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CONSTRUCTIVE APPROXIMATION

Polynomial Inequalities on Measurable Sets and Their Applications

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Abstract. We study Pólya- and Remez-type inequalities for univariate and multivariate polynomials and discuss their applications to Nikolskii-type inequalities and upper estimates of trigonometric integrals.

1. Introduction

Polynomial inequalities on measurable sets, that is, Pólya- and Remez-type inequalities, play an important role in many areas of Analysis. In the 1920s–1930s, Pólya and Remez initiated the study of polynomial inequalities on measurable sets in $\mathbf{R}^1$ by proving the following results:

Theorem 1.1 (Pólya [7], [46], [50]). *For a measurable set $E \subseteq \mathbf{R}^1$, $0 < |E| < \infty$, and a real polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$:*

$$|a_n| \le \tfrac{1}{2}(4/|E|)^n \sup_{x\in E} |P(x)|. \tag{1.1}$$

Equality in (1.1) *holds if and only if E is an interval $[a, a+\lambda]$ and $P(x) = AT_n(2(x-a)/\lambda - 1)$, where $a \in \mathbf{R}^1$, $A \in \mathbf{R}^1$, and $\lambda > 0$. Here and in the sequel*

$$T_n(x) = \tfrac{1}{2}((x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n)$$

is the Chebyshev polynomial of degree n.

Theorem 1.2 (Remez [11], [26], [40], [41], [47]). *For a measurable set $E \subseteq [a, b]$, $|E| > 0$, and a real polynomial P of degree n:*

$$\max_{x\in[a,b]} |P(x)| \le T_n(2(b-a)/|E| - 1) \sup_{x\in E} |P(x)|. \tag{1.2}$$

Equality in (1.2) *holds if and only if $E = [a, a+\lambda]$ and $P(x) = AT_n(2(x-a)/\lambda - 1)$*

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or $E = [b - \lambda, b]$ *and* $P(x) = AT_n(2(b-x)/\lambda - 1)$, *where* $A \in \mathbf{R}^1$ *and* $0 < \lambda < b - a$.

In the special case when E is an interval, inequalities (1.1) and (1.2) were obtained by Chebyshev [50, pp. 67–68]. These results, their generalizations, and applications in Analysis have received much attention since the 1970s [2]–[5], [8]–[16], [20]–[27], [29]–[33], [35], [36], [39]–[44], [49], [51].

In this paper, we establish new Pólya- and Remez-type inequalities for univariate and multivariate polynomials and apply them to some problems of Analysis.

The paper is organized as follows: Sections 2 and 3 contain inequalities on measurable sets for algebraic and trigonometric polynomials of a single variable. In particular, the main result of Section 2 is a combined version of Theorems 1.1 and 1.2. We also present a new Remez-type inequality for even polynomials. In Section 3, we obtain upper estimates of the constants in Remez-type inequalities for trigonometric and exponential polynomials. Note that the proofs of the main results in Sections 2 and 3 are based on the shift method developed in [7], [15], [22], [46]. In Section 4, we establish new Pólya- and Remez-type inequalities for multivariate polynomials. In Section 5, we discuss applications to Nikolskii-type inequalities for polynomials and entire functions of exponential type in rearrangement-invariant spaces. We also apply the Remez-type inequalities, established in Sections 2 and 4, to upper estimates of some trigonometric integrals.

1.1. *Notation and Definitions*

We use the following notation.

Let $\mathbf{R}^m$ be the m-dimensional Euclidean space; $\mathbf{C}^m := \mathbf{R}^m + i\mathbf{R}^m$ the m-dimensional complex space; $\mathbf{Z}^m$ the set of all integral lattice points in $\mathbf{R}^m$; $S^m := \{x \in \mathbf{R}^m : |x| = 1\}$ the unit $(m-1)$-dimensional sphere in $\mathbf{R}^m$; $K^m := \{x \in \mathbf{R}^m : 0 \le x_i \le 1,\ 1 \le i \le m\}$ the unit cube in $\mathbf{R}^m$; $|E| = |E|_k$, the k-dimensional Lebesgue measure of a k-measurable set $E \in \mathbf{R}^m$, $1 \le k \le m$; and χ_E the characteristic function of $E \subseteq \mathbf{R}^m$.

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{Z}^m, \alpha_i \ge 0, 1 \le i \le m$, we set $|\alpha| := \sum_{i=1}^m \alpha_i$, $x^\alpha := x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, $\partial^\alpha := (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_m)^{\alpha_m}$.

Let $\mathcal{P}^*_{n,m}$ be the class of all algebraic polynomials $P(x) = \sum_{\alpha_1=0}^n \cdots \sum_{\alpha_m=0}^n a_\alpha x^\alpha$ with real coefficients; $\mathcal{P}_{n,m}$ the subset of $\mathcal{P}^*_{n,m}$ of all algebraic polynomials $P(x) = \sum_{|\alpha|\le n} a_\alpha x^\alpha$ of degree n; and $\mathcal{T}_n$ the class of all trigonometric polynomials of a single variable of degree n with real coefficients.

Further, let $C(\Omega)$ be the real space of all real-valued continuous functions f on $\Omega \subseteq \mathbf{R}^m$ with the finite norm $\|f\|_{C(\Omega)} := \sup_{x\in\Omega} |f(x)|$, and let $L_p(\Omega)$, $0 \le p \le \infty$, be the space of all measurable functions f on $\Omega \subseteq \mathbf{R}^m$ such that $\|f\|_{L_0(\Omega)} := \exp(|\Omega|_m^{-1} \int_\Omega \ln |f(x)|\, dx) < \infty$, $\|f\|_{L_p(\Omega)} := (\int_\Omega |f(x)|^p\, dx)^{1/p} < \infty$ if $0 < p < \infty$, and $L_\infty(\Omega) := C(\Omega)$.

Throughout the paper $C, C_1, C_2, \dots$ denote positive constants independent of n, σ, λ, x, y, $|E|$, $b - a$, $|V|$, P, Q, g, F. The same symbol does not necessarily denote the same constant in different occurrences.

We shall often refer to the Remez inequalities $\|P\|_{C(\Omega)} \le Ch(|E|_k)\|P\|_{C(E)}$ as being the inequalities of first or second type, while the order of decreasing $h(|E|_k)$ is sharp for $|E|_k \to 0$ or $|E|_k \to |\Omega|_k$, respectively.

2. Pólya- and Remez-Type Inequalities for Univariate Polynomials

2.1. *Pólya- and Remez-Type Inequalities*

Inequalities (1.1) and (1.2) have been generalized in different directions in [2], [4], [7], [11], [21], [22], [23], [29], [30], [31]. In particular, Bernstein [7, p. 49] extended (2.1) to functions f satisfying the condition $\inf_{x\in[a,b]}|f^{(n)}(x)| > 0$:

$$\inf_{x\in[a,b]}|f^{(n)}(x)| \le \tfrac{1}{2}n!\,(4/|E|_1)^n\|f\|_{C(E)}. \tag{2.1}$$

Bachtin [4] independently proved (2.1) using a different method which is close to that used in [46]. Some Pólya-type inequalities, including a weaker form of (2.1), and their applications in Number Theory were obtained by Arkhipov, Karatsuba, and Chubarikov [2, pp. 12–26].

Theorem 1.2 had been little-known until the 1970s when it appeared in [26], and its multidimensional generalizations have been established in [15], [29], [31]. This is the reason why the theorem was proved independently by Dudley and Randol [20] and Brudnyi and the author [15]. The various proofs of Theorem 1.2 were given in [8], [11], [15], [20], [21], [26], [40], [41]. Apparently, the shortest proof was found by Remez [47] (see also Bojanov [8]).

The author [29], [30], [31] noticed that (1.2) implies a more general inequality

$$\|P\|_{C(E_2)} \le T_n(2|E_2|/|E_1|-1)\|P\|_{C(E_1)}, \qquad E_1 \subseteq E_2, \quad P \in \mathcal{P}_{n,1}, \tag{2.2}$$

and established some local versions of Nikolskii's inequality, based on (2.2).

In the late 1980s–early 1990s, Erdélyi [11], [21], [22], [23] has proved Remez's inequality for the generalized polynomials of the form $f(x) = C\prod_{i=1}^{n}|x-z_i|^{\alpha_i}$, $\alpha_i \in \mathbf{R}^1$, $z_i \in \mathbf{C}$, $1 \le i \le n$.

In this section we prove Theorems 2.1 and 2.2 which are combined versions of (1.1) and (1.2). As simple corollaries of these results, we obtain Pólya's and Remez's inequalities. A version of Remez's inequality for even polynomials (Theorem 2.3) and some corollaries are also presented.

Theorem 2.1. *For a measurable set $E \subseteq [a,b]$, $|E| > 0$, and a polynomial $P \in \mathcal{P}_{n,1}$:*

$$\begin{aligned}(2.3)\quad &\max(|P^{(k)}(a)|, |P^{(k)}(b)|)\\ &\qquad \le (2/|E|)^k T_n^{(k)}(2(b-a)/|E|-1)\|P\|_{C(E)}, \qquad k=0,1,\ldots,n.\end{aligned}$$

Equality in (2.3) *holds if and only if* $E=[a,a+\lambda]$ *and* $P(x) = AT_n(2(x-a)/\lambda-1)$ *or* $E=[b-\lambda,b]$ *and* $P(x) = AT_n(2(b-x)/\lambda-1)$, *where* $\lambda \in (0,b-a]$, $k+b-a-\lambda > 0$, *and* $A \in \mathbf{R}^1$.

It is easy to show by a linear substitution that Theorem 2.1 is equivalent to the following generalized Pólya inequality:

Theorem 2.2. *For a measurable set $E \subseteq [0,b]$, $|E| = \lambda > 0$, $b > 0$, and a polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$:*

$$|a_k| \le |A_k|\|P\|_{C(E)}, \tag{2.4}$$

where A_k, $0 \le k \le n$, *are the coefficients of the polynomial* $T_n(2(b-x)/\lambda - 1) = \sum_{k=0}^{n} A_k x^k$. *Equality in* (2.4) *holds if and only if* $E = [b-\lambda, b]$ *and* $P(x) = AT_n(2(b-x)/\lambda - 1)$, *where* $\lambda \in (0, b]$, $k + b - \lambda > 0$, *and* $A \in \mathbf{R}^1$.

Theorem 1.1 immediately follows from Theorem 2.2 for $k = n$, while Remez's inequality (1.2) is a consequence of (2.3) for $k = 0$, since for every $x \in [a, b]$:

$$\begin{aligned}|P(x)| &\le \min(T_n(2(x-a)/|E \cap [a,x]| - 1), T_n(2(b-x)/|E \cap [x,b]| - 1))\|P\|_{C(E)} \\ &\le T_n(2(b-a)/|E| - 1)\|P\|_{C(E)}.\end{aligned}$$

The following theorem is an analogue of Theorem 2.2 for even polynomials and $k = 0$.

Theorem 2.3. *For a measurable set* $E \subseteq [0, b]$, $|E| = \lambda/2 > 0$, *and an even polynomial* $P \in \mathcal{P}_{2n,1}$:

$$|P(0)| \le T_n\left(\frac{8b^2}{\lambda(4b-\lambda)} - 1\right)\|P\|_{C(E)}. \tag{2.5}$$

Equality in (2.5) *holds if and only if* $E = [b - \lambda/2, b]$ *and*

$$P(x) = AT_n\left(\frac{2x^2 - b^2 - (b-\lambda/2)^2}{b\lambda - \lambda^2/4}\right), \qquad A \in \mathbf{R}^1.$$

A new version of the Pólya inequality and a Remez inequality of second type are presented below.

Corollary 2.1. *For a measurable set* $E \subseteq [a, b]$, $|E| = \lambda > 0$, *and a polynomial* $P(x) = \sum_{k=0}^{n} a_k x^k$:

$$\sum_{k=0}^{n} |a_k|(b-a)^k \le T_n(4(b-a)/\lambda - 1)\|P\|_{C(E)}. \tag{2.6}$$

Equality in (2.6) *holds if and only if* E *and* P *are the extremal elements defined in Theorem* 2.1.

Corollary 2.2. *If* $\varepsilon \in [0, \frac{1}{2})$, *then for a measurable set* $E \subseteq [-b, b]$, *satisfying* $|E| = \lambda$, $(\frac{3}{2} - \varepsilon)b \le \lambda \le 2b$, *and a polynomial* $P \in \mathcal{P}_{n,1}$:

$$|P(0)| \le e^{4n(1-\lambda/2b)/(1-2\varepsilon)}\|P\|_{C(E)}. \tag{2.7}$$

Remark 2.1. An estimate $|P(0)| \le \exp(An(1 - \lambda/2b))\|P\|_{C(E)}$, where A is an absolute constant, follows from a poinwise Remez inequality of second type established by Erdélyi [22]. Inequality (2.7) shows that $A \le 4$ for $\varepsilon = 0$.

Remark 2.2. Brudnyi [14] announced that for $P \in \mathcal{P}_{n,1}$ and $E \subseteq [a, b]$, $|E| > 0$:

$$\|P^{(k)}\|_{C(a,b)} \le (4/|E| - 2/(b-a))^k T_n^{(k)}(2(b-a)/|E| - 1)\|P\|_{C(E)}.$$

Taking account of (2.3), there is very good reason to believe that the following V. A. Markov–Remez inequality is valid:

Conjecture 2.1. *For a measurable $E \subseteq [a, b]$ and a polynomial $P \in \mathcal{P}_{n,1}$:*

$$\|P^{(k)}\|_{C(a,b)} \le (2/|E|)^k T_n^{(k)}(2(b-a)/|E|-1)\|P\|_{C(E)}. \tag{2.8}$$

Equality in (2.8) *holds if and only if E and P are the extremal elements from Theorem* 2.1.

Note that there is no a simple proof of (2.8) even for $E = [a, b]$.

2.2. *Proofs of the Theorems and Corollaries*

Proof of Theorem 2.2. Let E be a subset of $[a, b]$, $|E| = \lambda > 0$, and let $P(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial. Without loss of generality, we may assume that E is a closed set, and $a_k \ne 0$ for a fixed k, $0 \le k \le n$. If $\lambda = b$, then (2.4) was proved by V. A. Markov [11, p. 254]. So we assume $0 < \lambda < b$.

Let $E = [b_1 - \lambda, b_1]$ be a subinterval of $[0, b]$. If $b_1 - \lambda > 0$, then using Chebyshev's inequality [50, p. 68], we obtain

$$\begin{aligned} |a_k| &\le (1/k!)|(d^k/dx^k)(T_n((2x-2b_1+\lambda)/\lambda))|_{x=0}\|P\|_{C(E)} \\ &\le (1/k!)(2/\lambda)^k T_n^{(k)}(2b/\lambda - 1)\|P\|_{C(E)} = |A_k|\|P\|_{C(E)}. \end{aligned} \tag{2.9}$$

If $b_1 - \lambda = 0$, then V. A. Markov's inequality [11, p. 254] shows that (2.9) remains valid. The equality in (2.9) holds if and only if $E = [b - \lambda, b]$ and $P = AT_n(2(b-x)/\lambda - 1)$.

Now, suppose that E is not a subinterval of $[0, b]$. It is known [11, p. 103] that $\{1, x, \ldots, x^n\}$ is a Descartes system on $(0, b)$. Thus using the necessity of Chebyshev–Bernstein's theorem [11, p. 94], we obtain that for a fixed k, $0 \le k \le n$, there exist the only polynomial $P_{k,E} \in \mathcal{P}_{n,1}$ with the kth coefficient a_k and a set of points $\{u_j\}_{j=1}^{n+1}$, $u_j \in E$, $1 \le j \le n+1$, $u_1 < u_2 < \cdots < u_{n+1}$, satisfying the relations

$$\begin{aligned} \|P_{k,E}\|_{C(E)} &= \inf_{c_i, i\ne k} \max_{x\in E} \left| a_k x^k - \sum_{i=0, i\ne k}^{n} c_i x^k \right|, \\ |P_{k,E}(u_j)| &= \|P_{k,E}\|_{C(E)}, \qquad 1 \le j \le n+1, \\ P_{k,E}(u_j) &= -P_{k,E}(u_{j+1}), \qquad 1 \le j \le n. \end{aligned} \tag{2.10}$$

Let us put $E_1 := \{x \in [a, b] : |P_{k,E}(x)| \le \|P_{k,E}\|_{C(E)}\}$. Then

$$E \subseteq E_1, \qquad |E| \le |E_1|, \qquad \|P_{k,E}\|_{C(E)} = \|P_{k,E}\|_{C(E_1)}. \tag{2.11}$$

Relations (2.11) and the sufficiency of Chebyshev–Bernstein's theorem [11, p. 94] yield $P_{k,E_1} = P_{k,E}$. If E_1 is a subinterval of $[a, b]$, then $|E_1| > |E|$ and $\|P_{k,E}\|_{C(E)} = \|P_{k,E_1}\|_{C(E_1)}$. Hence

$$\begin{aligned} |a_k| &\le (1/k!)|P_{k,E_1}(0)|\|P_{k,E}\|_{C(E)} \\ &\le (1/k!)(2/|E_1|)^k T_n^{(k)}(2b/|E_1| - 1)\|P\|_{C(E)} < |A_k|\|P\|_{C(E)}. \end{aligned} \tag{2.12}$$

If E_1 is not a subinterval of $[0, b]$, then the following properties of E_1 hold:

(a) there exists a family of intervals $\Phi = \{[a_j, b_j]\}_{j=1}^r$, $2 \le r \le n$, such that $E_1 = \cup_{j=1}^r [a_j, b_j]$, $0 \le a_1 \le b_1 < a_2 \le b_2 < \cdots < a_r \le b_r$;
(b) there exist n simple zeros $x_1, x_2, \dots, x_n$ of P_{k,E_1} that lie in E_1;
(c) there exist at least two intervals from Φ that contain zeros of P_{k,E_1}.

Property (b) follows from (2.10) and the definition of E_1, while property (a) is evident. To prove property (c) we assume that for some j, $1 \le j \le r$, all $x_i \in [a_j, b_j]$, $1 \le i \le n$. If $a_j > 0$, then $|P_{k,E_1}(a_j)| = \|P_{k,E_1}\|_{C(E_1)}$. Without loss of generality, we may assume that $P_{k,E_1}(a_j) > 0$. Then the inequality $P'_{k,E_1}(a_j) \ge 0$ is impossible since, in this case, either P_{k,E_1} or P'_{k,E_1} has a zero on $(-\infty, a_j]$. Thus P_{k,E_1} is decreasing on $(-\infty, a_j]$, and $(-\infty, a_j) \cap E_1 = \emptyset$ for $a_j \ge 0$. Similarly we prove that $(b_j, \infty) \cap E_1 = \emptyset$ for $b_j \le b$. Hence $E_1 = [a_j, b_j]$, and this contradicts our assumption that E_1 is not a subinterval of $[0, b]$. Property (c) follows.

Now we construct a set E_2 and a polynomial P^* satisfying certain conditions. Let us put $E_2 := \bigcup_{j=1}^r [a_j + \tau_j, b_j + \tau_j] = [b_r - |E_1|, b_r]$, where $\tau_j = \sum_{k=j}^{r-1}(a_{k+1} - b_k)$, $1 \le j \le r-1$; $\tau_r = 0$. Next, let $x_i^* = x_i + \tau_j$, where $j = j(i)$ is the index of an interval $[a_j, b_j]$ which contains x_i, $1 \le i \le n$. Note that $x_i^* \in E_2$ and $x_i^* \ge x_i$, $1 \le i \le n$, by properties (a) and (b). Moreover, there is an integer i such that $x_i^* > x_i$, by property (c). Let us put $P_1(y) = C_0 A \prod_{i=1}^n (y - x_i^*)$, where A is the leading coefficient of P_{k,E_1} and $C_0 = a_k/\alpha_k$. Here α_k is the kth coefficient of $(1/C_0)P_1$. Then P_1 is a polynomial of degree n, and its kth coefficient is a_k. Now applying Viete's theorem, we obtain $|a_k| < |\alpha_k|$, that is, $|C_0| < 1$. Next, note that for each $y \in E_2$ there exists $x \in E_1$ such that for all i, $|y - x_i^*| \le |x - x_i|$. Hence,

$$\|P_1\|_{C(E_2)} \le |C_0| \|P_{k,E_1}\|_{C(E_1)} < \|P_{k,E_1}\|_{C(E_1)}.$$

Thus

$$(2.13)\qquad |E_2| = |E_1| \ge \lambda, \qquad \|P_{k,E_2}\|_{C(E_2)} < \|P_{k,E_1}\|_{C(E_1)} = \|P_{k,E}\|_{C(E)}.$$

Next using (2.13) and Chebyshev–V. A. Markov's inequality for the interval E_2 and for the polynomial P_{k,E_2}, we obtain

$$(2.14)\qquad \begin{aligned} |a_k| &\le (1/k!)|P_{k,E_2}^{(k)}(0)| \|P_{k,E_2}\|_{C(E_2)} \\ &< (1/k!)(2/\lambda)^k T_n^{(k)}(2b/\lambda - 1)\|P_{k,E}\|_{C(E)} \le |A_k| \|P\|_{C(E)}. \end{aligned}$$

Finally, we deduce from inequalities (2.9), (2.12), and (2.14) that (2.4) holds, and the interval $E = [b - \lambda, b]$ is the only extremal set. ■

Proof of Theorem 2.3. The proof is similar to that of Theorem 2.2 with the following changes to yield. Note first that $\{x^{2k}\}_{k=1}^n$ is a Chebyshev system on $(0, b)$. Next, Chebyshev's theorem [11, p. 94] implies (2.5) for each interval $E = [a, a + \lambda/2] \subseteq [0, b]$. Then we take into account the relation $P'_{k,E_1}(0) = 0$ to prove property (c). Further, we replace the set E_2 and the polynomial P_1 in the proof of Theorem 2.2 by the set $E_3 = [b_r - \lambda_1/2, b_r]$ and by the polynomial

$$P_2(y) = A \prod_{i=1}^n (x_i^2/x_i^{*2})(y^2 - x_1^{*2})(y^2 - x_i^{*2}) \cdots (y^2 - x_n^{*2}),$$

respectively, where A is the leading coefficient of P_{0,E_1}. Then $P_2 \in \mathcal{P}_{n,1}$ and $|P_2(0)| = |P(0)|$. Finally, it remains to prove that

$$\|P_2\|_{C(E_3)} < \|P_{0,E_1}\|_{C(E_1)}. \tag{2.15}$$

Indeed, let $y \in E_3$, say $y \in [a_k + \tau_k, b_k + \tau_k]$. Then $x = y - \tau_k \in E_1$, and putting $\tau_{i,k} = \min(\tau_{j(i)}, \tau_k)$, where $j(i)$ is the index of an interval $[a_j, b_j]$ which contains x_i, we obtain

$$\begin{aligned} |P_2(y)| &= |A| \prod_{i=1}^{n} \frac{x_i^2 |y^2 - x_i^{*2}|}{x_i^{*2}} = \prod_{i=1}^{n} \frac{x_i^2 |(x + \tau_k)^2 - (x_i + \tau_{j(i)})^2|}{(x_i + \tau_{j(i)})^2} \\ &\le |A| \prod_{i=1}^{n} \frac{x_i^2 |(x + \tau_{i,k})^2 - (x_i + \tau_{i,k})^2|}{(x_i + \tau_{i,k})^2} < |A| \prod_{i=1}^{n} |x^2 - x_i^2| = |P_{0,E_1}(x)|. \end{aligned}$$

This yields (2.15). ■

Proof of Corollary 2.1. Without loss of generality, we may assume that $a = 0$. Then applying Theorem 2.2 and Descartes' rule of signs, we obtain

$$\sum_{k=0}^{n} |a_k| b^k \le \sum_{k=0}^{n} |A_k| b^k \|P\|_{C(E)} = T_n(4b/\lambda - 1)\|P\|_{C(E)}.$$

Hence (2.6) follows. ■

Proof of Corollary 2.2. Note first that $|-E \cap E| \ge 2\lambda - 2b$, and a polynomial $P_1(x) = \frac{1}{2}(P(x) + P(-x))$ satisfies the conditions: $P_1(0) = P(0)$ and P_1 is an even polynomial from $\mathcal{P}_{2N,1}$, where $N = [n/2]$. Next, setting $s = 2 - \lambda/b$ and using Theorem 2.3, we obtain

$$\begin{aligned} |P(0)| &\le T_N\left(\frac{2b^2}{(3b - \lambda)(\lambda - b)} - 1\right)\|P\|_{C(-E\cap E)} \le T_N\left(\frac{2}{1 - s^2} - 1\right)\|P\|_{C(E)} \\ &\le ((1 + s)/(1 - s))^N \|P\|_{C(E)} < e^{2ns/(1-2\varepsilon)}\|P\|_{C(E)}. \end{aligned}$$

This implies (2.7). ■

3. Remez-Type Inequalities for Trigonometric and Exponential Polynomials

3.1. *Remez-Type Inequalities*

The first Remez-type inequality for exponential polynomials of the form $Q(x) = \sum_{k=0}^{n} c_k e^{ikx}$, $c_k \in \mathbf{C}^1$, $0 \le k \le n$, and a measurable set $E \subseteq (-\pi, \pi]$, $|E| > 0$, was obtained by Stechkin and Ulyanov [49]:

$$\|Q\|_{C(-\pi,\pi)} \le (n + 1)\sin^{-n}(|E|/8n)\|Q\|_{C(E)}. \tag{3.1}$$

A similar result was proved independently by Ash and Welland [3]. Belov [5] established some estimates like (3.1) for generalized polynomials $Q(x) = \sum_{k=0}^{n} c_k e^{i\lambda_k x}$, $\lambda_k \in \mathbf{R}^1$,

$0 \le k \le n$, $\min_{k\ne m} |\lambda_k - \lambda_m| > 0$. A general inequality of first type for $Q(x) = \sum_{k=0}^{n} c_k e^{\mu_k x}$, $c_k, \mu_k \in \mathbf{C}^1$, $0 \le k \le n$, was obtained by Nazarov [44]: for a subinterval $I \subseteq (-\pi, \pi]$ and a set $E \subseteq I$, $|E| > 0$, there exists an absolute constant A such that

$$\|Q\|_{C(I)} \le \exp\left(|I| \max_{0\le k\le n} Re(\mu_k)\right)(A|I|/|E|)^n \|Q\|_{C(E)}. \tag{3.2}$$

In particular, for any $Q \in \mathcal{T}_n$, the following inequality of first type holds [44]:

$$\|Q\|_{C(-\pi,\pi)} \le (A/|E|)^{2n} \|Q\|_{C(E)}, \tag{3.3}$$

where $E \subseteq (-\pi, \pi]$ and $A \le 32e < 87$.

On the other hand, it is known [22], [37], [50, p. 90] that for a trigonometric polynomial $Q \in \mathcal{T}_n$ and an interval $E = [a, b] \subseteq (-\pi, \pi]$:

$$\|Q\|_{C(-\pi,\pi)} \le \tfrac{1}{2}(\tan^{2n}(|E|/8) + \cot^{2n}(|E|/8))\, \|Q\|_{C(E)}, \tag{3.4}$$

and equality in (3.4) holds if and only if

$$Q(x) = AT_n\left(\frac{\cos(x-(a+b)/2) - \cos^2((b-a)/4)}{\sin^2((b-a)/4)}\right), \qquad A \in \mathbf{R}^1. \tag{3.5}$$

Thus the best constant $A_n(|E|)$ in the inequality $\|Q\|_{C(-\pi,\pi)} \le A_n \|Q\|_{C(E)}$, $Q \in \mathcal{T}_n$, $E \subseteq (-\pi, \pi]$, satisfies the relations

$$\begin{aligned}(87/|E|)^{2n} &\ge A_n(|E|)\\ &\ge \tfrac{1}{2}(\tan^{2n}(|E|/8) + \cot^{2n}(|E|/8)) \sim \begin{cases}\tfrac{1}{2}(8/|E|)^{2n}, & |E| \to 0,\\ \exp(\tfrac{1}{2}n(2\pi - |E|)), & |E| \to 2\pi.\end{cases}\end{aligned}$$

Erdélyi [22] established a Remez inequality of second type and extended it to the generalized trigonometric polynomials. For $Q \in \mathcal{T}_n$ and $E \subseteq (-\pi, \pi]$, $|E| \ge 3\pi/2$, his result is

$$\|Q\|_{C(-\pi,\pi)} \le e^{An(2\pi - |E|)} \|Q\|_{C(E)}, \tag{3.6}$$

where A is an absolute constant. Borwein and Erdélyi [11, pp. 230–231] gave the estimate $A \le 4$, but their proof is incomplete; the correct version of the proof (with a larger upper bound) will be contained in the second print of book [11] (the private communication by T. Erdélyi).

Below we show that $A \le 17$ in (3.3) and $A \le 2$ in (3.6). These estimates are based on inequalities like (3.4) for special classes of polynomials and sets.

Theorem 3.1. *For a measurable set $E \subseteq (-\pi, \pi]$, $|E| \ge 3\pi/2$, and a trigonometric polynomial $Q \in \mathcal{T}_n$:*

$$\|Q\|_{C(-\pi,\pi)} \le e^{2n(2\pi - |E|)} \|Q\|_{C(E)}. \tag{3.7}$$

Theorem 3.2. *For a measurable set $E \subseteq (-\pi, \pi]$, $|E| > 0$, and a trigonometric polynomial $Q \in \mathcal{T}_n$:*

$$\|Q\|_{C(-\pi,\pi)} \le (17/|E|)^{2n} \|Q\|_{C(E)}. \tag{3.8}$$

These theorems are based on the following results:

Theorem 3.3. *For a measurable* $E \subseteq [-\pi, \pi]$, $|E| = \lambda > 0$, *and an even trigonometric polynomial* $Q \in \mathcal{T}_n$:

$$|Q(0)| \le \tfrac{1}{2}(\tan^{2n}(\lambda/8) + \cot^{2n}(\lambda/8))\|Q\|_{C(E)}. \tag{3.9}$$

Equality in (3.9) *holds if and only if* $E = [-\pi, -\pi + \lambda/2] \cup [\pi - \lambda/2, \pi]$ *and* $Q(x) = AT_n\left(-[\cos x + \cos^2(\lambda/4)]/\sin^2(\lambda/4)\right)$, *where* $\lambda \in (0, 2\pi)$ *and* $A \in \mathbf{R}^1$.

Theorem 3.4. *For a measurable set* $E \subseteq [a, a+\pi]$, $|E| = \lambda > 0$, $a \in \mathbf{R}^1$, *and a trigonometric polynomial* $Q \in \mathcal{T}_n$:

$$|Q(a)| \le T_{2n}(\cot(\lambda/4))\|Q\|_{C(E)}. \tag{3.10}$$

Equality in (3.10) *holds if and only if* $E = [a+\pi-\lambda, a+\pi]$ *and* $Q(x) = AT_n(-[\cos(x - a + \lambda/2) + \cos^2(\lambda/4)]/\sin^2(\lambda/4))$, *where* $\lambda \in (0, 2\pi)$, $a \in \mathbf{R}^1$, *and* $A \in \mathbf{R}^1$.

The following refinement of (3.1) and (3.2) for exponential polynomials of even degree can be easily derived from (3.8).

Corollary 3.1. *For a measurable set* $E \subseteq (-\pi, \pi]$, $|E| > 0$, *the following statements hold*:

(a) *if* $Q(x) = \sum_{k=-n}^{n} a_k \exp(ikx)$ *is a trigonometric polynomial with complex coefficients, then*

$$\|Q\|_{C(-\pi,\pi)} \le \sqrt{2}(17/|E|)^{2n}\|Q\|_{C(E)}, \tag{3.11}$$

(b) *if* $Q(x) = \sum_{k=0}^{2n} a_k \exp(ikx)$ *is an exponential polynomial, then* (3.11) *is valid.*

Remark 3.1. Erdélyi [22] established the inequality $|Q(0)| \le T_{2n}(\sin(\lambda/2))\|Q\|_{C(E)}$ which is equivalent to (3.9). We shall give the different proof of Theorem 3.3 which is similar to that of Theorem 3.4.

Remark 3.2. We believe that the following sharp Remez-type inequality is true:

Conjecture 3.1. *For a measurable set* $E \subseteq (-\pi, \pi]$, $|E| > 0$, *and* $Q \in \mathcal{T}_n$, *inequality* (3.4) *is valid. Equality in* (3.4) *holds if and only if* $E = [a, b]$ *and* Q *is polynomial* (3.5).

Remark 3.3. Stechkin and Ulyanov [49] posed the problem: What is the precise order of decreasing the best constant $B_n(|E|)$ in (3.1)? Conjecture 3.1 would imply that

$$C_1 \cot^{2n}(|E|/8) \le B_n(|E|) \le C_2 \cot^{2n}(|E|/8).$$

3.2. *Proofs of the Theorems*

First we prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1. Without loss of generality, we may assume that $|Q(0)| = \|Q\|_{C(-\pi,\pi)}$. Next, applying an approach used in [22], we have that $Q_1(x) = \frac{1}{2}(Q(x)+Q(-x))$ is an even polynomial of degree n, and a set $E_1 = E \cap -E$ satisfies the conditions: $|E_1| \geq 2|E|-2\pi$ and $\|Q_1\|_{C(E_1)} \leq \|Q\|_{C(E)}$. Further setting $\tau = 2\pi - |E|$, $0 \leq \tau \leq \pi/2$, and using (3.9), we obtain

$$\begin{aligned}\|Q\|_{C(-\pi,\pi)} &= |Q_1(0)| \leq \cot^{2n}(|E_1|/8)\|Q_1\|_{C(E_1)} \\ &\leq \cot^{2n}((\pi-\tau)/4)\|Q\|_{C(E)} = \left(1+\frac{\sqrt{2}\sin(\tau/4)}{\sin(\pi/4-\tau/4)}\right)^{2n}\|Q\|_{C(E)} \\ &\leq \left(1+\frac{\sqrt{2}\tau}{4\sin(\pi/8)}\right)^{2n}\|Q\|_{C(E)} < e^{1.85n\tau}\|Q\|_{C(E)}.\end{aligned}$$

This yields (3.7). ∎

Proof of Theorem 3.2. Let $a \in (-\pi,\pi]$ satisfy the equality $|Q(a)| = \|Q\|_{C(-\pi,\pi)}$, and let E^* be the 2π-periodic extension of E to $\mathbf{R}^1$, that is, χ_{E^*} is 2π-periodic on $\mathbf{R}^1$ and $E^* \cap (-\pi,\pi] = E$. Applying Theorem 3.4 to $E_1 = E^* \cap [a-\pi, a]$ and $E_2 = E^* \cap [a, a+\pi]$, and taking account of the elementary estimates

$$\begin{aligned}T_{2n}(\cot y) &\leq (2\cot y)^{2n} \leq \cot^{2n}(y/2), \qquad y \in (0,\pi/2], \\ \sin y &\geq 0.974y, \qquad y \in [0,\pi/8],\end{aligned}$$

we obtain

$$\begin{aligned}\|Q\|_{C(-\pi,\pi)} &= |Q(a)| \leq \min_{i=1,2} T_{2n}(\cot(|E|/4))\|Q\|_{C(E_i)} \\ &\leq T_{2n}(\cot(|E|/8))\|Q\|_{C(E)} \\ &\leq \cot^{2n}(|E|/16)\|Q\|_{C(E)} < (16.43/|E|)^{2n}\|Q\|_{C(E)}.\end{aligned} \tag{3.12}$$

Thus (3.8) follows from (3.12). ∎

To prove Theorems 3.3 and 3.4, we need several lemmas.

First we consider the following extremal problems: find

$$C_{n,i}(E) = \min_{Q \in M_{n,i}} \|Q\|_{C(E)}, \qquad i = 1, 2,$$

where E is a closed subset of $[0,\pi]$, $|E| > 0$; $M_{n,1} = \{Q \in \mathcal{T}_n : Q(0) = 1, Q(x) = Q(-x), x \in [0,\pi]\}$; and $M_{n,2} = \{Q \in \mathcal{T}_n : Q(0) = 1\}$.

Lemma 3.1. *The following properties hold:*

(a) *for a measurable set* $E \subseteq [0, \pi]$ *and an even polynomial* $Q \in \mathcal{T}_n$:

$$|Q(0)| \le (C_{n,1}(E))^{-1} \|Q\|_{C(E)}; \tag{3.13}$$

(b) *for a measurable set* $E \subseteq [0, \pi]$ *and a polynomial* $Q \in \mathcal{T}_n$:

$$|Q(0)| \le (C_{n,2}(E))^{-1} \|Q\|_{C(E)}; \tag{3.14}$$

(c) $C_{n,i}(E) > 0$, $i = 1, 2$.

Proof. Equation (3.14) trivially holds if $Q(0) = 0$. If $Q(0) \ne 0$, then $Q_1(x) = Q(x)/Q(0) \in M_{n,2}$. This yields $C_{n,2}(E) \le \|Q_1\|_{C(E)}$. Hence (3.14) follows. Similarly we prove (3.13). Statement (c) is an immediate consequence of (3.13), (3.14), and (3.1), though it can be proved independently by a compactness argument. ■

Let $K_{n,1} = \{(1 - \cos x) \cos kx\}_{k=0}^{n-1}$ and let $K_{2n,2} = \{\sin(x/2) \cos(k + \frac{1}{2}), \sin(x/2) \sin (k + \frac{1}{2})\}_{k=0}^{n-1}$ be the systems of trigonometric polynomials.

Lemma 3.2. *Let* $N = N(i) = in$, $i = 1, 2$. *Then the following properties hold for* $i = 1, 2$:

(a) $K_{N,i}$ *is a Chebyshev system on* $(0, \pi)$;

(b) *the error of approximation of* 1 *by polynomials from* span $K_{N,i}$ *in* $C(E)$ *is*

$$C_{n,i}(E) = \inf_{H \in \operatorname{span} K_{N,i}} \|1 - H\|_{C(E)}; \tag{3.15}$$

(c) *there exists the unique polynomial* $Q_{n,E} = Q_{n,E,i} \in M_{n,i}$ *such that*

$$C_{n,i}(E) = \|Q_{n,E}\|_{C(E)}; \tag{3.16}$$

(d) *a polynomial* $Q_{n,E} \in M_{n,i}$ *satisfies* (3.16) *if and only if there exist* $N + 1$ *distinct points* $y_k \in E$, $1 \le k \le N + 1$, *such that* $0 < y_1 < \cdots < y_{N+1} \le \pi$, $|Q_{n,E}(y_k)| = \|Q_{n,E}\|_{C(E)}$, $1 \le k \le N + 1$, *and* $Q_{n,E}(y_k) = -Q_{n,E}(y_{k+1})$, $1 \le k \le N$;

(e) $Q_{n,E}$ *has* N *zeros that lie in* $(0, \pi)$.

Proof. It is easy to show that a polynomial from span $K_{N,i}$ that has N distinct zeros on $(0, \pi)$ is identically zero. This proves statement (a) of the lemma. Next, note that $Q \in M_{n,i}$ if and only if $Q(x) = 1 - H$, where $H \in \operatorname{span} K_{N,i}$. Hence (3.15) follows. Statements (c) and (d) follow from (a) and (b), Lemma 3.1(c), and the approximation properties of Chebyshev systems [11, pp. 94, 98]. Finally, statement (e) of the lemma is an immediate consequence of (d). ■

Lemma 3.3. *If* $[c, d] \subseteq [0, \pi]$, *then*

$$\min_{d-c=\lambda/2} C_{n,1}([c, d]) = (\tfrac{1}{2}(\tan^{2n}(\lambda/8) + \cot^{2n}(\lambda/8)))^{-1}, \tag{3.17}$$

$$\min_{d-c=\lambda} C_{n,2}([c, d]) = (T_{2n}(\cot(\lambda/4)))^{-1}. \tag{3.18}$$

Minimum in (3.17) *and* (3.18) *attains if and only if* $[c, d] = [\pi - \lambda/2, \pi]$ *and* $[c, d] = [\pi - \lambda, \pi]$, *respectively.*

Proof. Applying the necessity of Lemma 3.2(d) for $i = 1$, $E = [c, d] \subseteq [0, \pi]$, and

$$Q_{n,[c,d]}(x) = T_n\left(\frac{2\cos x - \cos c - \cos d}{\cos c - \cos d}\right) \Big/ T_n\left(\frac{2 - \cos c - \cos d}{\cos c - \cos d}\right),$$

we obtain

$$(3.19)\quad \begin{aligned}\min_{d-c=\lambda/2} C_{n,1}([c, d]) &= \min_{d-c=\lambda/2}\left(T_n\left(\frac{2 - \cos c - \cos d}{\cos c - \cos d}\right)\right)^{-1}\\ &= 2\min_{d-c=\lambda/2}(((\cot(\lambda/8)\tan((c+d)/4))^n\\ &\qquad + (\tan(\lambda/8)\cot((c+d)/4))^n))^{-1}\\ &= (\tfrac{1}{2}(\tan^{2n}(\lambda/8) + \cot^{2n}(\lambda/8)))^{-1}.\end{aligned}$$

Thus (3.19) yields (3.17). Next, applying again the necessity of Lemma 3.2(d) for $i = 2$, $E = [c, d] \subseteq [0, \pi]$, and

$$\begin{aligned}&Q_{n,[c,d]}(x)\\ &\quad = T_n\left(\frac{\cos(x - (c+d)/2) - \cos^2(\lambda/2)}{\sin^2(\lambda/2)}\right) \Big/ T_n\left(\frac{\cos((c+d)/2) - \cos^2(\lambda/2)}{\sin^2(\lambda/2)}\right),\end{aligned}$$

we have

$$(3.20)\quad \begin{aligned}\min_{d-c=\lambda} C_{n,2}([c, d]) &= \min_{d-c=\lambda}\left(T_n\left(\frac{2(\cos(\lambda/2) - \cos((c+d)/2))}{1 - \cos(\lambda/2)} + 1\right)\right)^{-1}\\ &= (T_n(2\cos(\lambda/2)/\sin^2(\lambda/4) + 1))^{-1} = (T_{2n}(\cot(\lambda/4)))^{-1}.\end{aligned}$$

Thus (3.20) yields (3.18). ∎

Proof of Theorem 3.4. Without loss of generality, we may assume that $a = 0$ and E is a closed subset of $[0, \pi]$ of measure λ, $0 < \lambda < \pi$. Let $Q \in \mathcal{T}_n$ be a trigonometric polynomial. If $E = [a, b]$ is a subinterval of $[0, \pi]$, then (3.10) follows from (3.14) and (3.18), and the equality in (3.10) holds if and only if E and Q are the extremal elements defined in Theorem 3.4.

Suppose E is not a subinterval of $[0, \pi]$. Then by statements (c) and (d) of Lemma 3.2 for $i = 2$, there exists the only polynomial $Q_{n,E} = Q_{n,E,2}$ satisfying the alternating property on E. Letting $E_1 = \{x \in [0, \pi]: |Q_{n,E}|(x) \le \|Q_{n,E}\|_{C(E)}\}$ we obtain, from the sufficiency of Lemma 3.2(d) for $i = 2$, that $Q_{n,E_1} = Q_{n,E}$. If E_1 is a subinterval of $[0, \pi]$, then $|E_1| > |E|$ and $\|Q_{n,E_1}\|_{C(E_1)} = \|Q_{n,E}\|_{C(E)}$. Thus

$$(3.21)\qquad |Q(0)| < T_{2n}(\cot(\lambda/4))\|Q\|_{C(E)},$$

by (3.14) and (3.18).

Suppose now that E_1 is not a subinterval of $[0, \pi]$. Then the following properties of E_1 hold:

(a) $E_1 = \bigcup_{j=1}^r [a_j, b_j]$, $0 < a_1 \le b_1 < \cdots < a_r \le b_r \le \pi$, $r \ge 2$;
(b) there exist $2n$ simple zeros $x_1 < x_2 < \cdots < x_{2n}$ of Q_{n,E_1} that lie on E_1;
(c) there exist at least two subintervals of E_1 that contain zeros of Q_{n,E_1}.

Property (a) is evident, while (b) immediately follows from Lemma 3.2(e) for $i = 2$.

To prove property (c) we assume that $x_i \in [a_j, b_j]$, $1 \le i \le n$, for some j, $1 \le j \le r$. Then $|Q_{n,E_1}(a_j)| = \|Q_{n,E_1}\|_{C(E_1)}$. Further, the condition $Q_{n,E_1}(0) = 1$ yields $Q_{n,E_1}(a_j) > 0$ since if $Q_{n,E_1}(a_j) < 0$, then Q_{n,E_1} has $2n+1$ zeros on $[0, \pi]$. Next, $Q'_{n,E_1}(a_j) \le 0$. Indeed, if $Q'_{n,E_1}(a_j) > 0$, then there exists $x_0 \in (a_j, b_j)$ such that $Q_{n,E_1}(x_0) > Q_{n,E_1}(a_j) = \|Q_{n,E_1}\|_{C(E_1)}$. Then

$$Q_{n,E_1}(x) > \|Q_{n,E_1}\|_{C(E_1)}, \qquad x \in (0, a_j). \tag{3.22}$$

Indeed, if $Q'_{n,E_1}(a_j) = 0$, then Q_{n,E_1} is strictly decreasing on $[0, a_j]$. If $Q'_{n,E_1}(a_j) < 0$, then Q'_{n,E_1} has no more than one zero on $[0, a_j)$. Hence (3.22) follows. Thus $E_1 \cap [0, a_j) = \emptyset$, by (3.22). Similarly we prove that $E_1 \cap (b_j, 2\pi] = \emptyset$, that is, $E_1 = [a_j, b_j]$. This contradicts our assumption that E_1 is not an interval. Hence property (c) holds.

Now we construct a set E_2 and a polynomial Q_1 satisfying certain conditions. Let $E_2 = \bigcup_{j=1}^r [a_j + \tau_j, b_j + \tau_j] = [b_r - |E_1|, b_r]$, where $\tau_j = \sum_{k=j}^{r-1}(a_{k+1} - b_k)$, $1 \le j \le r-1$, $\tau_r = 0$. Next, let $x_i^* = x_i + \tau_j$, where $j = j(i)$ is the index of an interval $[a_j, b_j]$ which contains x_i, $1 \le i \le 2n$. Note that $x_i^* \in E_2$ and $x_i^* \ge x_i$, $1 \le i \le 2n$, by properties (a) and (b). Moreover, there is an integer i such that $x_i^* > x_i$, by property (c). Let us put $Q_1(y) = \prod_{i=1}^{2n}(\sin((x_i^* - y)/2))/(\sin(x_i^*/2))$. Then $Q_1 \in M_{n,2}$ and $\prod_{i=1}^{2n} |\sin(x_i/2)| < \prod_{i=1}^{2n} |\sin(x_i^*/2)|$. Next, note that for each $y \in E_2$ there is $x \in E_1$ such that $|\sin((x_i^* - y)/2)| \le |\sin((x_i - x)/2)|$, $1 \le i \le 2n$. Hence taking account of the representation $Q_{n,E_1}(x) = \prod_{i=1}^{2n}(\sin((x_i - x)/2))/(\sin(x_i/2))$, we obtain $\|Q_1\|_{C(E_2)} < \|Q_{n,E_1}\|_{C(E_1)} = \|Q_{n,E}\|_{C(E)}$. Thus

$$|E_2| = |E_1| \ge \lambda, \qquad C_{n,2}(E_2) < C_{n,2}(E). \tag{3.23}$$

Then for any polynomial $Q \in \mathcal{T}_n$ inequalities (3.14), (3.18), and (3.23) yield

$$\begin{aligned} |Q(0)| &\le (C_{n,2}(E))^{-1}\|Q\|_{C(E_0)} < (C_{n,2}(E_2))^{-1}\|Q\|_{C(E)} \\ &\le T_{2n}(\cot(\lambda/4))\|Q\|_{C(E)}. \end{aligned} \tag{3.24}$$

Finally, inequalities (3.21) and (3.24) show that (3.10) is valid for $a = 0$ and any measurable set $E \subseteq [0, \pi]$, and equality holds if and only if $E = [\pi - \lambda, \pi]$. ■

Proof of Theorem 3.3. The proof is similar to that of Theorem 3.4 with the following changes to yield. Without loss of generality, we may assume that E is closed and symmetric about the origin. If $\lambda = 2\pi$, then (3.9) trivially holds. So we assume $0 < \lambda < 2\pi$. Note that we shall use Lemma 3.1(a), equality (3.17), and Lemmas 3.1(c) and 3.2, for $i = 1$ instead of corresponding results for all polynomials. In particular, the extremal even polynomial $Q_{n,E_1} = Q_{n,E_1,1}$ has n simple zeros x_i, $1 \le i \le n$, that lie on E_1.

Next, we establish property (c) of E_1 that has a slightly different proof. Assume that $x_i \in [a_j, b_j]$, $1 \le i \le n$, for some j, $1 \le j \le r$. Then $Q_{n,E_1}(a_j) = \|Q_{n,E_1}\|_{C(E_1)}$ and $Q'_{n,E_1}(a_j) \le 0$. Moreover, if $Q'_{n,E_1}(a_j) = 0$, then taking account of $Q'_{n,E_1}(0) = 0$, we deduce that Q'_{n,E_1} is identically zero. This contradicts our assumption $\lambda < 2\pi$. Then the relations $Q'_{n,E_1}(a_j) < 0$ and $Q'_{n,E_1}(0) = 0$ imply that Q_{n,E_1} is strictly decreasing on $(0, a_j)$. Thus $E_1 \cap [0, a_j) = \emptyset$. Furthermore, assuming $b_j < \pi$ and using the similar argument, we obtain $|Q_{n,E_1}(b_j)| = \|Q_{n,E_1}\|_{C(E_1)}$ and $\operatorname{sgn}(Q_{n,E_1} Q'_{n,E_1}(b_j)) > 0$. Taking account of $Q'_{n,E_1}(0) = 0$, we deduce that $|Q_{n,E_1}|$ is strictly increasing on $(b_j, \pi]$. Thus $E_1 \cap (b_j, \pi] = \emptyset$, that is, $E_1 = [a_j, b_j]$, which contradicts our assumption that E_1 is not an interval. Hence property (c) holds.

Finally note that the polynomials Q_{n,E_1} and Q_1 can be written in the following forms:

$$Q_{n,E_1}(x) = \prod_{i=1}^{n}(\cos x - \cos x_i)/(1 - \cos x_i), \quad Q_1(y) = \prod_{i=1}^{n}(\cos y - \cos x_i^*)/(1 - \cos x_i^*).$$

This completes the proof of Theorem 3.3. ■

4. Pólya- and Remez-Type Inequalities for Multivariate Polynomials

4.1. *Pólya- and Remez-Type Inequalities*

A multidimensional generalization of Remez's inequality was obtained by Brudnyi and the author [15], [16]: for a convex body $V \subset \mathbf{R}^m$, a measurable set $E \subseteq V$, and a polynomial $P \in \mathcal{P}_{n,m}$:

$$\|P\|_{C(V)} \le T_n\left(\frac{1+\beta_m(|E|/|V|)}{1-\beta_m(|E|/|V|)}\right)\|P\|_{C(E)}, \tag{4.1}$$

where

$$\beta_m(t) := \sqrt[m]{1-t}. \tag{4.2}$$

The classes of all extremal bodies V, sets E, and polynomials P for which the equality in (4.1) holds, were found by the author [16]. A multidimensional generalization of (2.2) and local Nikolskii-type inequalities in rearrangement-invariant spaces were obtained in [31].

Taking account of the representation

$$T_n\left(\frac{1+\beta_m(t)}{1-\beta_m(t)}\right) = \frac{1}{2}\left(\left(\frac{1+\beta_{2m}(t)}{1-\beta_{2m}(t)}\right)^n + \left(\frac{1-\beta_{2m}(t)}{1+\beta_{2m}(t)}\right)^n\right),$$

it is easy to show that (4.1) implies the following inequalities of first and second type, respectively,

$$\|P\|_{C(V)} \le (C_1|V|/|E|)^n\|P\|_{C(E)}, \tag{4.3}$$

$$\begin{aligned} \|P\|_{C(V)} &\le \exp(C_2 n(1-|E|/|V|)^{1/2m})\,\|P\|_{C(E)}, \\ 1-2^{-2m} &\le |E|/|V| \le 1, \end{aligned} \tag{4.4}$$

where $C_1 \le 4m$ and $C_2 \le 4$. It is easy to verify that (4.3) also holds for a bounded domain $V \in \mathbf{R}^m$ with $C_1 \le 4m|\hat{V}|/|V|$, where $\hat{V}$ is the convex hull of V. Note that Nadirashvili [41] independently proved (4.3) in the case when V is a ball.

Kroó and Schmidt [39] noticed that (4.4) holds for a bounded domain in $\mathbf{R}^m$ satisfying the cone property, and refined (4.4) for a bounded domain $V \subset \mathbf{R}^m$ with C^2-boundary, replacing the exponent $1/2m$ with $1/(m+1)$. Recently A. Brudnyi [13] obtained an analogue of (4.3) for measurable subsets of an algebraic variety.

A multidimensional version of Pólya's inequality was established in [15]: for a measurable subset E of a ball $V = \{x \in \mathbf{R}^m : |x| \le 1\}$ and a polynomial $P(x) = \sum_{|\alpha|\le n} a_\alpha x^\alpha$:

$$(4.5) \qquad |a_\alpha| \le (C|V|/|E|)^n \|P\|_{C(E)}, \qquad |\alpha| \le n,$$

where C depends only on m and $|\alpha|$. It is easy to verify that (4.5) remains valid for every convex body V.

In this section we obtain nontrivial analogues of (4.5) and (4.3) for the unit cube in $\mathbf{R}^m$ and a polynomial from $\mathcal{P}^*_{n,m}$ (Theorem 4.1 and Corollary 4.1). Several multidimensional Remez-type inequalities are also presented. In particular, we establish the spherical versions of (1.2), (4.3), and (4.4) (Theorem 4.2 and Corollary 4.2) and prove a multidimensional analogue of Corollary 2.2 (Theorem 4.3).

Theorem 4.1. *For the unit cube K^m, the following statements hold*:

(a) *for a measurable set $E \subseteq K^m$, $|E| = \lambda > 0$; and a polynomial $P(x) = \sum_{\alpha_1=1}^n \cdots \sum_{\alpha_m=1}^n a_\alpha x^\alpha \in \mathcal{P}^*_{n,m}$*:

$$(4.6) \qquad |a_\alpha| \le (C/\lambda)^n \ln^{n(m-1)}(e/\lambda)\|P\|_{C(E)},$$

where $C \le 154^m m^{2m}$;

(b) *for $P_n(x) = (x_1 \dots x_m)^n \in \mathcal{P}^*_{n,m}$ and $E_t = \{x \in K^m : |P_n(x)| \le t\}$, where $t \in (0, 1]$ is a fixed number*

$$(4.7) \qquad a_{(n,\dots,n)} = \|P_n\|_{C(K^m)} \ge \frac{\ln^{n(m-1)}(e/|E_t|)\|P_n\|_{C(E_t)}}{((m-1)!|E_t|)^n}.$$

Let $\mathcal{T}_n(E) = \{Q \in \mathcal{T}_n : \|Q\|_{C(E)} \le 1\}$, and let $A_n(\tau) = \sup_{|E|\ge\tau} \sup_{Q\in\mathcal{T}_n(E)} \|Q\|_{C(-\pi,\pi)}$, where the first upper bound is taken over all measurable $E \subseteq (-\pi, \pi]$ with $|E|_1 \ge \tau$, $0 \le \tau \le 2\pi$. In other words, $A_n(\tau)$ is the least constant in the inequality $\|Q\|_{C(-\pi,\pi)} \le A\|Q\|_{C(E)}$ over all $Q \in \mathcal{T}_n$ and $|E|_1 \ge \tau$.

Theorem 4.2. *For a measurable set $E \subseteq S^m$, $|E|_{m-1} = \lambda > 0$, $m \ge 2$, and a polynomial $P \in \mathcal{P}_{n,m}$*:

$$(4.8) \qquad \|P\|_{C(S^m)} \le A_n(\varphi^{-1}(\lambda))\|P\|_{C(E)},$$

where

$$(4.9) \qquad \varphi(t) = 2\omega_m \int_0^{t/4} \cos^{m-2} u \, du.$$

Here $\omega_2 = 2$ and $\omega_m = |S^{m-1}|_{m-2} = 2\pi^{(m-1)/2}(\Gamma((m-1)/2))^{-1}$, $m \ge 3$.

Theorem 4.3. *For a centrally symmetric (with respect to the origin) convex body* $V \subset \mathbf{R}^m$, $m \geq 2$; *a measurable set* $E \subseteq V$ *satisfying* $|E| = \lambda$, $(1 - 2^{-2m})|V| \leq \lambda \leq |V|$; *and a polynomial* $P \in \mathcal{P}_{n,m}$:

$$|P(0)| \leq \exp(4n \sqrt[m]{1 - \lambda/|V|})\|P\|_{C(E)}. \tag{4.10}$$

Remark 4.1. The problem of finding the best constant and extremal sets and polynomials in the inequality $\|P\|_{C(V)} \leq C\|P\|_{C(E)}$ is solved only when V is a bounded convex cone [16]. Brudnyi and the author [15] conjectured that if V is a ball, then an extremal set E is convex.

The proofs of Theorems 4.1, 4.2, and 4.3 are based on the following idea [15], [16], [30]: first we apply a univariate Pólya- or Remez-type inequality to linear subsets of E and then make use of some estimates of linear measure of these subsets (Lemmas 4.1–4.3).

4.2. *Geometric Lemmas*

The proofs of the next two lemmas follow that of Lemma 3 in [15].

For a centrally symmetric body $V \subset \mathbf{R}^m$ we set

$$\gamma_m(V, \lambda) = \sup_{E \subseteq V, |E| \geq \lambda} \operatorname{ess\,inf}_l (|V \cap l|_1 / |E \cap l|_1),$$

where the ess inf is taken over almost all lines l passing through the origin.

Lemma 4.1. *For all* $\lambda \in (0, |V|]$:

$$\gamma_m(V, \lambda) = (1 - \beta_m(\lambda/|V|))^{-1}, \tag{4.11}$$

where β_m *is defined by* (4.2).

Proof. Let us introduce in $\mathbf{R}^m$, $m \geq 2$, a coordinate system by

$$x_1 = r \prod_{i=1}^{m-1} \cos\theta_i, \qquad x_j = r \sin\theta_{j-1} \prod_{i=j}^{m-1} \cos\theta_i, \qquad 2 \leq j \leq m, \tag{4.12}$$

where $r \in \mathbf{R}^1$, $|\theta_i| \leq \pi/2$, $1 \leq i \leq m - 1$, and the Jacobian is given by $J_1 = r^{m-1} \prod_{i=2}^{m} \cos^{i-1}\theta_i$ [48, pp. 314–318].

Let $|r| = F(\theta) = F(\theta_1, \ldots, \theta_{m-1})$ be the equation for the boundary of V. First we consider the set $E^* = \{(r, \theta) \in V : \beta_m(\lambda/|V|)F(\theta) \leq |r| \leq F(\theta)\}$. Then $|E^*| = \lambda$, and for every line l passing through the origin

$$|V \cap l|_1 / |E^* \cap l|_1 = (1 - \beta_m(\lambda/|V|))^{-1}. \tag{4.13}$$

Now we show that for a set $E \subseteq V$, $|E| \geq \lambda$:

$$\operatorname{ess\,inf}_l (|V \cap l|_1 / |E \cap l|_1) \leq (1 - \beta_m(\lambda/|V|))^{-1}. \tag{4.14}$$

Suppose there exists $E_0 \subseteq V$ such that $|E_0| \geq \lambda$ and

$$\operatorname{ess\,inf}_l (|V \cap l|_1 / |E_0 \cap l|_1) > (1 - \beta_m(\lambda/|V|))^{-1}. \tag{4.15}$$

Then from (4.13) and (4.15) we have that for almost every line l through the origin

$$|E_0 \cap l|_1 < |E^* \cap l|_1. \tag{4.16}$$

Next, (4.16) implies that for almost every line $l = l(\theta)$ through the origin

$$\begin{aligned}\int_{E_0 \cap l} |r|^{m-1} dr &\le 2\int_{(|V\cap l|_1 - |E_0 \cap l|_1)/2}^{|V\cap l|_1/2} r^{m-1}\, dr \\ &< 2\int_{(|V\cap l|_1 - |E^* \cap l|_1)/2}^{|V\cap l|_1/2} r^{m-1} dr = \int_{E^*\cap l} |r|^{m-1}\, dr.\end{aligned} \tag{4.17}$$

Finally, integrating (4.17) with respect to θ, we conclude that $|E_0| < |E^*|$ which contradicts our assumption $|E_0| \ge \lambda$. Hence (4.14) holds. Then (4.13) and (4.14) yield (4.11). ∎

Next we prove a spherical analogue of Lemma 4.1. Given a point $x_0 \in S^m$ we set

$$\Gamma_m(\lambda) = \inf_{E\subseteq S^m,\, |E|_{m-1}\ge\lambda} \operatorname{ess\,sup}_c |E \cap c|_1,$$

where the ess sup is taken over almost all great circles $c \subset S^m$ passing through x_0.

Lemma 4.2. *For all* $\lambda \in (0, |S^m|_{m-1}]$:

$$\Gamma_m(\lambda) = \varphi^{-1}(\lambda), \tag{4.18}$$

where φ *is defined by* (4.9).

Proof. Without loss of generality, we may assume that $x_0 = (0, \ldots, 0, 1)$. Let us introduce a coordinate system in $\mathbf{R}^m$, $m \ge 2$, by (4.12), where $r \ge 0$ and $|\theta_i| \le \pi/2$, $1 \le i \le m-2$, $|\theta_{m-1}| \le \pi$, with the Jacobian $J_2 = J_1$. Let

$$\begin{aligned}E^* &= \{x \in S^m\colon \theta_{m-1} \in [-\varphi^{-1}(\lambda)/4, \varphi^{-1}(\lambda)/4] \cup [-\pi, -\pi + \varphi^{-1}(\lambda)/4] \\ &\cup\ [\pi - \varphi^{-1}(\lambda)/4, \pi]\end{aligned}$$

be a spherical layer of height $2\sin(\varphi^{-1}(\lambda)/2)$ that is symmetric about the plane $\{x \in \mathbf{R}^m : x_m = 0\}$. Then $|E^*|_{m-1} = \lambda$ and for almost every great circle c passing through x_0:

$$|E^* \cap c|_1 = \varphi^{-1}(\lambda). \tag{4.19}$$

Next, we prove that for a set $E \subseteq S^m$, $|E|_{m-1} \ge \lambda$:

$$\operatorname{ess\,sup}_c |E \cap c|_1 \ge \varphi^{-1}(\lambda). \tag{4.20}$$

Suppose there exists $E_0 \subseteq S^m$ such that $|E_0|_{m-1} \ge \lambda$ and

$$\operatorname{ess\,sup}_c |E_0 \cap c|_1 < \varphi^{-1}(\lambda). \tag{4.21}$$

Then we deduce from (4.19) and (4.21) that for almost every great circle through x_0:

$$|E_0 \cap c|_1 < |E^* \cap c|_1. \tag{4.22}$$

Next, we obtain from (4.22) that for almost every $c = c(\theta_1, \dots, \theta_{m-2})$ through x_0:

$$\begin{aligned} \int_{E_0 \cap c} \left|\cos^{m-2}\theta_{m-1}\right| d\theta_{m-1} &\le 2\int_{-|E_0\cap c|_1/4}^{|E_0\cap c|_1/4} \left|\cos^{m-2}\theta_{m-1}\right| d\theta_{m-1} \\ &< 2\int_{-|E^*\cap c|_1/4}^{|E^*\cap c|_1/4} \left|\cos^{m-2}\theta_{m-1}\right| d\theta_{m-1} \\ &= \int_{E^*\cap c} \left|\cos^{m-2}\theta_{m-1}\right| d\theta_{m-1}. \end{aligned} \tag{4.23}$$

Integrating now (4.23) with respect to $\theta_1, \dots, \theta_{m-1}$, we conclude that $|E_0|_{m-1} < |E^*|_{m-1}$ which contradicts our assumption $|E_0|_{m-1} \ge \lambda$. Hence (4.20) is valid. Then (4.19) and (4.20) yield (4.18). ■

To prove Theorem 4.1, we need a more sophisticated result. Let $K^s = \{x \in \mathbf{R}^m : 0 < x_i \le 1, 1 \le i \le s; x_i = 0, s+1 \le i \le m\}$ be the s-dimensional unit cube in $\mathbf{R}^m$, and let $l_{x,s}$ be a line in $\mathbf{R}^m$ passing through $x \in \mathbf{R}^m$ and parallel to the sth coordinate axis.

Lemma 4.3. *For a measurable set $E \subseteq K^m$, $|E|_m > 0$, there exist sets $E_s \subseteq K^{m-s}$, $0 \le s \le m-1$ and numbers $t_s \in (0, 1]$, $1 \le s \le m-1$, such that $E_0 = E$ and for each $x \in E_{s+1}$:*

$$|E_s \cap l_{x,s+1}|_1 \ge t_{s+1}, \qquad 0 \le s \le m-2. \tag{4.24}$$

Moreover, the following estimate holds:

$$|E_{m-1}|_1 \ge \frac{|E|_m}{A_m t_1 \dots t_{m-1} \ln^{m-1}(e/|E|_m)}, \tag{4.25}$$

where $A_m \le 7^m m^{2m}$.

To prove Lemma 4.3, we need some technical estimates.

Lemma 4.4. *Let f be a nonincreasing function on $[0, 1]$ such that for all $x \in (0, 1]$, $f(x) = f(x-)$, and $0 < \sup_{x\in[0,1]} f(x) \le 1$. Then there exists $x_0 \in (0, 1]$ such that*

$$\int_0^1 f(x)\,dx \le x_0 f(x_0) \ln \frac{e}{x_0 f(x_0)}. \tag{4.26}$$

Proof. Note first that there is $x_0 \in (0, 1]$ such that $\max_{x\in[0,1]} x f(x) = x_0 f(x_0)$. Then $0 < x_0 f(x_0) \le 1$ and a function

$$f_1(x) = \begin{cases} 1, & 0 \le x \le x_0 f(x_0), \\ x_0 f(x_0)/x, & x_0 f(x_0) \le x < 1, \end{cases}$$

satisfies the inequality $f(x) \le f_1(x)$ for all $x \in [0, 1]$. Thus $\int_0^1 f(x)\,dx \le \int_0^1 f_1(x)\,dx = x_0 f(x_0) \ln e/x_0 f(x_0)$, and (4.26) follows. ■

Note that the equality in (4.26) holds if $x_0 \in [\varepsilon, 1]$ and $f(x) = 1$ on $[0, \varepsilon]$ and $f(x) = \varepsilon/x$ on $(\varepsilon, 1]$, where $\varepsilon \in (0, 1)$ is a fixed number.

Lemma 4.5. *If numbers $a \in (0, 1]$ and $b \in (0, 1]$ satisfy the inequality*

$$a \le b \ln(e/b), \tag{4.27}$$

then

$$a/\ln(e/a) \le c_0 b, \tag{4.28}$$

where $c_0 := 1/\alpha_0 = 2.2399\ldots$, and $\alpha_0 = 0.4464\ldots$ is the only solution of the equation $1/\alpha = \exp(\alpha/(1-\alpha))$ for $\alpha \in (0, 1)$.

Proof. Let $\alpha \in (0, 1 - e^{-1})$ be a number. If $0 < b < \exp(-\alpha/(1-\alpha))$, then (4.27) implies

$$\ln(e/a) \ge \ln(e/b) - \ln(\ln(e/b)) \ge \alpha \ln(e/b).$$

Hence

$$a/\ln(e/a) \le (1/\alpha)b. \tag{4.29}$$

Further, if $\exp(-\alpha/(1-\alpha)) \le b \le 1$, then

$$a/\ln(e/a) \le 1 \le b \exp(\alpha/(1-\alpha)). \tag{4.30}$$

Inequalities (4.29) and (4.30) show that (4.27) implies (4.28) with

$$c_0 := \inf_{\alpha \in (0, 1-e^{-1})} \max(1/\alpha, \exp(\alpha/(1-\alpha))) = 2.2399\ldots. \tag{4.31}$$

Hence Lemma 4.5 follows. ■

Lemma 4.6. *If $\{f_s\}_{s=0}^M$ and $\{t_s\}_{s=1}^M$ are sets of numbers from $(0, 1]$ satisfying the recurrence inequality*

$$f_{s-1}/\ln(e/f_{s-1}) \le c_0 t_s f_s, \qquad 1 \le s \le M, \tag{4.32}$$

then

$$f_s \ge \frac{f_0}{A_s t_1 \ldots t_s \ln^s(e/f_0)}, \qquad 1 \le s \le M, \tag{4.33}$$

where c_0 is defined by (4.31), and $A_s < 7^s s^{2s}$ are constants satisfying the recurrence relation

$$A_0 = 0, \qquad A_s = c_0 A_{s-1}(s + \ln A_{s-1}), \qquad 1 \le s \le M. \tag{4.34}$$

Proof. We shall prove (4.33) by induction. For $s = 1$, (4.33) follows from (4.32). Assume that for some s, $1 \le s \le M - 1$, (4.33) holds. Then we obtain from the hypotheses of induction and (4.32):

$$\begin{aligned} f_{s+1} &\ge \frac{f_s}{c_0 t_{s+1} \ln(e/f_s)} \ge \frac{f_0}{c_0 A_s t_1 \ldots t_{s+1} \ln^s(e/f_0) \ln(e f_0^{-1} A_s t_1 \ldots t_s \ln^s(e/f_0))} \\ &\ge \frac{f_0}{c_0 A_s t_1 \ldots t_{s+1} \ln^{s+1}(e/f_0)(s + 1 + \ln A_s)} = \frac{f_0}{A_{s+1} t_1 \ldots t_{s+1} \ln^{s+1}(e/f_0)}. \end{aligned}$$

Thus (4.33) holds for all s, $1 \le s \le M$. The estimate $A_s \le (3c_0)^s s^{2s} < 7^s s^{2s}$ can be easily derived from (4.34) by induction. ■

Proof of Lemma 4.3. We shall construct sets E_s inductively. Let us put $E_0 = E \subseteq K^m$. Assume that for some s, $0 \le s \le m-2$, E_s is constructed. Then $h_{s+1} = |l_{x,s+1} \cap E_s|_1$ is a measurable function on K^{m-s-1} such that $0 \le h_{s+1} \le 1$. Next, the function $f_{s+1}(t) = |\{x \in K^{m-s-1}\colon h_{s+1}(x) \ge t\}|_{m-s-1}$ satisfies the conditions of Lemma 4.4. Hence there exists $t_{s+1} \in (0, 1]$ such that

$$(4.35)\qquad |E_s|_{m-s} = \int_{K^{m-s-1}} h_{s+1}(x)\,dx = \int_0^1 f_{s+1}(t)\,dt \le t_{s+1} f_{s+1}(t_{s+1}) \ln \frac{e}{t_{s+1} f_{s+1}(t_{s+1})}.$$

Now we define E_{s+1} by $E_{s+1} = \{x \in K^{m-s-1} : h_{s+1}(x) \ge t_{s+1}\}$. Then (4.24) holds for all $x \in E_{s+1}$ and $|E_{s+1}|_{m-s-1} = f_{s+1}(t_{s+1})$. This shows that there exist sets $E_s \subseteq K^{m-s}$, $0 \le s \le m-1$ and numbers $t_s \in (0, 1]$, $1 \le s \le m-1$, satisfying (4.24). To prove (4.25), we first note that (4.35) yields the recurrence inequality

$$(4.36)\qquad |E_s|_{m-s} \le t_{s+1}|E_{s+1}|_{m-s-1} \ln \frac{e}{t_{s+1}|E_{s+1}|_{m-s-1}}, \qquad 0 \le s \le m-2.$$

Further using Lemma 4.5 for $a = |E_s|_{m-s-1}$ and $b = t_{s+1}|E_{s+1}|_{m-s-1}$, we obtain from (4.36) and (4.28):

$$(4.37)\qquad |E_s|_{m-s}/\ln(e/|E_s|_{m-s}) \le c_0 t_{s+1}|E_{s+1}|_{m-s-1}, \qquad 0 \le s \le m-2.$$

Now (4.25) follows from (4.37) and Lemma 4.6 for $f_s = |E_s|_{m-s}$. ■

4.3. *Proofs of the Theorems*

Proof of Theorem 4.1. First we prove statement (a). For a univariate polynomial $P(x) = \sum_{k=0}^n a_k x^k$ and a set $E' \subseteq [0, 1]$, (4.6) follows from (2.4). Indeed, we derive the estimate

$$(4.38)\qquad |a_k| \le (22/|E'|_1)^n \|P\|_{C(E')}, \qquad 0 \le k \le n.$$

from (2.4) by the straightforward calculations.

Next, let $P(x) = \sum_{\alpha_1=0}^n \cdots \sum_{\alpha_m=0}^n a_\alpha x_1^{\alpha_1} \cdots x_m^{\alpha_m} \in \mathcal{P}^*_{n,m}$. Then

$$\begin{aligned} P(x_1, \dots, x_m) &= \sum_{\alpha_1=0}^n x_1^{\alpha_1} P_{\alpha_1}(x_2, \dots, x_m), \\ P_{\alpha_1}(x_2, \dots, x_m) &= \sum_{\alpha_2=0}^n x_2^{\alpha_2} P_{\alpha_1\alpha_2}(x_3, \dots, x_m), \\ &\dots \\ P_{\alpha_1\dots\alpha_{m-1}}(x_m) &= \sum_{\alpha_m=0}^n a_\alpha x_m^{\alpha_m}, \end{aligned} \qquad (4.39)$$

where $P_{\alpha_1\dots\alpha_s}(x_{s+1}, \dots, x_m) \in \mathcal{P}^*_{n,m-s}$, $0 \le s \le m-1$.

Further, we prove the inequality

$$(4.40) \qquad |a_\alpha| \le \frac{22^{nm}\|P\|_{C(E)}}{(t_1 \dots t_{m-1}|E_{m-1}|_1)^n},$$

where E_s, $0 \le s \le m-1$, and t_s, $1 \le s \le m$, are numbers and sets from Lemma 4.3. It suffices to show that

$$(4.41) \qquad \max_{x \in E_s} |P_{\alpha_1 \dots \alpha_s}(x_{s+1}, \dots, x_m)| \le 22^{ns}(t_1 \dots t_s)^{-n}\|P\|_{C(E)}, \qquad 0 \le s \le m-1.$$

Indeed, (4.40) follows from (4.41) for $s = m-1$, (4.39), and (4.38) for $E' = E_{m-1}$.

We shall prove (4.41) by induction. It is trivial for $s = 0$. Assume that (4.41) holds for some s, $0 \le s \le m-2$. Then using (4.38) for any line $l_{x,s+1}$, $x \in E_{s+1}$, and taking account of the hypothesis of induction and (4.24), we have

$$\begin{aligned}\max_{x \in E_{s+1}} |P_{\alpha_1 \dots \alpha_{s+1}}(x_{s+2}, \dots, x_m)| &\le \frac{22^n \max_{x \in E_s} |P_{\alpha_1 \dots \alpha_s}(x_{s+1}, \dots, x_m)|}{\min_{x \in E_{s+1}} |l_{x,s+1} \cap E_s|_1^n} \\ &\le \frac{22^{n(s+1)}\|P\|_{C(E)}}{(t_1 \dots t_{s+1})^n}.\end{aligned}$$

Hence (4.41) is valid. Then (4.6) follows from (4.40) and (4.25).

Finally, statement (b) immediately follows from the relation

$$\left| x \in K^m : \prod_{s=1}^m x_s \le t^{1/n} \right| = t^{1/n} \sum_{s=0}^{m-1} \frac{\ln^s(t^{-1/n})}{s!}.$$

Thus Theorem 4.1 is established. ∎

Proof of Theorem 4.2. Let $E \subseteq S^m$ be a measurable set, $P \in \mathcal{P}_{n,m}$ a polynomial of degree n or less, and $x_0 = (0, \dots, 0, 1) \in S^m$. Let us introduce a coordinate system (4.12) in $\mathbf{R}^m$, $m \ge 2$, where $r \ge 0$ and $|\theta_i| \le \pi/2$, $1 \le i \le m-2$, $|\theta_{m-1}| \le \pi$. Then for any great circle c passing through x_0, the restriction of P to c is a trigonometric polynomial of a single variable θ_{m-1} of degree n or less. Hence

$$(4.42) \qquad |P(x_0)| \le A_n(|E \cap c|_1)\|P\|_{C(E)}.$$

Since $A_n(\tau)$ is a decreasing function, (4.42) and Lemma 4.2 imply

$$(4.43) \qquad |P(x_0)| \le A_n(\Gamma_m(\lambda))\|P\|_{C(E)} = A_n(\varphi^{-1}(\lambda))\|P\|_{C(E)}.$$

Finally, we note that (4.43) holds for each $x_0 \in S^m$. Thus (4.8) follows. ∎

Proof of Theorem 4.3. By Lemma 4.1, for any $\varepsilon \in (0, \frac{1}{2})$ there exists a line l_ε in $\mathbf{R}^m$, passing through the origin such that

$$(4.44) \qquad |E \cap l_\varepsilon|_1/|V \cap l_\varepsilon|_1 \ge 1 - \sqrt[m]{1 - \lambda/|V|} - \varepsilon/2 \ge \tfrac{3}{4} - \varepsilon/2.$$

Further, the restriction of P to l_ε is a polynomial of a single variable. Using now Corollary 2.2 and relations (4.44), we obtain

$$|P(0)| \le \exp((4n\sqrt[m]{1 - \lambda/|V|})/(1 - 2\varepsilon))\|P\|_{C(E)}.$$

This yields (4.10). ∎

4.4. *Some Corollaries*

The following is the Remez inequality of first type for polynomials from $\mathcal{P}^*_{n,m}$.

Corollary 4.1. *For a measurable set $E \subseteq K^m$, $|E| = \lambda > 0$, and a polynomial $P \in \mathcal{P}^*_{n,m}$:*

$$\|P\|_{C(K^m)} \le (C/\lambda)^n \ln^{n(m-1)}(e/\lambda)\|P\|_{C(E)}, \tag{4.45}$$

where $C \le 154^m m^{2m}$.

Proof. Note first that the class $\mathcal{P}^*_{n,m}$ is invariant with respect to all linear transformations. Hence Theorem 4.1 implies a more general inequality

$$|P(v_j)| \le (C|\Pi|/|E|)^n \ln^{n(m-1)}(e|\Pi|/\lambda)\|P\|_{C(E)}, \qquad 1 \le j \le 2^m, \tag{4.46}$$

where $C \le 154^m m^{2m}$. Here, $\Pi = \{x \in \mathbf{R}^m : a_i \le x_i \le b_i, 1 \le i \le m\}$ is a rectangular parallelepiped in $\mathbf{R}^m$ with vertices v_j, $1 \le j \le 2^m$; E is a measurable subset of Π; and $P \in \mathcal{P}^*_{n,m}$.

Next, for each $x \in K^m$ there exist N parallelepipeds Π_i, $1 \le i \le N$, $1 \le N \le 2^m$, such that x is a vertex of all Π_i, $1 \le i \le N$, and

$$K^m = \bigcup_{i=1}^N \Pi_i, \qquad |K^m| = \sum_{i=1}^N |\Pi_i|, \qquad |E| = \sum_{i=1}^N |\Pi_i \cap E|. \tag{4.47}$$

Applying (4.46) to each Π_i, we obtain from (4.47):

$$\begin{aligned}|P(x)| &\le \min_{1\le i\le N}(C|\Pi_i|/|\Pi_i \cap E|)^n \ln^{n(m-1)}(e|\Pi_i|/|\Pi_i \cap E|)\|P\|_{C(\Pi_i\cap E)}\\ &\le (C|K^m|/|E|)^n \ln^{n(m-1)}(e|K^m|/|E|)\|P\|_{C(E)}.\end{aligned}$$

This establishes (4.45). ∎

The following are the Remez-type inequalities of first and second types for polynomials on the unit sphere.

Corollary 4.2. *Let $E \subseteq S^m$, let $|E| = \lambda > 0$ be a measurable set, and let $P \in \mathcal{P}_{n,m}$, $m \ge 2$, be a polynomial. Then:*

(a) *for all $\lambda \in (0, |S^m|_{m-1}]$:*

$$\|P\|_{C(S^m)} \le (17\omega_m 2^{-(m-2)/2}/\lambda)^{2n}\|P\|_{C(E)}; \tag{4.48}$$

(b) *for all $\lambda > \lambda_0 = 2\omega_m \int_0^{3\pi/8} \cos^{m-2} u\,du$:*

$$\|P\|_{C(S^m)} \le \exp(19n(1 - \lambda/|S^m|_{m-1})^{1/(m-1)})\|P\|_{C(E)}. \tag{4.49}$$

Proof. Inequality (4.48) follows from Theorems 3.2 and 4.2 and the estimate

$$\begin{aligned}\varphi(t) &= 2\omega_m \int_0^{t/4} \cos^{m-2} u\, du \\ &\geq 2\omega_m \int_0^{t/8} \cos^{m-2} u\, du \geq 2^{-(m-2)/2}\omega_m t, \qquad t \in (0, 2\pi].\end{aligned}$$

To prove (4.49), we note first that if $\lambda > \lambda_0$, then $\varphi^{-1}(\lambda) > 3\pi/2$. Further,

$$\begin{aligned}|S_m|_{m-1} - \lambda &= 2\omega_m \int_0^{(2\pi-\varphi^{-1}(\lambda))/4} \sin^{m-2} u\, du \\ &\geq 2^{-m+1}\pi^{-m+2}(m-1)^{-1}\omega_m(2\pi - \varphi^{-1}(\lambda))^{m-1}.\end{aligned}$$

Taking account of the inequalities $|S^m|_{m-1} < \omega_m$ and $(m-1)^{1/(m-1)} < 3^{1/3}$, we obtain $2\pi - \varphi^{-1}(\lambda) \leq 2\cdot 3^{1/3}\pi(1 - \lambda/|S^m|_{m-1})^{1/(m-1)}$. Now (4.49) follows from Theorems 3.1 and 4.3. ∎

Remark 4.2. Remez-type inequalities (4.8), (4.45), and (4.48) can be applied to the problem of finding the limit distribution of a polynomial on the unit sphere or on the unit cube of large dimension that is useful in statistics and statistical physics.

5. Applications

Applications of Pólya- and Remez-type inequalities in various areas of Analysis have received much attention since the 1960s [2], [3], [9]–[12], [14], [20]–[27], [31]–[33], [35], [36], [39], [42]–[44], [49], [51].

In this section we present several applications of Pólya- and Remez-type inequalities to some problems of Analysis. In particular, we obtain some new Nikolskii-type inequalities in rearrangement-invariant spaces for polynomials and entire functions of exponential type with a convex spectrum (Theorems 5.1 and 5.2). Finally, we establish some estimates of trigonometric integrals (Theorems 5.3 and 5.4).

5.1. *Rearrangement-Invariant Spaces*

Here, we define rearrangements of functions and rearrangement-invariant spaces. We consider measurable functions f defined on the k-dimensional set $\Omega \subseteq \mathbf{R}^m$, equipped with the k-dimensional Lebesgue measure $|E|_k$, $1 \leq k \leq m$, for every measurable $E \subseteq \Omega$. For example, $k = m-1$ if $\Omega = S^m$, and $k = m$ if $\Omega = \mathbf{R}^m$, or Ω is a bounded domain in $\mathbf{R}^m$.

For every f on the bounded set $\Omega \subset \mathbf{R}^m$, we define its *increasing rearrangement* $f^* : [0, |\Omega|_k] \to [0, \infty]$ by $f^*(t) := f^*(t, \Omega) := \sup\{\tau \geq 0 : E_\tau \leq t\}$, where $E_\tau := |\{x \in \Omega : |f(x)| \leq \tau\}|_k$.

Similarly, for every function f on $\Omega \subseteq \mathbf{R}^m$ we define its *decreasing rearrangement* f_* by $f_*(t) := \inf\{\tau \geq 0 : I_\tau \leq t\}$, where $I_\tau := |\{x \in \Omega : |f(x)| > \tau\}|_k$.

We say that a linear real space $F(\Omega)$ of k-measurable functions defined on $\Omega \subseteq \mathbf{R}^m$ is a *rearrangement-invariant space* (RIS) if there is a nonnegative functional $\|\cdot\|_{F(\Omega)}$

on $F(\Omega)$ with the properties:

(i) $\|f\|_{F(\Omega)} = 0$ if and only if $f = 0$;
(ii) $\|cf\|_{F(\Omega)} = |c|\|f\|_{F(\Omega)}$ for a scalar c;
(iii) if $g \in F(\Omega)$ and $f_*(t) \leq g_*(t)$ for all $t \in [0, |\Omega|_k)$, then $f \in F(\Omega)$ and $\|f\|_{F(\Omega)} \leq \|g\|_{F(\Omega)}$.

Note that if Ω is the bounded set, the property (iii) is equivalent to that with the condition $f_*(t) \leq g_*(t)$ replaced by $f^*(t) \leq g^*(t)$.

The *fundamental function* of $F(\Omega)$ is defined by $\psi_F(t) := \|\chi_E\|_{F(\Omega)}$, where $E \subseteq \Omega$, $|E|_k = t, 0 \leq t \leq |\Omega|_k$.

Let Ω be a bounded set and $\omega : \Omega \to [0, |\Omega|_k]$ a measure-preserving transformation which is one-to-one and onto. Then every RIS $F(\Omega)$ generates the RIS $\tilde{F}(0, |\Omega|_k) := \{h = f(\omega\cdot) : f \in F(\Omega)\}$ with $\|h\|_{\tilde{F}(0,|\Omega|_k)} = \|h \circ \omega^{-1}\|_{F(\Omega)}$ and $\psi_{\tilde{F}} = \psi_F$. It is clear that $f^* \in \tilde{F}(0, |\Omega|_k)$ for every $f \in F(\Omega)$ and

$$\|f\|_{F(\Omega)} = \|f^*\|_{\tilde{F}(0,|\Omega|_k)}. \tag{5.1}$$

If $F(\Omega)$ is a normed RIS (NRIS), that is, $\|\cdot\|_{F(\Omega)}$ satisfies the triangle inequality, then it has the following properties [38]:

(a) If $\hat{\psi}_F$ is the least concave majorant of ψ_F, then $\frac{1}{2}\hat{\psi}_F(t) \leq \psi_F(t) \leq \hat{\psi}_F(t)$ for all $t \geq 0$.
(b) Let $F^1(\Omega)$ be the associated space of all measurable functions g on Ω with the finite norm $\|g\|_{F^1(\Omega)} = \sup_{\|f\|_{F(\Omega)}\leq 1} \int_\Omega f(x)g(x)\,dx$. Then $F^1(\Omega)$ is the NRIS with the fundamental function $\psi_{F^1} = t/\psi_F(t)$. In particular,

$$\left|\int_\Omega f(x)g(x)\,dx\right| \leq \|f\|_{F(\Omega)}\|g\|_{F^1(\Omega)}, \qquad f \in F(\Omega), \quad g \in F^1(\Omega). \tag{5.2}$$

It easy to see that spaces $L_p(\Omega)$, $0 \leq p < 1$, are RISs, while $C(\Omega)$, $L_p(\Omega)$, $1 \leq p \leq \infty$, the Orlicz, Lorentz, and Marcinkiewicz spaces, are NRISs (see [19], [38]).

5.2. *Nikolskii-Type Inequalities for Multivariate Polynomials*

Daugavet [17], [18] obtained an algebraic analogue of the Nikolskii inequality (see [50, p. 235]) for multivariate polynomials, by adapting the bridge method of Nikolskii:

$$\|P\|_{L_q(\Omega)} \leq Cn^{\sigma(\Omega)}\|P\|_{L_p(\Omega)}, \qquad 1 \leq p \leq q \leq \infty, \tag{5.3}$$

where Ω is a bounded domain in $\mathbf{R}^m$, $P \in \mathcal{P}_{n,m}$, and $\sigma(\Omega) = 2m(1/p - 1/q)$ if Ω satisfies the cone properties, and $\sigma(\Omega) = (m+1)(1/p - 1/q)$ if Ω has the smooth boundary. In the one-dimensional case (5.3) was established by Lebed, Potapov, and Timan (see [50, p. 236]).

The first results connecting Remez- and Nikolskii-type inequalities were given in [29], [31]. In particular, estimate (5.3) for a convex Ω and $0 < p \leq q \leq \infty$, was obtained in [29], [31] as an easy corollary of (4.1). Moreover, (4.1) implies a more general inequality [29], [31]:

$$\|P\|_{C(V)} \leq \frac{8}{\psi_F(|V|_m(n+1)^{-2m})}\|P\|_{F(V)}, \tag{5.4}$$

where V is a convex body in $\mathbf{R}^m$ and $F(V)$ is an RIS with the fundamental function ψ_F. It is easy to verify that (5.4) remains valid for a bounded domain $V \subset \mathbf{R}^m$ satisfying the cone condition. Note also that Kroó and Schmidt [39] independently proved (5.3) for smooth domains, by using a Remez-type inequality.

Below we consider two versions of (5.3) and (5.4) with no "boundary effect," unlike these inequalities.

Theorem 5.1.

(a) *If n is large enough, then for any RIS $F(S^m)$ and a polynomial $P \in \mathcal{P}_{n,m}$:*

$$\|P\|_{C(S^m)} \le \frac{e^{19}}{\psi_F(|S^m|_{m-1} n^{1-m})} \|P\|_{F(S^m)}. \tag{5.5}$$

(b) *For a centrally symmetric (with respect to the origin) body $V \subset \mathbf{R}^m$, any RIS $F(V)$, and a polynomial $P \in \mathcal{P}_{n,m}$, $n \ge 1$:*

$$|P(0)| \le \frac{e}{\psi_F(|V|_m (4n)^{-m})} \|P\|_{F(V)}. \tag{5.6}$$

Proof. We first prove (5.6). Given $t \in [1 - 2^{-2m}|V|, |V|]$, we consider a set

$$E_t = \{x \in V : |P(x)| \le P^*(t)\}.$$

It is easy to see that $|E_t| = t$ and $\|P\|_{C(E_t)} = P^*(t)$. Now applying (4.10) to P and $E = E_t$, we obtain the estimate

$$P^*(t) \ge \exp(-4n \sqrt[m]{1 - t/|V|})|P(0)|. \tag{5.7}$$

Next, setting $a_n = 1 - (4n)^{-m}$ and using (5.1) and (5.7), we have

$$\begin{aligned}\|P\|_{F(V)} &= \|P^*\|_{\tilde{F}(0,|V|)} \ge \|P^*\|_{\tilde{F}(a_n|V|,|V|)} \ge P^*(a_n|V|)\psi_F(|V|(4n)^{-m}) \\ &\ge e^{-1}|P(0)|\psi_F(|V|(4n)^{-m}).\end{aligned}$$

This yields (5.6). Inequality (5.5) can be established similarly, if we apply (4.49) instead of (4.10). ■

Remark 5.1. Inequality (5.6) plays an important role in the limit theorems of approximation theory [35], [36]. Another application of (5.6) to entire functions of exponential type is presented below.

5.3. *A Nikolskii-Type Inequality for Entire Functions of Exponential Type*

Let V be a centrally symmetric (with respect to the origin) body in $\mathbf{R}^m$, and let $V^* := \{y \in \mathbf{R}^m : \sup_{x \in V} |\sum_{i=1}^m x_i y_i| \le 1\}$ be a polar of V. We say that an entire function g has exponential type σV, $\sigma > 0$, if for every $\varepsilon > 0$ there exists a constant A_ε satisfying the inequality $|g(z)| \le A_\varepsilon \exp((\sigma + \varepsilon) \sup_{x \in V} |\sum_{i=1}^m x_i z_i|)$ for all $z \in \mathbf{C}^m$. We denote by $B_{\sigma V}$, $\sigma > 0$, the class of all entire functions of exponential type σV.

An extension of the Nikolskii inequality (see [50, p. 235]) to an NRIS was given in [28], [29]. In particular, for $g \in B_{\sigma Q^m} \cap F(\mathbf{R}^m)$, the following inequality is valid:

$$\|g\|_{C(\mathbf{R}^m)} \leq \frac{C}{\psi_F((\pi/\sigma)^m)} \|g\|_{F(\mathbf{R}^m)}, \tag{5.8}$$

where $Q^m := \{x \in \mathbf{R}^m : |x_i| \leq 1, 1 \leq i \leq m\}$ is the cube, and C depends only on m. A different proof of (5.8) was given by Berkolaiko and Ovchinnikov [6]. On the other hand, Nessel and Wilmes [45] (see also Aliev [1]) obtained the Nikolskii-type inequality for $g \in B_{\sigma V} \cap L_p(\mathbf{R}^m)$:

$$\|g\|_{L_q(\mathbf{R}^m)} \leq ((s/(2\pi))^m |V^*|_m)^{1/p-1/q} \|g\|_{L_p(\mathbf{R}^m)}, \qquad 1 \leq p \leq q \leq \infty, \tag{5.9}$$

where $s := \inf\{k \in \mathbf{Z}^1 : k \geq p/2\}$.

The following theorem is a combined version of inequalities (5.8) and (5.9).

Theorem 5.2. *If $F(\mathbf{R}^m)$ is an NRIS, then for $g \in B_{\sigma V} \cap F(\mathbf{R}^m)$, $\sigma > 0$:*

$$\|g\|_{C(\mathbf{R}^m)} \leq \frac{C}{\psi_F(|V^*|_m (4\sigma)^{-m})} \|g\|_{F(\mathbf{R}^m)}, \tag{5.10}$$

where $C \leq e$ is an absolute constant.

To prove the theorem we need two lemmas.

Lemma 5.1. *Let $F(\mathbf{R}^m)$ be an NRIS. If $g \in B_{\sigma V} \cap F(\mathbf{R}^m)$, then $g \in B_{\sigma V} \cap C(\mathbf{R}^m)$.*

Proof. We first show that for every $\varepsilon > 0$ the function $h_\varepsilon(x) = (\sin \varepsilon|x|/\varepsilon|x|)^{m+2}$ belongs to any NRIS $F(\mathbf{R}^m)$. Indeed, for the cubes $Q_k^m := \{x \in \mathbf{R}^m : |x_i - k_i| \leq \frac{1}{2}, 1 \leq i \leq m\}$, $k = (k_1, \ldots, k_m) \in \mathbf{Z}^m$, we have

$$\|h_\varepsilon\|_{F(\mathbf{R}^m)} \leq \sum_{k \in \mathbf{Z}^m} \|h_\varepsilon\|_{F(Q_k^m)} \leq C\psi_F(1) \left(1 + \sum_{|k|>0} |k|^{-m-2}\right) < \infty. \tag{5.11}$$

Next, for a fixed $x \in \mathbf{R}^m$ the function $g(y)h(x-y)$ belongs to $B_{\sigma_1 Q^m}$, where σ_1 depends only on V. Hence using the Nikolskii inequality (see [50, p. 235]) and taking account of (5.2) and (5.11), we obtain that for all $x \in \mathbf{R}^m$, $y \in \mathbf{R}^m$:

$$|g(y)h_1(x-y)| \leq C \int_{\mathbf{R}^m} |g(u)h_1(x-u)|\, du \leq C\|g\|_{F(\mathbf{R}^m)} \|h_1\|_{F'(\mathbf{R}^m)} = C_1.$$

Finally choosing $y = x$, we conclude that $g \in C(\mathbf{R}^m)$. ∎

Lemma 5.2. *Let $F(\mathbf{R}^m)$ be an NRIS. Then for every $g \in B_{\sigma V} \cap F(\mathbf{R}^m)$ there exists a sequence of polynomials $P_n \in \mathcal{P}_{n,m}$, $n = 1, 2, \ldots$, such that the following relations hold:*

$$\lim_{n\to\infty} \|g - P_n\|_{C((a_n/\sigma)V^*)} = 0, \tag{5.12}$$

$$\lim_{n\to\infty} \|g - P_n\|_{F((a_n/\sigma)V^*)} = 0, \tag{5.13}$$

where $a_n = n - \sqrt{n}$.

Proof. Lemma 5.1 shows that $g \in C(\mathbf{R}^m)$. Let $P_n \in \mathcal{P}_{n,m}$ satisfy the condition

$$\|g - P_n\|_{C((a_n/\sigma)V^*)} = \inf_{P\in\mathcal{P}_{n,m}} \|g - P\|_{C((a_n/\sigma)V^*)}, \qquad n = 1, 2, \ldots.$$

Next, we shall use the following estimate proved in [35]:

$$\|g - P_n\|_{C((a_n/\sigma)V^*)} \le C_1 n^\gamma \exp(-C_2 n^\beta)\|g\|_{C(\mathbf{R}^m)}, \tag{5.14}$$

where C_1, C_2, γ and $\beta > 0$ are constants that depend only on m. Then (5.12) follows from (5.14). Further, taking account of property (a) of the NRIS, we obtain from (5.14):

$$\lim_{n\to\infty} \|g - P_n\|_{F((a_n/\sigma)V^*)} \le \lim_{n\to\infty} \psi_F((a_n/\sigma)^m |V^*|_m)\|g - P_n\|_{C((a_n/\sigma)V^*)} = 0.$$

Hence (5.13) follows. ∎

Proof of Theorem 5.2. Let P_n, $n = 1, 2, \ldots$, be the sequence of polynomials from Lemma 5.2. Then using Theorem 5.1(b) and Lemma 5.2, we obtain

(5.15)

$$\begin{aligned} |g(0)| &\le \limsup_{n\to\infty} |g(0) - P_n(0)| + \limsup_{n\to\infty} \frac{e}{\psi_F((a_n/(4n\sigma))^m |V^*|_m)} \|P_n\|_{F((a_n/\sigma)V^*)} \\ &\le \frac{e}{\psi_F(|V^*|_m (4\sigma)^{-m})}\|g\|_{F(\mathbf{R}^m)} + \frac{e}{\psi_F((4\sigma)^{-m}|V^*|_m)} \lim_{n\to\infty} \|g - P_n\|_{F((a_n/\sigma)V^*)} \\ &= \frac{e}{\psi_F(|V^*(4\sigma)^{-m})}\|g\|_{F(\mathbf{R}^m)}. \end{aligned}$$

Lemma 5.1 and (5.15) yield (5.10). ∎

Remark 5.2. The following example shows that Theorem 5.2 cannot be essentially improved. Let $g_0(x) = (\int_{(\sigma/2)V} \cos(\sum_{j=1}^m x_j y_j)\, dy)^2$ be the function from $B_{\sigma V} \cap L_1(\mathbf{R}^m)$. Then g_0 belongs to any NRIS $F(\mathbf{R}^m)$, and

$$\|g_0\|_{F(\mathbf{R}^m)} \le C(m)\psi_F(|V^*|(4\sigma)^{-m})\|g_0\|_{C(\mathbf{R}^m)}.$$

5.4. *Upper Estimates of Trigonometric Integrals*

Upper estimates of trigonometric integrals

$$I_{m,P} = \int_{K^m} \exp(2\pi i P(x))\, dx,$$

where P is a polynomial in m variables and $K^m = [0, 1]^m$, play an important role in some areas of Number Theory, Analysis, Probability, and Mathematical Statistics. In 1980 Vinogradov [51] came up with the idea of an estimate of $I_{1,P}$ which is based on a Pólya-type inequality. Arkhipov, Karatsuba, and Chubarikov [2] developed this approach and established the following result: for $P(x) = \sum_{k=0}^n a_k x^k$, $n \ge 2$:

$$|I_{1,P}| \le \min\left(1, 32\left(\max_{1\le k\le n} |a_k|\right)^{-1/n}\right). \tag{5.16}$$

Applying the same approach and using Corollary 2.1, it is possible to obtain a refinement of (5.16).

Theorem 5.3. *For* $P(x) = \sum_{k=0}^{n} a_k x^k$, $n \geq 2$:

$$|I_{1,P}| \leq \min\left(1, 20\left(\sum_{k=1}^{n} k|a_k|\right)^{-1/n}\right). \tag{5.17}$$

Proof. For a fixed number $t > 0$ and a set $E_t = \{x \in [0,1] : |P'(x)| \leq t\}$, we have

$$|I_{1,P}| \leq \left|\int_{E_t} \exp(2\pi i P(x))\,dx\right| + \left|\int_{[0,1]-E_t} \exp(2\pi i P(x))\,dx\right| = I^* + I^{**}. \tag{5.18}$$

Next, applying Corollary 2.1 to the polynomial P', to the interval $[a, b] = [0, 1]$, and to the set $E = E_t$, we obtain

$$\sum_{k=1}^{n} k|a_k| \leq T_{n-1}(4/|E_t| - 1)\|P'\|_{C(E_t)} \leq (8/|E_t|)^{n-1} t.$$

This shows that

$$I^* \leq |E_t| \leq 8\left(t/\sum_{k=1}^{n} k|a_k|\right)^{1/(n-1)}. \tag{5.19}$$

The inequality

$$I^{**} \leq \sqrt{2}(n-1)/t \tag{5.20}$$

was proved in [2, p. 15]. Thus (5.18), (5.19), and (5.20) yield

$$\begin{aligned} |I_{1,P}| &\leq \min\left(1, \inf_{t>0}\left(8\left(t/\sum_{k=1}^{n} k|a_k|\right)^{1/(n-1)} + \sqrt{2}(n-1)/t\right)\right) \\ &\leq \min\left(1, 20\left(\sum_{k=1}^{n} k|a_k|\right)^{-1/n}\right). \end{aligned}$$

Hence (5.17). ■

Using (5.17) and induction in m, the authors of [2] obtained the following multidimensional version of (5.17):

$$|I_{m,P}| \leq \min\left(1, 32\left(\max_{|\alpha|>0} |a_\alpha|\right)^{-1/n} \ln^{m-1}\left(\max_{|\alpha|>0} |a_\alpha| + 2\right)\right), \tag{5.21}$$

where a_α, $|\alpha| > 0$, are the coefficients of $P \in \mathcal{P}^*_{n.m}$. Below we establish some new estimates of $I_{m,P}$.

Theorem 5.4. *Let P be a polynomial in m variables, $m \geq 1$:*

(a) *If $P(x) = \sum_{|\alpha| \leq n} a_\alpha x^\alpha \in \mathcal{P}_{n,m}$, then for $n \geq 2$:*

$$|I_{m,P}| \leq \min\left(1, C\left(\max_{1 \leq |\alpha| \leq n} |a_\alpha|\right)^{-1/n}\right), \tag{5.22}$$

where C depends only on m and n.

(b) *If $P(x) = \sum_{\alpha_1=0}^{n} \cdots \sum_{\alpha_m=0}^{n} a_\alpha x^\alpha \in \mathcal{P}^*_{n,m}$, then for $n \geq 2$:*

$$|I_{m,P}| \leq \min\left(1, 20\left(\max_{|\alpha|>0} |a_\alpha|\right)^{-1/(n)} \ln^{m-1}\left(\max_{|\alpha>0} |a_\alpha| + 2\right)\right). \tag{5.23}$$

To prove the theorem we need a multidimensional generalization of (5.20).

Lemma 5.3. *For a polynomial $P \in \mathcal{P}_{n,m}$, $n \geq 2$, and a set $E_{t,j} = \{x \in K^m : |\partial P(x)/\partial x_j| \leq t\}$, $1 \leq j \leq m$, $t > 0$, we have*

$$\left|\int_{K^m - E_{t,j}} \exp(2\pi i P(x))\, dx\right| \leq \sqrt{2}(n-1)/t. \tag{5.24}$$

Proof. Let $P(x) = P_{x^*}(x_j)$ be a polynomial of a single variable x_j and let $E_{t,x^*} = \{x_j \in [0,1] : |dP_{x^*}(x_j)/dx_j| \leq t\}$, where $x^* = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m)$ is a fixed point of K^{m-1}, $1 \leq j \leq m$. Then applying (5.20) to P_{x^*}, we obtain

$$\begin{aligned}\left|\int_{K^m - E_{t,j}} \exp(2\pi i P(x)) x\right| &\leq \int_{K^{m-1}} \left|\int_{[0,1]-E_{t,x^*}} \exp(2\pi i P_{x^*}(x_j))\, dx_j\right| dx^* \\ &\leq \sqrt{2}(n-1)/t.\end{aligned}$$

Thus (5.24). ■

Proof of Theorem 5.4. We first prove statement (a). Let $a_\alpha = a_{(\alpha_1,\dots,\alpha_m)}$ be a coefficient of $P \in \mathcal{P}_{n,m}$ such that $|a_\alpha| > 0$, $|\alpha| > 0$, and $\alpha_j > 0$ for some j, $1 \leq j \leq m$. Then for every $t > 0$ we have

$$|I_{m,P}| \leq \left|\int_{E_{t,j}} \exp(2\pi i P(x))\, dx\right| + \left|\int_{K^m - E_{t,j}} \exp(2\pi i P(x))\, dx\right| = I^* + I^{**}, \tag{5.25}$$

where $E_{t,j}$ is defined in Lemma 5.3. Next, by Lemma 5.3:

$$I^{**} \leq \sqrt{2}(n-1)/t. \tag{5.26}$$

Further applying Pólya-type inequality (4.5) to $\partial P(x)/\partial x_j \in \mathcal{P}_{n-1,m}$ and to $V = K^m$, we obtain

$$I^* \leq |E_{t,j}| \leq C(t/|a_\alpha|)^{1/(n-1)}.$$

Together with (5.25) and (5.26), this implies

$$|I_{m,P}| \le \min\left(1, \inf_{t>0}(\sqrt{2}(n-1)/t + C(t/|a_\alpha|)^{1/(n-1)})\right) \le \min(1, C|a_\alpha|^{-1/n}).$$

This yields (5.22). The proof of (5.23) is similar to that of (5.21) if we use inequality (5.17) instead of (5.16). ■

Remark 5.3. Estimate (5.22) immediately implies that the special integral of Tarry's problem [2]:

$$\theta = \int_{\mathbf{R}^{N-1}} \left| \int_{K^m} \exp(2\pi i P(x))\, dx \right|^{2L} d\bar{a}$$

converges if $2L > n(N-1)$. Here $\bar{a} = (a_\alpha)_{0<|\alpha|\le m}$ is the vector of all coefficients of $P \in \mathcal{P}_{n,m}$ but $a_{(0,\ldots,0)}$, and $N = \dim \mathcal{P}_{n,m} = \binom{n+m}{m}$. Using a more sophisticated approach, the authors of [2] obtained that θ converges if $2L > (m+1)\binom{n+m}{m+1}$.

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