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CONSTRUCTIVE APPROXIMATION

Orthogonal Polynomials with Ultra-Exponential Weight Functions: An Explicit Solution to the Ditkin–Prudnikov Problem

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Abstract

New sequences of orthogonal polynomials with ultra-exponential weight functions are discovered. In particular, we give an explicit solution to the Ditkin–Prudnikov problem (1966). The 3-term recurrence relations, explicit representations, generating functions and Rodrigues-type formulae are derived. The method is based on differential properties of the involved special functions and their representations in terms of the Mellin–Barnes and Laplace integrals. A notion of the composition polynomial orthogonality is introduced. The corresponding advantages of this orthogonality to discover new sequences of polynomials and their relations to the corresponding multiple orthogonal polynomial ensembles are shown.

Keywords Orthogonal polynomials · Modified Bessel functions · Meijer G-function · Mellin transform · Associated Laguerre polynomials · Multiple orthogonal polynomials

Mathematics Subject Classification 33C47 · 33C45 · 33C10 · 44A15 · 42C05

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1 Introduction and preliminary results

Throughout the text, $\mathbb{N}$ will denote the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}$ and $\mathbb{C}$ the field of the real and complex numbers, respectively. The notation $\mathbb{R}_+$ corresponds to the set of all positive real numbers. The present investigation is primarily targeted at analysis of sequences of orthogonal polynomials with respect to the weight functions related to the modified Bessel functions of the second kind or Macdonald functions $K_\nu(x)$ [5, Vol. II]. The problem was posed by Ditkin and Prudnikov in the seminal work of 1966 [4] to find a new sequence of orthogonal polynomials $(P_n)_{n\in\mathbb{N}_0}$, satisfying the orthogonality conditions

$$\int_0^\infty 2K_0(2\sqrt{x})P_m(x)P_n(x)\mathrm{d}x = \delta_{n,m}\ , \quad n, m \in \mathbb{N}_0, \tag{1.1}$$

where $\delta_{n,m}$ represents the Kronecker symbol, and related to the weight $2K_0(2\sqrt{x})$ which can be defined in terms of the Mellin–Barnes integral (see [10, relation (8.4.23.1), Vol. III]).

$$2K_0(2\sqrt{x}) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma^2(s)x^{-s}\mathrm{d}s\ , \quad x, \gamma \in \mathbb{R}_+, \tag{1.2}$$

where $\Gamma(z)$ is the Euler gamma function [5, Vol. I]. The first four polynomials are

$$P_0(x) = 1, \quad P_1(x) = \frac{1}{\sqrt{3}}(x-1), \quad P_2(x) = \sqrt{\frac{3}{41}}\left(\frac{x^2}{4} - \frac{8}{3}x + \frac{5}{3}\right),$$
$$P_3(x) = \sqrt{\frac{41}{2841}}\left(\frac{x^3}{36} - \frac{177}{164}x^2 + \frac{267}{41}x - \frac{131}{41}\right).$$

Later in 1993 [9] Prudnikov formulated the problem in terms of more general ultra-exponential weight functions $\rho_{0,k-1},\ k \in \mathbb{N}$ (see Definition 1 below), and in [13] it was announced in terms of the scaled Macdonald function

$$\rho_\nu(x) = 2x^{\nu/2}K_\nu(2\sqrt{x}),\ x \in \mathbb{R}_+, \quad \nu \geq 0. \tag{1.3}$$

This function has the Mellin–Barnes integral representation in the form

$$\rho_\nu(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(\nu+s)\Gamma(s)x^{-s}\mathrm{d}s\ , \quad x, \gamma \in \mathbb{R}_+, \tag{1.4}$$

and more general ultra-exponential weight functions can be represented, in turn, in terms of Meijer G-functions [15]. Namely, the problem is to find a sequence of orthogonal polynomials $\left(P_n^\nu\right)_{n\in\mathbb{N}_0}$ $(P_n^0 \equiv P_n)$, satisfying the following orthogonality conditions

$$\int_0^\infty P_n^\nu(x)P_m^\nu(x)\rho_\nu(x)\mathrm{d}x = \delta_{n,m}, \quad n, m \in \mathbb{N}_0. \tag{1.5}$$

As was shown in [13] and [2] it is more natural to investigate multiple orthogonal polynomials for two Macdonald weights ρ_ν and $\rho_{\nu+1}$ since it gives explicit formulas, differential properties, recurrence relations and Rodrigues formulas. Nevertheless, there is still an attractive original problem: to understand the nature of such polynomial sequences and their relation to classical systems of orthogonal polynomials and associated multiple orthogonal polynomial ensembles.

On the other hand, the operational calculus associated with the differential operator $\frac{\mathrm{d}}{\mathrm{d}t}$ gives rise to the Laplace transform

$$F(x) = \int_0^\infty e^{-xt} f(t)\mathrm{d}t, \quad x \in \mathbb{R}_+, \tag{1.6}$$

having the exponential function as a kernel, which is the weight function for classical Laguerre polynomials [1], being represented in terms of the Mellin–Barnes integral [10, relation (8.4..3.1), Vol. III]

$$\mathrm{e}^{-x} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s) x^{-s} \mathrm{d}s \ , \quad x, \gamma \in \mathbb{R}_+. \tag{1.7}$$

Meanwhile, the operator $\frac{\mathrm{d}}{\mathrm{d}t} t \frac{\mathrm{d}}{\mathrm{d}t}$ which is also called the Laguerre derivative [3], leads to the Meijer transform [15], involving the weight $2K_0(2\sqrt{x})$ which is given by (1.2), namely,

$$G(x) = \int_0^\infty 2K_0(2\sqrt{xt}) g(t)\mathrm{d}t, \quad x \in \mathbb{R}_+. \tag{1.8}$$

This transform is an important example of the so-called Mellin type convolution transforms, which are extensively investigated in [15]. Moreover, we will employ the Mellin transform technique developed in [15] in order to investigate various properties of the scaled Macdonald functions and more general ultra-exponential weights. Specifically, the Mellin transform is defined, for instance, in $L_{\mu,p}(\mathbb{R}_+),\ 1 \le p \le 2$ (see details in [12]) by the integral

$$f^*(s) = \int_0^\infty f(x) x^{s-1} \mathrm{d}x, \quad s \in \mathbb{C}, \tag{1.9}$$

which is convergent in mean with respect to the norm in $L_q(\mu - i\infty, \mu + i\infty),\ \mu \in \mathbb{R},\ q = p/(p-1)$. Moreover, the Parseval equality holds for $f \in L_{\mu,p}(\mathbb{R}_+),\ g \in L_{1-\mu,q}(\mathbb{R}_+)$

$$\int_0^\infty f(x) g(x)\mathrm{d}x = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} f^*(s) g^*(1-s)\mathrm{d}s. \tag{1.10}$$

The inverse Mellin transform is given accordingly

$$f(x) = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} f^*(s) x^{-s} \mathrm{d}s, \tag{1.11}$$

where the integral converges in mean with respect to the norm in $L_{\mu,p}(\mathbb{R}_+)$

$$||f||_{\mu,p} = \left(\int_0^\infty |f(x)|^p x^{\mu p - 1} \mathrm{d}x \right)^{1/p}.$$

In particular, letting $\mu = 1/p$ we get the usual space $L_p(\mathbb{R}_+;\ \mathrm{d}x)$. Recalling the Meijer transform (1.8) one can treat it as an analog of the Laplace transform (1.6) in the operational calculus associated with the Laguerre derivative. Consequently, the corresponding analog of the classical Laguerre polynomials would be important to investigate, discovering the mentioned Ditkin–Prudnikov polynomial sequence. Finally, we note in this section that in [8] some non-orthogonal polynomial systems were investigated which share the same canonical regular form with Ditkin–Prudnikov polynomial sequence $(P_n)_{n \in \mathbb{N}_0}$. An analogous relation occurs, for instance, between the Bernoulli polynomials, which also happen to be non-orthogonal, and the (orthogonal) Legendre polynomials.

2 Properties of the Scaled Macdonald Functions

We begin with

Definition 1 Let $x, \gamma \in \mathbb{R}_+,\ \nu \geq 0,\ k \in \mathbb{N}_0$. The function $\rho_{\nu,k}(x)$ is called the ultra-exponential weight function and it is expressed in terms of the following Mellin–Barnes integral

$$\rho_{\nu,k}(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(\nu + s) \, [\Gamma(s)]^k \, x^{-s} \mathrm{d}s. \tag{2.1}$$

It is easily seen from the reciprocal formulas (1.9), (1.11) for the Mellin transform that the case $k = 0$ corresponds to the weight function $\rho_{\nu,0}(x) = x^\nu e^{-x}$, which is related to the associated classical Laguerre polynomials $L_n^\nu(x)$ [1]

$$\int_0^\infty L_n^\nu(x) L_m^\nu(x) e^{-x} x^\nu \mathrm{d}x = \delta_{n,m}, \quad n, m \in \mathbb{N}_0 \tag{2.2}$$

and $k = 1$ gives the function $\rho_{\nu,1} \equiv \rho_\nu$, which is associated with the Prudnikov polynomials P_n^ν under orthogonality conditions (1.5). As mentioned above the weights $\rho_{\nu,k}$ can be expressed in terms of the Meijer G-functions (see [7]). Concerning the scaled Macdonald function ρ_ν, we employ the Parseval equality (1.10) to the integral (1.4) to derive the Laplace integral representation for this weight function which will be used later. In fact, we obtain

$$\rho_\nu(x) = \int_0^\infty t^{\nu-1} e^{-t-x/t} \mathrm{d}t, \quad x > 0, \ \nu \in \mathbb{R}. \tag{2.3}$$

The direct Mellin transform (1.9) gives the moments of ρ_ν. Specifically, we obtain

$$\int_0^\infty \rho_\nu(x) x^\mu \mathrm{d}x = \Gamma(\mu + \nu + 1)\Gamma(\mu + 1). \tag{2.4}$$

Moreover, the asymptotic behavior of the modified Bessel function at infinity and near the origin [5, Vol. II] gives the corresponding values for the scaled Macdonald function $\rho_\nu, \ \nu \in \mathbb{R}$. To be Precise we have

$$\rho_\nu(x) = O\left(x^{(\nu-|\nu|)/2}\right), \ x \to 0, \ \nu \neq 0, \quad \rho_0(x) = O(\log x), \ x \to 0,$$
$$\rho_\nu(x) = O\left(x^{\nu/2-1/4} e^{-2\sqrt{x}}\right), \ x \to +\infty.$$

Returning to the Mellin–Barnes integral (1.4), we multiply both sides of this equality by $x^{-\nu}$ and then differentiate with respect to x under the integral sign. This is possible via the absolute and uniform convergence by $x \geq x_0 > 0$, which can be established using the Stirling asymptotic formula for the gamma function [5, Vol. I]. Therefore we deduce

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-\nu}\rho_\nu(x)\right] = -\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(\nu + s + 1)\Gamma(s) x^{-s-\nu-1} \mathrm{d}s,$$

where the reduction formula $\Gamma(z + 1) = z\Gamma(z)$ for the gamma function is applied. Multiplying the latter equality by $x^{\nu+1}$ and differentiating again, we involve a simple change of variables and the analyticity on the right half-plane $\mathrm{Re}s > 0$ of the integrand to end up with the second order differential equation for ρ_ν

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\nu+1}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-\nu}\rho_\nu(x)\right]\right] = \rho_\nu(x). \tag{2.5}$$

Further, denoting the operator of the Laguerre derivative by $\beta = DxD$ and its companion $\theta = xDx$ (see [11]), where D is the differential operator $D = \frac{\mathrm{d}}{\mathrm{d}x}$, we calculate the nth power, employing amazing Viskov-type identities [14]

$$\beta^n = (DxD)^n = D^n x^n D^n, \quad \theta^n = (xDx)^n = x^n D^n x^n, \quad n \in \mathbb{N}_0. \tag{2.6}$$

Equalities (2.6) can be proved by the method of mathematical induction. We show how to establish (2.6), using the Mellin transform technique for a class of functions f whose Mellin transforms (1.9) $f^*(s), \ s = \gamma + i\tau$ belong to the Schwartz space as a function of τ. As is known, this space is a topological vector space of functions φ

such that $\varphi \in C^{\infty}(\mathbb{R})$ and $x^m \varphi^{(n)}(x) \to 0,\ |x| \to \infty,\ m, n \in \mathbb{N}_0$. This means that one can differentiate under the integral sign in (1.11) infinitely many times. Hence

$$\begin{aligned}(\beta^n f)(x) = (DxD)^n f &= \frac{1}{2\pi i}(DxD)^{n-1}\int_{\gamma-i\infty}^{\gamma+i\infty} s^2 f^*(s)x^{-s-1}\mathrm{d}s\\ &= \frac{1}{2\pi i}(DxD)^{n-2}\int_{\gamma-i\infty}^{\gamma+i\infty}[s(s+1)]^2 f^*(s)x^{-s-2}\mathrm{d}s\\ &= \cdots = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}[(s)_n]^2 f^*(s)x^{-s-n}\mathrm{d}s,\end{aligned}$$

where

$$(s)_n = s(s+1)\dots(s+n-1) = \frac{\Gamma(s+n)}{\Gamma(s)} \tag{2.7}$$

is the Pochhammer symbol [5]. On the other hand,

$$\begin{aligned}\left(D^n x^n D^n\right) f &= \frac{(-1)^n}{2\pi i} D^n \int_{\gamma-i\infty}^{\gamma+i\infty}(s)_n f^*(s)x^{-s}\mathrm{d}s\\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}[(s)_n]^2 f^*(s)x^{-s-n}\mathrm{d}s,\end{aligned}$$

which proves the first identity in (2.6). Analogously,

$$\begin{aligned}(\theta^n f)(x) = (xDx)^n f &= \frac{1}{2\pi i}(xDx)^{n-1}\int_{\gamma-i\infty}^{\gamma+i\infty}(1-s) f^*(s)x^{1-s}\mathrm{d}s\\ &= \cdots = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}(1-s)_n f^*(s)x^{n-s}\mathrm{d}s\\ &= \frac{x^n}{2\pi i} D^n \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)x^{n-s}\mathrm{d}s = \left(x^n D^n x^n\right) f.\end{aligned}$$

This proves the second identity in (2.6). In particular, we easily find the values

$$\begin{aligned}(\beta^n \rho_0)(x) &= (DxD)^n \rho_0 = \rho_0(x),\\ (\beta^n \rho_1)(x) &= (DxD)^n \rho_1 = \rho_1(x) - n\rho_0(x), \quad n \in \mathbb{N}_0, \qquad (2.8)\\ (\theta^n 1)(x) &= (xDx)^n 1 = n!x^n,\\ (\theta^n x^k)(x) &= (xDx)^n x^k = \frac{(n+k)!}{k!}x^{n+k}, \quad n, k \in \mathbb{N}_0. \qquad (2.9)\end{aligned}$$

The quotient of the scaled Macdonald functions ρ_ν, $\rho_{\nu+1}$ is given by the important Ismail integral representation [6]

$$\frac{\rho_\nu(x)}{\rho_{\nu+1}(x)} = \frac{1}{\pi^2}\int_0^\infty \frac{y^{-1}\mathrm{d}y}{(x+y)\left[J^2_{\nu+1}(2\sqrt{y}) + Y^2_{\nu+1}(2\sqrt{y})\right]}, \tag{2.10}$$

where $J_\nu(z), Y_\nu(z)$ are Bessel functions of the first and second kind, respectively [5]. Another interesting integral representation for the scaled Macdonald function ρ_ν is given via [10, relation (2.19.4.13), Vol. II] in terms of the associated Laguerre polynomials. Namely, we have

$$\frac{(-1)^n x^n}{n!}\,\rho_\nu(x) = \int_0^\infty t^{\nu+n-1} e^{-t-x/t} L_n^\nu(t)\mathrm{d}t, \quad n \in \mathbb{N}_0. \tag{2.11}$$

Meanwhile, an important property for the scaled Macdonald functions can be obtained in terms of the Riemann–Liouville fractional integral [15]

$$\left(I_-^\nu f\right)(x) = \frac{1}{\Gamma(\nu)}\int_x^\infty (t-x)^{\nu-1} f(t)\mathrm{d}t. \tag{2.12}$$

In fact, appealing to [5, relation (2.16.3.8), Vol. II]

$$2^{\alpha-1}x^{\alpha+\nu}\Gamma(\alpha)K_{\nu+\alpha}(x) = \int_x^\infty t^{1+\nu}(t^2-x^2)^{\alpha-1}K_\nu(t)\mathrm{d}t, \tag{2.13}$$

making simple changes of variables and letting $\nu = 0$, we derive the formula

$$\rho_\alpha(x) = \left(I_-^\alpha \rho_0\right)(x). \tag{2.14}$$

Moreover, the index law for fractional integrals immediately implies

$$\rho_{\nu+\mu}(x) = \left(I_-^\nu \rho_\mu\right)(x) = \left(I_-^\mu \rho_\nu\right)(x). \tag{2.15}$$

The corresponding definition of the fractional derivative presumes the relation $D_-^\mu = -DI_-^{1-\mu}$. Hence for the ordinary nth derivative of ρ_ν we get

$$D^n\rho_\nu(x) = (-1)^n\rho_{\nu-n}(x), \quad n \in \mathbb{N}_0. \tag{2.16}$$

Another way to get this formula is to differentiate n-times the integral (1.4), to use the definition of the Pochhammer symbol (2.7) and to make a simple change of variables.

In the meantime, the Mellin–Barnes integral (1.4) and reduction formula for the gamma function yield

$$\rho_{\nu+1}(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(\nu+s+1)\Gamma(s)x^{-s}\mathrm{d}s$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(\nu+s)(\nu+s)\Gamma(s)x^{-s}\mathrm{d}s = \nu\rho_\nu(x)\\
&\quad+\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(\nu+s)\Gamma(s+1)x^{-s}\mathrm{d}s\\
&= \nu\rho_\nu(x) + x\rho_{\nu-1}(x).
\end{aligned}
$$

Hence we deduce the following recurrence relation for the scaled Macdonald functions

$$
\rho_{\nu+1}(x) = \nu\rho_\nu(x) + x\rho_{\nu-1}(x), \quad \nu \in \mathbb{R}. \tag{2.17}
$$

In the operator form it can be written as follows

$$
\rho_{\nu+1}(x) = (\nu - xD)\,\rho_\nu(x), \tag{2.18}
$$

and more generally

$$
\rho_{\nu+n}(x) = \prod_{k=0}^{n-1}(\nu+n-k-1-xD)\,\rho_\nu(x), \quad n \in \mathbb{N}_0. \tag{2.19}
$$

Further, recalling the definition of the operator θ, identities (2.6), and the Rodrigues formula for Laguerre polynomials, we obtain

$$
\theta^n\{x^\nu e^{-x}\} = n!x^{n+\nu}e^{-x}L_n^\nu(x),\ n \in \mathbb{N}_0. \tag{2.20}
$$

This formula permits us to derive an integral representation for the product $\rho_\nu f_n$, where f_n is an arbitrary polynomial of degree n

$$
f_n(x) = \sum_{k=0}^{n} f_{n,k}x^k. \tag{2.21}
$$

In fact, considering the operator equality and using (2.20), we write

$$
f_n(-\theta)\left\{x^\nu e^{-x}\right\} = x^\nu e^{-x}\sum_{k=0}^{n} f_{n,k}(-1)^k k!x^k L_k^\nu(x) = x^\nu e^{-x}q_{2n}^\nu(x),
$$

where

$$
q_{2n}^\nu(x) = \sum_{k=0}^{n} f_{n,k}(-1)^k k!x^k L_k^\nu(x) \tag{2.22}
$$

will be called the associated polynomial of degree $2n$. Then, integrating by parts in the following integral and eliminating the integrated terms, we find

$$\int_0^\infty t^{-1}e^{-x/t} f_n(-\theta)\left\{t^\nu e^{-t}\right\} \mathrm{d}t = \int_0^\infty f_n(\theta)\left\{t^{-1}e^{-x/t}\right\} t^\nu e^{-t}\mathrm{d}t.$$

Meanwhile,

$$\theta^k \left\{t^{-1}e^{-x/t}\right\} = (tDt)^k \left\{t^{-1}e^{-x/t}\right\} = x^k t^{-1}e^{-x/t}. \tag{2.23}$$

Hence, appealing to (2.3) and (2.22), we establish the following integral representation of an arbitrary polynomial f_n in terms of its associated polynomial q_{2n}^ν

$$f_n(x) = \frac{1}{\rho_\nu(x)}\int_0^\infty t^{\nu-1}e^{-t-x/t} q_{2n}^\nu(t)\mathrm{d}t. \tag{2.24}$$

The following lemma gives the so-called linear polynomial independence of the scaled Macdonald functions.

Lemma 1 *Let $n, m \in \mathbb{N}_0, \nu \geq 0,\ f_n,\ g_m$ be polynomials of degree at most $n,\ m$, respectively. Let*

$$f_n(x)\rho_\nu(x) + g_m(x)\rho_{\nu+1}(x) = 0 \tag{2.25}$$

for all $x > 0$. Then $f_n \equiv 0,\ g_m \equiv 0$.

Proof The proof will be based on the Ismail integral representation (2.10) of the quotient $\rho_\nu/\rho_{\nu+1}$. In fact, let $r \geq \max\{n, m+1\}$. Since $\rho_{\nu+1} > 0$, we divide (2.25) by $\rho_{\nu+1}$ and then differentiate r times the obtained equality. Thus we arrive at the relation

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left[f_n(x)\frac{\rho_\nu(x)}{\rho_{\nu+1}(x)}\right] = 0, \quad x > 0. \tag{2.26}$$

Meanwhile, integral representation (2.10) says

$$\begin{aligned}\frac{\rho_\nu(x)}{\rho_{\nu+1}(x)} &= \frac{1}{\pi^2}\int_0^\infty \frac{s^{-1}\mathrm{d}s}{(x+s)(J_{\nu+1}^2(2\sqrt{s}) + Y_{\nu+1}^2(2\sqrt{s}))}\\ &= \frac{1}{\pi^2}\int_0^\infty e^{-xy}\mathrm{d}y\int_0^\infty \frac{e^{-sy}\, s^{-1}\mathrm{d}s}{J_{\nu+1}^2(2\sqrt{s}) + Y_{\nu+1}^2(2\sqrt{s})},\end{aligned} \tag{2.27}$$

where the interchange of the order of integration is allowed by Fubini theorem, taking into account the asymptotic behavior of Bessel functions at infinity and near zero [5, Vol. II]. Further, assuming f_n by formula (2.21), we substitute it in the left-hand side of (2.26) together with the right-hand side of the latter equality in (2.27). Then, differentiating under the integral sign, which is possible via the absolute and uniform convergence, we deduce

$$
\begin{aligned}
&\frac{d^r}{dx^r}\left[f_n(x)\frac{\rho_\nu(x)}{\rho_{\nu+1}(x)}\right]\\
&=\frac{1}{\pi^2}\frac{d^r}{dx^r}\sum_{k=0}^{n} f_{n,k}x^k\int_0^\infty e^{-xy}dy\int_0^\infty \frac{e^{-sy}\, s^{-1}ds}{J^2_{\nu+1}(2\sqrt{s})+Y^2_{\nu+1}(2\sqrt{s})}\\
&=\frac{1}{\pi^2}\sum_{k=0}^{n} f_{n,k}(-1)^k\frac{d^r}{dx^r}\int_0^\infty \frac{d^k}{dy^k}\left[e^{-xy}\right]dy\int_0^\infty \frac{e^{-sy}\, s^{-1}ds}{J^2_{\nu+1}(2\sqrt{s})+Y^2_{\nu+1}(2\sqrt{s})}\\
&=\frac{1}{\pi^2}\sum_{k=0}^{n} f_{n,k}(-1)^k\int_0^\infty \frac{\partial^{k+r}}{\partial y^k\partial x^r}\left[e^{-xy}\right]dy\int_0^\infty \frac{e^{-sy}\, s^{-1}ds}{J^2_{\nu+1}(2\sqrt{s})+Y^2_{\nu+1}(2\sqrt{s})}\\
&=\frac{1}{\pi^2}\sum_{k=0}^{n} f_{n,k}(-1)^{k+r}\int_0^\infty \frac{d^k}{dy^k}\left[y^r e^{-xy}\right]dy\int_0^\infty \frac{e^{-sy}\, s^{-1}ds}{J^2_{\nu+1}(2\sqrt{s})+Y^2_{\nu+1}(2\sqrt{s})}.
\end{aligned}
$$

Now, integrating k times by parts in the outer integral with respect to y on the right-hand side of the latter equality, we eliminate integrated terms and then differentiate under the integral sign in the inner integral with respect to s owing to the same arguments as above. Hence we get, combining with (2.26),

$$
\begin{aligned}
&\frac{1}{\pi^2}\sum_{k=0}^{n} f_{n,k}(-1)^{k+r}\int_0^\infty \frac{d^k}{dy^k}\left[y^r e^{-xy}\right]dy\int_0^\infty \frac{e^{-sy}\, s^{-1}ds}{J^2_{\nu+1}(2\sqrt{s})+Y^2_{\nu+1}(2\sqrt{s})}\\
&=\frac{1}{\pi^2}\int_0^\infty y^r e^{-xy}\int_0^\infty \frac{e^{-sy}\, s^{-1}}{J^2_{\nu+1}(2\sqrt{s})+Y^2_{\nu+1}(2\sqrt{s})}\left(\sum_{k=0}^{n} f_{n,k}(-1)^{k+r}s^k\right)ds=0,\ x>0.
\end{aligned}
$$

Consequently, cancelling twice the Laplace transform (1.6) via its injectivity for integrable continuous functions [12], and taking into account the positivity of the function

$$\frac{s^{-1}}{J^2_{\nu+1}(2\sqrt{s})+Y^2_{\nu+1}(2\sqrt{s})}$$

on $\mathbb{R}_+$, we conclude that

$$\sum_{k=0}^{n} f_{n,k}(-1)^k s^k \equiv 0,\quad s>0.$$

Hence $f_{n,k}=0,\ k=0,\ldots,n$ and therefore $f_n\equiv 0$. Returning to the original equality (2.25), we find immediately that $g_m\equiv 0$. Lemma 1 is proved. □

Remark 1 An alternative proof of Lemma 1 would follow from the existence of a multiple orthogonal polynomial sequence with respect to the vector of weight functions $(\rho_\nu, \rho_{\nu+1})$ (see [13]).

Let $\alpha \in \mathbb{R}$ and

$$S_n^{\nu,\alpha}(x) = \frac{d^n}{dx^n}\left[x^{n+\alpha}\rho_\nu(x)\right], \quad n \in \mathbb{N}_0. \tag{2.28}$$

According to [13], the sequence of functions $\left(S_n^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$ generates multiple orthogonal polynomials related to the scaled Macdonald functions $\rho_\nu,\ \rho_{\nu+1}$. In order to obtain an integral representation for functions $S_n^{\nu,\alpha}$, we again employ (1.4), Parseval equality (1.10) for the Mellin transform, and the Mellin–Barnes integral representation for the Laguerre polynomials (see [10, relation (8.4.33.3), Vol. III]). Then, motivating the differentiation under the integral sign by the absolute and uniform convergence and using the reflection formula for the gamma function, we obtain the following chain of equalities

$$\begin{aligned}
\frac{d^n}{dx^n}\left[x^{n+\alpha}\rho_\nu(x)\right] &= \frac{1}{2\pi i}\frac{d^n}{dx^n}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s+n+\alpha)\Gamma(s+n+\nu+\alpha)x^{-s}ds\\
&= \frac{(-1)^n}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s+n+\alpha)\Gamma(s+n+\nu+\alpha)(s)_n\, x^{-s-n}ds\\
&= \frac{(-1)^n}{2\pi i}\int_{\gamma+n-i\infty}^{\gamma+n+i\infty}\Gamma(s+\alpha)\Gamma(s+\nu+\alpha)\frac{\Gamma(s)}{\Gamma(s-n)}\, x^{-s}ds\\
&= \frac{(-1)^n x^\alpha}{2\pi i}\int_{\gamma+n+\alpha-i\infty}^{\gamma+n+\alpha+i\infty}\Gamma(s)\Gamma(s+\nu)\frac{\Gamma(s-\alpha)}{\Gamma(s-\alpha-n)}\, x^{-s}ds\\
&= \frac{x^\alpha}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s)\Gamma(s+\nu)\frac{\Gamma(1+\alpha+n-s)}{\Gamma(1+\alpha-s)}\, x^{-s}ds\\
&= \frac{x^{\alpha+\nu}}{2\pi i}\int_{\gamma+\nu-i\infty}^{\gamma+\nu+i\infty}\Gamma(s-\nu)\Gamma(s)\frac{\Gamma(1+\alpha+\nu+n-s)}{\Gamma(1+\alpha+\nu-s)}\, x^{-s}ds\\
&= x^{\alpha+\nu}n!\int_0^\infty e^{-t-x/t}\left(\frac{x}{t}\right)^{-\nu}L_n^{\nu+\alpha}(t)\frac{dt}{t}.
\end{aligned}$$

Thus, combining with (2.28), we established the following integral representation for $S_n^{\nu,\alpha}(x)$

$$S_n^{\nu,\alpha}(x) = x^\alpha n!\int_0^\infty e^{-t-x/t}t^{\nu-1}L_n^{\nu+\alpha}(t)dt, \quad x > 0. \tag{2.29}$$

Now, employing recurrence relations and differential properties for the Laguerre polynomials [1], in particular, the identity $\frac{d}{dt}L_n^\alpha(t) = -L_{n-1}^{\alpha+1}(t)$, we integrate by parts in (2.29) and differentiate with respect to x under the integral sign by virtue of the absolute and uniform convergence by $x \geq x_0 > 0$ to deduce the corresponding relations for the sequence $S_n^{\nu,\alpha}$. Indeed, we have, for instance, for $\nu > 0, \alpha \in \mathbb{R}$

$$
\begin{aligned}
S_n^{\nu+1,\alpha-1}(x) &= x^{\alpha-1} n! \int_0^\infty e^{-t-x/t} t^\nu L_n^{\nu+\alpha}(t)\mathrm{d}t \\
&= x^{\alpha-1} n! \left[\nu \int_0^\infty e^{-t-x/t} t^{\nu-1} L_n^{\nu+\alpha}(t)\mathrm{d}t + x \int_0^\infty e^{-t-x/t} t^{\nu-2} L_n^{\nu+\alpha}(t)\mathrm{d}t \right. \\
&\quad \left. - \int_0^\infty e^{-t-x/t} t^\nu L_{n-1}^{\nu+\alpha+1}(t)\mathrm{d}t \right] = \frac{\nu}{x} S_n^{\nu,\alpha}(x) \\
&\quad + \frac{1}{x} S_n^{\nu-1,\alpha+1}(x) - \frac{n}{x} S_{n-1}^{\nu+1,\alpha}(x).
\end{aligned}
$$

Hence we obtain the identity

$$
x S_n^{\nu+1,\alpha-1}(x) = \nu\, S_n^{\nu,\alpha}(x) + S_n^{\nu-1,\alpha+1}(x) - n\, S_{n-1}^{\nu+1,\alpha}(x), \quad x > 0,\ n \in \mathbb{N}_0. \tag{2.30}
$$

Differentiating (2.29) by x, we get

$$
\frac{\mathrm{d}}{\mathrm{d}x} S_n^{\nu,\alpha}(x) = \alpha x^{\alpha-1} n! \int_0^\infty e^{-t-x/t} t^{\nu-1} L_n^{\nu+\alpha}(t)\mathrm{d}t - x^\alpha n! \int_0^\infty e^{-t-x/t} t^{\nu-2} L_n^{\nu+\alpha}(t)\mathrm{d}t,
$$

or,

$$
x \frac{\mathrm{d}}{\mathrm{d}x} S_n^{\nu,\alpha}(x) = \alpha S_n^{\nu,\alpha}(x) - S_n^{\nu-1,\alpha+1}(x), \quad x > 0,\ n \in \mathbb{N}_0. \tag{2.31}
$$

On the other hand, integrating again by parts in (2.29) under the same conditions, we find

$$
\begin{aligned}
S_n^{\nu,\alpha}(x) &= \frac{x^\alpha n!}{\nu} \int_0^\infty e^{-t-x/t} t^\nu L_n^{\nu+\alpha}(t)\mathrm{d}t + \frac{x^\alpha n!}{\nu} \int_0^\infty e^{-t-x/t} t^\nu L_{n-1}^{\nu+\alpha+1}(t)\mathrm{d}t \\
&\quad - \frac{x^{\alpha+1} n!}{\nu} \int_0^\infty e^{-t-x/t} t^{\nu-2} L_n^{\nu+\alpha}(t)\mathrm{d}t = \frac{1}{\nu} S_n^{\nu+1,\alpha-1}(x) \\
&\quad + \frac{1}{\nu} S_{n-1}^{\nu+1,\alpha}(x) - \frac{1}{\nu} S_n^{\nu-1,\alpha+1}(x),
\end{aligned}
$$

or,

$$
\nu S_n^{\nu,\alpha}(x) = S_n^{\nu+1,\alpha-1}(x) + S_{n-1}^{\nu+1,\alpha}(x) - S_n^{\nu-1,\alpha+1}(x), \quad x > 0,\ n \in \mathbb{N}_0. \tag{2.32}
$$

Combining with (2.30) gives the following identity

$$
(x-1) S_n^{\nu+1,\alpha-1}(x) = (1-n) S_{n-1}^{\nu+1,\alpha}(x), \quad x > 0,\ n \in \mathbb{N}_0. \tag{2.33}
$$

Meanwhile, from (2.28) and (2.17) we have

$$\begin{aligned} S_n^{\nu-1,\alpha+1}(x) &= \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^{n+\alpha+1}\rho_{\nu-1}(x)\right] \\ &= \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^{n+\alpha}\left[\rho_{\nu+1}(x)-\nu\rho_\nu(x)\right]\right] \\ &= S_n^{\nu+1,\alpha}(x)-\nu S_n^{\nu,\alpha}(x). \end{aligned}$$

Therefore from (2.32) we have

$$S_n^{\nu+1,\alpha}(x) = S_{n-1}^{\nu+1,\alpha}(x) + S_n^{\nu+1,\alpha-1}(x), \tag{2.34}$$

and from (2.33) we find

$$(x-1)S_n^{\nu+1,\alpha}(x) = (x-n)S_{n-1}^{\nu+1,\alpha}(x), \quad x>0,\ n\in\mathbb{N}_0. \tag{2.35}$$

Moreover, recalling again (2.17), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}x}S_{n-1}^{\nu+1,\alpha}(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^{n+\alpha-1}\rho_{\nu+1}(x)\right] = \nu S_n^{\nu,\alpha-1}(x) + S_n^{\nu-1,\alpha}(x). \tag{2.36}$$

Finally, employing the 3-term recurrence relation for Laguerre polynomials

$$\begin{aligned} (n+1)L_{n+1}^{\nu+\alpha}(x) = (2n+1+\nu+\alpha-x)L_n^{\nu+\alpha}(x) \\ -(n+\nu+\alpha)L_{n-1}^{\nu+\alpha}(x), \end{aligned} \tag{2.37}$$

we return to (2.29) to obtain the following identity

$$\begin{aligned} S_{n+1}^{\nu,\alpha}(x) = (2n+1+\nu+\alpha)S_n^{\nu,\alpha}(x) - n(n+\nu+\alpha)S_{n-1}^{\nu,\alpha}(x) \\ -xS_n^{\nu+1,\alpha-1}(x),\ x>0,\ n\in\mathbb{N}_0. \end{aligned} \tag{2.38}$$

3 Prudnikov's Orthogonal Polynomials

Our goal in this section is to find an explicit expression for Prudnikov's orthogonal polynomial sequence $\left(P_n^\nu\right)_{n\in\mathbb{N}_0}$, $\nu\ge 0$. We will do even more, defining the Prudnikov orthogonality (1.5) in a more general setting for the sequence $\left(P_n^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$, $\alpha>-1$,

$$\int_0^\infty P_n^{\nu,\alpha}(x)P_m^{\nu,\alpha}(x)x^\alpha\rho_\nu(x)\mathrm{d}x = \delta_{n,m}, \quad n,m\in\mathbb{N}_0. \tag{3.1}$$

Here $P_n^\nu \equiv P_n^{\nu,0}$. Writing it in terms of coefficients

$$P_n^{\nu,\alpha}(x) = \sum_{k=0}^n a_{n,k}x^k, \tag{3.2}$$

we know that it is of degree exactly n because this sequence is regular; i.e., its leading coefficient $a_{n,n} \equiv a_n \neq 0$ (see [8]). Furthermore, as follows from the general theory of orthogonal polynomials [1], up to a normalization factor the orthogonality (3.1) is equivalent to the following n conditions

$$\int_0^\infty P_n^{\nu,\alpha}(x) x^{m+\alpha} \rho_\nu(x) \mathrm{d}x = 0, \quad m = 0, 1, \ldots, n-1. \tag{3.3}$$

Moreover, the sequence $\left(P_n^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$ satisfies the 3-term recurrence relation in the form

$$x P_n^{\nu,\alpha}(x) = A_{n+1} P_{n+1}^{\nu,\alpha}(x) + B_n P_n^{\nu,\alpha}(x) + A_n P_{n-1}^{\nu,\alpha}(x), \tag{3.4}$$

where $P_{-1}^{\nu,\alpha}(x) \equiv 0$ and

$$A_{n+1} = \frac{a_n}{a_{n+1}}, \quad B_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}, \quad b_n \equiv a_{n,n-1}. \tag{3.5}$$

The associated polynomial sequence with $\left(P_n^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$ (see (2.22)), which will be used below, has the form

$$Q_{2n}(x) = \sum_{k=0}^n a_{n,k} (-1)^k k! x^k L_k^\nu(x). \tag{3.6}$$

It follows from the orthogonality (3.1)

$$\int_0^\infty \left[P_n^{\nu,\alpha}(x)\right]^2 x^\alpha \rho_\nu(x) \mathrm{d}x = 1.$$

However, using properties of the scaled Macdonald functions from the previous section one can calculate the following values

$$\psi_n^{\nu,\alpha} = \int_0^\infty \left[P_n^{\nu,\alpha}(x)\right]^2 x^\alpha \rho_{\nu+1}(x) \mathrm{d}x, \quad n \in \mathbb{N}_0, \ \nu \geq 0, \ \alpha > -1.$$

In fact, appealing to (3.1), (3.2), (3.6), (2.16), (2.17) and integrating by parts, we derive

$$\begin{aligned}\psi_n^{\nu,\alpha} &= \nu + \int_0^\infty \left[P_n^{\nu,\alpha}(x)\right]^2 x^{\alpha+1} \rho_{\nu-1}(x) \mathrm{d}x \\ &= \nu + \alpha + 1 + 2 \int_0^\infty P_n^{\nu,\alpha}(x) \frac{\mathrm{d}}{\mathrm{d}x}\left[P_n^{\nu,\alpha}(x)\right] x^{\alpha+1} \rho_\nu(x) \mathrm{d}x = 2n + 1 + \nu + \alpha\end{aligned}$$

since

$$\int_0^\infty P_n^{\nu,\alpha}(x) x^{n+\alpha} \rho_\nu(x) \mathrm{d}x = \frac{1}{a_n}. \tag{3.7}$$

Therefore we find the formula

$$\psi_n^{\nu,\alpha} = 2n + 1 + \nu + \alpha. \tag{3.8}$$

In the meantime, taking the corresponding integral representation (2.11) for the product $x^m \rho_\nu(x)$, we substitute its right-hand side in (3.3) and change the order of integration by Fubini's theorem. Thus we obtain

$$\int_0^\infty t^{\nu+m-1} e^{-t} L_m^\nu(t) \int_0^\infty P_n^{\nu,\alpha}(x) e^{-x/t} x^\alpha \mathrm{d}x \mathrm{d}t = 0, \quad m = 0, 1, \ldots, n-1. \tag{3.9}$$

But the inner integral with respect to x can be treated, involving the differential operator θ (see (2.6)). Indeed, using (3.2) and (2.23), we have

$$\begin{aligned} \frac{1}{t}\int_0^\infty P_n^{\nu,\alpha}(x) e^{-x/t} x^\alpha \mathrm{d}x &= \sum_{k=0}^n a_{n,k} \theta^k \left\{ \frac{1}{t} \int_0^\infty e^{-x/t} x^\alpha \mathrm{d}x \right\} \\ &= \Gamma(1+\alpha) \sum_{k=0}^n a_{n,k} \theta^k \left\{ t^\alpha \right\} = \Gamma(1+\alpha) P_n^{\nu,\alpha}(\theta)\{t^\alpha\}, \end{aligned}$$

where the interchange of the differential operator θ^k and integration is guaranteed due to the uniform convergence by $t \in [1/M, M],\ M > 0$ of the integral

$$\frac{1}{t}\int_0^\infty e^{-x/t} x^{\alpha+k} \mathrm{d}x, \quad k = 0, 1, \ldots, n.$$

Moreover, the Rodrigues formula for Laguerre polynomials and Viskov-type identity (2.6) for the operator θ imply

$$t^{\nu+m} e^{-t} L_m^\nu(t) = \frac{1}{m!} \theta^m \left\{ t^\nu e^{-t} \right\}.$$

Substituting these values in (3.9), it becomes

$$\int_0^\infty \theta^m \left\{ t^\nu e^{-t} \right\} P_n^{\nu,\alpha}(\theta)\{t^\alpha\} \mathrm{d}t = 0, \quad m = 0, 1, \ldots, n-1.$$

After m times integration by parts in the latter integral, we end up with the following orthogonality conditions

$$\int_0^\infty t^\nu e^{-t} \theta^m P_n^{\nu,\alpha}(\theta)\{t^\alpha\} \mathrm{d}t = 0, \quad m = 0, 1, \ldots, n-1. \tag{3.10}$$

Analogously, the orthogonality (3.1) is equivalent to the equality

$$\int_0^\infty t^\nu e^{-t} P_m^{\nu,\alpha}(\theta) P_n^{\nu,\alpha}(\theta)\{t^\alpha\} \mathrm{d}t = \frac{\delta_{m,n}}{\Gamma(1+\alpha)}, \quad \alpha > -1. \tag{3.11}$$

Definition 2 The orthogonality (3.11) is called the composition orthogonality of the sequence $\left(P_n^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$ in the sense of Laguerre.

Thus we proved the following theorem.

Theorem 1 *The Prudnikov orthogonality (3.1) is equivalent to the composition orthogonality (3.11) in the sense of Laguerre; i.e., Prudnikov's orthogonal polynomials are Laguerre polynomials in the sense of composition orthogonality (3.11).*

Meanwhile, in terms of the associated polynomial (3.6) the orthogonality conditions (3.10) can be rewritten, using the commutativity property

$$\theta^m P_n^{\nu,\alpha}(\theta)\{t^\alpha\} = P_n^{\nu,\alpha}(\theta)\theta^m\{t^\alpha\}$$

and the Rodrigues formula for Laguerre polynomials. Then, integrating by parts an appropriate number of times and taking into account (2.9), we get

$$\begin{aligned}0 &= \int_0^\infty t^\nu e^{-t}\theta^m P_n^{\nu,\alpha}(\theta)\{t^\alpha\}\mathrm{d}t = \int_0^\infty t^\nu e^{-t} P_n^{\nu,\alpha}(\theta)\theta^m\{t^\alpha\}\mathrm{d}t\\ &= (1+\alpha)_m \int_0^\infty P_n^{\nu,\alpha}(-\theta)\left\{t^\nu e^{-t}\right\} t^{m+\alpha}\mathrm{d}t = (1+\alpha)_m \int_0^\infty t^{\nu+\alpha+m} e^{-t} Q_{2n}(t)\mathrm{d}t,\end{aligned}$$

or, finally,

$$\int_0^\infty t^{\nu+\alpha+m} e^{-t} Q_{2n}(t)\mathrm{d}t = 0, \quad m = 0, 1, \ldots, n-1. \tag{3.12}$$

On the other hand, developing the polynomial $Q_{2n}(t)$ in terms of the Laguerre polynomials $L_n^{\nu+\alpha}(x)$, we find

$$Q_{2n}(x) = \sum_{j=0}^{2n} c_{n,j} L_j^{\nu+\alpha}(x), \tag{3.13}$$

where

$$c_{n,k} = \frac{k!}{\Gamma(k+\nu+\alpha+1)} \int_0^\infty t^{\nu+\alpha} e^{-t} Q_{2n}(t) L_k^{\nu+\alpha}(t)\mathrm{d}t, \tag{3.14}$$

and orthogonality conditions (3.12) immediately imply that

$$c_{n,j} = 0, \quad j = 0, 1, \ldots, n-1. \tag{3.15}$$

Therefore, the expansion (3.13) becomes

$$Q_{2n}(x) = \sum_{j=n}^{2n} c_{n,j} L_j^{\nu+\alpha}(x). \tag{3.16}$$

In the meantime, expanding $(-1)^m m! x^m L_m^\nu(x)$ via the Laguerre polynomials $L_k^{\nu+\alpha}(x)$ as well, we obtain

$$(-1)^m m! x^m L_m^\nu(x) = \sum_{k=0}^{2m} d_{m,k} L_k^{\nu+\alpha}(x), \tag{3.17}$$

where coefficients $d_{m,k}$ are calculated accordingly by the formula (see [10, relation (2.19.14.8), Vol. II])

$$\begin{aligned} d_{m,k} &= \frac{(-1)^m\ m!\ k!}{\Gamma(k+\nu+\alpha+1)} \int_0^\infty e^{-t} t^{\nu+\alpha+m} L_m^\nu(t)\ L_k^{\nu+\alpha}(t) \mathrm{d}t \\ &= \frac{(-1)^{m+k}\ m!}{(m-k)!} (1+\nu)_m\ (\nu+\alpha+1+k)_{m-k} \\ &\quad \times {}_3F_2\left(-m,\ \nu+\alpha+m+1,\ m+1;\ 1+\nu,\ m+1-k;\ 1\right), \end{aligned} \tag{3.18}$$

where ${}_3F_2(a, b, c; d, e; z)$ is the generalized hypergeometric function [10, Vol. III]. It is easily seen from the orthogonality of the Laguerre polynomials $L_k^{\nu+\alpha}(x)$ that

$$d_{m,k} = 0, \quad k > 2m.$$

Moreover, the associated polynomial (3.6) Q_{2n} has the representation

$$\begin{aligned} Q_{2n}(x) &= \sum_{m=0}^{n} a_{n,m} \sum_{k=0}^{2m} d_{m,k} L_k^{\nu+\alpha}(x) \\ &= \sum_{m=0}^{n} a_{n,m} \left[\sum_{k=0}^{m} d_{m,2k} L_{2k}^{\nu+\alpha}(x) + \sum_{k=0}^{m-1} d_{m,2k+1} L_{2k+1}^{\nu+\alpha}(x) \right] \\ &= \sum_{k=0}^{n} L_{2k}^{\nu+\alpha}(x) \left(\sum_{m=k}^{n} a_{n,m}\ d_{m,2k} \right) + \sum_{k=0}^{n-1} L_{2k+1}^{\nu+\alpha}(x) \left(\sum_{m=k}^{n-1} a_{n,m+1}\ d_{m+1,2k+1} \right). \end{aligned} \tag{3.19}$$

Lemma 2 *Coefficients $d_{m,k},\ m, k \in \mathbb{N}_0$, satisfy the following recurrence relation*

$$\begin{aligned} d_{m+1,k} = &-mk(k-1)(m+\nu)d_{m-1,k-2} + mk(m+\nu)(1+2\alpha+3k+2(m+\nu))d_{m-1,k-1} \\ &- m(m+\nu)\left(\alpha^2+3k^2+(\nu+1)(2(1+m)+\nu)\right. \\ &\left.+ \alpha(3+4k+2(m+\nu)) + k(5+4(m+\nu))\right) d_{m-1,k} \\ &+ m(m+\nu)(1+\alpha+k+\nu)(2(1+m)+\alpha+k+\nu)d_{m-1,k+1} \\ &+ k(k-1)d_{m,k-2} \\ &- k(3k+2\alpha+\nu)d_{m,k-1} \\ &+ \big((1+\alpha)(1+\alpha+4k)+3(k^2-m^2)+2m(k-1) \\ &+ \nu(1+\alpha+3k-m)\big) d_{m,k} - (1+\alpha+k+\nu)(2(1+m)+\alpha+k+\nu)\big) d_{m,k+1}. \end{aligned} \tag{3.20}$$

Proof In fact, recalling the 3-term recurrence relation (2.37) for Laguerre polynomials and, as its direct consequence, the following equality

$$xL_n^{\nu+\alpha+1}(x) = (n+\nu+\alpha)L_{n-1}^{\nu+\alpha}(x) - (n-x)L_n^{\nu+\alpha}(x), \tag{3.21}$$

we derive from (3.18) via integration by parts

$$\begin{aligned}
d_{m+1,k} &= \frac{(-1)^{m+1}\,(m+1)!\,k!}{\Gamma(k+\nu+\alpha+1)} \int_0^\infty e^{-t} t^{\nu+\alpha+m+1} L_{m+1}^{\nu}(t)\; L_k^{\nu+\alpha}(t)\mathrm{d}t \\
&= \frac{(-1)^{m+1}\,(m+1)!\,k!(\nu+\alpha+m+1)}{\Gamma(k+\nu+\alpha+1)} \int_0^\infty e^{-t} t^{\nu+\alpha+m} L_{m+1}^{\nu}(t)\; L_k^{\nu+\alpha}(t)\mathrm{d}t \\
&\quad + \frac{(-1)^{m}\,(m+1)!\,k!}{\Gamma(k+\nu+\alpha+1)} \int_0^\infty e^{-t} t^{\nu+\alpha+m+1} L_{m}^{\nu+1}(t)\; L_k^{\nu+\alpha}(t)\mathrm{d}t \\
&\quad + \frac{(-1)^{m}\,(m+1)!\,k!}{\Gamma(k+\nu+\alpha+1)} \int_0^\infty e^{-t} t^{\nu+\alpha+m+1} L_{m+1}^{\nu}(t)\; L_{k-1}^{\nu+\alpha+1}(t)\mathrm{d}t \\
&= \frac{(-1)^{m+1}\,m!\,k!(\nu+\alpha+m+1)}{\Gamma(k+\nu+\alpha+1)} \int_0^\infty e^{-t} t^{\nu+\alpha+m} \\
&\qquad \times \left[(2m+1+\nu-t)L_m^{\nu}(t) - (m+\nu)L_{m-1}^{\nu}(t)\right]\; L_k^{\nu+\alpha}(t)\mathrm{d}t \\
&\quad + \frac{(-1)^{m}\,(m+1)!\,k!}{\Gamma(k+\nu+\alpha+1)} \int_0^\infty e^{-t} t^{\nu+\alpha+m} \\
&\qquad \times \left[(m+\nu)L_{m-1}^{\nu}(t) - (m-t)L_m^{\nu}(t)\right] L_k^{\nu+\alpha}(t)\mathrm{d}t \\
&\quad + \frac{(-1)^{m}\,m!\,k!}{\Gamma(k+\nu+\alpha+1)} \int_0^\infty e^{-t} t^{\nu+\alpha+m}
\end{aligned}$$

$$\begin{aligned}
&\qquad \times \left[(2m+1+\nu-t)L_m^{\nu}(t) - (m+\nu)L_{m-1}^{\nu}(t)\right] \\
&\quad \times \left[(k+\nu+\alpha-1)L_{k-2}^{\nu+\alpha}(t) - (k-t-1)L_{k-1}^{\nu+\alpha}(t)\right] \mathrm{d}t \\
&= -(\nu+\alpha+m+1)(2m+1+\nu)d_{m,k} \\
&\quad + \frac{(-1)^{m}\,m!\,k!(\nu+\alpha+m+1)}{\Gamma(k+\nu+\alpha+1)} \int_0^\infty e^{-t} t^{\nu+\alpha+m} L_m^{\nu}(t) \left[(2k+1+\nu+\alpha)L_k^{\nu+\alpha}(t)\right. \\
&\quad \left. -(k+\nu+\alpha)L_{k-1}^{\nu+\alpha}(t) - (k+1)L_{k+1}^{\nu+\alpha}(t)\right] \mathrm{d}t \\
&\quad + \frac{(-1)^{m}\,m!\,k!(\nu+\alpha+m+1)(m+\nu)}{\Gamma(k+\nu+\alpha+1)} \int_0^\infty e^{-t} t^{\nu+\alpha+m-1} L_{m-1}^{\nu}(t) \\
&\qquad \times \left[(2k+1+\nu+\alpha)L_k^{\nu+\alpha}(t)\right. \\
&\quad \left. -(k+\nu+\alpha)L_{k-1}^{\nu+\alpha}(t) - (k+1)L_{k+1}^{\nu+\alpha}(t)\right] \mathrm{d}t \\
&\quad -m(m+1)d_{m,k} + \frac{(-1)^{m}\,(m+1)!\,k!(m+\nu)}{\Gamma(k+\nu+\alpha+1)} \int_0^\infty e^{-t} t^{\nu+\alpha+m-1} L_{m-1}^{\nu}(t) \\
&\qquad \times \left[(2k+1+\nu+\alpha)L_k^{\nu+\alpha}(t)\right. \\
&\quad \left. -(k+\nu+\alpha)L_{k-1}^{\nu+\alpha}(t) - (k+1)L_{k+1}^{\nu+\alpha}(t)\right] \mathrm{d}t \\
&\quad + \frac{(-1)^{m}\,(m+1)!\,k!}{\Gamma(k+\nu+\alpha+1)} \int_0^\infty e^{-t} t^{\nu+\alpha+m} L_m^{\nu}(t) \\
&\quad \times \left[(2k+1+\nu+\alpha)L_k^{\nu+\alpha}(t) - (k+\nu+\alpha)L_{k-1}^{\nu+\alpha}(t) - (k+1)L_{k+1}^{\nu+\alpha}(t)\right] \mathrm{d}t \\
&\quad + \frac{k(k-1)(2m+1+\nu)}{k+\nu+\alpha} \left[d_{m,k-2} - d_{m,k-1}\right]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{(-1)^m\ m!\ k!(k+\nu+\alpha-1)}{\Gamma(k+\nu+\alpha+1)}\int_0^\infty e^{-t}t^{\nu+\alpha+m}L_m^\nu(t)\\
&\quad\times\left[(2k-3+\nu+\alpha)L_{k-2}^{\nu+\alpha}(t)-(k+\nu+\alpha-2)L_{k-3}^{\nu+\alpha}(t)\right.\\
&\left.-(k-1)L_{k-1}^{\nu+\alpha}(t)\right]\mathrm{d}t\\
&+\frac{(-1)^m\ m!\ k!(2m+\nu+k)}{\Gamma(k+\nu+\alpha+1)}\int_0^\infty e^{-t}t^{\nu+\alpha+m}L_m^\nu(t)\\
&\quad\times\left[(2k-1+\nu+\alpha)L_{k-1}^{\nu+\alpha}(t)-(k+\nu+\alpha-1)L_{k-2}^{\nu+\alpha}(t)\right.\\
&\left.-kL_k^{\nu+\alpha}(t)\right]\mathrm{d}t\\
&-\frac{(-1)^m\ m!\ k!(k+\nu+\alpha-1)(m+\nu)}{\Gamma(k+\nu+\alpha+1)}\int_0^\infty e^{-t}t^{\nu+\alpha+m-1}L_{m-1}^\nu(t)\\
&\quad\times\left[(2k-3+\nu+\alpha)L_{k-2}^{\nu+\alpha}(t)-(k+\nu+\alpha-2)L_{k-3}^{\nu+\alpha}(t)\right.\\
&\left.-(k-1)L_{k-1}^{\nu+\alpha}(t)\right]\mathrm{d}t\\
&+\frac{(-1)^m\ m!\ k!(m+\nu)(k-1)}{\Gamma(k+\nu+\alpha+1)}\int_0^\infty e^{-t}t^{\nu+\alpha+m-1}L_{m-1}^\nu(t)\\
&\quad\times\left[(2k-1+\nu+\alpha)L_{k-1}^{\nu+\alpha}(t)-(k+\nu+\alpha-1)L_{k-2}^{\nu+\alpha}(t)\right.\\
&\left.-kL_k^{\nu+\alpha}(t)\right]\mathrm{d}t\\
&-\frac{(-1)^m\ m!\ k!(m+\nu)}{\Gamma(k+\nu+\alpha+1)}\int_0^\infty e^{-t}t^{\nu+\alpha+m-1}L_{m-1}^\nu(t)\,[(2k-1+\nu+\alpha)\\
&\quad\times\left[(2k-1+\nu+\alpha)L_{k-1}^{\nu+\alpha}(t)-(k+\nu+\alpha-1)L_{k-2}^{\nu+\alpha}(t)\right.\\
&\left.-kL_k^{\nu+\alpha}(t)\right]-(k+\nu+\alpha-1)\\
&\quad\times\left[(2k-3+\nu+\alpha)L_{k-2}^{\nu+\alpha}(t)-(k+\nu+\alpha-2)L_{k-3}^{\nu+\alpha}(t)\right.\\
&\left.-(k-1)L_{k-1}^{\nu+\alpha}(t)\right]-k\left[(2k+1+\nu+\alpha)L_k^{\nu+\alpha}(t)-(k+\nu+\alpha)L_{k-1}^{\nu+\alpha}(t)\right.\\
&\left.\left.-(k+1)L_{k+1}^{\nu+\alpha}(t)\right]\right]\mathrm{d}t\\
&-\frac{(-1)^m\ m!\ k!}{\Gamma(k+\nu+\alpha+1)}\int_0^\infty e^{-t}t^{\nu+\alpha+m}L_m^\nu(t)\,[(2k-1+\nu+\alpha)\\
&\quad\times\left[(2k-1+\nu+\alpha)L_{k-1}^{\nu+\alpha}(t)-(k+\nu+\alpha-1)L_{k-2}^{\nu+\alpha}(t)\right.\\
&\left.-kL_k^{\nu+\alpha}(t)\right]-(k+\nu+\alpha-1)\left[(2k-3+\nu+\alpha)L_{k-2}^{\nu+\alpha}(t)\right.\\
&\quad-(k+\nu+\alpha-2)L_{k-3}^{\nu+\alpha}(t)\\
&\left.-(k-1)L_{k-1}^{\nu+\alpha}(t)\right]-k\left[(2k+1+\nu+\alpha)L_k^{\nu+\alpha}(t)-(k+\nu+\alpha)L_{k-1}^{\nu+\alpha}(t)\right.\\
&\left.\left.-(k+1)L_{k+1}^{\nu+\alpha}(t)\right]\right]\mathrm{d}t\\
&=(\nu+\alpha+m+1)(2(k-m)+\alpha)d_{m,k}-k(\nu+\alpha+m+1)d_{m,k-1}\\
&\quad-(\nu+\alpha+m+1)(k+\nu+\alpha+1)d_{m,k+1}\\
&\quad-m(\nu+\alpha+m+1)(m+\nu)(2k+1+\nu+\alpha)d_{m-1,k}\\
&\quad+mk(\nu+\alpha+m+1)(m+\nu)d_{m-1,k-1}\\
&\quad+m(\nu+\alpha+m+1)(m+\nu)(k+\nu+\alpha+1)d_{m-1,k+1}\\
&\quad-m(m+1)d_{m,k}-(m+1)m(2k+1+\nu+\alpha)(m+\nu)d_{m-1,k}\\
&\quad+mk(m+1)(m+\nu)d_{m-1,k-1}\\
&\quad+m(m+1)(m+\nu)(k+\nu+\alpha+1)d_{m-1,k+1}\\
&\quad+(m+1)(2k+1+\nu+\alpha)d_{m,k}-k(m+1)d_{m,k-1}
\end{aligned}
$$

$$
\begin{aligned}
&-(m+1)(k+\nu+\alpha+1)d_{m,k+1}\\
&+\frac{k(k-1)(2m+1+\nu)}{k+\nu+\alpha}\left[d_{m,k-2}-d_{m,k-1}\right]\\
&-\frac{k(k-1)(2k-3+\nu+\alpha)}{k+\nu+\alpha}d_{m,k-2}\\
&+\frac{k(k-1)(k-2)}{k+\nu+\alpha}d_{m,k-3}+\frac{k(k-1)(k+\nu+\alpha-1)}{k+\nu+\alpha}d_{m,k-1}\\
&+\frac{k(2m+\nu+k)(2k-1+\nu+\alpha)}{k+\nu+\alpha}d_{m,k-1}-\frac{k(k-1)(2m+\nu+k)}{k+\nu+\alpha}d_{m,k-2}\\
&-k(2m+\nu+k)d_{m,k}\\
&+\frac{mk(k-1)(m+\nu)(2k-3+\nu+\alpha)}{k+\nu+\alpha}d_{m-1,k-2}\\
&-\frac{k(k-1)(k-2)m(m+\nu)}{k+\nu+\alpha}d_{m-1,k-3}\\
&-\frac{km(k-1)(m+\nu)(k+\nu+\alpha-1)}{k+\nu+\alpha}d_{m-1,k-1}\\
&-\frac{mk(k-1)(m+\nu)(2k-1+\nu+\alpha)}{k+\nu+\alpha}d_{m-1,k-1}\\
&+\frac{k(k-1)^2m(m+\nu)}{k+\nu+\alpha}d_{m-1,k-2}\\
&+km(k-1)(m+\nu)d_{m-1,k}+\frac{mk(m+\nu)(2k-1+\nu+\alpha)^2}{k+\nu+\alpha}d_{m-1,k-1}\\
&-\frac{mk(k-1)(m+\nu)(2k-1+\nu+\alpha)}{k+\nu+\alpha}d_{m-1,k-2}\\
&-mk(m+\nu)(2k-1+\nu+\alpha)d_{m-1,k}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{k(k-1)m(m+\nu)(2k-3+\nu+\alpha)}{k+\nu+\alpha}d_{m-1,k-2}\\
&+\frac{k(k-1)(k-2)m(m+\nu)}{k+\nu+\alpha}d_{m-1,k-3}\\
&+\frac{mk(k-1)(m+\nu)(k-1+\nu+\alpha)}{k+\nu+\alpha}d_{m-1,k-1}\\
&-mk(m+\nu)(2k+1+\nu+\alpha)d_{m-1,k}+k^2m(m+\nu)d_{m-1,k-1}\\
&+mk(m+\nu)(k+1+\nu+\alpha)d_{m-1,k+1}\\
&-\frac{k(2k-1+\nu+\alpha)^2}{k+\nu+\alpha}d_{m,k-1}\\
&+\frac{k(k-1)(2k-1+\nu+\alpha)}{k+\nu+\alpha}d_{m,k-2}+k(2k-1+\nu+\alpha)d_{m,k}\\
&+\frac{k(k-1)(2k-3+\nu+\alpha)}{k+\nu+\alpha}d_{m,k-2}-\frac{k(k-1)(k-2)}{k+\nu+\alpha}d_{m,k-3}\\
&-\frac{k(k-1)(k-1+\nu+\alpha)}{k+\nu+\alpha}d_{m,k-1}\\
&+k(2k+1+\nu+\alpha)d_{m,k}-k^2d_{m,k-1}-k(k+1+\nu+\alpha)d_{m,k+1}.
\end{aligned}
$$

Hence after simplification we get (3.20). □

On by other hand, taking into account orthogonality conditions (3.15), we have

$$Q_{2n}(x) = \sum_{j=0}^{2n} c_{n,j} L_j^{\nu+\alpha}(x) = \sum_{j=0}^{n} c_{n,2j}\; L_{2j}^{\nu+\alpha}(x) + \sum_{j=0}^{n-1} c_{n,2j+1}\; L_{2j+1}^{\nu+\alpha}(x),$$

and by the uniqueness of the expansion of the associated polynomial Q_{2n} by Laguerre polynomials we find from (3.19)

$$c_{n,2j} = \sum_{m=j}^{n} a_{n,m}\; d_{m,2j}, \quad c_{n,2j+1} = \sum_{m=j+1}^{n} a_{n,m}\; d_{m,2j+1}. \tag{3.22}$$

We observe via (3.18) that $d_{0,0} = 1,\ d_{m,2j} \neq 0,\ m = j, \ldots, n,\ d_{m,2j+1} \neq 0, m = j+1, \ldots, n$. But from (3.15) we get for $n \in \mathbb{N}$

$$\begin{aligned} c_{2n,2j} &= 0, \quad j = 0, 1, \ldots n-1; \quad c_{2n,2j+1} = 0,\ j = 0, 1, \ldots n-1, \\ c_{2n+1,2j} &= 0, \quad j = 0, 1, \ldots n; \quad c_{2n+1,2j+1} = 0,\ j = 0, 1, \ldots n-1. \end{aligned}$$

Consequently, equalities (3.22) represent for the polynomial sequence $\left(P_{2n}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$ $\left(\left(P_{2n+1}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}\right)$ linear homogeneous systems of $2n$ $(2n+1)$ equations with $2n+1$ $(2(n+1))$ unknowns. However, if we assume that the free coefficient $a_{2n,0}$ $(a_{2n+1,0})$ is known, we come out with linear non-homogeneous systems of $2n$ $(2n+1)$ equations with $2n$ $(2n+1)$ unknowns. It can be solved uniquely by Cramer's rule with nonzero determinant. In fact, we have the following non-homogeneous systems of $2n,\ 2n+1$ linear equations to determine the sequences $\left(P_{2n}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$, $\left(P_{2n+1}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$, respectively,

$$\begin{pmatrix} d_{1,0} & d_{2,0} & \cdots & \cdots & d_{n-1,0} & d_{n,0} & \cdots & d_{2n,0} \\ d_{1,1} & d_{2,1} & \cdots & \cdots & \cdots & \cdots & \cdots & d_{2n,1} \\ d_{1,2} & d_{2,2} & \cdots & \cdots & \cdots & \cdots & \cdots & d_{2n,2} \\ 0 & d_{2,3} & \cdots & \cdots & \cdots & \cdots & \cdots & d_{2n,3} \\ \vdots & d_{2,4} & \cdots & \cdots & \cdots & \cdots & \cdots & d_{2n,4} \\ \vdots & 0 & d_{3,5} & \cdots & \cdots & \cdots & \cdots & d_{2n,5} \\ \vdots & \vdots & d_{3,6} & \cdots & \cdots & \cdots & \cdots & d_{2n,6} \\ \vdots & \vdots & 0 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & d_{n-1,2n-3} & d_{n,2n-3} & \cdots & d_{2n,2n-3} \\ \vdots & \ddots & \ddots & 0 & d_{n-1,2(n-1)} & d_{n,2(n-1)} & \cdots & d_{2n,2(n-1)} \\ 0 & \cdots & \cdots & 0 & 0 & d_{n,2n-1} & \cdots & d_{2n,2n-1} \end{pmatrix} \begin{pmatrix} a_{2n,1} \\ a_{2n,2} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_{2n,2n-1} \\ a_{2n,2n} \end{pmatrix} = \begin{pmatrix} -a_{2n,0} \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \tag{3.23}$$

$$\begin{pmatrix} d_{1,0} & d_{2,0} & \cdots & \cdots & d_{n,0} & \cdots & d_{2n+1,0} \\ d_{1,1} & d_{2,1} & \cdots & \cdots & \cdots & \cdots & d_{2n+1,1} \\ d_{1,2} & d_{2,2} & \cdots & \cdots & \cdots & \cdots & d_{2n+1,2} \\ 0 & d_{2,3} & \cdots & \cdots & \cdots & \cdots & d_{2n+1,3} \\ \vdots & d_{2,4} & \cdots & \cdots & \cdots & \cdots & d_{2n+1,4} \\ \vdots & 0 & d_{3,5} & \cdots & \cdots & \cdots & d_{2n+1,5} \\ \vdots & \vdots & d_{3,6} & \cdots & \cdots & \cdots & d_{2n+1,6} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & d_{n,2n-1} & \cdots & d_{2n+1,2n-1} \\ 0 & \cdots & \cdots & 0 & d_{n,2n} & \cdots & d_{2n+1,2n} \end{pmatrix} \begin{pmatrix} a_{2n+1,1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_{2n+1,2n} \\ a_{2n+1,2n+1} \end{pmatrix} = \begin{pmatrix} -a_{2n+1,0} \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}. \tag{3.24}$$

Denoting by D_{2n}, D_{2n+1} the corresponding nonzero determinants of the systems (3.23), (3.24)

$$D_{2n} = \begin{vmatrix} d_{1,0} & d_{2,0} & \cdots & \cdots & \cdots & \cdots & \cdots & d_{2n,0} \\ d_{1,1} & d_{2,1} & \cdots & \cdots & \cdots & \cdots & \cdots & d_{2n,1} \\ d_{1,2} & d_{2,2} & \cdots & \cdots & \cdots & \cdots & \cdots & d_{2n,2} \\ 0 & d_{2,3} & \cdots & \cdots & \cdots & \cdots & \cdots & d_{2n,3} \\ \vdots & d_{2,4} & \cdots & \cdots & \cdots & \cdots & \cdots & d_{2n,4} \\ \vdots & 0 & d_{3,5} & \cdots & \cdots & \cdots & \cdots & d_{2n,5} \\ \vdots & \vdots & d_{3,6} & \cdots & \cdots & \cdots & \cdots & d_{2n,6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & d_{n-1,2n-3} & \cdots & \cdots & d_{2n,2n-3} \\ 0 & \cdots & \cdots & 0 & d_{n-1,2(n-1)} & \cdots & \cdots & d_{2n,2(n-1)} \\ 0 & \cdots & \cdots & 0 & 0 & d_{n,2n-1} & \cdots & d_{2n,2n-1} \end{vmatrix}, \tag{3.25}$$

$$D_{2n+1} = \begin{vmatrix} d_{1,0} & d_{2,0} & \cdots & \cdots & \cdots & \cdots & d_{2n+1,0} \\ d_{1,1} & d_{2,1} & \cdots & \cdots & \cdots & \cdots & d_{2n+1,1} \\ d_{1,2} & d_{2,2} & \cdots & \cdots & \cdots & \cdots & d_{2n+1,2} \\ 0 & d_{2,3} & \cdots & \cdots & \cdots & \cdots & d_{2n+1,3} \\ \vdots & d_{2,4} & \cdots & \cdots & \cdots & \cdots & d_{2n+1,4} \\ \vdots & 0 & d_{3,5} & \cdots & \cdots & \cdots & d_{2n+1,5} \\ \vdots & \vdots & d_{3,6} & \cdots & \cdots & \cdots & d_{2n+1,6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & d_{n,2n-1} & \cdots & d_{2n+1,2n-1} \\ 0 & \cdots & \cdots & 0 & d_{n,2n} & \cdots & d_{2n+1,2n} \end{vmatrix}, \tag{3.26}$$

we apply Cramer's rule to get the expressions for the coefficients of the sequences $\left(P_{2n}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$, $\left(P_{2n+1}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$ in terms of the related free coefficients. Precisely, denoting by

$$D_{2n,1} = \begin{vmatrix} d_{2,1} & \dots & \dots\dots\dots\dots & & d_{2n,1} \\ d_{2,2} & \dots & \dots\dots\dots\dots & & d_{2n,2} \\ d_{2,3} & \dots & \dots\dots\dots\dots & & d_{2n,3} \\ d_{2,4} & \dots & \dots\dots\dots\dots & & d_{2n,4} \\ 0 & d_{3,5} & \dots\dots\dots\dots & & d_{2n,5} \\ \vdots & d_{3,6} & \dots\dots\dots\dots & & d_{2n,6} \\ \vdots & 0 & \vdots \quad \vdots \quad \ddots \quad \ddots & & \vdots \\ 0 & \dots & \dots\dots\dots\dots & & d_{2n,2n-3} \\ 0 & \dots & \dots\dots\dots\dots & & d_{2n,2(n-1)} \\ 0 & \dots & \dots\dots\dots\dots & & d_{2n,2n-1} \end{vmatrix}, \tag{3.27}$$

$$D_{2n,k} = \begin{vmatrix} d_{1,1} & d_{2,1} & \dots & \dots & d_{k-1,1} & d_{k+1,1} & \dots & d_{2n,1} \\ d_{1,2} & d_{2,2} & \dots & \dots & \dots & \dots & \dots & d_{2n,2} \\ 0 & d_{2,3} & \dots & \dots & \dots & \dots & \dots & d_{2n,3} \\ \vdots & d_{2,4} & \dots & \dots & \dots & \dots & \dots & d_{2n,4} \\ \vdots & 0 & d_{3,5} & \dots & \dots & \dots & \dots & d_{2n,5} \\ \vdots & \vdots & d_{3,6} & \dots & \dots & \dots & \dots & d_{2n,6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & d_{2n,2n-3} \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & d_{2n,2(n-1)} \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & d_{2n,2n-1} \end{vmatrix}, \quad k = 2, \dots, 2n-1, \tag{3.28}$$

$$D_{2n,2n} = \begin{vmatrix} d_{1,1} & d_{2,1} & \dots & \dots\dots\dots\dots & & d_{2n-1,1} \\ d_{1,2} & d_{2,2} & \dots & \dots\dots\dots\dots & & d_{2n-1,2} \\ 0 & d_{2,3} & \dots & \dots\dots\dots\dots & & d_{2n-1,3} \\ \vdots & d_{2,4} & \dots & \dots\dots\dots\dots & & d_{2n-1,4} \\ \vdots & 0 & d_{3,5} & \dots\dots\dots\dots & & d_{2n-1,5} \\ \vdots & \vdots & d_{3,6} & \dots\dots\dots\dots & & d_{2n-1,6} \\ \vdots & \vdots & \vdots & \vdots \quad \vdots \quad \ddots \quad \ddots & & \vdots \\ 0 & \dots & \dots & \dots\dots\dots\dots & & d_{2n-1,2n-3} \\ 0 & \dots & \dots & \dots\dots\dots\dots & & d_{2n-1,2(n-1)} \\ 0 & \dots & \dots & \dots\dots\dots\dots & & d_{2n-1,2n-1} \end{vmatrix}, \tag{3.29}$$

$$D_{2n+1,1} = \begin{vmatrix} d_{2,1} & \dots & \dots\dots\dots\dots & d_{2n+1,1} \\ d_{2,2} & \dots & \dots\dots\dots\dots & d_{2n+1,2} \\ d_{2,3} & \dots & \dots\dots\dots\dots & d_{2n+1,3} \\ d_{2,4} & \dots & \dots\dots\dots\dots & d_{2n+1,4} \\ 0 & d_{3,5} & \dots\dots\dots\dots & d_{2n+1,5} \\ \vdots & d_{3,6} & \dots\dots\dots\dots & d_{2n+1,6} \\ \vdots & \vdots & \vdots \;\; \vdots \;\; \ddots \;\; \ddots & \vdots \\ 0 & \dots & \dots\dots\dots\dots & d_{2n+1,2n-1} \\ 0 & \dots & \dots\dots\dots\dots & d_{2n+1,2n} \end{vmatrix}, \tag{3.30}$$

$$D_{2n+1,k} = \begin{vmatrix} d_{1,1} & d_{2,1} & \dots & \dots & d_{k-1,1} & d_{k+1,1} & \dots & d_{2n+1,1} \\ d_{1,2} & d_{2,2} & \dots & \dots & \dots & \dots & \dots & d_{2n+1,2} \\ 0 & d_{2,3} & \dots & \dots & \dots & \dots & \dots & d_{2n+1,3} \\ \vdots & d_{2,4} & \dots & \dots & \dots & \dots & \dots & d_{2n+1,4} \\ \vdots & 0 & d_{3,5} & \dots & \dots & \dots & \dots & d_{2n+1,5} \\ \vdots & \vdots & d_{3,6} & \dots & \dots & \dots & \dots & d_{2n+1,6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & d_{2n+1,2n-1} \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & d_{2n+1,2n} \end{vmatrix}, \quad k = 2, \dots, 2n, \tag{3.31}$$

$$D_{2n+1,2n+1} = \begin{vmatrix} d_{1,1} & d_{2,1} & \dots & \dots\dots\dots\dots & d_{2n,1} \\ d_{1,2} & d_{2,2} & \dots & \dots\dots\dots\dots & d_{2n,2} \\ 0 & d_{2,3} & \dots & \dots\dots\dots\dots & d_{2n,3} \\ \vdots & d_{2,4} & \dots & \dots\dots\dots\dots & d_{2n,4} \\ \vdots & 0 & d_{3,5} & \dots\dots\dots\dots & d_{2n,5} \\ \vdots & \vdots & d_{3,6} & \dots\dots\dots\dots & d_{2n,6} \\ \vdots & \vdots & \vdots & \vdots \;\; \vdots \;\; \ddots \;\; \ddots & \vdots \\ 0 & \dots & \dots & \dots\dots\dots\dots & d_{2n,2n-1} \\ 0 & \dots & \dots & \dots\dots\dots\dots & d_{2n,2n} \end{vmatrix}, \tag{3.32}$$

we obtain the values for coefficients of the sequences $\left(P_{2n}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$ $\left(P_{2n+1}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$, respectively,

$$a_{2n,k} = (-1)^k a_{2n,0} \, \frac{D_{2n,k}}{D_{2n}}, \quad k = 1, \dots, 2n, \tag{3.33}$$

$$a_{2n+1,k} = (-1)^k a_{2n+1,0} \, \frac{D_{2n+1,k}}{D_{2n+1}}, \quad k = 1, \dots, 2n+1. \tag{3.34}$$

Moreover, returning to (3.5), we immediately obtain the values of the coefficients for the 3-term recurrence relation (3.4). Indeed, we have

$$A_{2n+1} = -\frac{a_{2n,0}}{a_{2n+1,0}} \frac{D_{2n,2n}\ D_{2n+1}}{D_{2n+1,2n+1}\ D_{2n}}, \tag{3.35}$$

$$A_{2n} = -\frac{a_{2n-1,0}}{a_{2n,0}} \frac{D_{2n-1,2n-1}\ D_{2n}}{D_{2n,2n}\ D_{2n-1}}, \tag{3.36}$$

$$B_{2n} = \frac{D_{2n+1,2n}}{D_{2n+1,2n+1}} - \frac{D_{2n,2n-1}}{D_{2n,2n}}, \tag{3.37}$$

$$B_{2n+1} = \frac{D_{2(n+1),2n+1}}{D_{2(n+1),2(n+1)}} - \frac{D_{2n+1,2n}}{D_{2n+1,2n+1}}. \tag{3.38}$$

In order to find free coefficients of the even and odd Prudnikov's sequences, we appeal to the identity (3.7) and values (2.4) of the moments for ρ_ν. Thus using (3.33), we derive from (3.7) for the sequence $\left(P_{2n}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$

$$\frac{D_{2n}}{a_{2n,0}\ D_{2n,2n}} = \frac{a_{2n,0}}{D_{2n}} \sum_{m=0}^{2n} (-1)^m D_{2n,m} \Gamma(2n+m+\alpha+\nu+1)\Gamma(2n+m+\alpha+1),$$
$$D_{2n,0} \equiv D_{2n}.$$

Hence, taking into account the positive sign of the leading coefficient a_{2n}, we get the value of $a_{2n,0}$ in the form

$$a_{2n,0} = \frac{D_{2n}}{\left[D_{2n,2n}\right]^{1/2}} \times \left[\sum_{m=0}^{2n} (-1)^m D_{2n,m} \Gamma(2n+m+\alpha+\nu+1)\Gamma(2n+m+\alpha+1)\right]^{-1/2}, \quad D_{2n,0} \equiv D_{2n}. \tag{3.39}$$

Analogously, we obtain the value $a_{2n+1,0}$ for the odd sequence $\left(P_{2n+1}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$, namely,

$$a_{2n+1,0} = -\frac{D_{2n+1}}{\left[D_{2n+1,2n+1}\right]^{1/2}} \times \left[\sum_{m=0}^{2n+1} (-1)^m D_{2n+1,m} \Gamma(2(n+1)+m+\alpha+\nu)\Gamma(2(n+1)+m+\alpha)\right]^{-1/2}, \tag{3.40}$$

where $D_{2n+1,0} \equiv D_{2n+1}$. Leading coefficients for the Prudnikov sequences have the values, accordingly,

$$a_{2n} = \left[D_{2n,2n}\right]^{1/2} \left[\sum_{m=0}^{2n} (-1)^m D_{2n,m} \Gamma(2n+m+\alpha+\nu+1)\Gamma(2n+m+\alpha+1)\right]^{-1/2} \tag{3.41}$$

$$a_{2n+1} = \left[D_{2n+1,2n+1}\right]^{1/2} \times \left[\sum_{m=0}^{2n+1} (-1)^m D_{2n+1,m}\Gamma(2(n+1)+m+\alpha+\nu)\Gamma(2(n+1)+m+\alpha)\right]^{-1/2}. \tag{3.42}$$

Thus we proved the following theorem.

Theorem 2 *Let $\nu \geq 0$, $\alpha > -1$, $n \in \mathbb{N}_0$. Prudnikov's sequences of orthogonal polynomials $\left(P_{2n}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$, $\left(P_{2n+1}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$ have explicit values with coefficients calculated by formulas (3.33), (3.34), respectively, where the determinants D_{2n}, D_{2n+1}, $D_{2n,k}$, $D_{2n+1,k}$ are defined by (3.25)- (3.32) and free coefficients $a_{2n,0}, a_{2n+1,0}$ by (3.37), (3.38). Moreover, the 3-term recurrence relation (3.4) holds with coefficients (3.35)-(3.38).*

Remark 2 It would be an interesting problem to study algebraic properties of the determinants (3.25)- (3.32) whose entries satisfy the recurrence relation (3.20).

Corollary 1 *Coefficients (3.14) are calculated by formulas*

$$c_{2n,2j} = \frac{a_{2n,0}}{D_{2n}} \sum_{m=j}^{2n} (-1)^m D_{2n,m}\, d_{m,2j},\ \ j = n, \ldots, 2n, \tag{3.43}$$

$$c_{2n+1,2j} = \frac{a_{2n+1,0}}{D_{2n+1}} \sum_{m=j}^{2n+1} (-1)^m D_{2n+1,m}\, d_{m,2j},\ \ j = n+1, \ldots, 2n+1, \tag{3.44}$$

$$c_{2n,2j+1} = \frac{a_{2n,0}}{D_{2n}} \sum_{m=j+1}^{2n} (-1)^m D_{2n,m}\, d_{m,2j+1},\ \ j = n, \ldots, 2n-1, \tag{3.45}$$

$$c_{2n+1,2j+1} = \frac{a_{2n+1,0}}{D_{2n+1}} \sum_{m=j+1}^{2n+1} (-1)^m D_{2n+1,m}\, d_{m,2j+1},\ \ j = n, \ldots, 2n. \tag{3.46}$$

where values D_{2n}, D_{2n+1}, $D_{2n,k}$, $D_{2n+1,k}$ are defined by (3.25)-(3.32) and free coefficients $a_{2n,0}, a_{2n+1,0}$ by (3.39), (3.40).

Our goal now is to find an analog of the Rodrigues formula for Prudnikov's polynomials. To do this, we recall the representation (2.24) of an arbitrary polynomial in terms of its associated polynomial and representations (2.28), (2.29), (3.15), (3.16) to write the following equalities for the sequence $\left(P_n^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$

$$P_n^{\nu,\alpha}(x) = \frac{x^{-\alpha}}{\rho_\nu(x)} \sum_{j=n}^{2n} \frac{c_{n,j}}{j!} S_j^{\nu,\alpha}(x) = \frac{x^{-\alpha}}{\rho_\nu(x)} \sum_{j=n}^{2n} \frac{c_{n,j}}{j!} \frac{\mathrm{d}^j}{\mathrm{d}x^j}\left[x^{j+\alpha}\rho_\nu(x)\right]$$
$$= \frac{x^{-\alpha}}{\rho_\nu(x)} \sum_{j=0}^{n} \frac{c_{n,j+n}}{(j+n)!} \frac{\mathrm{d}^{j+n}}{\mathrm{d}x^{j+n}}\left[x^{j+n+\alpha}\rho_\nu(x)\right].$$

Therefore for sequences $\left(P_{2n}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$, $\left(P_{2n+1}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$ we have, correspondingly,

$$P_{2n}^{\nu,\alpha}(x) = \frac{x^{-\alpha}}{\rho_\nu(x)} \sum_{j=0}^{2n} \frac{c_{2n,j+2n}}{(j+2n)!} S_{j+2n}^{\nu,\alpha}(x), \tag{3.47}$$

$$P_{2n+1}^{\nu,\alpha}(x) = \frac{x^{-\alpha}}{\rho_\nu(x)} \sum_{j=0}^{2n+1} \frac{c_{2n+1,j+2n+1}}{(j+2n+1)!} S_{j+2n+1}^{\nu,\alpha}(x). \tag{3.48}$$

In the meantime, the sums in (3.47), (3.48) can be treated as follows

$$\sum_{j=0}^{2n} \frac{c_{2n,j+2n}}{(j+2n)!} S_{j+2n}^{\nu,\alpha}(x) = \sum_{j=0}^{n} \frac{c_{2n,2(j+n)}}{(2(j+n))!} S_{2(j+n)}^{\nu,\alpha}(x) + \sum_{j=0}^{n-1} \frac{c_{2n,2(j+n)+1}}{(2(j+n)+1)!} S_{2(j+n)+1}^{\nu,\alpha}(x),$$

$$\sum_{j=0}^{2n+1} \frac{c_{2n+1,j+2n+1}}{(j+2n+1)!} S_{j+2n+1}^{\nu,\alpha}(x) = \sum_{j=0}^{n} \frac{c_{2n+1,2(j+n+1)}}{(2(j+n+1))!} S_{2(j+n+1)}^{\nu,\alpha}(x) + \sum_{j=0}^{n} \frac{c_{2n+1,2(j+n)+1}}{(2(j+n)+1)!} S_{2(j+n)+1}^{\nu,\alpha}(x).$$

Hence, employing the theory of multiple orthogonal polynomials associated with the scaled Macdonald functions and the related Rodrigues formulas (see details in [13], [2]), we find the following expressions

$$S_{2(j+n)}^{\nu,\alpha}(x) = x^\alpha \left[A_{j+n,j+n-1}^{\alpha}(x)\rho_\nu(x) + B_{j+n,j+n-1}^{\alpha}(x)\rho_{\nu+1}(x) \right], \tag{3.49}$$

$$S_{2(j+n)+1}^{\nu,\alpha}(x) = x^\alpha \left[A_{j+n,j+n}^{\alpha}(x)\rho_\nu(x) + B_{j+n,j+n}^{\alpha}(x)\rho_{\nu+1}(x) \right], \tag{3.50}$$

where A-polynomials in front of ρ_ν are of degree $j+n$ as well as B-polynomial in (3.50), while B-polynomial in (3.49) is of degree $j+n-1$. These polynomials are explicitly calculated in [2]. Therefore formulas (3.47), (3.48) become, respectively,

$$P_{2n}^{\nu,\alpha}(x) = \sum_{j=0}^{n} \frac{c_{2n,2(j+n)}}{(2(j+n))!} A_{j+n,j+n-1}^{\alpha}(x) + \sum_{j=0}^{n-1} \frac{c_{2n,2(j+n)+1}}{(2(j+n)+1)!} A_{j+n,j+n}^{\alpha}(x) + \frac{\rho_{\nu+1}(x)}{\rho_\nu(x)} \left[\sum_{j=0}^{n} \frac{c_{2n,2(j+n)}}{(2(j+n))!} B_{j+n,j+n-1}^{\alpha}(x) + \sum_{j=0}^{n-1} \frac{c_{2n,2(j+n)+1}}{(2(j+n)+1)!} B_{j+n,j+n}^{\alpha}(x) \right], \tag{3.51}$$

$$P_{2n+1}^{\nu,\alpha}(x) = \sum_{j=0}^{n}\left[\frac{c_{2n+1,2(j+n+1)}}{(2(j+n+1))!}A_{j+n+1,j+n}^{\alpha}(x) + \frac{c_{2n+1,2(j+n)+1}}{(2(j+n)+1)!}A_{j+n,j+n}^{\alpha}(x)\right]$$
$$+\frac{\rho_{\nu+1}(x)}{\rho_\nu(x)}\sum_{j=0}^{n}\left[\frac{c_{2n+1,2(j+n+1)}}{(2(j+n+1))!}B_{j+n+1,j+n}^{\alpha}(x)\right.$$
$$\left.+\frac{c_{2n+1,2(j+n)+1}}{(2(j+n)+1)!}B_{j+n,j+n}^{\alpha}(x)\right]. \tag{3.52}$$

But Lemma 1 presumes immediately the following identities from (3.51), (3.52)

$$P_{2n}^{\nu,\alpha}(x) = \sum_{j=n}^{2n}\frac{c_{2n,2j}}{(2j)!}A_{j,j-1}^{\alpha}(x) + \sum_{j=n}^{2n-1}\frac{c_{2n,2j+1}}{(2j+1)!}A_{j,j}^{\alpha}(x), \tag{3.53}$$
$$P_{2n+1}^{\nu,\alpha}(x) = \sum_{j=n}^{2n}\left[\frac{c_{2n+1,2(j+1)}}{(2(j+1))!}A_{j+1,j}^{\alpha}(x) + \frac{c_{2n+1,2j+1}}{(2j+1)!}A_{j,j}^{\alpha}(x)\right], \tag{3.54}$$

giving explicit expressions of Prudnikov's polynomials in terms of the multiple orthogonal polynomials for the scaled Macdonald functions, and two more relations between multiple B-polynomials

$$\sum_{j=n}^{2n}\frac{c_{2n,2j}}{(2j)!}B_{j,j-1}^{\alpha}(x) + \sum_{j=n}^{2n-1}\frac{c_{2n,2j+1}}{(2j+1)!}B_{j,j}^{\alpha}(x) \equiv 0,$$
$$\sum_{j=n}^{2n}\left[\frac{c_{2n+1,2(j+1)}}{(2(j+1))!}B_{j+1,j}^{\alpha}(x) + \frac{c_{2n+1,2j+1}}{(2j+1)!}B_{j,j}^{\alpha}(x)\right] \equiv 0.$$

On the other hand,

$$P_n^{\nu,\alpha}(x) = \frac{x^{-\alpha}}{\rho_\nu(x)}\sum_{j=n}^{2n}\frac{c_{n,j}}{j!}S_j^{\nu,\alpha}(x) = \frac{x^{-\alpha}}{\rho_\nu(x)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\sum_{j=0}^{n}\frac{c_{n,j+n}}{(j+n)!}S_j^{\nu,n+\alpha}(x). \tag{3.55}$$

Hence, recalling integral representations (2.3), (2.29), and the explicit formula for Laguerre polynomials [1], we obtain

$$S_j^{\nu,n+\alpha}(x) = x^{n+\alpha}j!\sum_{k=0}^{j}\frac{(-1)^k}{k!}\binom{j+n+\nu+\alpha}{j-k}\int_0^\infty e^{-t-x/t}t^{\nu+k-1}\mathrm{d}t$$
$$= x^{n+\alpha}j!\sum_{k=0}^{j}\frac{(-1)^k}{k!}\binom{j+n+\nu+\alpha}{j-k}\rho_{\nu+k}(x). \tag{3.56}$$

The problem now is to express $\rho_{\nu+k},\ k \in \mathbb{N}_0$ in terms of ρ_ν and $\rho_{\nu+1}$. To do this, we use the Mellin–Barnes representation (1.4) and the definition (2.7) of the Pochhammer symbol to derive

$$
\begin{aligned}
\rho_{\nu+k}(x) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+\nu+k)\Gamma(s)x^{-s}\mathrm{d}s \\
&= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} (s+\nu)_k\Gamma(s+\nu)\Gamma(s)x^{-s}\mathrm{d}s \\
&= \frac{(-1)^k x^{\nu+k}}{2\pi i}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+\nu)\Gamma(s)x^{-s-\nu}\mathrm{d}s \\
&= (-1)^k x^{\nu+k}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[x^{-\nu}\rho_\nu(x)\right].
\end{aligned}
$$

Then, employing the Leibniz formula and (2.16), we find

$$
\rho_{\nu+k}(x) = \sum_{m=0}^{k}\binom{k}{m}(\nu)_{k-m}\, x^m \rho_{\nu-m}(x). \tag{3.57}
$$

Meanwhile, employing the identity from [2] for the scaled Macdonald functions, specifically,

$$
x^m\rho_{\nu-m}(x) = x^{m/2}r_m(2\sqrt{x};\nu)\rho_\nu(x) + x^{(m-1)/2}r_{m-1}(2\sqrt{x};\nu-1)\rho_{\nu+1}(x), \quad m\in\mathbb{N}_0, \tag{3.58}
$$

where $r_{-1}(z;\nu)=0$,

$$
x^{m/2}r_m(2\sqrt{x};\nu) = (-1)^m\sum_{i=0}^{[m/2]}(\nu+i-m+1)_{m-2i}(m-2i+1)_i\frac{x^i}{i!},
$$

formula (3.57) takes the final expression

$$
\begin{aligned}
\rho_{\nu+k}(x) = \rho_\nu(x)&\sum_{m=0}^{k}\sum_{i=0}^{[m/2]}(-1)^m\,(\nu+i-m+1)_{m-2i}\,(m-2i+1)_i\,(\nu)_{k-m}\binom{k}{m}\frac{x^i}{i!} \\
&+\rho_{\nu+1}(x)\sum_{m=0}^{k-1}\sum_{i=0}^{[m/2]}(-1)^m\,(\nu+i-m)_{m-2i}\,(m-2i+1)_i\,(\nu)_{k-m-1}\binom{k}{m+1}\frac{x^i}{i!}.
\end{aligned} \tag{3.59}
$$

Substituting the right-hand side of the equality (3.59) into (3.56), we get finally

$$
\begin{aligned}
S_j^{\nu,n+\alpha}(x) = x^{n+\alpha}j!\Bigg[\rho_\nu(x)&\sum_{k=0}^{j}\sum_{m=0}^{k}\sum_{i=0}^{[m/2]}\frac{(-1)^{k+m}}{k!}\binom{j+n+\nu+\alpha}{j-k} \\
&\times(\nu+i-m+1)_{m-2i}\,(m-2i+1)_i\,(\nu)_{k-m}\binom{k}{m}\frac{x^i}{i!}
\end{aligned}
$$

$$+\rho_{\nu+1}(x)\sum_{k=0}^{j}\sum_{m=0}^{k-1}\sum_{i=0}^{[m/2]}\frac{(-1)^{k+m}}{k!}\binom{j+n+\nu+\alpha}{j-k}$$
$$\times\,(\nu+i-m)_{m-2i}\,(m-2i+1)_i\,(\nu)_{k-m-1}\binom{k}{m+1}\frac{x^i}{i!}\Bigg].$$

Thus, returning to (3.55), we end up with the so-called Rodrigues type formula for the Prudnikov orthogonal polynomials $P_n^{\nu,\alpha}$

$$P_n^{\nu,\alpha}(x)=\frac{x^{-\alpha}}{\rho_\nu(x)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Bigg[x^{n+\alpha}\Bigg[\rho_\nu(x)\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{m=0}^{k}\sum_{i=0}^{[m/2]}\frac{(-1)^{k+m}}{(j+n)!}$$
$$\times c_{n,j+n}(n+\nu+\alpha+k+1)_{j-k}(\nu+i-m+1)_{m-2i}$$
$$\times\,(m-2i+1)_i(\nu)_{k-m}\binom{j}{k}\binom{k}{m}\frac{x^i}{i!}+\rho_{\nu+1}(x)\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{m=0}^{k-1}\sum_{i=0}^{[m/2]}$$
$$\frac{(-1)^{k+m}}{(j+n)!}\,c_{n,j+n}(n+\nu+\alpha+k+1)_{j-k}$$
$$\times\,(\nu+i-m)_{m-2i}\,(m-2i+1)_i\,(\nu)_{k-m-1}\binom{j}{k}\binom{k}{m+1}\frac{x^i}{i!}\Bigg]\Bigg]. \tag{3.60}$$

Theorem 3 *Prudnikov's orthogonal polynomials $P_n^{\nu,\alpha}$ can be obtained from the Rodrigues type formula (3.60), where connection coefficients $c_{n,j+n}$ are calculated in Corollary 1. Moreover, Prudnikov's sequences $\left(P_{2n}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$, $\left(P_{2n+1}^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$ are expressed in terms of multiple orthogonal polynomials related to the scaled Macdonald functions by equalities (3.53), (3.54), respectively, where the polynomials $A_{j,j-1}^{\alpha}$, $A_{j,j}^{\alpha}$ are calculated explicitly in [2] by formulas*

$$A_{j,j-1}^{\alpha}(x)=(\alpha+1)_{2j}\sum_{m=0}^{j}\binom{2j}{2m}\frac{x^m}{(\alpha+1)_{2m}}$$
$$\times{}_3F_2\left(-2(j-m),\ m-\nu,\ m+1;\ 2m+1+\alpha,\ 2m+1;\ 1\right),$$
$$A_{j,j}^{\alpha}(x)=(\alpha+1)_{2j+1}\sum_{m=0}^{j}\binom{2j+1}{2m}\frac{x^m}{(\alpha+1)_{2m}}$$
$$\times{}_3F_2\left(-2(j-m)-1,\ m-\nu,\ m+1;\ 2m+1+\alpha,\ 2m+1;\ 1\right).$$

Further, the generating function for polynomials $P_n^{\nu,\alpha}$ can be defined as usual by the equality

$$G(x,z)=\sum_{n=0}^{\infty}P_n^{\nu,\alpha}(x)\frac{z^n}{n!},\quad x>0,\ z\in\mathbb{C}, \tag{3.61}$$

where $|z| < h_x$ and $h_x > 0$ is a convergence radius of the power series. Then returning to (3.55) and employing (2.28), we have from (3.61)

$$G(x,z) = \frac{x^{-\alpha}}{\rho_\nu(x)} \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{j=n}^{2n} \frac{c_{n,j}}{j!} \frac{\mathrm{d}^j}{\mathrm{d}x^j} \left[x^{j+\alpha} \rho_\nu(x) \right]$$
$$= \frac{1}{\rho_\nu(x)} \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{j=n}^{2n} \sum_{k=0}^{j} (-1)^k \frac{c_{n,j}}{k!} \binom{j+\alpha}{j-k} x^k \rho_{\nu-k}(x).$$

Hence substituting the value of $x^k \rho_{\nu-k}(x)$ by formula (3.58), we get, finally, the expression for the generating function for the Prudnikov sequence $\left(P_n^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$, namely,

$$G(x,z) = \sum_{n=0}^{\infty} \sum_{j=n}^{2n} \sum_{k=0}^{j} \frac{(-1)^k\, c_{n,j}}{n!\, k!} \binom{j+\alpha}{j-k} x^{k/2} r_k(2\sqrt{x};\nu) z^n$$
$$+ \frac{\rho_{\nu+1}(x)}{\rho_\nu(x)} \sum_{n=0}^{\infty} \sum_{j=n}^{2n} \sum_{k=0}^{j} \frac{(-1)^k c_{n,j}}{n!\, k!} \binom{j+\alpha}{j-k} x^{(k-1)/2} r_{k-1}(2\sqrt{x};\nu-1) z^n,$$

where $c_{n,j}$ are defined in Corollary 1.

4 Orthogonal Polynomials with Ultra-Exponential Weights

In this section we will consider a sequence of polynomials $\left(Q_n^{\nu,k}\right)_{n\in\mathbb{N}_0}$, which is orthogonal with respect to the weight function (2.1) $x^\alpha \rho_{\nu,k}(x)$

$$\int_0^\infty Q_n^{\nu,\alpha}(x) Q_m^{\nu,\alpha}(x) \rho_{\nu,k}(x) x^\alpha \mathrm{d}x = \delta_{m,n}, \;\; \nu \geq 0, \;\; \alpha > -1. \tag{4.1}$$

The function $\rho_{\nu,k}$ satisfies some interesting properties. In fact, recalling the Mellin–Barnes integral representation (2.1), we write

$$\rho_{\nu+1,k}(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(\nu+1+s) \left[\Gamma(s)\right]^k x^{-s} \mathrm{d}s$$
$$= \frac{\nu}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(\nu+s) \left[\Gamma(s)\right]^k x^{-s} \mathrm{d}s$$
$$+ \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(\nu+s) s \left[\Gamma(s)\right]^k x^{-s} \mathrm{d}s$$
$$= \nu \rho_{\nu,k}(x) - x D \rho_{\nu,k}(x).$$

Hence, as in (2.18)

$$\rho_{\nu+1,k}(x) = (\nu - xD)\rho_{\nu,k}(x). \tag{4.2}$$

Further,

$$\begin{aligned} D\,(xD)^{k-1}\left(x^{\nu+1}D\left(x^{-\nu}\rho_{\nu,k}(x)\right)\right) &= \frac{(-1)^{k+1}}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(\nu+s+1)s^k\,[\Gamma(s)]^k\,x^{-s-1}\mathrm{d}s \\ &= (-1)^{k+1}\rho_{\nu,k}(x). \end{aligned}$$

Thus we derive the following $k+1$th order differential equation for the function $\rho_{\nu,k}$, generalizing equation (2.5) for $\rho_{\nu,1} \equiv \rho_\nu$

$$(-1)^{k+1}D\,(xD)^{k-1}\left(x^{\nu+1}D\left(x^{-\nu}\rho_{\nu,k}(x)\right)\right) = \rho_{\nu,k}(x),\ k\in\mathbb{N},\quad D\equiv\frac{\mathrm{d}}{\mathrm{d}x}. \tag{4.3}$$

The integral recurrence relation for functions $\rho_{\nu,k}$ follows from the Parseval equality (1.10). To be precise, we obtain

$$\rho_{\nu,k+1}(x) = \int_0^\infty e^{-x/t}\rho_{\nu,k}(t)\frac{\mathrm{d}t}{t},\quad k\in\mathbb{N}_0. \tag{4.4}$$

An analog of the integral representation (2.11) for $\rho_{\nu,k}$ can be deduced in the following manner. In fact, the Mellin–Barnes integral for Laguerre polynomials (see [10, relation (8.4.33.3), Vol. III])

$$n!\,e^{-x}L_n^\nu(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s)\frac{\Gamma(1+n+\nu-s)}{\Gamma(1+\nu-s)}x^{-s}\mathrm{d}s,$$

integral (2.1) with the Parseval identity (1.10), and the reflection formula for the gamma function imply the equality for $k\in\mathbb{N}$

$$\begin{aligned} x^n\rho_{\nu,k}(x) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s+\nu+n)\,[\Gamma(s+n)]^k\,x^{-s}\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s+\nu+n)\Gamma(s+n)\Gamma(1-s-n)\frac{[\Gamma(s+n)]^{k-1}}{\Gamma(1-s-n)}x^{-s}\mathrm{d}s \\ &= \frac{(-1)^n}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s+\nu+n)\Gamma(s)\Gamma(1-s)\frac{[\Gamma(s+n)]^{k-1}}{\Gamma(1-s-n)}x^{-s}\mathrm{d}s \\ &= (-1)^n n!\int_0^\infty t^{\nu+n-1}e^{-t}L_n^\nu(t)\varphi_n\left(\frac{x}{t}\right)\mathrm{d}t, \end{aligned}$$

where

$$\varphi_{n,k}(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s)\,[\Gamma(s+n)]^{k-1}\,x^{-s}\mathrm{d}s. \tag{4.5}$$

Therefore we obtain the integral representation

$$x^n \rho_{\nu,k}(x) = (-1)^n n! \int_0^\infty t^{\nu+n-1} e^{-t} L_n^\nu(t) \varphi_{n,k}\left(\frac{x}{t}\right) \mathrm{d}t. \tag{4.6}$$

Differentiating (4.5) n times by x, where the differentiation under the integral sign is possible due to the absolute and uniform convergence, we take into account the reduction formula for the gamma function and (2.1) to obtain

$$\begin{aligned} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \varphi_{n,k}(x) &= \frac{(-1)^n}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} (s)_n \Gamma(s) \left[\Gamma(s+n)\right]^{k-1} x^{-s-n} \mathrm{d}s \\ &= \frac{(-1)^n}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \left[\Gamma(s+n)\right]^{k} x^{-s-n} \mathrm{d}s = (-1)^n \rho_{0,k-1}(x). \end{aligned} \tag{4.7}$$

Consequently, after differentiating both sides of (4.6) n times we find an analog of the representation (2.29), namely,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[x^n \rho_{\nu,k}(x)\right] = n! \int_0^\infty t^{\nu-1} e^{-t} L_n^\nu(t) \rho_{0,k-1}\left(\frac{x}{t}\right) \mathrm{d}t. \tag{4.8}$$

Now, returning to (4.1), we substitute the function $\rho_{\nu,k}$ with the integral (4.4) and interchange the order of integration by Fubini's theorem. Then, employing again the Viskov-type identities (2.6) for the differential operator θ, we derive for $k \in \mathbb{N}$

$$\begin{aligned} \delta_{m,n} &= \int_0^\infty Q_n^{\nu,\alpha}(x) Q_m^{\nu,\alpha}(x) \rho_{\nu,k}(x) x^\alpha \mathrm{d}x \\ &= \int_0^\infty \rho_{\nu,k-1}(t) \frac{1}{t} \int_0^\infty e^{-x/t} Q_n^{\nu,\alpha}(x) Q_m^{\nu,\alpha}(x) x^\alpha \mathrm{d}x \mathrm{d}t \\ &= \Gamma(1+\alpha) \int_0^\infty \rho_{\nu,k-1}(t) Q_n^{\nu,\alpha}(\theta) Q_m^{\nu,\alpha}(\theta)\{t^\alpha\} \mathrm{d}t. \end{aligned}$$

Hence it leads to

Theorem 4 *Let $k \in \mathbb{N}$, $\nu \geq 0, \alpha > -1$. The orthogonality (4.1) for the sequence of polynomials $\left(Q_n^{\nu,\alpha}\right)_{n\in\mathbb{N}_0}$ with the weight $x^\alpha \rho_{\nu,k}(x)$ is the composition orthogonality of the same sequence with respect to the weight $\rho_{\nu,k-1}$, namely*

$$\int_0^\infty \rho_{\nu,k-1}(t) Q_n^{\nu,\alpha}(\theta) Q_m^{\nu,\alpha}(\theta)\{t^\alpha\} \mathrm{d}t = \frac{\delta_{m,n}}{\Gamma(1+\alpha)}, \quad m, n \in \mathbb{N}_0. \tag{4.9}$$

In particular, for $k = 2$ this sequence is compositionally orthogonal in the sense of Prudnikov.

Further, up to a normalization constant equality, (4.1) is equivalent to the following n conditions

$$\int_0^\infty Q_n^{\nu,\alpha}(x) \rho_{\nu,k}(x) x^{\alpha+m} \mathrm{d}x = 0, \quad m = 0, 1, \ldots, n-1, \quad n \in \mathbb{N}. \tag{4.10}$$

Hence the composition orthogonality (4.9) implies, with the integration by parts and properties of the operator θ,

$$\int_0^\infty \theta^m \left\{\rho_{\nu,k-1}(t)\right\} Q_n^{\nu,\alpha}(\theta)\left\{t^\alpha\right\} \mathrm{d}t = 0,\ m = 0, 1, \ldots, n-1,\ k, n \in \mathbb{N}. \tag{4.11}$$

Writing $Q_n^{\nu,\alpha}$ in the explicit form

$$Q_n^{\nu,\alpha}(x) = \sum_{j=0}^{n} a_{n,j} x^j \equiv Q_n^{\nu,\alpha,0}(x),$$

we have

$$Q_n^{\nu,\alpha}(\theta)\left\{t^\alpha\right\} = \sum_{j=0}^{n} a_{n,j}\theta^j\left\{t^\alpha\right\} = t^\alpha \sum_{j=0}^{n} a_{n,j}(1+\alpha)_j\, t^j = t^\alpha Q_n^{\nu,\alpha,1}(t),$$

where

$$Q_n^{\nu,\alpha,1}(t) = \frac{1}{\Gamma(1+\alpha)} \sum_{j=0}^{n} a_{n,j}\Gamma(1+\alpha+j)\, t^j. \tag{4.12}$$

On the other hand, employing (4.4) for $k \geq 2$ and observing that owing to the Viskov-type identities (2.6) $(\theta_t \equiv tDt,\ \beta_y \equiv DyD)$

$$\theta_t^m\left\{e^{-ty}\right\} = (-1)^m \beta_y^m\left\{e^{-ty}\right\}, \quad m \in \mathbb{N}_0, \tag{4.13}$$

we deduce, integrating by parts,

$$\begin{aligned}
\theta_t^m\left\{\rho_{\nu,k-1}(t)\right\} &= \theta_t^m\left\{\int_0^\infty e^{-ty}\rho_{\nu,k-2}\left(\frac{1}{y}\right)\frac{\mathrm{d}y}{y}\right\}\\
&= (-1)^m \int_0^\infty \beta_y^m\left\{e^{-ty}\right\}\ \rho_{\nu,k-2}\left(\frac{1}{y}\right)\frac{\mathrm{d}y}{y}\\
&= (-1)^m \int_0^\infty e^{-ty}\beta_y^m\left\{\rho_{\nu,k-2}\left(\frac{1}{y}\right)\frac{1}{y}\right\}\mathrm{d}y,
\end{aligned}$$

where the differentiation under integral sign is allowed via the absolute and uniform convergence. Thus, returning to (4.11), we plug in the latter expressions and change the order of integration by Fubini's theorem to write it in the form

$$\int_0^\infty Q_n^{\nu,\alpha,2}\left(\frac{1}{y}\right)\beta_y^m\left\{\rho_{\nu,k-2}\left(\frac{1}{y}\right)\frac{1}{y}\right\} y^{-\alpha-1}\mathrm{d}y = 0,\ m = 0, 1, \ldots, n-1,\ n \in \mathbb{N}, \tag{4.14}$$

where

$$Q_n^{\nu,\alpha,2}(x) = \frac{1}{\Gamma(1+\alpha)} \sum_{j=0}^{n} a_{n,j} [\Gamma(1+\alpha+j)]^2 \, x^j. \tag{4.15}$$

Meanwhile, recalling (2.1), we get

$$\begin{aligned}
\beta_y^m \left\{ \rho_{\nu,k-2}\left(\frac{1}{y}\right) \frac{1}{y} \right\} &= \left(D^m y^m D^m\right) \left\{ \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+\nu)\, [\Gamma(s)]^{k-2}\, y^{s-1} \mathrm{d}s \right\} \\
&= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} [(s-1)\dots(s-m)]^2 \, \Gamma(s+\nu)\, [\Gamma(s)]^{k-2}\, y^{s-m-1} \mathrm{d}s \\
&= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} [(s)_m]^2 \, \Gamma(s+m+\nu)\, [\Gamma(s+m)]^{k-2}\, y^{s-1} \mathrm{d}s \\
&= \frac{y^{-m}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s+\nu)}{\Gamma^2(s-m)} [\Gamma(s)]^k \, y^{s-1} \mathrm{d}s.
\end{aligned}$$

Therefore we find from (4.14)

$$\int_0^\infty Q_n^{\nu,\alpha,2}(y)\, \Phi_{\nu,k,m}^{(2)}(y)\, y^\alpha \mathrm{d}y = 0, \; m = 0, 1, \dots, n-1, \; n \in \mathbb{N}, \tag{4.16}$$

where

$$\Phi_{\nu,k,m}^{(2)}(y) \equiv \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s+m+\nu)}{\Gamma^2(s)} [\Gamma(s+m)]^k \, y^{-s} \mathrm{d}s, \quad k \geq 2. \tag{4.17}$$

But it is easily seen from the properties of the Mellin transform [15] and (2.1) that

$$\Phi_{\nu,k,m}^{(2)}(y) = y^m D^m y^m D^m y^m \left\{\rho_{k-2}(y)\right\}, \quad k \geq 2. \tag{4.18}$$

Now, recalling (4.4), we have

$$y^m D^m y^m D^m y^m \left\{\rho_{k-2}(y)\right\} = y^m D^m y^m D^m y^m \left\{ \int_0^\infty e^{-yu} \rho_{\nu,k-3}\left(\frac{1}{u}\right) \frac{du}{u} \right\}. \tag{4.19}$$

Hence, modifying the formula (4.13), we obtain

$$y^m D_y^m y^m D_y^m y^m \left\{e^{-yu}\right\} = (-1)^m D_u^m u^m D_u^m u^m D_u^m \left\{e^{-yu}\right\}. \tag{4.20}$$

Therefore, integrating by parts, we get from (4.18), (4.19), (4.20)

$$\Phi_{\nu,k,m}^{(2)}(y) = \int_0^\infty e^{-yu} D_u^m u^m D_u^m u^m D_u^m \left\{ \rho_{\nu,k-3}\left(\frac{1}{u}\right) \frac{1}{u} \right\} du. \tag{4.21}$$

Moreover, in a similar manner as above we derive

$$
\begin{aligned}
& D_u^m u^m D_u^m u^m D_u^m \left\{ \rho_{\nu,k-3}\left(\frac{1}{u}\right) \frac{1}{u} \right\} \\
&= D_u^m u^m D_u^m u^m D_u^m \left\{ \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+\nu)\, [\Gamma(s)]^{k-3}\, u^{s-1} \mathrm{d}s \right\} \\
&= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} [(s-1)\dots(s-m)]^3\, \Gamma(s+\nu)\, [\Gamma(s)]^{k-3}\, u^{s-m-1} \mathrm{d}s \\
&= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} [(s)_m]^3\, \Gamma(s+m+\nu)\, [\Gamma(s+m)]^{k-3}\, u^{s-1} \mathrm{d}s \\
&= \frac{u^{-m}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s+\nu)}{\Gamma^3(s-m)}\, [\Gamma(s)]^k\, u^{s-1} \mathrm{d}s. \qquad (4.22)
\end{aligned}
$$

So, substituting the right-hand side of the last equality in (4.22) into (4.21) and the obtained expression into (4.16), we find after the interchange of the order of integration and simple change of variables the following orthogonality conditions

$$\int_0^\infty Q_n^{\nu,\alpha,3}(u)\, \Phi_{\nu,k,m}^{(3)}(u)\, u^\alpha du = 0,\ m = 0, 1, \dots, n-1,\ n \in \mathbb{N},$$

where

$$\Phi_{\nu,k,m}^{(3)}(u) \equiv \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s+m+\nu)}{\Gamma^3(s)}\, [\Gamma(s+m)]^k\, u^{-s} \mathrm{d}s, \quad k \ge 3,$$

and

$$Q_n^{\nu,\alpha,3}(u) = \frac{1}{\Gamma(1+\alpha)} \sum_{j=0}^n a_{n,j} [\Gamma(1+\alpha+j)]^3\, u^j.$$

Continuing this process by virtue of the same technique, involving the Mellin and Laplace transforms and the Mellin–Barnes integrals, after the kth step we end up with the equalities

$$\int_0^\infty Q_n^{\nu,\alpha,k}(x)\, \Phi_{\nu,k,m}^{(k)}(x)\, x^\alpha \mathrm{d}x = 0,\ m = 0, 1, \dots, n-1,\ n \in \mathbb{N},$$

where

$$\Phi_{\nu,k,m}^{(k)}(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} [(s)_m)]^k\, \Gamma(s+m+\nu) x^{-s} \mathrm{d}s,$$

and

$$Q_n^{\nu,\alpha,k}(x) = \frac{1}{\Gamma(1+\alpha)} \sum_{j=0}^{n} a_{n,j} [\Gamma(1+\alpha+j)]^k \, x^j.$$

On the other hand,

$$\begin{aligned}\Phi_{\nu,k,m}^{(k)}(x) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} [(s)_m)]^k \, \Gamma(s+m+\nu) x^{-s} \mathrm{d}s = (-1)^{km} \left\{x^m D^m\right\}^k \left(x^{\nu+m} e^{-x}\right) \\ &= (-1)^{km} m! \left\{x^m D^m\right\}^{k-1} \left(x^{\nu+m} e^{-x} L_m^{\nu}(x)\right).\end{aligned}$$

Consequently, the orthogonality (4.10) is equivalent to the following conditions

$$\int_0^{\infty} Q_n^{\nu,\alpha,k}(x) \; x^{\alpha} \left\{x^m D^m\right\}^k \left(x^{\nu+m} e^{-x}\right) \mathrm{d}x = 0, \;\; m = 0, 1, \ldots, n-1, \;\; n \in \mathbb{N}, \;\; k \in \mathbb{N}_0.$$

Moreover, we see that $\{x^m D^m\}^k \left(x^{\nu+m} e^{-x}\right) = x^{\nu} e^{-x} p_{m(k+1)}(x)$, where $p_{m(k+1)}$ is a polynomial of degree $m(k+1)$ whose coefficients can be calculated explicitly via properties of the Pochhammer symbol and the Laguerre polynomials. Thus it can be reduced to the orthogonality with respect to the measure $x^{\nu+\alpha} e^{-x} \mathrm{d}x$ and ideas of the previous section can be applied. We leave all details to the interested reader. Besides, further developments, an analog of Lemma 1 and relations with the multiple orthogonal polynomial ensemble from [7] will be a promising investigation.

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