat{\mathbf{g}}(\omega)^* e^{i\omega\cdot\alpha}\, d\omega \\ &= (2\pi)^{-s} \int_{[-\pi,\pi]^s} [\hat{\mathbf{h}}, \hat{\mathbf{g}}](\omega) e^{i\omega\cdot\alpha}\, d\omega. \end{aligned} \tag{2.47}$$

Hence

$$[\hat{\mathbf{h}}, \hat{\mathbf{g}}](\omega) = \sum_{\alpha \in \mathbb{Z}^s} \langle \mathbf{h}, \mathbf{g}(\cdot + \alpha)\rangle e^{-i\alpha\cdot\omega}, \tag{2.48}$$

whence the assertion follows. ∎

Now define for any finitely supported mask $\mathbf{A}$ and any $\mathbf{g} \in L_2^N(\mathbb{R}^s)$ the operator

$$T_{\mathbf{A}}\mathbf{g} := \sum_{\alpha \in \mathbb{Z}^s} \mathbf{A}_\alpha \mathbf{g}(M \cdot - \alpha). \tag{2.49}$$

Note that

$$(T_{\mathbf{A}}\mathbf{g})^\wedge(\omega) = m^{-1}\mathbf{A}(e^{-iM^{-T}\omega})\hat{\mathbf{g}}(M^{-T}\omega). \tag{2.50}$$

Moreover, since M is expanding there exists some $\rho \in (0, 1)$ and some norm $\|\cdot\|$ on $\mathbb{R}^s$ such that

$$\|M^{-1}x\| \le \rho\|x\|, \qquad x \in \mathbb{R}^s. \tag{2.51}$$

Let

$$B_r := \{x \in \mathbb{R}^s : \|x\| \le r\}.$$

Lemma 2.5. *Suppose* $\operatorname{supp}(\mathbf{A}) \subseteq B_r$ *and* $\operatorname{supp}(\mathbf{g}) \subseteq B_R$ *where*

$$R := \frac{\rho r'}{1 - \rho} \tag{2.52}$$

for some $r' \ge r$. *Then*

$$\operatorname{supp}(T_{\mathbf{A}}^k\mathbf{g}) \subseteq B_R, \qquad k \in \mathbb{N}. \tag{2.53}$$

Proof. Clearly, $Mx - \alpha \in B_R$ for $\alpha \in B_r$ implies $x \in M^{-1}(B_R + B_r) \subseteq M^{-1}B_{R+r}$ so that, by (2.51),

$$\|x\| \le \rho(R + r) = \rho\left(\frac{\rho r'}{1-\rho} + \frac{(1-\rho)r}{1-\rho}\right) \le R,$$

i.e., $x \in B_R$. Thus, when $\operatorname{supp}(\mathbf{g}) \subseteq B_R$ one has $\operatorname{supp}(T_{\mathbf{A}}\mathbf{g}) \subseteq B_R$, which completes the proof. ∎

Now suppose $\mathbf{h}$ is an admissible $\mathbf{A}$-refinable function and let $\mathbf{B}$ be a finitely supported mask which is dual to $\mathbf{A}$. Moreover, let $\mathbf{g} \in L_2^N(\mathbb{R}^s)$ be some compactly supported function which is dual to $\mathbf{h}$. Recall that the existence of such a $\mathbf{g}$ is asserted by Proposition 2.4(iii).

Lemma 2.6. *For* $\mathbf{h}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{g}$ *as above one has*

$$[\hat{\mathbf{h}}, \widehat{T_{\mathbf{B}}^n\mathbf{g}}] = \mathbf{I}, \tag{2.54}$$

i.e., $\mathbf{h}$ *and* $T_{\mathbf{B}}^n\mathbf{g}$ *are dual for all* $n \in \mathbb{N}$. *Moreover, there exists some bounded domain* $\Omega \subset \mathbb{R}^s$ *such that*

$$\operatorname{supp}(T_{\mathbf{B}}^n\mathbf{g}) \subseteq \Omega, \qquad n \in \mathbb{N}. \tag{2.55}$$

Proof. By assumption and Lemma 2.4 one has $[\hat{\mathbf{h}}, \hat{\mathbf{g}}^{(0)}] = \mathbf{I}$, where we set

$$\mathbf{g}^{(n)} := T_{\mathbf{B}}^n\mathbf{g}.$$

Thus by (2.47), (2.51), and (2.3), one has

$$\begin{aligned}
\langle \mathbf{h}, \mathbf{g}^{(n+1)}(\cdot + \beta)\rangle &= (2\pi)^{-s}\int_{[-\pi,\pi]^s} e^{i\beta\cdot\omega}[\hat{\mathbf{h}}, \hat{\mathbf{g}}^{(n+1)}](\omega)\,d\omega \\
&= m^{-2}(2\pi)^{-s}\int_{[-\pi,\pi]^s} e^{i\beta\cdot\omega} \\
&\quad \times \sum_{\alpha\in\mathbb{Z}^s} \mathbf{A}(e^{-iM^{-T}(\omega+2\pi\alpha)})\hat{\mathbf{h}}(M^{-T}(\omega+2\pi\alpha)) \\
&\quad \times \hat{\mathbf{g}}^{(n)}(M^{-T}(\omega+2\pi\alpha))^*\mathbf{B}(e^{-iM^{-T}(\omega+2\pi\alpha)})^*\,d\omega \\
&= m^{-2}(2\pi)^{-s}\int_{[-\pi,\pi]^s} e^{i\beta\cdot\omega} \\
&\quad \times \sum_{e\in\hat{E}} \mathbf{A}(e^{i(M^{-T}\omega)}\zeta_e)[\hat{\mathbf{h}}, \hat{\mathbf{g}}^{(n)}](M^{-T}(\omega+2\pi e)\mathbf{B}(e^{i(M^{-T}\omega)}\zeta_e)^* d\omega.
\end{aligned}$$

Since for $z = e^{-iM^{-T}\omega}$ one has $\mathbf{B}(\zeta_e z)^* = \mathbf{B}^*(\zeta_{-e}z^{-1})$ relation (2.39) in Proposition 2.6 gives, by induction,

$$\langle \mathbf{h}, \mathbf{g}^{(n+1)}(\cdot + \beta)\rangle = (2\pi)^{-s}\int_{[-\pi,\pi]^s} \mathbf{I}e^{i\beta\cdot\omega}\,d\omega = \delta_{0,\beta}\mathbf{I},$$

which proves (2.54). The rest follows from Lemma 2.5. ∎

As mentioned above, our ultimate goal is to find a converse to Proposition 2.6, i.e., given an admissible $\mathbf{A}^0$-refinable function $\mathbf{h}_0$, when does the associated DBS $\mathbf{A}^e, \mathbf{B}^e, e \in E$, give rise to a $\mathbf{B}^0$-refinable admissible function $\mathbf{g}_0$ which is dual to $\mathbf{h}_0$. A first step would be to ask the following question: Suppose there exists some compactly supported $\mathbf{g}_0$ which is $\mathbf{B}^0$-refinable: Is $\mathbf{g}_0$ dual to $\mathbf{h}_0$?

Let us briefly discuss this question for the scalar case $N = 1$ first, in order to bring out the relevant issues which need to be addressed. To this end, note first that, by Corollary 2.1, $b(1) = m$ so that Proposition 2.3 in [DM3] ensures that the products

$$G_k(u) := \prod_{j=1}^{k} m^{-1} b(e^{-i(M^{-T})^j u}) \tag{2.56}$$

converge uniformly on compact sets to some entire function which we denote by $G(u)$. On the other hand, since g_0 has compact support it also belongs to $L_1(\mathbb{R}^s)$ so that $\hat{g}_0$ is continuous and $\hat{g}_0(0) = 1$. Therefore $G = \hat{g}_0$. Now pick any compactly supported function $g \in L_2(\mathbb{R}^s)$ which is dual to h. By Proposition 2.4(iii) such a function exists. Since by duality $\hat{h}(0)\overline{\hat{g}(0)} = 1$ we conclude that also $\hat{g}(0) = 1$. Thus, by (2.50), $\hat{g}^{(n)} := (T_{\mathbf{b}}^n g)^\wedge$ converge uniformly on compact sets to $\hat{g}_0$. By Lemma 2.6, there exists for every $\varepsilon > 0$ and any $n \in \mathbb{N}$ some $l \in \mathbb{N}$ such that

$$\left| 1 - \sum_{\alpha \in B_l \cap \mathbb{Z}^s} \hat{h}(u + 2\pi\alpha)\overline{\hat{g}^{(n)}(u + 2\pi\alpha)} \right| < \varepsilon \qquad \text{for all} \quad u \in [0, 2\pi]^s.$$

Hence g_0 would be seen to be dual to h if we could ensure that

$$\sum_{\alpha \in B_l \cap \mathbb{Z}^s} \hat{h}(u + 2\pi\alpha)\overline{\hat{g}^{(n)}(u + 2\pi\alpha)}$$

get uniformly close to $\sum_{\alpha \in B_l \cap \mathbb{Z}^s} \hat{h}(u+2\pi\alpha)\overline{\hat{g}_0(u + 2\pi\alpha)}$ and hence to $[\hat{h}, \hat{g}_0]$. Evidently this requires more information about the convergence of the products G_k in (2.56). This, in turn, is closely related to convergence of certain *stationary subdivision schemes* which we will study in the following section.

3. Stationary Subdivision

As pointed out at the end of the last section the convergence of certain sequences played a crucial role for proving the existence of dual refinable functions. These convergence issues are closely related to the notion of *stationary subdivision schemes*. An extensive treatment of such schemes is given in [CDM], however, for the special scaling matrix $M = 2I$ and mainly for the scalar case $N = 1$. In this section we will present some results on stationary subdivision adapted to the present more general setting. Since we find this subject of interest in its own right we will allow the discussion to go at times beyond the particular needs of the present context. As before, M will be some fixed expanding scaling matrix. In analogy to scaling by powers of two one expects the refinement equation (2.2) to be closely related to the subdivision operator $S = S_{\mathbf{A},M}\colon \ell_p^N(\mathbb{Z}^s) \to \ell_p^N(\mathbb{Z}^s)$ defined by

$$(S_{\mathbf{A},M}\mathbf{c})_\alpha := \sum_{\beta \in \mathbb{Z}^s} \mathbf{A}^T_{\alpha - M\beta} \mathbf{c}_\beta. \tag{3.1}$$

Again, we will be mainly interested in finitely supported matrix-valued masks $\mathbf{A}$ but, unless otherwise stated, it will be assumed that the bi-infinite matrix $(\mathbf{A}_{\alpha-M\beta})_{\alpha,\beta\in\mathbb{Z}^s}$ maps $\ell_p^N(\mathbb{Z}^s)$ into itself for p under consideration.

In this regard, we note the following proposition.

Proposition 3.1. *Let* $\mathbf{A} = \{\mathbf{A}_\alpha\}_{\alpha\in\mathbb{Z}^s} \in \ell_1^{N\times N}(\mathbb{Z}^s)$, *then* $S_{\mathbf{A},M}\colon \ell_p^N(\mathbb{Z}^s) \to \ell_p^N(\mathbb{Z}^s)$ *is a bounded linear operator.*

Proof. The proof is based on Young's inequality (see [F, Theorem 6.18]) which states that for any $d \in \ell_1(\mathbb{Z}^s)$, $c \in \ell_p(\mathbb{Z}^s)$, then the convolution $d * c$, defined to have components

$$(d * c)_\alpha := \sum_{\beta\in\mathbb{Z}^s} d_\beta c_{\alpha-\beta}, \qquad \alpha \in \mathbb{Z}^s,$$

is in $\ell_p(\mathbb{Z}^s)$ and, moreover,

$$\|d * c\|_{\ell_p(\mathbb{Z}^s)} \le \|d\|_{\ell_1(\mathbb{Z}^s)} \|c\|_{\ell_p(\mathbb{Z}^s)}.$$

To bound the norm of $S_{\mathbf{A},M}$ acting on $\ell_p^N(\mathbb{Z}^s)$ we must extend Young's inequality to a vector setting. To this end, let $\mathbf{A} = \{\mathbf{A}_\alpha\}_{\alpha\in\mathbb{Z}^s} \in \ell_1^{N\times N}(\mathbb{Z}^s)$ and define $\mathbf{A} * \mathbf{c}$, $\mathbf{c} \in \ell_p^N(\mathbb{Z}^s)$, as the vector $((\mathbf{A} * \mathbf{c})_1, \ldots, (\mathbf{A} * \mathbf{c})_N)$, where $(\mathbf{A} * \mathbf{c})_i := \sum_{j=1}^N A_{i,j} * c_j$. Then

$$\|\mathbf{A} * \mathbf{c}\|^p_{\ell_p^N(\mathbb{Z}^s)} := \sum_{i=1}^N \|(\mathbf{A} * \mathbf{c})_i\|^p_{\ell_p(\mathbb{Z}^s)},$$

and by Minkowski's inequality, Young's inequality, and Hölder's inequality

$$\begin{aligned}
\|(\mathbf{A} * \mathbf{c})_i\|_{\ell_p(\mathbb{Z}^s)} &\le \sum_{j=1}^N \|A_{i,j} * \mathbf{c}_j\|_{\ell_p(\mathbb{Z}^s)} \\
&\le \sum_{j=1}^N \|A_{i,j}\|_{\ell_1(\mathbb{Z}^s)} \|c_j\|_{\ell_p(\mathbb{Z}^s)} \\
&\le \left(\sum_{j=1}^N \|A_{i,j}\|^q_{\ell_1(\mathbb{Z}^s)}\right)^{1/q} \left(\sum_{j=1}^N \|c_j\|^p_{\ell_p(\mathbb{Z}^s)}\right)^{1/p},
\end{aligned}$$

where $1/p + 1/q = 1$. Therefore, we get

$$\|\mathbf{A} * \mathbf{c}\|_{\ell_p^N(\mathbb{Z}^s)} \le \left(\sum_{i=1}^N \left(\sum_{j=1}^N \|A_{i,j}\|^q_{\ell_1(\mathbb{Z}^s)}\right)^{p-1}\right)^{1/p} \|\mathbf{c}\|_{\ell_p^N(\mathbb{Z}^s)}.$$

We call the sum in the upper inequality $\kappa_{p,1}(\mathbf{A})$ which is finite whenever $\mathbf{A} \in \ell_1^{N\times N}(\mathbb{Z}^s)$.

Going back to the subdivision operator, we introduce for each $e \in E := \mathbb{Z}^s/M\mathbb{Z}^s$ the submask

$$\mathbf{A}_{e,M} := \{\mathbf{A}_{e+M\alpha}\}_{\alpha\in\mathbb{Z}^s},$$

and observe that $S_{\mathbf{A},M}\mathbf{c}$ acts on the sublattice $e + M\mathbb{Z}^s$ as the convolution $\mathbf{A}^T_{e,M} * \mathbf{c}$. In fact, for $\alpha \in \mathbb{Z}^s$ one has

$$(S_{\mathbf{A},M}\mathbf{c})_{e+M\alpha} = \sum_{\beta\in\mathbb{Z}^s} \mathbf{A}^T_{e+M\alpha-M\beta}\mathbf{c}_\beta = (\mathbf{A}^T_{e,M} * \mathbf{c})_\alpha.$$

Moreover, we have

$$\|S_{\mathbf{A},M}\mathbf{c}\|^p_{\ell^N_p(\mathbb{Z}^s)} = \sum_{e\in E} \|\mathbf{A}^T_{e,M} * \mathbf{c}\|^p_{\ell^N_p(\mathbb{Z}^s)},$$

and so

$$\|S_{\mathbf{A},M}\mathbf{c}\|^p_{\ell^N_p(\mathbb{Z}^s)} \le \left(\sum_{e\in E} \kappa_{p,1}(\mathbf{A}^T_{e,M})^p\right) \|\mathbf{c}\|^p_{\ell^N_p(\mathbb{Z}^s)}.$$

Since $\kappa_{p,1}(\mathbf{A}^T_{e,M}) \le \kappa_{p,1}(\mathbf{A}^T) < \infty$, $e \in E$, we see that $S_{\mathbf{A},M}$ is indeed a bounded linear operator on $\ell^N_p(\mathbb{Z}^s)$. ■

Next, we wish to connect this form of stationary subdivision to a class of scalar subdivision schemes that was recently studied in [BM]. Under an additional restriction on the dilation matrix M we will show that the scheme $S_{\mathbf{A},M}$ in (3.1) above is isometrically isomorphic to a *homogeneous essentially stationary subdivision scheme* in the sense of [BM]. To explain this, we choose any $s \times s$ integer matrix R such that $|\det R| = N$ and choose representers $\mu_1, \dots, \mu_N$ of the cosets $\mathbb{Z}^s/R\mathbb{Z}^s$. Therefore, every $\alpha \in \mathbb{Z}^s$ has a unique representation as

$$\alpha = \mu_i + R\gamma, \qquad 1 \le i \le N, \quad \gamma \in \mathbb{Z}^s.$$

This fact allows us to introduce an isometry from $\ell^N_p(\mathbb{Z}^s)$ onto $\ell_p(\mathbb{Z}^s)$. The isometry $Q\colon\ \ell^N_p(\mathbb{Z}^s) \to \ell_p(\mathbb{Z}^s)$ which we have in mind is defined as

$$Q(\mathbf{c})_{\mu_i+R\gamma} := (c_i)_\gamma, \qquad 1 \le i \le N, \quad \gamma \in \mathbb{Z}^s,$$

where

$$\mathbf{c} = (c_1, \dots, c_N)^T, \qquad c_i \in \ell_p(\mathbb{Z}^s), \quad 1 \le i \le N.$$

The inverse of Q is given by

$$Q^{-1}(c) = (d_1, \dots, d_N)^T, \qquad c = \{c_\alpha\}_{\alpha\in\mathbb{Z}^s},$$

where

$$(d_i)_\gamma := c_{\mu_i+R\gamma}, \qquad \gamma \in \mathbb{Z}^s.$$

Since the norm on $\ell^N_p(\mathbb{Z}^s)$ is given by

$$\|\mathbf{c}\|_{\ell^N_p(\mathbb{Z}^s)} := \left(\sum_{j=1}^N \|c_j\|^p_{\ell_p(\mathbb{Z}^s)}\right)^{1/p},$$

it is clear that Q is an isometry since

$$\begin{aligned}\|Q(\mathbf{c})\|^p_{\ell_p(\mathbb{Z}^s)} &= \sum_{i=1}^N \sum_{\gamma\in\mathbb{Z}^s} |(c_i)_\gamma|^p \\ &= \sum_{i=1}^N \|c_i\|^p_{\ell_p(\mathbb{Z}^s)} = \|\mathbf{c}\|^p_{\ell^N_p(\mathbb{Z}^s)}.\end{aligned}$$

Using this isometry we consider the subdivision scheme

$$\mathcal{S} := Q S_{\mathbf{A},M} Q^{-1} \tag{3.2}$$

on $\ell_p(\mathbb{Z}^s)$. We wish to point out that when $V := RMR^{-1}$ is a matrix with *integer entries* (when M is an integer multiple of the identity matrix this would be the case) then $\mathcal{S}$ is a homogeneous essentially stationary subdivision scheme in the terminology of [BM]. Specifically, if we represent $\mathcal{S}$ in the form

$$(\mathcal{S}c)_\alpha = \sum_{\beta\in\mathbb{Z}^s} \mathcal{S}_{\alpha,\beta} c_\beta, \qquad c = \{c_\alpha\}_{\alpha\in\mathbb{Z}^s} \in \ell_p(\mathbb{Z}^s),$$

then:

(i) For some $k \in \mathbb{Z}^s$

$$\mathcal{S}_{\alpha+Vk,\beta+k} = \mathcal{S}_{\alpha,\beta}, \qquad \alpha, \beta \in \mathbb{Z}^s.$$

(ii) There is a norm $\|\cdot\|$ on $\mathbb{R}^s$ such that

$$\mathcal{S}_{\alpha,\beta} = 0 \qquad \text{if} \quad \|\alpha - V\beta\| > 1.$$

(This notion was introduced in [BM] only for the case $V = 2I$.)

To prove these properties, we consider the relationship between translation operators acting on $\ell_p^N(\mathbb{Z}^s)$ and $\ell_p(\mathbb{Z}^s)$. Every $y \in \mathbb{Z}^s$ induces a translation operator $\mathcal{E}_y\colon\ \ell_p(\mathbb{Z}^s) \to \ell_p(\mathbb{Z}^s)$ given by

$$(\mathcal{E}_y c)_\alpha := c_{\alpha+y}, \qquad \alpha \in \mathbb{Z}^s, \quad c = \{c_\alpha\}_{\alpha\in\mathbb{Z}^s}.$$

Likewise, on $\ell_p^N(\mathbb{Z}^s)$, the translation operator determined by y is defined by

$$(E_y \mathbf{c})_\alpha := \mathbf{c}_{\alpha+y}, \qquad \alpha \in \mathbb{Z}^s, \quad \mathbf{c} = (c_1, \dots, c_N)^T,$$

where $c_i \in \ell_p(\mathbb{Z}^s)$, $i = 1, \dots, N$, and $\mathbf{c}_\alpha = ((c_1)_\alpha, \dots, (c_N)_\alpha)^T$. According to (3.1), we see that for $\mathbf{c} \in \ell_p^N(\mathbb{Z}^s)$, $\alpha \in \mathbb{Z}^s$, $y \in \mathbb{Z}^s$, we have

$$\begin{aligned}(E_{My} S_{\mathbf{A},M} E_y^{-1}\mathbf{c})_\alpha &= (S_{\mathbf{A},M} E_y^{-1}\mathbf{c})_{\alpha+My} = \sum_{\beta\in\mathbb{Z}^s} \mathbf{A}^T_{\alpha+My-M\beta}\mathbf{c}_{\beta-y} \\ &= \sum_{\beta\in\mathbb{Z}^s} \mathbf{A}^T_{\alpha-M\beta}\mathbf{c}_\beta = (S_{\mathbf{A},M}\mathbf{c})_\alpha.\end{aligned}$$

That is,

$$E_{My} S_{\mathbf{A},M} = S_{\mathbf{A},M} E_y. \tag{3.3}$$

Moreover, there is a simple relationship between shifts on $\ell_p^N(\mathbb{Z}^s)$ and $\ell_p(\mathbb{Z}^s)$. In fact, it is straightforward to see that

$$Q E_y Q^{-1} = \mathcal{E}_{Ry}, \qquad y \in \mathbb{Z}^s. \tag{3.4}$$

In fact, given $c \in \ell_p(\mathbb{Z}^s)$, we have

$$Q^{-1}(c) = (d_1, \dots, d_N)^T,$$

where $d_i \in \ell_p(\mathbb{Z}^s)$ are defined by $(d_i)_\gamma = c_{\mu_i+R\gamma}$, $1 \le i \le N$, $\gamma \in \mathbb{Z}^s$. Hence for $\alpha \in \mathbb{Z}^s$ we have

$$\begin{aligned}\mathbf{b}_\alpha &= ((b_1)_\alpha, \dots, (b_N)_\alpha)^T := \left(E_y(Q^{-1}(c))\right)_\alpha \\ &= ((d_1)_{\alpha+y}, \dots, (d_N)_{\alpha+y})^T = (c_{\mu_1+R(\alpha+y)}, \dots, c_{\mu_N+R(\alpha+y)})^T.\end{aligned}$$

Therefore, for $1 \le j \le N$, $\gamma \in \mathbb{Z}^s$, we obtain

$$(Q\mathbf{b})_{\mu_j+R\gamma} = (b_j)_\gamma = c_{\mu_j+R(\gamma+y)} = c_{\mu_j+R\gamma+Ry}.$$

That is,

$$QE_yQ^{-1} = \mathcal{E}_{Ry},$$

as claimed in (3.4).

Let us now confirm property (i). To this end, we solve for $S_{\mathbf{A},M}$ in formula (3.2) and substitute it into (3.3). This provides us the formula

$$(QE_{My}Q^{-1})\mathcal{S} = \mathcal{S}(QE_yQ^{-1}),$$

and so (3.4) yields the equation

$$\mathcal{E}_{RMy}\mathcal{S} = \mathcal{S}\mathcal{E}_{Ry}, \qquad y \in \mathbb{Z}^s.$$

Invoking our hypothesis, we obtain for any $k \in \mathbb{Z}^s$ such that $R^{-1}k \in \mathbb{Z}^s$

$$\mathcal{E}_{Vk}\mathcal{S} = \mathcal{S}\mathcal{E}_k. \tag{3.5}$$

Since such k obviously exists this proves the result, since (3.5) is evidently equivalent to (i).

To prove (ii) we introduce for each $\beta \in \mathbb{Z}^s$, first the sequence $\delta_\beta \in \ell_p(\mathbb{Z}^s)$ given by

$$(\delta_\beta)_\alpha := \begin{cases} 0, & \alpha \ne \beta, \\ 1, & \alpha = \beta, \end{cases}$$

and then for $j, 1 \le j \le N$, the vector $\mathbf{e}_j \in \mathbb{R}^N$, defined by $(\mathbf{e}_j)_i = 0, i \ne j$ and $(\mathbf{e}_j)_i = 1$, if $i = j$. Then it follows that

$$Q(\delta_\beta \mathbf{e}_j) = \delta_{\mu_j+R\beta} \qquad 1 \le j \le N, \quad \beta \in \mathbb{Z}^s. \tag{3.6}$$

Returning to the form (3.1) and using the hypothesis that $\#\{\alpha : \mathbf{A}_\alpha \ne \mathbf{0}\} < \infty$, we conclude that for some matrix norm $\|\cdot\|$ and $1 \le j \le N$

$$S_{\mathbf{A},M}(\delta_\beta \mathbf{e}_j)_\alpha = \mathbf{0} \qquad \text{if} \quad \|\alpha - M\beta\| > 1. \tag{3.7}$$

That is, in view of (3.6),

$$Q^{-1}(\mathcal{S}(\delta_{\mu_j+R\beta}))_\alpha = \mathbf{0} \qquad \text{if} \quad \|\alpha - M\beta\| > 1,$$

or equivalently, for $1 \le i, j \le N$,

$$\mathcal{S}(\delta_{\mu_j+R\beta})_{\mu_i+R\alpha} = 0 \qquad \text{if} \quad \|\alpha - M\beta\| > 1.$$

Thus for any $\hat{\alpha}, \hat{\beta} \in \mathbb{Z}^s$ we choose $\alpha, \beta \in \mathbb{Z}^s$, $1 \le i, j \le N$, such that $\mu_j + R\beta = \hat{\beta}$, $\mu_i + R\alpha = \hat{\alpha}$, and conclude that

$$\mathcal{S}_{\hat{\alpha},\hat{\beta}} = \mathcal{S}(\delta_{\hat{\beta}})_{\hat{\alpha}} = 0,$$

provided that $\|R^{-1}(\hat{\alpha}-\mu_i)-MR^{-1}(\hat{\beta}-\mu_j)\| > 1$. Set $\rho := \max\{\|R^{-1}\mu_i - MR^{-1}\mu_j\| : 1 \le i, j \le N\}$. Then, whenever $\|R^{-1}\hat{\alpha} - MR^{-1}\hat{\beta}\| > 1+\rho$, we get that $\mathcal{S}_{\hat{\alpha},\hat{\beta}} = 0$. Hence, in (ii) we can choose the norm

$$|x| := \frac{1}{\rho+1}\|R^{-1}x\|, \qquad x \in \mathbb{R}^s.$$

We now study various notions of convergence relative to the vector subdivision schemes discussed so far. To this end, we first explain what we mean by *convergence* of $S_{\mathbf{A},M}$. For $\mathbf{f} \in L^N_{1,loc}(\mathbb{R}^s)$ let

$$\mu^j_M(\mathbf{f})_\alpha := m^j \int_{M^{-j}(\alpha+[0,1]^s)} \mathbf{f}(x)\,dx \tag{3.8}$$

and also set $\mu^j_M(\mathbf{f}) := \{\mu^j_M(\mathbf{f})_\alpha\}_{\alpha\in\mathbb{Z}^s}$. The same arguments as in [JM1] yield the following fact.

Proposition 3.2. *For any $g \in \mathcal{L}_p(\mathbb{R}^s)$ satisfying*

$$u\Sigma g = 1, \tag{3.9}$$

and any $f \in L_p(\mathbb{R}^s)$ we have

$$\lim_{k\to\infty} \|f - sc^k_M(\mu^k_M(\mathbf{f})\Sigma g)\|_{L_p(\mathbb{R}^s)} = 0. \tag{3.10}$$

Moreover, when g has compact support, we have

$$\|\mathbf{c}\Sigma g\|_{L_p(\mathbb{R}^s)} \le \kappa\|\mathbf{c}\|_{\ell_p(\mathbb{Z}^s)} \tag{3.11}$$

for some constant κ.

Any function $g \in L_p(\mathbb{R}^s)$ with compact support and stable integer translates which satisfies (3.9) is called a *p-test function*. A consequence of this proposition is the fact that whenever $f \in L_p(\mathbb{R}^s)$ and $\lim_{k\to\infty} m^{-k/p}\|\mu^k_M(f)\|_{L_p(\mathbb{R}^s)} = 0$ then $f = 0$. To see this just choose any p-test function then by (3.11) and a change of variable of integration we have

$$\begin{aligned}\|sc^k_M(\mu^k_M(f)\Sigma g)\|_{L_p(\mathbb{R}^s)} &= m^{-k/p}\|\mu^k_M(f)\Sigma g\|_{L_p(\mathbb{R}^s)}\\ &\le \kappa m^{-k/p}\|\mu^k_M(f)\|_{\ell_p(\mathbb{Z}^s)},\end{aligned}$$

and so

$$\lim_{k\to\infty} sc^k_M(\mu^k_M(f)\Sigma g) = 0.$$

Combining this fact with (3.10), proves that $f = 0$, as claimed.

Also, we should keep in mind for future use the simple fact that

$$m^{-k/p}\|\mu^k_M(\mathbf{f})\|_{\ell^N_p(\mathbb{Z}^s)} \le \|\mathbf{f}\|_{L^N_p(\mathbb{R}^s)}, \tag{3.12}$$

which follows directly from Hölder's inequality.

We say that $S_{\mathbf{A},M}$ *converges in ℓ_p relative to some p-test function g* if for every $\mathbf{c} \in \ell^N_p(\mathbb{Z}^s)$ there exists a function $\mathbf{f_c} \in L^N_p(\mathbb{R}^s)$ such that

$$\lim_{k\to\infty} \|sc^k_M\left((S^k_{\mathbf{A},M}\mathbf{c})\Sigma g\right) - \mathbf{f_c}\|_{L^N_p(\mathbb{R}^s)} = 0. \tag{3.13}$$

Here we assume also that for some $\mathbf{c}$ the function $f_{\mathbf{c}}$ does not vanish identically.

Proposition 3.3. *The scheme $S_{\mathbf{A},M}$ converges relative to some p-test function g in ℓ_p if and only if*

$$\lim_{k\to\infty} m^{-k/p}\|S^k_{\mathbf{A},M}\mathbf{c} - \mu^k_M(\mathbf{f_c})\|_{\ell^N_p(\mathbb{Z}^s)} = 0\,. \tag{3.14}$$

Proof. By stability (2.6) we have

$$\begin{aligned} m^{-k/p}\|S^k_{\mathbf{A},M}\mathbf{c} - \mu^k_M(\mathbf{f_c})\|_{\ell^N_p(\mathbb{Z}^s)} &\le d\|sc^k_M((S^k_{\mathbf{A},M}\mathbf{c} - \mu^k_M(\mathbf{f_c}))\Sigma g)\|_{L^N_p(\mathbb{R}^s)} \\ &\le d(\|sc^k_M((S^k_{\mathbf{A},M}\mathbf{c})\Sigma g) - \mathbf{f_c}\|_{L^N_p(\mathbb{R}^s)} + \|\mathbf{f_c} - sc^k_M(\mu^k_M(\mathbf{f_c})\Sigma g)\|_{L^N_p(\mathbb{R}^s)}). \end{aligned} \tag{3.15}$$

By (3.13), and Proposition 3.2, both terms on the right-hand side of (3.15) tend to zero as k tends to infinity which proves that (3.14) follows from the convergence of $S_{\mathbf{A},M}$.

Conversely, we have

$$\begin{aligned} &\|sc^k_M\left((S^k_{\mathbf{A},M}\mathbf{c})\Sigma g\right) - \mathbf{f_c}\|_{L^N_p(\mathbb{R}^s)} \\ &\qquad \le \|sc^k_M\left((S^k_{\mathbf{A},M}\mathbf{c} - \mu^k_M(\mathbf{f_c}))\Sigma g\right)\|_{L^N_p(\mathbb{R}^s)} + \|sc^k_M\left(\mu^k_M(\mathbf{f_c})\Sigma g\right) - \mathbf{f_c}\|_{L^N_p(\mathbb{R}^s)}\,. \end{aligned}$$

Again Proposition 3.2 ensures that the second term on the right-hand side tends to zero. By the properties of g, the first term is easily estimated by a constant times $m^{-k/p}\|S^k_{\mathbf{A},M}\mathbf{c} - \mu^k_M(\mathbf{f}_c)\|_{\ell^N_p(\mathbb{Z}^s)}$ which completes the proof. ∎

We observe next that the special choice of g in the above notion of convergence is not relevant so that we may drop the reference to g when talking of convergence. In fact, an immediate consequence of Proposition 3.3 is the following fact.

Corollary 3.1. *If $S_{\mathbf{A},M}$ converges relative to some p-test function then it converges relative to any other p-test function as well.*

Note that the particular choice $g = \chi_{[0,1]^s}$ shows that the convergence concept studied in [CDM] agrees with the present one for the special case $N = 1$, $M = I$, and $p = \infty$.

Our next goal is to extend Proposition 2.1 in [CDM] to matrix subdivision schemes. A first step can be formulated as follows.

Proposition 3.4. *Suppose that in addition to the hypotheses in Proposition* 3.3 $\mathbf{A}$ *has finite support. Then there exists a common eigenvector $\mathbf{y} \in \mathbb{C}^N$ of the matrices $\mathbf{A}_e(\mathbf{1})^T$, $e \in E$, with eigenvalue* 1*. In particular, the limit function $\mathbf{f_c}$ produced by the scheme $S_{\mathbf{A},M}$ satisfies*

$$\mathbf{A}_e(\mathbf{1})^T\mathbf{f_c}(x) = \mathbf{f_c}(x), \qquad e \in E, \quad x \in \mathbb{R}^s, \quad \mathbf{c} \in \ell^N_p(\mathbb{Z}^s). \tag{3.16}$$

Proof. In view of (3.1), we can write

$$\begin{aligned} &m^{-k/p}\|S^k_{\mathbf{A},M}\mathbf{c} - \mu^k_M(\mathbf{f_c})\|_{\ell^N_p(\mathbb{Z}^s)} \\ &\qquad \ge m^{-k/p}\left(\sum_{\alpha\in\mathbb{Z}^s}\left|\left(\sum_{\beta\in\mathbb{Z}^s}\mathbf{A}^T_{\alpha-M\beta}\right)\mu^{k-1}_M(\mathbf{f_c})_\alpha - \mu^k_M(\mathbf{f_c})_\alpha\right|^p_p\right)^{1/p} \end{aligned} \tag{3.17}$$

$$
- m^{-k/p}\left(\sum_{\alpha\in\mathbb{Z}^s}\left|\sum_{\beta\in\mathbb{Z}^s}\mathbf{A}^T_{\alpha-M\beta}\left(\mu_M^{k-1}(\mathbf{f_c})_\beta-\mu_M^k(\mathbf{f_c})_\alpha)\right)\right|_p^p\right)^{1/p}
$$

$$
- m^{-k/p}\left(\sum_{\alpha\in\mathbb{Z}^s}\left|\sum_{\beta\in\mathbb{Z}^s}\mathbf{A}^T_{\alpha-M\beta}\left((S^{k-1}_{\mathbf{A},M}\mathbf{c})_\beta-\mu_M^{k-1}(\mathbf{f_c})_\beta\right)\right|_p^p\right)^{1/p}.
$$

The last sum above is bounded by

$$
m^{-k/p}\|S_{\mathbf{A},M}\|_p\|S^{k-1}_{\mathbf{A},M}\mathbf{c}-\mu_M^{k-1}(\mathbf{f_c})\|_{\ell_p^N(\mathbb{Z}^s)},
$$

where $\|S_{\mathbf{A},M}\|_p$ denotes the norm of the subdivision operator acting on $\ell_p^N(\mathbb{Z}^s)$. According to Proposition 3.1 and Proposition 3.3, this quantity tends to zero as $k\to\infty$.

As for the next to last sum we estimate it from above in the following manner. Let $|\mathbf{A}^T_{\alpha-M\beta}|_p$ stand for the norm of the matrix $\mathbf{A}^T_{\alpha-M\beta}$ acting on $\mathbb{R}^s$ with ℓ_p-norm. Then this term is bounded by

$$
m^{-k/p}\left(\sum_{\alpha\in\mathbb{Z}^s}\left(\sum_{\beta\in\mathbb{Z}^s}|\mathbf{A}^T_{\alpha-M\beta}|_p\,|\mu_M^{k-1}(\mathbf{f_c})_\beta-\mu_M^k(\mathbf{f_c})_\alpha|_p\right)^p\right)^{1/p}
$$

$$
= m^{-k/p}\left(\sum_{e\in E}\sum_{\gamma\in\mathbb{Z}^s}\left(\sum_{\beta\in\mathbb{Z}^s}|\mathbf{A}^T_{e+M\beta}|_p|\mu_M^{k-1}(\mathbf{f_c})_{\gamma-\beta}-\mu_M^k(\mathbf{f_c})_{e+M\gamma}|_p\right)^p\right)^{1/p}.
$$

Let

$$
\Gamma := \{\beta\in\mathbb{Z}^s\colon\ \exists e\in E\ \ni\ |\mathbf{A}^T_{e+M\beta}|_p\neq 0\}
$$

and

$$
\theta := \max\{|\mathbf{A}^T_\alpha|_p : \alpha\in\Gamma\}.
$$

Observe that necessarily the right-hand side above can be bounded by

$$
\theta m^{-k/p}\left(\sum_{e\in E}\sum_{\gamma\in\mathbb{Z}^s}\left(\sum_{\beta\in\Gamma}|\mu_M^{k-1}(\mathbf{h}^k_{\beta,e})_\gamma|_p\right)^p\right)^{1/p},
$$

where

$$
\mathbf{h}^k_{\beta,e} := \mathbf{f_c}(-M^{-k+1}\beta+\cdot)-\mathbf{f_c}(M^{-k}e+\cdot).
$$

Since

$$
\sum_{\gamma\in\mathbb{Z}^s}\left(\sum_{\beta\in\Gamma}|\mu_M^k(\mathbf{h}^k_{\beta,e})_\gamma|_p\right)^p\le m^k(\#\Gamma)^p\int_{\mathbb{R}^s}\sum_{\beta\in\Gamma}|\mathbf{h}^k_{\beta,e}(x)|_p^p\,dx,
$$

we get that the next to the last sum is bounded by

$$
\theta(\#\Gamma)\left(\int_{\mathbb{R}^s}\sum_{e\in E}\sum_{\beta\in\Gamma}|\mathbf{h}^k_{\beta,e}(x)|_p^p\,dx\right)^{1/p},
$$

which can be estimated by

$$
\theta m(\#\Gamma)^2\sup_{|h|_2\le\delta_k}\|\mathbf{f_c}(\cdot+h)-\mathbf{f_c}\|_{L_p^N(\mathbb{R}^s)},
$$

where

$$\delta_k := \max\{|M^{-k}\alpha|_2 \colon \alpha \in \mathbb{Z}^s, \mathbf{A}_\alpha^T \neq \mathbf{0}\}.$$

Since $\lim_{k\to\infty} \delta_k = 0$ we conclude that likewise the second to last term in the lower bound in (3.17) tends to zero as $k \to \infty$.

To interpret these observations made so far, we set $\mathbf{y}^k = \{\mathbf{y}_\alpha^k\}_{\alpha\in\mathbb{Z}^s}$ where for $\alpha \in \mathbb{Z}^s$

$$\mathbf{y}_\alpha^k := \left(\sum_{\beta\in\mathbb{Z}^s} \mathbf{A}_{\alpha-M\beta}^T\right) \mu_M^k(\mathbf{f_c})_\alpha - \mu_M^k(\mathbf{f_c})_\alpha.$$

Thus, we conclude by what has been said above that

$$\lim_{k\to\infty} m^{-k/p} \|\mathbf{y}^k\|_{\ell_p^N(\mathbb{Z}^s)} = 0.$$

Next, observe that for each $e \in E$ and $\alpha \in \mathbb{Z}^s$ we have

$$\begin{aligned}\mathbf{y}_{e+M\alpha}^k &= \mathbf{A}_e^T(\mathbf{1})\mu_M^k(\mathbf{f_c})_{e+M\alpha} - \mu_M^k(\mathbf{f_c})_{e+M\alpha} \\ &= \mu_M^k\left(\mathbf{A}_e^T(\mathbf{1})\mathbf{f_c} - \mathbf{f_c}\right)_\alpha + \mu_M^k(\mathbf{h}(\cdot) - \mathbf{h}(M^{-k}e + \cdot))_\alpha,\end{aligned}$$

where we set

$$\mathbf{h} := \mathbf{f_c} - \mathbf{A}_e^T(\mathbf{1})\mathbf{f_c}.$$

Therefore, we obtain the inequality

$$m^{-k/p}\|\mu_M^k(\mathbf{A}_e^T(\mathbf{1})\mathbf{f_c} - \mathbf{f_c})\|_{\ell_p^N(\mathbb{Z}^s)} \leq m^{-k/p}\|\mathbf{y}^k\|_{\ell_p^N(\mathbb{Z}^s)} + \|\mathbf{h}(\cdot) - \mathbf{h}(M^{-k}e + \cdot)\|_{L_p^N(\mathbb{R}^s)}$$

and conclude that

$$\mathbf{A}_e^T(\mathbf{1})\mathbf{f_c}(x) = \mathbf{f_c}(x), \qquad x \in \mathbb{R}^s, \text{ a.e., } e \in E, \ \mathbf{c} \in \ell_p^N(\mathbb{Z}^s). \tag{3.18}$$

This proves the claim. ∎

Now consider the sequences $\mathbf{d}^j := \{\delta_\alpha \mathbf{e}^j\}_{\alpha\in\mathbb{Z}^s} \in \ell_p^N(\mathbb{Z}^s)$, where $(\mathbf{e}^j)_l := \delta_{j,l}$, $j, l = 1, \ldots, N$, and let $\mathbf{g}_j \in L_p^N(\mathbb{R}^s)$ denote the $\ell_p(\mathbb{Z}^s)$-limit of $S_{\mathbf{A},M}^k \mathbf{d}^j$. Moreover, let $\mathbf{G}(x)$ denote the matrix-valued function whose columns are formed by the vectors $\mathbf{g}_j(x)$, $j = 1, \ldots, N$.

Proposition 3.5. *Under the assumptions of Proposition* 3.4 *the function* $\mathbf{G}$ *has compact support and the limit functions* $\mathbf{f_c}$ *produced by* $S_{\mathbf{A},M}$ *have the form*

$$\mathbf{f_c}(x) = \sum_{\alpha\in\mathbb{Z}^s} \mathbf{G}(x - \alpha)\mathbf{c}_\alpha. \tag{3.19}$$

Proof. The first part of the claim follows in a similar way as the claim of Lemma 2.5. As for the rest, writing

$$\mathbf{c}_\alpha = \sum_{\beta\in\mathbb{Z}^s}\sum_{j=1}^N (\mathbf{c}_\beta)_j \delta_{\alpha-\beta}\mathbf{e}^j = \sum_{\beta\in\mathbb{Z}^s}\sum_{j=1}^N (\mathbf{c}_\beta)_j \mathbf{d}_{\alpha-\beta}^j,$$

we see that

$$(S_{\mathbf{A},M}^k \mathbf{c})_\alpha = \sum_{\beta\in\mathbb{Z}^s}\sum_{j=1}^N (\mathbf{c}_\beta)_j (S_{\mathbf{A},M}^k \mathbf{d}_{\cdot-\beta}^j)_\alpha,$$

whence the assertion follows. ∎

Corollary 3.2. *Suppose the hypotheses of Proposition* 3.4 *are satisfied. Then for any common eigenvector* $\mathbf{y}$ *of the matrices* $\mathbf{A}_e(\mathbf{1})$, $e \in E$, *we have*

$$\mathbf{y} = \sum_{\alpha\in\mathbb{Z}^s} \mathbf{G}(x-\alpha)\mathbf{y}. \tag{3.20}$$

Proof. Consider the particular sequence $\mathbf{c}$, defined by

$$\mathbf{c}_\alpha := \begin{cases} \mathbf{y} & \text{if } \alpha \in \mathbb{Z}^s,\ |\alpha|_2 \le R, \\ 0 & \text{if } |\alpha|_2 > R. \end{cases}$$

Clearly $\mathbf{c} \in \ell_p^N(\mathbb{Z}^s)$. Thus, on account of the finite support of $\mathbf{A}$, for R sufficiently large one has, by assumption on $\mathbf{y}$,

$$(S^k_{\mathbf{A},M}\mathbf{c})_\alpha = \begin{cases} \mathbf{y} & \text{for } |M^{-k}\alpha|_2 \le R/2, \\ 0 & \text{for } |M^{-k}\alpha|_2 \ge 2R. \end{cases}$$

Again since $\mathbf{G}$ has compact support we conclude from (3.19) that for $|x|_2 \le R/2$

$$\mathbf{y} = \sum_{\alpha\in\mathbb{Z}^s} \mathbf{G}(x-\alpha)\mathbf{y}, \qquad |x| \le R/2.$$

Since R was arbitrarily large the assertion follows. ∎

We are now ready to prove a central result in this section.

Theorem 3.1. *Suppose that* $S_{\mathbf{A},M}$ *converges in* $\ell_p(\mathbb{Z}^s)$ *and that* $\mathbf{A}$ *has finite support. Then* (*the matrix-valued function*) $\mathbf{H}(x) := \mathbf{G}(x)^T$ *is a solution of the refinement equation* (2.2), *i.e.*,

$$\mathbf{H}(x) = \sum_{\alpha\in\mathbb{Z}^s} \mathbf{A}_\alpha \mathbf{H}(Mx-\alpha), \qquad x \in \mathbb{R}^s. \tag{3.21}$$

Proof. Define the mapping $\tilde{T}_{\mathbf{A}}$ by

$$(\tilde{T}_{\mathbf{A}}\mathbf{G}) := \sum_{\alpha\in\mathbb{Z}^s} \mathbf{G}(Mx-\alpha)\mathbf{A}^T_\alpha. \tag{3.22}$$

One easily verifies that for $k = 1, 2, \ldots$

$$\sum_{\alpha\in\mathbb{Z}^s} (\tilde{T}^k_{\mathbf{A}}\mathbf{G})(x-\alpha)\mathbf{c}_\alpha = \sum_{\alpha\in\mathbb{Z}^s} \mathbf{G}(M^k x-\alpha)(S^k_{\mathbf{A},M}\mathbf{c})_\alpha. \tag{3.23}$$

Note that, since $\mathbf{G}$ has, by Proposition 3.5, compact support, the convergence of $S_{\mathbf{A},M}$ ensures, on account of Proposition 3.3, that the right-hand side of this identity has for the special choice $\mathbf{c} = \mathbf{d}^j$ the same $\ell_p(\mathbb{Z}^s)$-limit as k tends to infinity as the expressions

$$\sum_{\alpha\in\mathbb{Z}^s} \mathbf{G}(M^k x-\alpha)\mu^k_M(\mathbf{g}_j)_\alpha,$$

where as above $\mathbf{f}_{\mathbf{d}^j} = \mathbf{g}_j$. Moreover, for almost every $x \in \mathbb{R}^s$ one has that $|\mathbf{g}_j(x) - \mu^k_\alpha(\mathbf{g}_j)|_2 \to 0$ if $|x - M^{-k}\alpha|_2 \to 0$ as $k \to \infty$. Thus

$$\sum_{\alpha\in\mathbb{Z}^s} \mathbf{G}(M^k x-\alpha)(\mu^k_M(\mathbf{g}_j)_\alpha - \mathbf{f}_{\mathbf{d}^j}(x))$$

tends to zero as $k \to \infty$. But by Proposition 3.4, each $\mathbf{f}_{\mathbf{d}^j}(x) = \mathbf{g}_j(x)$ is a common eigenvector of the matrices $\mathbf{A}_e(\mathbf{1})$, $e \in E$, so that, by Corollary 3.2,

$$\sum_{\alpha \in \mathbb{Z}^s} \mathbf{G}(M^k x - \alpha)\mathbf{g}_j(x) = \mathbf{g}_j(x), \qquad j = 1, \ldots, N.$$

Since the left-hand side of (3.23) reduces for $\mathbf{c} = \mathbf{d}^j$ to $(\tilde{T}^k_{\mathbf{A}}\mathbf{G})(x)$, we conclude that

$$\lim_{k \to \infty} (\tilde{T}^k_{\mathbf{A}}\mathbf{G})(x) = \mathbf{G}(x), \tag{3.24}$$

which immediately gives

$$(\tilde{T}_{\mathbf{A}}\mathbf{G})(x) = \mathbf{G}(x).$$

By definition of $\mathbf{H}$ and $\tilde{T}_{\mathbf{A}}$, this proves the claim. ■

So far each row of $\mathbf{G}$, i.e., each column of $\mathbf{H}$ is a solution to the refinement equation (2.2). More can be said under the additional assumption of stability.

Theorem 3.2. *Now suppose that for $S_{\mathbf{A},M}$ as above one of the columns of $\mathbf{H}$ say $\mathbf{h}_j$ is stable. Then $\mathbf{H}$ satisfying* (3.21) *is a rank-one matrix. More precisely, there exists a unique $\mathbf{h} \in L^N_p(\mathbb{R}^s)$ of compact support and a unique $\mathbf{y} \in \mathbb{R}^N$ of unit length such that*

$$\mathbf{H}(x) = \mathbf{h}(x)\mathbf{y}^T \tag{3.25}$$

and

$$\sum_{\alpha \in \mathbb{Z}^s} \mathbf{h}(x - \alpha)^T \mathbf{y} = 1. \tag{3.26}$$

The vector $\mathbf{y}$ in (3.25) *and* (3.26) *is the unique common eigenvector of the matrices $\mathbf{A}_e(\mathbf{1})^T$, $e \in E$.*

Proof. From Proposition 2.1 we know that if (2.2) has a stable solution, then $m^{-1}\mathbf{A}(\mathbf{1})$ has a unique eigenvector (of unique length) with eigenvalue one. Therefore, and on account of Proposition 3.4, the matrices $\mathbf{A}_e(\mathbf{1})$, $e \in E$, have a unique common eigenvector $\mathbf{y}$ of unit length. Again by Proposition 3.4, the functions $\mathbf{g}_j = \mathbf{f}_{\mathbf{d}^j}$ must have the form

$$\mathbf{g}_j(x) = h_j(x)\mathbf{y},$$

where $\mathbf{h}(x) = (h_j(x), \ldots, h_N(x))^T$ has compact support. Thus

$$\mathbf{G}(x) = \mathbf{y}\mathbf{h}(x)^T, \qquad \text{so that} \quad \mathbf{H}(x) = \mathbf{h}(x)\mathbf{y}^T, \tag{3.27}$$

which proves (3.25) as well as the assertions on $\mathbf{y}$. Furthermore, by Corollary 3.2,

$$\begin{aligned} 1 &= \mathbf{y}^T\mathbf{y} = \sum_{\alpha \in \mathbb{Z}^s} \mathbf{y}^T \mathbf{G}(x - \alpha)\mathbf{y} \\ &= \sum_{\alpha \in \mathbb{Z}^s} \mathbf{y}^T \mathbf{y}\mathbf{h}(x - \alpha)^T \mathbf{y} \\ &= \sum_{\alpha \in \mathbb{Z}^s} \mathbf{h}(x - \alpha)^T \mathbf{y}, \end{aligned}$$

which finishes the proof. ■

Corollary 3.3. *Under the assumptions of Theorem* 3.2 *the limit function* $\mathbf{f_c}$ *has the form*

$$\mathbf{f_c}(x) = f_{\mathbf{c}}(x)\mathbf{y}, \tag{3.28}$$

where

$$f_{\mathbf{c}}(x) = \sum_{\alpha\in\mathbb{Z}^s} \mathbf{h}(x-\alpha)^T \mathbf{c}_\alpha. \tag{3.29}$$

Proof. By Theorem 3.2 and Proposition 3.4, $\mathbf{f_c}$ must have the form $\mathbf{f_c}(x) = f_{\mathbf{c}}(x)\mathbf{y}$ for some scalar-valued function $f_{\mathbf{c}}$. From Proposition 3.5 and (3.27) we infer

$$f_{\mathbf{c}}(x)\mathbf{y} = \sum_{\alpha\in\mathbb{Z}^s} \mathbf{y}\mathbf{h}(x-\alpha)^T \mathbf{c}_\alpha.$$

The assertion is now an immediate consequence of the normalization of $\mathbf{y}$. ∎

In view of the results in [CDM] we are led to the following result.

Theorem 3.3. *Suppose that* $\mathbf{A}$ *has finite support and that* $\mathbf{h} \in L_p^N(\mathbb{R}^s)$ *is a compactly supported* ℓ_p*-stable solution of the refinement equation* (2.2). *Then* $S_{\mathbf{A},M}$ *converges in* ℓ_p.

Proof. By Proposition 2.1, there exists a unique unit vector $\mathbf{y}$ satisfying (2.16) and a normalization for $\mathbf{h}$ such that the integer translates of $g := \mathbf{y}^T\mathbf{h}$ sum to one. Clearly g has ℓ_p-stable integer translates and for any $\mathbf{c} \in \ell_p^N(\mathbb{Z}^s)$ the function $f_{\mathbf{c}}(x) := \sum_{\alpha\in\mathbb{Z}^s} \mathbf{c}_\alpha^T \mathbf{h}(x-\alpha)$ is in $L_p(\mathbb{R}^s)$. Thus

$$\begin{aligned}
&\left\| f_{\mathbf{c}} - \sum_{\alpha\in\mathbb{Z}^s} \mu_M^k(f_{\mathbf{c}})_\alpha g(M^k\cdot-\alpha)\right\|_{L_p(\mathbb{R}^s)}\\
&\qquad = \left\| \sum_{\alpha\in\mathbb{Z}^s} (\mu_M^k(f_{\mathbf{c}}\mathbf{y}^T)_\alpha - (S_{\mathbf{A},M}^k\mathbf{c})_\alpha^T)\mathbf{h}(M^k\cdot-\alpha)\right\|_{L_p(\mathbb{R}^s)}\\
&\qquad \geq \kappa\, m^{-k/p}\|\mu_M^k(f_{\mathbf{c}}\mathbf{y}) - S_{\mathbf{A},M}^k\mathbf{c}\|_{\ell_p^N(\mathbb{Z}^s)},
\end{aligned}$$

by stability of $\mathbf{h}$. Since, by Proposition 3.2, the left-hand side of the above inequality tends to zero as k tends to infinity the assertion follows from Proposition 3.3 for $\mathbf{f_c} = f_{\mathbf{c}}\mathbf{y}$. ∎

4. Biorthogonal Wavelets

In this section we collect a few consequences of the above results. Throughout the remainder we will assume that $\mathbf{h}$ is admissible (i.e., belongs to $L_p^N(\mathbb{R}^s)$, has compact support and linearly independent integer translates), and is $\mathbf{A}$-refinable.

By Proposition 2.4(ii) and (iii), $\mathbf{A}$ has finite support. Combining Propositions 2.3 and 2.5 ensures the existence of a finitely supported mask $\mathbf{B}$ such that $\mathbf{A}$ and $\mathbf{B}$ are dual, i.e.,

$$m\mathbf{I} = \sum_{e\in E} \mathbf{A}_e(z^{-1})(\mathbf{B}_e)^*(z). \tag{4.1}$$

Theorem 4.1. *There exists an admissible* $\mathbf{B}$*-refinable function* $\tilde{\mathbf{h}} \in L_q^N(\mathbb{R}^s)$, $1/p + 1/q = 1$, *which is dual to* $\mathbf{h}$ *if and only if the stationary subdivision scheme* $S_{\mathbf{B},M}$ *converges in* ℓ_q.

Proof. Suppose that $\tilde{\mathbf{h}} \in L_q^N(\mathbb{R}^s)$ is an admissible $\mathbf{B}$-refinable function which is dual to $\mathbf{h}$. By Proposition 2.4, $\tilde{\mathbf{h}}$ is linearly independent. By Theorem 3.3, $S_{\mathbf{B},M}$ converges in ℓ_q.

Conversely, suppose that $\mathbf{B}$ satisfies (4.1) and that $S_{\mathbf{B},M}$ converges in ℓ_q. Let $\mathbf{H} = \mathbf{G}^T \in L_p^{N\times N}(\mathbb{R}^s)$ and let $\tilde{\mathbf{H}} = \tilde{\mathbf{G}}^T \in L_q^{N\times N}(\mathbb{R}^s)$ be the functions from Theorem 3.1 associated with the schemes $S_{\mathbf{A},M}$, $S_{\mathbf{B},M}$, respectively. Let $(\delta\mathbf{I})_\alpha := \delta_{0,\alpha}\mathbf{I}$ and consider

$$\begin{aligned}\langle \mathbf{H}, \tilde{\mathbf{H}}(\cdot - \alpha)\rangle &= \langle \mathbf{G}^T, \tilde{\mathbf{G}}(\cdot-\alpha)^T - (sc_M^k(S_{\mathbf{B},M}^k\delta\mathbf{I})_{\cdot - M^k\alpha}\Sigma g)^T\rangle \\ &\quad + \langle (\mathbf{G}^T - sc_M^k(S_{\mathbf{A},M}^k\delta\mathbf{I})\Sigma g)^T, (sc_M^k(S_{\mathbf{B},M}^k\delta\mathbf{I})_{\cdot - M^k\alpha}\Sigma g)^T\rangle \\ &\quad + \langle (sc_M^k(S_{\mathbf{A},M}^k\delta\mathbf{I})\Sigma g)^T, (sc_M^k(S_{\mathbf{B},M}^k\delta\mathbf{I})_{\cdot - M^k\alpha}\Sigma g)^T\rangle \\ &=: I(k) + II(k) + III(k).\end{aligned}$$

By Theorem 3.3, $S_{\mathbf{A},M}$ converges in ℓ_p. Since Proposition 3.1 ensures the uniform boundedness of the operators $S_{\mathbf{B},M}^k$ the terms $I(k)$ and $II(k)$ tend to zero when $k \to \infty$ provided that g is a p- as well as a q-test function. Here we will exploit Corollary 3.1 which allows us to choose any suitable test function. In fact, we choose

$$g := \chi_\Omega.$$

where Ω is a tile of measure 1 satisfying (2.23). g is a p-test function for *any* p which, in particular, satisfies

$$\langle g, g(\cdot - \alpha)\rangle = \delta_{0,\alpha}, \qquad \alpha \in \mathbb{Z}^s. \tag{4.2}$$

From (4.2) we readily conclude that

$$III(k) = m^{-k}\sum_{\gamma\in\mathbb{Z}^s}(S_{\mathbf{A},M}^k\delta\mathbf{I})_\gamma^T\overline{(S_{\mathbf{B},M}^k\delta\mathbf{I})}_{\gamma - M^k\alpha}.$$

We claim that

$$III(k) = \delta_{0,\alpha}\mathbf{I}. \tag{4.3}$$

In fact, for $k = 1$, we obtain

$$m^{-1}\sum_{\gamma\in\mathbb{Z}^s}(S_{\mathbf{A},M}\delta\mathbf{I})_\gamma^T\overline{(S_{\mathbf{B},M}\delta\mathbf{I})}_{\gamma - M\alpha} = \sum_{\gamma\in\mathbb{Z}^s}\mathbf{A}_\gamma^T\mathbf{B}_{\gamma - M\alpha}^* = \delta_{0,\alpha}\mathbf{I}, \tag{4.4}$$

where we have used that $\mathbf{A}$ and $\mathbf{B}$ are dual (see (4.1)). Using induction on k we get

$$\begin{aligned}&m^{-k}\sum_{\gamma\in\mathbb{Z}^s}(S_{\mathbf{A},M}^k\delta\mathbf{I})_\gamma^T\overline{(S_{\mathbf{B},M}^k\delta\mathbf{I})}_{\gamma - M^k\alpha} \\ &\quad = m^{-k}\sum_{\gamma\in\mathbb{Z}^s}\left(\sum_{\beta,\nu\in\mathbb{Z}^s}(S_{\mathbf{A},M}^{k-1}\delta\mathbf{I})_\beta^T\mathbf{A}_{\gamma - M\beta}\mathbf{B}_{\gamma - M(M^{k-1}\alpha+\nu)}^*\overline{(S_{\mathbf{B},M}^{k-1}\delta\mathbf{I})}_\nu\right) \\ &\quad = m^{-k+1}\sum_{\beta,\nu\in\mathbb{Z}^s}(S_{\mathbf{A},M}^{k-1}\delta\mathbf{I})_\beta^T\left(m^{-1}\sum_{\gamma\in\mathbb{Z}^s}\mathbf{A}_{\gamma - M\beta}\mathbf{B}_{\gamma - M\nu}^*\right)\overline{(S_{\mathbf{B},M}^{k-1}\delta\mathbf{I})}_{\nu - M^{k-1}\alpha} \\ &\quad = m^{-k+1}\sum_{\beta\in\mathbb{Z}^s}(S_{\mathbf{A},M}^{k-1}\delta\mathbf{I})_\beta^T\overline{(S_{\mathbf{B},M}^{k-1}\delta\mathbf{I})}_{\beta - M^{k-1}\alpha} = \delta_{0,\alpha}\mathbf{I},\end{aligned}$$

where we have used (4.4) and the induction hypothesis. This confirms (4.3) and hence

$$\langle \mathbf{H}, \tilde{\mathbf{H}}(\cdot - \alpha)\rangle = \delta_{0,\alpha}\mathbf{I}. \tag{4.5}$$

Since $\mathbf{h}$ is, by assumption, linearly independent and hence stable Theorem 3.2 states that $\mathbf{H}(x) = \mathbf{h}(x)\mathbf{y}^T$ where $\mathbf{y}$ is the unique common eigenvector of unit length of the matrices $\mathbf{A}_e(\mathbf{1})$, $e \in E$. Thus (4.5) becomes

$$\int_{\mathbb{R}^s} \mathbf{h}(x)\mathbf{y}^T\overline{\tilde{\mathbf{G}}(x-\alpha)}\,dx = \delta_{0,\alpha}\mathbf{I},$$

which means that

$$\tilde{\mathbf{h}}(x) := \tilde{\mathbf{G}}(x)^T\overline{\mathbf{y}}$$

is dual to $\mathbf{h}$. Since by Theorem 3.1, (3.21),

$$\tilde{\mathbf{G}}^T = sc_M(\mathbf{B}\Sigma\tilde{\mathbf{G}}^T),$$

multiplication by $\overline{\mathbf{y}}$ yields

$$\tilde{\mathbf{h}} = sc_M(\mathbf{B}\Sigma\tilde{\mathbf{h}}),$$

i.e., $\tilde{\mathbf{h}}$ is $\mathbf{B}$-refinable. By Proposition 3.5, $\tilde{\mathbf{h}}$ has compact support and Proposition 2.4(i) ensures that $\tilde{\mathbf{h}}$ is linearly independent and hence stable. Thus Theorem 3.2 implies that $\tilde{\mathbf{H}}$ is also a rank-one matrix, namely

$$\tilde{\mathbf{H}}(x) = \tilde{\mathbf{h}}(x)\overline{\mathbf{y}}^T$$

(see also Proposition 2.8 (2.45)). Since we have established all the asserted properties of $\tilde{\mathbf{h}}$ the proof is complete. ■

We will confine the remaining discussion to the case $p = q = 2$. Given an admissible $\mathbf{A}^0$-refinable function $\mathbf{h}$, we adhere to the notation used in Section 2 and apply Proposition 2.7 to conclude that there exists an $(mN \times mN)$-matrix $\mathbf{A}(z)$ which contains the subsymbols $\mathbf{A}_e^0$, $e \in E$, as its first block row which has constant determinant on $(\mathbb{C}\backslash\{0\})^s$ and whose entries are Laurent polynomials. Let $\mathbf{B}(z) = (\mathbf{B}_{e'}^e)_{e,e'\in E}$ be defined by (2.43). By construction, the mask $\mathbf{B}^0$ defined by $\mathbf{B}^0(z) = \sum_{e\in E} z^e\mathbf{B}_e^0(z)$ is dual to $\mathbf{A}^0$. By Theorem 4.1, we get a dual $\mathbf{B}^0$-refinable function $\tilde{\mathbf{h}}$ if and only if $S_{\mathbf{B}^0,M}$ converges in ℓ_2. If this is the case, the masks $\mathbf{A}^e$, $\mathbf{B}^e$, $e \in E_* := E\backslash\{0\}$, corresponding to the subsequent block rows of $\mathbf{A}(z)$ and $\mathbf{B}(z)$, respectively, give rise to a biorthogonal system as described in Proposition 2.6.

Of course, it is not clear whether the completion $\mathbf{A}(z)$ of the first block row given by $\mathbf{A}^0$ gives rise to a $\mathbf{B}^0$ such that $S_{\mathbf{B}^0,M}$ converges in ℓ_2. A possible strategy for dealing with this difficulty can be sketched as follows. One can try to employ the concept from [CDP] to modify the masks $\mathbf{A}^e$, $e \in E_*$, so as to ensure convergence of $S_{\mathbf{B}^0,M}$. It remains then to show that if $S_{\mathbf{B}^0,M}$ converges in ℓ_2 then the limits actually belong to some Sobolev space $H^t(\mathbb{R}^s)$ of positive index $t > 0$. One can then resort to the general stability criterion from [D] to conclude that the dilates of the corresponding biorthogonal systems from Proposition 2.6 form Riesz bases for $L_2(\mathbb{R}^s)$. This issue will be taken up in a forthcoming paper.

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References

[BM] M. D. BUHMANN, C. A. MICCHELLI (1994): *Using* 2*-slanted matrices for subdivision*. Proc. London Math. Soc., **69**:428–448.

[CDM] A. S. CAVARETTA, W. DAHMEN, C. A. MICCHELLI (1991): Stationary Subdivision. Memoirs Amer. Math. Soc., Vol. **93**, No. 453.

[CDP] J. M. CARNICER, W. DAHMEN, J. M. PEÑA (1996): *Local decompositions of refinable spaces*. Appl. Comput. Harmonic Anal., **3**:127–153.

[CL] C. K. CHUI, C. LI (1994): *A general framework of multivariate wavelets with duals*. Appl. Comput. Harmonic Anal., **1**:368–390.

[CD] A. COHEN, I. DAUBECHIES (1993): *Nonseparable bidimensional wavelet bases*. Revista Mat. Iberoamericana, **9**:51–137.

[CDF] A. COHEN, I. DAUBECHIES, J.-C. FEAUVEAU (1992): *Biorthogonal bases of compactly supported wavelets*. Comm. Pure Appl. Math., **45**:485–560.

[CS] A. COHEN, SCHLENKER (1993): *Compactly supported bi-dimensional wavelet bases with hexagonal symmetry*. Const. Approx., **9**:209–236.

[DDL] S. DAHLKE, W. DAHMEN, V. LATOUR (1994): *Smooth refinable functions by convolution products*. Appl. Comput. Harmonic Anal., **2**:68–84.

[D] W. DAHMEN (1996): *Stability of multiscale transformations*. IGPM Report 109, Nov. 1994. J. Fourier Anal. Appl., **2**.

[DM1] W. DAHMEN, C.A. MICCHELLI (1984): *Subdivision algorithms for box spline surfaces*. Comput. Aided Geom. Design, **1**:115–129.

[DM2] W. DAHMEN, C.A. MICCHELLI (1985): *On the local linear independence of the translates of a box spline*. Studia Math., **82**:243–263.

[DM3] W. DAHMEN, C. A. MICCHELLI (1993): *Continuous refinement equations and subdivision*. Adv. Comput. Math., **1**:1–37.

[DGH] G. C. DONOVAN, J. S. GERONIMO, D. P. HARDIN (1994): *Intertwining multiresolution analyses and the construction of piecewise polynomial wavelets*. Preprint 94-017, Department of Mathematics, Vanderbilt University.

[F] G. B. FOLLAND (1984): Real Analysis. New York: Wiley.

[G] T. N. T. GOODMAN (1993): *Construction of wavelets with multiplicity*. Preprint.

[GL] T. N. T. GOODMAN, S. L. LEE (1994): *Wavelets of multiplicity r*. Trans. Amer. Math. Soc., **342**:307–324.

[GM] K. H. GRÖCHENIG, W. R. MADYCH (1992): *Multiresolution analysis, Haar bases, and self-similar tilings of* $\mathbb{R}^n$. IEEE Trans. Inform. Theory, **38**:556–568.

[H] L. HERVÉ (1994): *Multi-resolution analysis of multiplicity d, application to dyadic interpolation*. Appl. Comput. Harmonic Anal., **1**:299–315.

[J] R. Q. JIA (to appear): *Subdivision schemes in L_p spaces*. Adv. in Comput. Math.

[JM1] R. Q. JIA, C. A. MICCHELLI (1991): *Using the refinement equation for the construction of pre-wavelets II: Powers of two*. In: Curves and Surfaces (P. J. Laurent, A. LeMéhauté, L. L. Schumaker, eds.). New York: Academic Press, pp. 209–246.

[JM2] R. Q. JIA, C. A. MICCHELLI (1992a): *On linear independence of integer translates of a finite number of functions*. Proc. Edinburgh Math. Soc., **36**:69–85.

[JM3] R. Q. JIA, C.A. MICCHELLI (1992b): *Using the refinement equation for the construction of pre-wavelets V: Extensibility of trigonometric polynomials*. Computing, **48**:61–72.

[KV] A. KOROIU, R. VAILANCOURT (1994): *Nonseparable biorthogonal wavelet bases on* $L^2(\mathbb{R}^n)$. Preprint.

[L] T. L. LAM (1978): Serre's Conjecture. Lecture Notes in Mathematics, Vol. 635. Berlin: Springer-Verlag.

[LC] R. LONG, D. CHEN (1994): *Biorthogonal wavelet bases on* $\mathbb{R}^d$. Preprint.

[M] J. L. MERRIEN (1992): *A family of Hermite interpolants by bisection algorithms*. Numer. Algor., **2**:187–200.

[M1] C. A. MICCHELLI (1992a): *Using the refinement equation for the construction of pre-wavelets*. In Approximation Theory, Spline Functions and Applications (S. P. Singh, ed.). Amsterdam: Kluwer Academic, pp. 213–222.

[M2] C. A. MICCHELLI (1992b): *Banded matrices with banded inverses*. J. Comput. Appl. Math., **41**:281–300.

[MX] C. A. MICCHELLI, Y. S. XU (1994): *Using the matrix refinement equation for the construction of wavelets on invariant sets*. Appl. Comput. Harmonic Anal., **4**:391–401.

[P1] G. PLONKA (1994): *Approximation order of shift-invariant subspaces of $L^2(\mathbb{R})$ generated by refinable functions*. Preprint, Universität Rostock.

[P2] G. PLONKA (1995): *Two scale symbol and autocorrelation symbol for B-splines with multiple knots*. Adv. in Comput. Math., **3**:1–22.

[SB] J. STOER, R.BULIRSCH (1980): Introduction to Numerical Analysis. New York: Springer-Verlag.

[SS] G. STRANG, V. STRELA (1994): *Short wavelets and matrix dilation equations*. Preprint.

[S] R. S. STRICHARTZ (1993): *Wavelets and self-affine tilings*. Wavelets and self-affine tilings, Const. Approx., **9**:327–346.

[U] K. URBAN (1995): *On divergence free wavelets*. Adv. in Comput. Math., **4**:51–82.

[W] VAN DER WAERDEN (1970): Modern Algebra, Vol. II, 7th ed. New York: Frederich Ungar.

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