Constructive Approximation (2019) 50:293–321
https://doi.org/10.1007/s00365-018-9449-z

CONSTRUCTIVE APPROXIMATION

Asymptotic Evaluations for Some Sequences of Positive Linear Operators

Dumitru Popa[1]

Received: 4 December 2017 / Revised: 29 June 2018 / Accepted: 17 July 2018 /
Published online: 17 September 2018

Abstract
We prove asymptotic evaluations for univariate and multivariate positive linear operators. Our proofs are different from what has been used so far. As applications of our results, we find the full asymptotic evaluation for the iterates of the univariate Cesàro and Volterra operators. Moreover, we find asymptotic evaluations for the iterates of multivariate Cesàro and Volterra type operators on the k-dimensional unit cube, k-dimensional unit triangle, etc.

Keywords Korovkin approximation theorem · Positive linear operators · Asymptotic evaluations for univariate and multivariate positive operators · Iterates · Cesàro and Volterra operator

Mathematics Subject Classification 41A35 · 41A36 · 41A25 · 41A65

1 Introduction, Notation, and Background

The study of the limit behavior of the iterates of Bernstein's operators and other classes of positive linear operators has been considered by many mathematicians. Without any claim of completeness, we mention [7–9]. In this paper, we prove a general convergence result for some sequences of positive linear operators, see Theorem 1 and Corollary 2. Moreover, we obtain the full asymptotic evaluation for some univariate operators, see Theorem 2, and as application of this result, we deduce the full asymptotic evaluation for the Cesàro and Volterra type operators, Corollaries 5 and 7. We continue by showing other results of the same kind for multivariate positive

Communicated by Wolfgang Dahmen.

✉ Dumitru Popa
dpopa@univ-ovidius.ro

1 Department of Mathematics, Ovidius University of Constanta, Bd. Mamaia 124, 900527 Constanta, Romania

linear operators, see Theorems 3 and 4. We then apply these general results to obtain the asymptotic evaluations for various kinds of Cesàro and Volterra type multivariate operators, see for example Corollaries 12, 13, 17, 19, 22.

We now fix some notation and terminology used in this paper. Let T be a compact metric space and X a real Banach space. We denote by $C(T,X)$ the real Banach space of the all X-valued continuous functions on T equipped with the uniform norm, $\|f\| = \sup_{t\in T}\|f(t)\|$ and $C(T) = C(T,\mathbb{R})$. For every $\varphi \in C(T)$, $x \in X$, we define $\varphi \otimes x : T \to X$ by $(\varphi \otimes x)(t) := \varphi(t)x$, $\forall t \in T$ and write $C(T) \otimes X = \left\{ \sum_{i=1}^{n} \varphi_i \otimes x_i \mid \varphi_i \in C(T), x_i \in X, i = 1, \ldots, n, n \in \mathbb{N} \right\}$ to denote their tensor product, see [4, page 20] or [12, page 11]. We will use that $C(T) \otimes X$ is dense in $C(T,X)$ and that, by a result of Grothendieck, $C(T,X) = C(T)\widehat{\otimes}_\varepsilon X$, the completion of $C(T) \otimes X$ with respect to the injective tensor norm, see [4, page 48], [5, Example 6 pages 224-225], or [12, pages 49-50]. Let also $V : C(T) \to C(K)$ be a bounded linear operator and X a real Banach space. We define $V_X : C(T,X) \to C(K,X)$ by $V_X(\varphi \otimes x) = V(\varphi) \otimes x$, $\forall \varphi \in C(T)$, $\forall x \in X$, and then extend by the linearity and continuity. Since $C(T,X) = C(T)\widehat{\otimes}_\varepsilon X$, by the general theory, $V_X = V\widehat{\otimes}_\varepsilon I_X$, the injective tensor product ($I_X : X \to X$ is the identity operator of X, that is, $I_X(x) = x$), and thus $\|V_X\| = \|V\|\,\|I_X\| = \|V\|$, see again [4, Proposition 4.1, page 46], [5, page 228], or [12, Proposition 3.2, page 47]. Hereafter, we call the operator V_X the vector extension of the bounded linear operator V. For example, if $\mathcal{C} : C[0,1] \to C[0,1]$, $\mathcal{C}\varphi(t) = \int_0^1 \varphi(tu)\,du = \begin{cases} \frac{1}{t}\int_0^t \varphi(u)\,du, & t \neq 0 \\ \varphi(0), & t = 0 \end{cases}$, is the Cesàro operator and X is a real Banach space, then its vector extension $\mathcal{C}_X : C([0,1],X) \to C([0,1],X)$ is defined by $\mathcal{C}_X f(t) = \int_0^1 f(tu)\,du = \begin{cases} \frac{1}{t}\int_0^t f(u)\,du, & t \neq 0 \\ f(0), & t = 0 \end{cases}$, $f \in C([0,1],X)$; similarly, if $\mathcal{V} : C[0,1] \to C[0,1]$, $\mathcal{V}\varphi(t) = \int_0^t \varphi(u)\,du = t\int_0^1 \varphi(tu)\,du$, is the Volterra operator and X is a real Banach space, then its vector extension $\mathcal{V}_X : C([0,1],X) \to C([0,1],X)$ is defined by $\mathcal{V}_X f(t) = \int_0^t f(u)\,du = t\int_0^1 f(tu)\,du$, $f \in C([0,1],X)$. Since the applications of our general results are to the iterates of positive linear operators, we recall that, as is usual, if $V : C(T) \to C(T)$ is a bounded linear operator, we write V^n to denote the composition $\underbrace{V \circ V \circ \cdots \circ V}_{n\text{-times}}$ and if $V_X : C(T,X) \to C(T,X)$ is its vector extension, then V_X^n denotes the composition $\underbrace{V_X \circ V_X \circ \cdots \circ V_X}_{n\text{-times}}$. A function $f \in C(T)$ is called positive, and we write, as usual, $f \geq 0$ if $f(t) \geq 0$, $\forall t \in T$, and also if $f, g \in C(T)$, the notation $f \leq g$ means $g - f \geq 0$. An operator $V : C(T) \to C(K)$ is called positive if $f \geq 0$ implies $V(f) \geq 0$. We will use the simple result that a positive linear operator $V : C(T) \to C(K)$ is increasing; that is, if $f \leq g$, then $V(f) \leq V(g)$, and that $|V(f)| \leq V(|f|)$. If A is a set, we write $\mathbf{1}$ to denote the constant function $\mathbf{1} : A \to \mathbb{R}$, $\mathbf{1}(x) = 1$, and we write as is usual $e_j : [0,1] \to \mathbb{R}$, $e_j(x) = x^j$, $j \in \mathbb{N}\cup\{0\}$. If $k \in \mathbb{N}$, $k \geq 2$, we consider $p_i : \mathbb{R}^k \to \mathbb{R}$, $p_i(t_1, \ldots, t_k) = t_i$, $i = 1, \ldots, k$, the canonical projections. We will use that if $V : C(T) \to C(K)$ is a positive linear operator, then $\|V\| = \|V(\mathbf{1})\|$. If $\varphi \in C(T)$, $f \in C(T,X)$, we define $\varphi \otimes f : T \to X$

by $(\varphi \otimes f)(t) = \varphi(t) f(t)$, $\forall t \in T$. Let us note the following obvious equality: $\psi \otimes (\varphi \otimes x) = (\psi\varphi) \otimes x$, $\psi, \varphi \in C(T)$, $x \in X$. We need the following:

Remark 1 Let $V : C(T) \to C(K)$ be a bounded linear operator, $\psi \in C(T)$, X a real Banach space, and define $U : C(T) \to C(K)$ by $U(\varphi) = V(\psi\varphi)$. Then $U_X(f) = V_X(\psi \otimes f)$, $\forall f \in C(T, X)$.

Proof It is obvious that U is bounded linear; hence U_X is well defined. Let us define $L : C(T, X) \to C(K, X)$, $L(f) = V_X(\psi \otimes f)$. If $\varphi \in C(T)$, $x \in X$, we have $L(\varphi \otimes x) = V_X(\psi \otimes (\varphi \otimes x)) = V_X((\psi\varphi) \otimes x) = V(\psi\varphi) \otimes x = U(\varphi) \otimes x = U_X(\varphi \otimes x)$. By linearity, we deduce that $L = U_X$ on $C(T) \otimes X$, and since $C(T) \otimes X$ is dense in $C(T, X)$, by the continuity, $L = U_X$ on $C(T, X)$, which ends the proof. □

All notation and concepts concerning approximation theory used and not defined are standard, see [1], and the notation and concepts from Banach space theory are also standard, see [4], or [12].

2 The Convergence

In this section, T, K are compact metric spaces and X is a real Banach space. We need the following technical result, see also [7, proof of Theorem 1], [10, Lemma 1], [11, Lemma 1].

Lemma 1 *Let $a \in T$ be an accumulation point of T and $\varphi : T \to \mathbb{R}$ a continuous function such that $\varphi(t) > 0$, $\forall t \in T - \{a\}$.*

(i) *If $g : T \to \mathbb{R}$ is a continuous function such that $g(a) = 0$, then $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$, such that $|g(t)| < \varepsilon + \delta_\varepsilon \varphi(t)$, $\forall t \in T$.*
(ii) *If $f : T \to \mathbb{R}$ is a continuous function, then $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$, such that $|f(t) - f(a)| < \varepsilon + \delta_\varepsilon \varphi(t)$, $\forall t \in T$.*

Proof Since a is an accumulation point of T and T is a metric space, there exists a sequence $(t_n)_{n \in \mathbb{N}} \subset T - \{a\}$ such that $t_n \to a$. Then, by the continuity of φ, $\varphi(t_n) \to \varphi(a)$, and since by the hypothesis $\varphi(t_n) > 0$, $\forall n \in \mathbb{N}$, we deduce that $\varphi(a) \geq 0$.

(i) Let us suppose that (i) is not true. This means that $\exists \varepsilon_0 > 0$ such that $\forall \delta > 0$ there exist $t_\delta \in T$ such that $|g(t_\delta)| \geq \varepsilon_0 + \delta\varphi(t_\delta)$. In particular, for $\delta = n \in \mathbb{N}$, there exist $t_n \in T$ such that $|g(t_n)| \geq \varepsilon_0 + n\varphi(t_n)$, $\forall n \in \mathbb{N}$. Since T is compact, there exist $t \in T$ and a subsequence $(k_n)_{n \in \mathbb{N}}$ such that $t_{k_n} \to t$. We can have two cases: The first case is $t = a$, that is, $t_{k_n} \to a$. Since $\varphi(t) \geq 0$, $\forall t \in T$, we deduce that $\left|g\left(t_{k_n}\right)\right| \geq \varepsilon_0$, $\forall n \in \mathbb{N}$, and passing to the limit and using that $g(a) = 0$, we obtain $0 \geq \varepsilon_0$, which is impossible. The second case is $t \neq a$, that is, $t \in T - \{a\}$. Now note that $|g(t_n)| \leq \|g\|$, and thus $\|g\| \geq \varepsilon_0 + k_n\varphi\left(t_{k_n}\right)$, $\forall n \in \mathbb{N}$, or $0 \leq \varphi\left(t_{k_n}\right) \leq \frac{\|g\| - \varepsilon_0}{k_n}$, $\forall n \in \mathbb{N}$. Passing to the limit, we obtain $\varphi\left(t_{k_n}\right) \to 0$, and since φ is continuous, $\varphi(t) = 0$. But this is impossible since $t \in T - \{a\}$, and by the hypothesis, $\varphi(v) > 0$ for every $v \in T - \{a\}$, in particular $\varphi(t) > 0$.

(ii) Apply (i) to the function $g : T \to \mathbb{R}$, $g(t) = f(t) - f(a)$.

□

Corollary 1 *Let $a \in T$ be an accumulation point of T and $\varphi : T \to \mathbb{R}$ a continuous function such that $\varphi(t) > 0$, $\forall t \in T - \{a\}$.*

(i) *If $g : T \to \mathbb{R}$ is a continuous function such that $g(a) = 0$, then $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that for any positive linear operator $V : C(T) \to C(K)$, we have $\|V(g)\| \le \varepsilon \|V(\mathbf{1})\| + \delta_\varepsilon \|V(\varphi)\|$.*

(ii) *If $f : T \to \mathbb{R}$ is a continuous function, then $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that for any positive linear operator $V : C(T) \to C(K)$, we have $\|V(f) - f(a) V(\mathbf{1})\| \le \varepsilon \|V(\mathbf{1})\| + \delta_\varepsilon \|V(\varphi)\|$.*

(iii) *If $V : C(T) \to C(K)$ is a positive linear operator such that $V(\varphi) = 0$, then $V(f) = f(a) V(\mathbf{1})$, $\forall f \in C(T)$.*

Proof (i) Let $\varepsilon > 0$. From Lemma 1(i) there exists $\delta_\varepsilon > 0$ such that $|g| < \varepsilon \cdot \mathbf{1} + \delta_\varepsilon \varphi$. Since V is positive linear, we obtain $|V(g)| \le V(|g|) \le \varepsilon V(\mathbf{1}) + \delta_\varepsilon V(\varphi)$ in $C(K)$; that is, $|V(g)(k)| \le \varepsilon V(\mathbf{1})(k) + \delta_\varepsilon V(\varphi)(k) \le \varepsilon \|V(\mathbf{1})\| + \delta_\varepsilon \|V(\varphi)\|$, $\forall k \in K$, and thus $\|V(g)\| \le \varepsilon \|V(\mathbf{1})\| + \delta_\varepsilon \|V(\varphi)\|$.

(ii) Apply (i) to the function $g : T \to \mathbb{R}$, $g(t) = f(t) - f(a)$.

(iii) Let $f \in C(T)$. For every $\varepsilon > 0$, by (ii) $\exists \delta_\varepsilon > 0$ such that

$$\|V(f) - f(a) V(\mathbf{1})\| \le \varepsilon \|V(\mathbf{1})\| + \delta_\varepsilon \|V(\varphi)\|,$$

and since $V(\varphi) = 0$, we obtain $\|V(f) - f(a) V(\mathbf{1})\| \le \varepsilon \|V(\mathbf{1})\|$. Passing to the limit for $\varepsilon \to 0$, we get $\|V(f) - f(a) V(\mathbf{1})\| \le 0$, $V(f) - f(a) V(\mathbf{1}) = 0$.

□

The next result is a large extension of Theorem 1 in [7].

Theorem 1 *Let $a \in T$ be an accumulation point of T and $\varphi : T \to \mathbb{R}$ a continuous function such that $\varphi(t) > 0$, $\forall t \in T - \{a\}$. Let $V_n : C(T) \to C(K)$ be a sequence of positive linear operators such that the sequence $(V_n(\mathbf{1}))_{n \in \mathbb{N}}$ is (norm) bounded in $C(K)$ and $\lim_{n\to\infty} V_n(\varphi) = 0$ uniformly. Then:*

(i) *for every $g \in C(T)$ with $g(a) = 0$, we have $\lim_{n\to\infty} V_n(g) = 0$ uniformly.*

(ii) *for every $f \in C(T)$, we have $\lim_{n\to\infty} [V_n(f) - f(a) V_n(\mathbf{1})] = 0$ uniformly.*

Proof (i) Let $\varepsilon > 0$. From Corollary 1(i) there exists $\delta_\varepsilon > 0$ such that

$$\|V_n(g)\| \le \varepsilon \|V_n(\mathbf{1})\| + \delta_\varepsilon \|V_n(\varphi)\|, \forall n \in \mathbb{N}.$$

Since $(V_n(\mathbf{1}))_{n \in \mathbb{N}}$ is bounded in $C(K)$, there exists $M > 0$ such that $\|V_n(\mathbf{1})\| \le M$, $\forall n \in \mathbb{N}$. Also from $\lim_{n\to\infty} V_n(\varphi) = 0$ uniformly, $\exists n_\varepsilon \in \mathbb{N}$ such that $\|V_n(\varphi)\| \le \frac{\varepsilon}{\delta_\varepsilon}$, $\forall n \ge n_\varepsilon$. We deduce that $\|V_n(g)\| \le \varepsilon(M+1)$, $\forall n \ge n_\varepsilon$; that is, $\lim_{n\to\infty} V_n(g) = 0$ uniformly.

(ii) Let $f \in C(T)$. Then $g = f - f(a) \cdot \mathbf{1} \in C(T)$ and $g(a) = 0$. We apply now (i). ◻

We prove now that the result in Theorem 1 can be extended to the vector case.

Corollary 2 *Let $a \in T$ be an accumulation point of T and $\varphi : T \to \mathbb{R}$ a continuous function such that $\varphi(t) > 0$, $\forall t \in T - \{a\}$. Let $V_n : C(T) \to C(K)$ be a sequence of positive linear operators such that the sequence $(V_n(\mathbf{1}))_{n\in\mathbb{N}}$ is (norm) bounded in $C(K)$, $\lim_{n\to\infty} V_n(\varphi) = 0$ uniformly and $V_{X,n} : C([0,1], X) \to C(K, X)$ their vector extensions. Then for every $f \in C(T, X)$, we have $\lim_{n\to\infty} \left[V_{X,n}(f) - V_n(\mathbf{1}) \otimes f(a)\right] = 0$ uniformly.*

Proof Let $\varphi = \sum_{i=1}^{k} \varphi_i \otimes x_i \in C(T) \otimes X$. Let also $n \in \mathbb{N}$. Then $V_n(\mathbf{1}) \otimes \varphi(a) = V_n(\mathbf{1}) \otimes \left(\sum_{i=1}^{k} \varphi_i(a) x_i\right) = \sum_{i=1}^{k} \varphi_i(a) V_n(\mathbf{1}) \otimes x_i$, $V_{X,n}(\varphi) = \sum_{i=1}^{k} V_{X,n}(\varphi_i \otimes x_i) = \sum_{i=1}^{k} V_n(\varphi_i) \otimes x_i$. We get $V_{X,n}(\varphi) - V_n(\mathbf{1}) \otimes \varphi(a) = \sum_{i=1}^{k} [V_n(\varphi_i) - \varphi_i(a) V_n(\mathbf{1})] \otimes x_i$ and

$$\left\| V_{X,n}(\varphi) - V_n(\mathbf{1}) \otimes \varphi(a) \right\| \leq \sum_{i=1}^{k} \| V_n(\varphi_i) - \varphi_i(a) V_n(\mathbf{1}) \| \, \| x_i \| .$$

From Theorem 1 and the above inequality, we simply deduce that

$$\lim_{n\to\infty} \left[V_{X,n}(\varphi) - V_n(\mathbf{1}) \otimes \varphi(a) \right] = 0 \text{ uniformly}$$

Let $f \in C(T, X)$ and $\varphi = \sum_{i=1}^{k} \varphi_i \otimes x_i$. Then for every $n \in \mathbb{N}$,

$$\left\| V_{X,n}(f) - V_{X,n}(\varphi) \right\| \leq \left\| V_{X,n} \right\| \| f - \varphi \| = \| V_n \| \, \| f - \varphi \| = \| V_n(\mathbf{1}) \| \, \| f - \varphi \| .$$

Let us note that

$$\begin{aligned}
&\left\| V_{X,n}(f) - V_n(\mathbf{1}) \otimes f(a) \right\| \\
&\quad \leq \left\| V_{X,n}(f) - V_{X,n}(\varphi) \right\| + \left\| V_{X,n}(\varphi) - V_n(\mathbf{1}) \otimes \varphi(a) \right\| \\
&\qquad + \| V_n(\mathbf{1}) \otimes \varphi(a) - V_n(\mathbf{1}) \otimes f(a) \| \\
&\quad \leq \| V_n(\mathbf{1}) \| \, \| f - \varphi \| + \left\| V_{X,n}(\varphi) - V_n(\mathbf{1}) \otimes \varphi(a) \right\| + \| V_n(\mathbf{1}) \| \, \| \varphi(a) - f(a) \| \\
&\quad \leq 2 \| V_n(\mathbf{1}) \| \, \| f - \varphi \| + \left\| V_{X,n}(\varphi) - V_n(\mathbf{1}) \otimes \varphi(a) \right\| .
\end{aligned}$$

There exist $M > 0$ such that $\| V_n(\mathbf{1}) \| \leq M$, $\forall n \in \mathbb{N}$. Now let $f \in C(T, X)$ and $\varepsilon > 0$. Then there exists $\varphi \in C(T) \otimes X$ such that $\| f - \varphi \| \leq \frac{\varepsilon}{4M}$. We have

$$\left\| V_{X,n}(f) - V_n(\mathbf{1}) \otimes f(a) \right\| \leq \frac{\varepsilon}{2} + \left\| V_{X,n}(\varphi) - V_n(\mathbf{1}) \otimes \varphi(a) \right\| , \ \forall n \in \mathbb{N}.$$

Since, by the first part, $\lim_{n\to\infty}\left[V_{X,n}(\varphi)-V_n(\mathbf{1})\otimes\varphi(a)\right]=0$ uniformly, there exists $n_\varepsilon\in\mathbb{N}$ such that $\forall n\geq n_\varepsilon$, we have $\left\|V_{X,n}(\varphi)-V_n(\mathbf{1})\otimes\varphi(a)\right\|\leq\frac{\varepsilon}{2}$. We deduce then that $\forall n\geq n_\varepsilon$, we have $\left\|V_{X,n}(f)-V_n(\mathbf{1})\otimes f(a)\right\|\leq\varepsilon$. We are done. □

To end this section we give two concrete examples.

Corollary 3 *Let* $\mathcal{A}_X,\mathcal{B}_X:C\left([0,1]^2,X\right)\to C\left([0,1]^2,X\right)$ *be the operators defined by* $\mathcal{A}_X(f)(t_1,t_2)=t_1\iint_{[0,1]^2}f(t_1x,t_2y)\,\mathrm{d}x\mathrm{d}y$, $\mathcal{B}_X(f)(t_1,t_2)=t_2\iint_{[0,1]^2}f(t_1x,t_2y)\mathrm{d}x\mathrm{d}y$. *Then for every* $f\in C\left([0,1]^2,X\right)$, *we have* $\lim_{n\to\infty}\left(n!\mathcal{A}_X^n(f)(t_1,t_2)-t_1^nf(0,0)\right)=0$, $\lim_{n\to\infty}\left(n!\mathcal{B}_X^n(f)(t_1,t_2)-t_2^nf(0,0)\right)=0$ *uniformly with respect to* $(t_1,t_2)\in[0,1]^2$.

Proof By induction we can prove that for every $n\in\mathbb{N}$,

$$\mathcal{A}^n(p_1)=\frac{p_1^{n+1}}{(n+1)!},\quad\mathcal{A}^n(p_2)=\frac{p_1^{n-1}P}{2^nn!},\quad\mathcal{A}(\mathbf{1})=p_1,\quad\mathcal{A}^n(\mathbf{1})=\mathcal{A}^{n-1}(p_1)=\frac{p_1^n}{n!},$$

where $P(t_1,t_2)=t_1t_2$. Now let us observe that the function $s:[0,1]^2\to[0,\infty)$, $s(t_1,t_2)=t_1+t_2$ is continuous and $s(t_1,t_2)>0$, $\forall(t_1,t_2)\in[0,1]^2-\{(0,0)\}$. Then $\mathcal{A}^n(s)=\mathcal{A}^n(p_1)+\mathcal{A}^n(p_2)=\frac{p_1^{n+1}}{(n+1)!}+\frac{p_1^{n-1}P}{2^nn!}$ and $\lim_{n\to\infty}\frac{\|\mathcal{A}^n(s)\|}{\|\mathcal{A}^n(\mathbf{1})\|}=\lim_{n\to\infty}\frac{\frac{1}{2^nn!}+\frac{1}{(n+1)!}}{\frac{1}{n!}}=0$. From Corollary 2 applied for $V_n=\frac{\mathcal{A}^n}{\|\mathcal{A}^n(\mathbf{1})\|}$ and using that $(\mathcal{A}^n)_X=\mathcal{A}_X^n$, we deduce that $\lim_{n\to\infty}\left(\frac{\mathcal{A}_X^n(f)(t_1,t_2)}{\|\mathcal{A}^n(\mathbf{1})\|}-\frac{\mathcal{A}^n(\mathbf{1})(t_1,t_2)}{\|\mathcal{A}^n(\mathbf{1})\|}f(0,0)\right)=0$; that is, $\lim_{n\to\infty}\left(n!\mathcal{A}_X^n(f)(t_1,t_2)-t_1^nf(0,0)\right)=0$ uniformly with respect to $(t_1,t_2)\in[0,1]^2$. Similarly, for every $n\in\mathbb{N}$, we have

$$\mathcal{B}^n(p_1)=\frac{p_2^{n-1}P}{2^nn!},\mathcal{B}^n(p_2)=\frac{p_2^{n+1}}{(n+1)!},\mathcal{B}(\mathbf{1})=p_2,\mathcal{B}^n(\mathbf{1})=\mathcal{B}^{n-1}(p_2)=\frac{p_2^n}{n!}$$

and $\lim_{n\to\infty}\frac{\|\mathcal{B}^n(s)\|}{\|\mathcal{B}^n(\mathbf{1})\|}=0$. We apply now Corollary 2 for $V_n=\frac{\mathcal{B}^n}{\|\mathcal{B}^n(\mathbf{1})\|}$. □

3 The Full Asymptotic Evaluation for One Variable

In this section, K is a compact metric space and X is a real Banach space. In the next result we give the full asymptotic evaluation for some sequences of positive linear operators. It is a natural completion of Corollary 2.

Theorem 2 *Let* $V_n:C[0,1]\to C(K)$ *be a sequence of positive linear operators,* $V_{X,n}:C([0,1],X)\to C(K,X)$ *their vector extensions, k a natural number such that:*

(i) $V_n(e_k)\neq0$, $\forall n\in\mathbb{N}$;

(ii) *there exists* $\varphi_k \in C[0,1]$ *with* $\varphi_k(t) > 0$, $\forall t \in [0,1] - \{0\}$ *and such that* $\lim\limits_{n\to\infty} \frac{V_n(e_k \varphi_k)}{\|V_n(e_k)\|} = 0$ *uniformly. Then for every function* $f : [0,1] \to X$ *that is* k *-times differentiable at* 0, *we have*

$$\lim_{n\to\infty} \frac{V_{X,n}(f)(t) - \sum\limits_{i=0}^{k} \frac{V_n(e_i)(t)}{i!} f^{(i)}(0)}{\|V_n(e_k)\|} = 0 \text{ uniformly with respect to } t \in [0,1].$$

Moreover, if $\lim\limits_{n\to\infty} \frac{V_n(e_k)}{\|V_n(e_k)\|} = u_k$ *uniformly, then*

$$\lim_{n\to\infty} \frac{V_{X,n}(f)(t) - \sum\limits_{i=0}^{k-1} \frac{V_n(e_i)(t)}{i!} f^{(i)}(0)}{\|V_n(e_k)\|} = \frac{u_k(t)}{k!} f^{(k)}(0)$$

uniformly with respect to $t \in [0,1]$.

Proof Since f is k-times differentiable at 0,

$$\lim_{t\to 0} \frac{f(t) - \sum\limits_{i=0}^{k-1} \frac{t^i}{i!} f^{(i)}(0)}{t^k} = \frac{1}{k!} \cdot f^{(k)}(0),$$

see [2, Theorem 1, page 21]. Thus the function $g : [0,1] \to X$,

$$g(t) = \begin{cases} \frac{f(t) - \sum\limits_{i=0}^{k} \frac{t^i}{i!} f^{(i)}(0)}{t^k}, & t \neq 0 \\ 0, & t = 0 \end{cases},$$

is continuous, and for all $t \in [0,1]$, the following relation holds: $f(t) = \sum\limits_{i=0}^{k} \frac{t^i}{i!} f^{(i)}(0) + t^k g(t)$. This means that $f = \sum\limits_{i=0}^{k} \frac{1}{i!} e_i \otimes f^{(i)}(0) + e_k \otimes g$ in $C([0,1], X)$. Let $n \in \mathbb{N}$. Since all $V_{X,n}$ are linear, we have

$$\begin{aligned} V_{X,n}(f) &= \sum_{i=0}^{k} \frac{1}{i!} V_{X,n}\left(e_i \otimes f^{(i)}(0)\right) + V_{X,n}(e_k \otimes g) \\ &= \sum_{i=0}^{k} \frac{1}{i!} V_n(e_i) \otimes f^{(i)}(0) + V_{X,n}(e_k \otimes g) \text{ in } C(K, X), \end{aligned}$$

and thus

$$\frac{\left\| V_{X,n}(f) - \sum\limits_{i=0}^{k} \frac{1}{i!} V_n(e_i) \otimes f^{(i)}(0) \right\|}{\|V_n(e_k)\|} = \frac{\|V_{X,n}(e_k \otimes g)\|}{\|V_n(e_k)\|}. \tag{1}$$

Let $U_n : C[0,1] \to C(K)$ be the operator defined by $U_n(\varphi) = \frac{V_n(e_k \cdot \varphi)}{\|V_n(e_k)\|}$ (see the hypothesis (i)). Then $\|U_n(\mathbf{1})\| = 1, \forall n \in \mathbb{N}$. Moreover, by the hypothesis (ii) $\lim_{n\to\infty} \frac{\|V_n(e_k\varphi_k)\|}{\|V_n(e_k)\|} = 0$; that is, $\lim_{n\to\infty} U_n(\varphi_k) = 0$ uniformly. From Corollary 2 it follows that for every $f \in C([0,1], X)$, $\lim_{n\to\infty} \left[U_{X,n}(f) - U_n(\mathbf{1}) \otimes f(0)\right] = 0$ uniformly. In particular, $\lim_{n\to\infty} \left[U_{X,n}(g) - U_n(\mathbf{1}) \otimes g(0)\right] = 0$ uniformly; that is, since $g(0) = 0$, $\lim_{n\to\infty} U_{X,n}(g) = 0$ uniformly. By Remark 1, this is equivalent to $\lim_{n\to\infty} \frac{\|V_{X,n}(e_k \otimes g)\|}{\|V_n(e_k)\|} = 0$, which, by (1), ends the proof. □

By taking $\varphi_k = e_j$ in Theorem 2, we obtain:

Corollary 4 *Let $V_n : C[0,1] \to C(K)$ be a sequence of positive linear operators, $V_{X,n} : C([0,1], X) \to C(K, X)$ their vector extensions, k a natural number such that:*

(i) $V_n(e_k) \neq 0, \forall n \in \mathbb{N}$;

(ii) *there exists $j \in \mathbb{N}$ such that $\lim_{n\to\infty} \frac{\|V_n(e_{k+j})\|}{\|V_n(e_k)\|} = 0$. Then for every function $f : [0,1] \to X$ that is k-times differentiable at 0, we have*

$$\lim_{n\to\infty} \frac{V_{X,n}(f)(t) - \sum_{i=0}^{k} \frac{V_n(e_i)(t)}{i!} f^{(i)}(0)}{\|V_n(e_k)\|} = 0 \text{ uniformly with respect to } t \in [0,1].$$

Moreover, if $\lim_{n\to\infty} \frac{V_n(e_k)}{\|V_n(e_k)\|} = u_k$ uniformly, then

$$\lim_{n\to\infty} \frac{V_{X,n}(f)(t) - \sum_{i=0}^{k-1} \frac{V_n(e_i)(t)}{i!} f^{(i)}(0)}{\|V_n(e_k)\|} = \frac{u_k(t)}{k!} \cdot f^{(k)}(0)$$

uniformly with respect to $t \in [0,1]$.

4 The Full Asymptotic Evaluations for the Cesàro and Volterra Type Operators

In this section X is a real Banach space. As an application of Corollary 4, we indicate the full asymptotic evaluations for the Cesàro and Volterra type operators. We begin with a result that is a large extension of Theorem 3 in [6].

Corollary 5 *Let $\varphi : [0,1] \to [0,\infty)$ be a continuous non-null function, $\mathcal{C}_{X,\varphi} : C([0,1], X) \to C([0,1], X)$ the Cesàro type operator defined by*

$$\mathcal{C}_{X,\varphi} f(t) = \int_0^1 \varphi(s) f(st)\, ds,$$

and k a natural number. Then for every function $f : [0,1] \to X$ that is k-times differentiable at 0*, we have*

$$\lim_{n\to\infty} \frac{\mathcal{C}^n_{X,\varphi}(f)(t) - \sum_{i=0}^{k-1} \left(\int_0^1 s^i \varphi(s)\,\mathrm{d}s\right)^n \frac{t^i f^{(i)}(0)}{i!}}{\left(\int_0^1 s^k \varphi(s)\,\mathrm{d}s\right)^n} = \frac{t^k}{k!} f^{(k)}(0)$$

uniformly with respect to $t \in [0,1]$.

Proof Let $i \in \mathbb{N} \cup \{0\}$ and define $\lambda_i = \int_0^1 s^i \varphi(s)\,\mathrm{d}s$. Let us note that $\lambda_{i+1} < \lambda_i$, $\forall i \in \mathbb{N} \cup \{0\}$. Indeed, if $\lambda_{i+1} \geq \lambda_i$, that is, $\int_0^1 s^{i+1}\varphi(s)\,\mathrm{d}s \geq \int_0^1 s^i \varphi(s)\,\mathrm{d}s$, then $\int_0^1 s^i(1-s)\varphi(s)\,\mathrm{d}s \leq 0$. Since $s^i(1-s)\varphi(s) \geq 0, \forall s \in [0,1]$ $(\varphi(s) \geq 0)$, we have $\int_0^1 s^i(1-s)\varphi(s)\,\mathrm{d}s \geq 0$; that is, $\int_0^1 s^i(1-s)\varphi(s)\,\mathrm{d}s = 0$. A well-known property assures us that $s^i(1-s)\varphi(s) = 0$, $\forall s \in [0,1]$, whence $\varphi(s) = 0$, $\forall s \in (0,1)$. By continuity $\varphi(s) = 0$, $\forall s \in [0,1]$, which is impossible. We have $\mathcal{C}_\varphi(e_i) = \lambda_i e_i$ and by induction on n, $\mathcal{C}^n_\varphi(e_i) = \lambda_i^n e_i$, $\forall n \in \mathbb{N}$. Then $\frac{\left\|\mathcal{C}^n_\varphi(e_{i+1})\right\|}{\left\|\mathcal{C}^n_\varphi(e_i)\right\|} = \left(\frac{\lambda_{i+1}}{\lambda_i}\right)^n$ and thus $\lim_{n\to\infty} \frac{\left\|\mathcal{C}^n_\varphi(e_{i+1})\right\|}{\left\|\mathcal{C}^n_\varphi(e_i)\right\|} = 0$. Also $\frac{\mathcal{C}^n_\varphi(e_i)}{\left\|\mathcal{C}^n_\varphi(e_i)\right\|} = e_i$, $\forall n \in \mathbb{N}$. From Corollary 4, we have

$$\lim_{n\to\infty} \frac{\mathcal{C}^n_{X,\varphi}(f) - \sum_{i=0}^{k-1} \frac{1}{i!}\mathcal{C}^n_\varphi(e_i) \otimes f^{(i)}(0)}{\left\|\mathcal{C}^n_\varphi(e_k)\right\|} = \frac{1}{k!} e_k \otimes f^{(k)}(0) \text{ uniformly,}$$

which after simple calculations gives us the statement. □

In the case of the Cesàro operator, that is, $\varphi = e_0$ in Corollary 5, we get:

Corollary 6 *Let $\mathcal{C}_X : C([0,1], X) \to C([0,1], X)$ be the Cesàro operator*

$$\mathcal{C}_X(f)(t) = \begin{cases} \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s, & t \neq 0 \\ f(0), & t = 0 \end{cases} = \int_0^1 f(st)\,\mathrm{d}s$$

and k be a natural number. Then for every function $f : [0,1] \to X$ that is k-times differentiable at 0*, we have*

$$\lim_{n\to\infty} (k+1)^n \left(\mathcal{C}^n_X(f)(t) - \sum_{i=0}^{k-1} \frac{t^i f^{(i)}(0)}{(i+1)^n\, i!}\right) = \frac{t^k f^{(k)}(0)}{k!}$$

uniformly with respect to $t \in [0,1]$.

In the case of the Volterra type operators, we have the following asymptotic evaluation.

Corollary 7 *Let $\varphi : [0,1] \to [0,\infty)$ be a continuous non-null function, $\mathcal{V}_{X,\varphi} : C([0,1], X) \to C([0,1], X)$ the Volterra type operator defined by*

$$\mathcal{V}_{X,\varphi} f(t) = t \int_0^1 \varphi(s) f(st)\, \mathrm{d}s,$$

and k a natural number. Then for every function $f : [0,1] \to X$ that is k-times differentiable at 0, we have

$$\lim_{n\to\infty} \frac{\mathcal{V}^n_{X,\varphi}(f)(t) - \sum\limits_{i=0}^{k} \left(\int_0^1 s^i \varphi(s)\, \mathrm{d}s\right) \cdots \left(\int_0^1 s^{i+n-1} \varphi(s)\, \mathrm{d}s\right) \frac{t^{n+i}}{i!} f^{(i)}(0)}{\left(\int_0^1 s^k \varphi(s)\, \mathrm{d}s\right) \cdots \left(\int_0^1 s^{k+n-1}\varphi(s)\, \mathrm{d}s\right)} = 0$$

uniformly with respect to $t \in [0,1]$ and thus

$$\lim_{n\to\infty} \frac{\mathcal{V}^n_{X,\varphi}(f)(t) - \sum\limits_{i=0}^{k-1} \left(\int_0^1 s^i \varphi(s)\, \mathrm{d}s\right) \cdots \left(\int_0^1 s^{i+n-1} \varphi(s)\, \mathrm{d}s\right) \frac{t^{n+i}}{i!} f^{(i)}(0)}{\left(\int_0^1 s^k \varphi(s)\, \mathrm{d}s\right) \cdots \left(\int_0^1 s^{k+n-1}\varphi(s)\, \mathrm{d}s\right)}$$
$$= \begin{cases} 0 & \text{if} t \neq 1, \\ \frac{1}{k!} f^{(k)}(0) & \text{if} t = 1. \end{cases}$$

Proof Let $i \in \mathbb{N} \cup \{0\}$ and define $\lambda_i = \int_0^1 s^i \varphi(s)\, \mathrm{d}s$. We have shown in Corollary 5 that $\lambda_{i+1} < \lambda_i$, $\forall i \in \mathbb{N} \cup \{0\}$. We have $\mathcal{V}_\varphi(e_i) = \lambda_i e_{i+1}$ and by induction on n, $\mathcal{V}^n_\varphi(e_i) = \lambda_i \lambda_{i+1} \cdots \lambda_{i+n-1} e_{n+i}$, $\forall n \in \mathbb{N}$. Then $\frac{\left\|\mathcal{V}^n_\varphi(e_{i+1})\right\|}{\left\|\mathcal{V}^n_\varphi(e_i)\right\|} = \left(\frac{\lambda_{i+1}\cdots\lambda_{i+n}}{\lambda_i\lambda_{i+1}\cdots\lambda_{i+n-1}}\right)^n = \left(\frac{\lambda_{i+n}}{\lambda_i}\right)^n \leq \left(\frac{\lambda_{i+1}}{\lambda_i}\right)^n$ and thus $\lim\limits_{n\to\infty} \frac{\left\|\mathcal{V}^n_\varphi(e_{i+1})\right\|}{\left\|\mathcal{V}^n_\varphi(e_i)\right\|} = 0$. From Corollary 4, we have

$$\lim_{n\to\infty} \frac{\mathcal{V}^n_{X,\varphi}(f) - \sum\limits_{i=0}^{k} \frac{1}{i!} \mathcal{V}^n_\varphi(e_i) \otimes f^{(i)}(0)}{\left\|\mathcal{V}^n_\varphi(e_k)\right\|} = 0 \text{ uniformly},$$

which after simple calculations gives us the statement. The second part follows from the first, the equality $\frac{\mathcal{V}^n_\varphi(e_k)}{\left\|\mathcal{V}^n_\varphi(e_k)\right\|}(t) = t^{n+k}$, $\forall t \in [0,1]$, $\forall n \in \mathbb{N}$, and the limit $\lim\limits_{n\to\infty} t^{n+k} = \begin{cases} 0 & \text{if} t \neq 1, \\ 1 & \text{if} t = 1. \end{cases}$ □

In the case of the Volterra operator, that is, $\varphi = e_0$ in Corollary 7, we get:

Corollary 8 *Let $\mathcal{V}_X : C([0,1], X) \to C([0,1], X)$ be the Volterra operator*

$$\mathcal{V}_X(f)(t) = \int_0^t f(s)\, \mathrm{d}s = t \int_0^1 f(st)\, \mathrm{d}s$$

and k a natural number. Then for every function $f : [0,1] \to X$ that is k-times differentiable at 0*, we have*

$$\lim_{n\to\infty}\left[(n+k)!\left(\mathcal{V}_X^n(f)(t)-\sum_{i=0}^{k-1}\frac{t^{n+i}f^{(i)}(0)}{(n+i)!}\right)-t^{n+k}f^{(k)}(0)\right]=0$$

uniformly with respect to $t \in [0,1]$ and thus

$$\lim_{n\to\infty}(n+k)!\left(\mathcal{V}_X^n(f)(t)-\sum_{i=0}^{k-1}\frac{t^{n+i}f^{(i)}(0)}{(n+i)!}\right)=\begin{cases}0 & \text{if} t\neq 1,\\ f^{(k)}(0) & \text{if} t=1.\end{cases}$$

5 The Asymptotic Evaluation for Multivariate Differentiable Functions

To avoid repetition in this section, we consider $k \geq 2$ a natural number, $\Lambda_k \subset [0,\infty)^k$ a compact set such that $0 \in \Lambda_k$ and 0 is an accumulation point of Λ_k, and $D \subset \mathbb{R}^k$ is an open set such that $\Lambda_k \subset D$. Also K is a compact metric space and X is a real Banach space.

Theorem 3 *Let $V_n : C(\Lambda_k) \to C(K)$ be a sequence of positive linear operators with the following properties:*

(i) *for every $i = 1,\ldots,k$ and every $n \in \mathbb{N}$, $V_n(p_i) \neq 0$.*
(ii) *for every $i = 1,\ldots,k$, there exist $\varphi_i \in C(\Lambda_k)$ with $\varphi_i(\mathbf{t}) > 0$, $\forall \mathbf{t} = (t_1,\ldots,t_k) \in \Lambda_k - \{0\}$ and such that $\lim_{n\to\infty}\frac{V_n(p_i\cdot\varphi_i)}{\|V_n(p_i)\|} = 0$ uniformly. Then for every function $f : D \to X$ differentiable at* 0*, we have*

$$\lim_{n\to\infty}\frac{V_{X,n}(f)(\mathbf{t})-V_n(\mathbf{1})(\mathbf{t})f(0)-\sum\limits_{i=1}^{k}V_n(p_i)(\mathbf{t})\frac{\partial f}{\partial x_i}(0)}{\sum\limits_{i=1}^{k}\|V_n(p_i)\|}=0$$

uniformly with respect to $\mathbf{t} = (t_1,\ldots,t_k) \in \Lambda_k$.

Proof Since f is differentiable at 0, we have

$$\lim_{(t_1,\ldots,t_k)\to(0,\ldots,0)}\frac{f(t_1,\ldots,t_k)-f(0,\ldots,0)-\sum\limits_{i=1}^{k}\frac{\partial f}{\partial t_i}(0)t_i}{\sum\limits_{i=1}^{k}|t_i|}=0.$$

Thus the function $g : D \to X$,

$$g(t_1, \ldots, t_k) = \begin{cases} \dfrac{f(t_1,\ldots,t_k) - f(0,\ldots,0) - \sum_{i=1}^{k} \frac{\partial f}{\partial t_i}(0) t_i}{\sum_{i=1}^{k} |t_i|}, & (t_1, \ldots, t_k) \neq (0, \ldots, 0) \\ 0, & (t_1, \ldots, t_k) = (0, \ldots, 0) \end{cases}$$

is continuous, and for all $(t_1, \ldots, t_k) \in D$, the following relation holds:

$$f(t_1, \ldots, t_k) = f(0, \ldots, 0) + \sum_{i=1}^{k} \frac{\partial f}{\partial t_i}(0)\, t_i + \left(\sum_{i=1}^{k} |t_i| \right) g(t_1, \ldots, t_k).$$

In particular, for all $(t_1, \ldots, t_k) \in \Lambda_k (\subset [0, \infty)^k)$, the following relation holds:

$$f(t_1, \ldots, t_k) = f(0, \ldots, 0) + \sum_{i=1}^{k} \frac{\partial f}{\partial t_i}(0)\, t_i + \left(\sum_{i=1}^{k} t_i \right) g(t_1, \ldots, t_k).$$

This means that $f = \mathbf{1} \otimes f(0) + \sum_{i=1}^{k} p_i \otimes \frac{\partial f}{\partial t_i}(0) + \sum_{i=1}^{k} p_i \otimes g$ in $C(\Lambda_k, X)$. Let $n \in \mathbb{N}$. Since all $V_{X,n}$ are linear, we have

$$\begin{aligned} V_{X,n}(f) &= V_{X,n}(\mathbf{1} \otimes f(0)) + \sum_{i=1}^{k} V_{X,n}\left(p_i \otimes \frac{\partial f}{\partial t_i}(0) \right) + \sum_{i=1}^{k} V_{X,n}(p_i \otimes g) \\ &= V_n(\mathbf{1}) \otimes f(0) + \sum_{i=1}^{k} V_n(p_i) \otimes \frac{\partial f}{\partial t_i}(0) + \sum_{i=1}^{k} V_{X,n}(p_i \otimes g), \end{aligned}$$

and thus

$$\begin{aligned} &\left\| V_{X,n}(f) - V_n(\mathbf{1}) \otimes f(0) - \sum_{i=1}^{k} V_n(p_i) \otimes \frac{\partial f}{\partial t_i}(0) \right\| \\ &\quad = \left\| \sum_{i=1}^{k} V_{X,n}(p_i \otimes g) \right\| \leq \sum_{i=1}^{k} \left\| V_{X,n}(p_i \otimes g) \right\|. \end{aligned} \tag{2}$$

For every $i = 1, \ldots, k$, let $U_{n,i} : C(\Lambda_k) \to C(K)$ be the operator defined by $U_{n,i}(f) = \frac{V_n(p_i \cdot f)}{\|V_n(p_i)\|}$ (see the hypothesis (i)). We have $\left\| U_{n,i}(\mathbf{1}) \right\| = 1$, and by the hypothesis (ii), $\lim_{n\to\infty} U_{n,i}(\varphi_i) = \lim_{n\to\infty} \frac{V_n(p_i \cdot \varphi_i)}{\|V_n(p_i)\|} = 0$ uniformly; From Corollary 2, for every $f \in C(\Lambda_k, X)$, $\lim_{n\to\infty} \left[U_{X,n,i}(f) - U_{n,i}(\mathbf{1}) \otimes f(0) \right] = 0$ uniformly. In particular, $\lim_{n\to\infty} \left[U_{X,n,i}(g) - U_{n,i}(\mathbf{1}) \otimes g(0) \right] = 0$ uniformly; that is, since $g(0, \ldots, 0) = 0$, $\lim_{n\to\infty} U_{X,n,i}(g) = 0$ uniformly. By Remark 1, this is equivalent to $\lim\limits_{n\to\infty}\frac{\|V_{X,n}(p_i\otimes g)\|}{\|V_n(p_i)\|}=0$. This means that $\forall\varepsilon>0$, $\exists n_\varepsilon\in\mathbb{N}$ such that $\forall n\geq n_\varepsilon$, $\forall i=1,\ldots,k$, we have $\frac{\|V_{X,n}(p_i\otimes g)\|}{\|V_n(p_i)\|}<\varepsilon$. From (2) we deduce that $\forall\varepsilon>0$, $\exists n_\varepsilon\in\mathbb{N}$ such that $\forall n\geq n_\varepsilon$ we have

$$\left\|V_{X,n}\left(f\right)-V_n\left(\mathbf{1}\right)\otimes f\left(0\right)-\sum_{i=1}^{k}V_n\left(p_i\right)\otimes\frac{\partial f}{\partial t_i}\left(0\right)\right\|<\varepsilon\sum_{i=1}^{k}\left\|V_n\left(p_i\right)\right\|,$$

which ends the proof. □

Corollary 9 *Let $V_n:C\left(\Lambda_k\right)\to C\left(K\right)$ be a sequence of positive linear operators with the following properties:*

(i) *for every $i=1,\ldots,k$ and every $n\in\mathbb{N}$, $V_n\left(p_i\right)\neq 0$.*
(ii) *for every $i,j=1,\ldots,k$, we have $\lim\limits_{n\to\infty}\frac{\|V_n(p_i\cdot p_j)\|}{\|V_n(p_i)\|}=0$.*

Then for every function $f:D\to X$ differentiable at 0, we have

$$\lim_{n\to\infty}\frac{V_{X,n}\left(f\right)\left(\mathbf{t}\right)-V_n\left(\mathbf{1}\right)\left(\mathbf{t}\right)f\left(0\right)-\sum\limits_{i=1}^{k}V_n\left(p_i\right)\left(\mathbf{t}\right)\frac{\partial f}{\partial t_i}\left(0\right)}{\sum\limits_{i=1}^{k}\left\|V_n\left(p_i\right)\right\|}=0$$

uniformly with respect to $\mathbf{t}=\left(t_1,\ldots,t_k\right)\in\Lambda_k$.

Proof Let us note that the function $s:\Lambda_k\to[0,\infty)$, $s\left(t_1,\ldots,t_k\right)=t_1+\cdots+t_k$, is continuous and $s\left(t_1,\ldots,t_k\right)>0$, $\forall\left(t_1,\ldots,t_k\right)\in\Lambda_k-\{(0,\ldots,0)\}$. Since $s=p_1+\cdots+p_k$, for every $i=1,\ldots,k$, we have $V_n\left(p_is\right)=\sum\limits_{j=1}^{k}V_n\left(p_ip_j\right)$, $\forall n\in\mathbb{N}$, and from (ii) $\lim\limits_{n\to\infty}\frac{V_n(p_is)}{\|V_n(p_i)\|}=0$ uniformly. We apply Theorem 3. □

6 The Asymptotic Evaluation for Multivariate Twice Differentiable Functions

As in the preceding section, $k\geq 2$ is a natural number, $\Lambda_k\subset[0,\infty)^k$ a compact set such that $0\in\Lambda_k$ and 0 is an accumulation point of Λ_k, and $D\subset\mathbb{R}^k$ is an open set such that $\Lambda_k\subset D$. Also K is a compact metric space and X is a real Banach space.

Theorem 4 *Let $V_n:C\left(\Lambda_k\right)\to C\left(K\right)$ be a sequence of positive linear operators with the following properties:*

(i) *for every $i=1,\ldots,k$ and every $n\in\mathbb{N}$, $V_n\left(p_i^2\right)\neq 0$.*
(ii) *for every $i=1,\ldots,k$, there exist $\varphi_i\in C\left(\Lambda_k\right)$ with $\varphi_i\left(\mathbf{t}\right)>0$, $\forall\mathbf{t}=\left(t_1,\ldots,t_k\right)\in\Lambda_k-\{0\}$ and such that $\lim\limits_{n\to\infty}\frac{V_n\left(p_i^2\cdot\varphi_i\right)}{\|V_n\left(p_i^2\right)\|}=0$ uniformly.*

Then for every function $f : D \to X$ twice differentiable at 0*, we have*

$$\lim_{n\to\infty} \frac{V_{X,n}(f)(\mathbf{t})-V_n(\mathbf{1})(\mathbf{t})f(0)-\sum_{i=1}^{k} V_n(p_i)(\mathbf{t})\frac{\partial f}{\partial t_i}(0)-\frac{1}{2}\sum_{i,j=1}^{k} V_n\left(p_i p_j\right)(\mathbf{t})\frac{\partial^2 f}{\partial t_i \partial t_j}(0)}{\sum_{i=1}^{k}\left\| V_n\left(p_i^2\right)\right\|} = 0$$

uniformly with respect to $\mathbf{t} = (t_1, \ldots, t_k) \in \Lambda_k$.

Proof Since f is twice differentiable at 0,

$$\lim_{(t_1,\ldots,t_k)\to(0,\ldots,0)} \frac{f(t_1,\ldots,t_k) - f(0,\ldots,0) - \sum_{i=1}^{k} t_i \frac{\partial f}{\partial t_i}(0) - \frac{1}{2}\sum_{i,j=1}^{k} t_i t_j \frac{\partial^2 f}{\partial t_i \partial t_j}(0)}{\sum_{i=1}^{k} t_i^2} = 0$$

see [3, Théorème 5.6.3, page 78]. Thus the function $g : \Lambda_k \to X$,

$$g(t_1,\ldots,t_k) = \begin{cases} \dfrac{f(t_1,\ldots,t_k)-f(0,\ldots,0)-\sum_{i=1}^{k} t_i \frac{\partial f}{\partial t_i}(0)-\frac{1}{2}\sum_{i,j=1}^{k} t_i t_j \frac{\partial^2 f}{\partial t_i \partial t_j}(0)}{\sum_{i=1}^{k} t_i^2}, & (t_1,\ldots,t_k) \neq (0,\ldots,0), \\ 0, & (t_1,\ldots,t_k) = (0,\ldots,0), \end{cases}$$

is continuous, and for all $(t_1, \ldots, t_k) \in \Lambda_k$, the following relation holds:

$$\begin{aligned} f(t_1,\ldots,t_k) &= f(0,\ldots,0) + \sum_{i=1}^{k} t_i \frac{\partial f}{\partial t_i}(0) + \frac{1}{2}\sum_{i,j=1}^{k} t_i t_j \frac{\partial^2 f}{\partial t_i \partial t_j}(0) \\ &\quad + \left(\sum_{i=1}^{k} t_i^2\right) g(t_1,\ldots,t_k). \end{aligned}$$

This means that

$$\begin{aligned} f &= \mathbf{1} \otimes f(0) + \sum_{i=1}^{k} p_i \otimes \frac{\partial f}{\partial t_i}(0) + \frac{1}{2}\sum_{i,j=1}^{k} \left(p_i p_j\right) \otimes \frac{\partial^2 f}{\partial t_i \partial t_j}(0) \\ &\quad + \sum_{i=1}^{k} p_i^2 \otimes g \text{ in } C(\Lambda_k, X). \end{aligned}$$

Let $n \in \mathbb{N}$. Since all $V_{X,n}$ are linear, we have

$$\begin{aligned} V_{X,n}(f) &= V_{X,n}(\mathbf{1} \otimes f(0)) + \sum_{i=1}^{k} V_{X,n}\left(p_i \otimes \frac{\partial f}{\partial t_i}(0)\right) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{k} V_{X,n}\left(\left(p_i p_j\right) \otimes \frac{\partial^2 f}{\partial t_i \partial t_j}(0)\right) + \sum_{i=1}^{k} V_{X,n}\left(p_i^2 \otimes g\right) \\ &= V_n(\mathbf{1}) \otimes f(0) + \sum_{i=1}^{k} V_n(p_i) \otimes \frac{\partial f}{\partial t_i}(0) + \frac{1}{2}\sum_{i,j=1}^{k} V_n\left(p_i p_j\right) \otimes \frac{\partial^2 f}{\partial t_i \partial t_j}(0) \end{aligned}$$

$$+\sum_{i=1}^{k} V_{X,n}\left(p_i^2 \otimes g\right) \text{ in } C(K,X),$$

and thus

$$\left\| V_{X,n}(f) - V_n(\mathbf{1}) \otimes f(0) - \sum_{i=1}^{k} V_n(p_i) \otimes \frac{\partial f}{\partial t_i}(0) - \frac{1}{2}\sum_{i,j=1}^{k} V_n\left(p_i p_j\right) \otimes \frac{\partial^2 f}{\partial t_i \partial t_j}(0) \right\|$$

$$= \left\| \sum_{i=1}^{k} V_{X,n}\left(p_i^2 \otimes g\right) \right\| \leq \sum_{i=1}^{k} \left\| V_{X,n}\left(p_i^2 \otimes g\right) \right\|. \tag{3}$$

For every $i = 1, \ldots, k$, let $U_{n,i} : C(\Lambda_k) \to C(K)$ be the operator defined by $U_{n,i}(f) = \frac{V_n(p_i^2 \cdot f)}{\|V_n(p_i^2)\|}$. We have $\|U_{n,i}(\mathbf{1})\| = 1$ (see the hypothesis (i)) and by the hypothesis (ii), $\lim_{n\to\infty} U_{n,i}(\varphi_i) = \lim_{n\to\infty} \frac{V_n(p_i^2 \cdot \varphi_i)}{\|V_n(p_i^2)\|} = 0$ uniformly. Since $g(0) = 0$ from Theorem 1, we deduce that $\lim_{n\to\infty} U_{n,i}(g) = 0$ uniformly, or $\lim_{n\to\infty} \frac{V_n(p_i^2 \cdot g)}{\|V_n(p_i^2)\|} = 0$ uniformly. From Corollary 2, it follows that for every $f \in C(\Lambda_k, X)$, we have $\lim_{n\to\infty} \left[U_{X,n,i}(f) - U_{n,i}(\mathbf{1}) \otimes f(0)\right] = 0$ uniformly. In particular, $\lim_{n\to\infty} \left[U_{X,n,i}(g) - U_{n,i}(\mathbf{1}) \otimes g(0)\right] = 0$ uniformly; that is, since $g(0) = 0$, $\lim_{n\to\infty} U_{X,n,i}(g) = 0$ uniformly. By Remark 1, this is equivalent to $\lim_{n\to\infty} \frac{\|V_{X,n}(p_i^2 \otimes g)\|}{\|V_n(p_i)\|} = 0$. This means that $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$, $\forall i = 1, \ldots, k$, we have $\frac{\|V_n(p_i^2 \otimes g)\|}{\|V_n(p_i^2)\|} < \varepsilon$. Then from (3), we deduce that $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$, we have

$$\left\| V_{X,n}(f) - V_n(\mathbf{1}) \otimes f(0) - \sum_{i=1}^{k} V_n(p_i) \otimes \frac{\partial f}{\partial t_i}(0) - \frac{1}{2}\sum_{i,j=1}^{k} V_n\left(p_i p_j\right) \otimes \frac{\partial^2 f}{\partial t_i \partial t_j}(0) \right\|$$

$$< \varepsilon \sum_{i=1}^{k} \left\| V_n\left(p_i^2\right) \right\|,$$

which ends the proof. □

Corollary 10 *Let $V_n : C(\Lambda_k) \to C(K)$ be a sequence of positive linear operators with the following properties:*

(i) *for every $i = 1, \ldots, k$ and every $n \in \mathbb{N}$, $V_n\left(p_i^2\right) \neq 0$;*

(ii) *for every $i, j = 1, \ldots, k$, we have $\lim_{n\to\infty} \frac{\|V_n(p_i^2 p_j)\|}{\|V_n(p_i^2)\|} = 0$.*

Then for every function $f : D \to X$ *twice differentiable at* 0*, we have*

$$\lim_{n\to\infty} \frac{V_{X,n}(f)(\mathbf{t}) - V_n(\mathbf{1})(\mathbf{t}) f(0) - \sum_{i=1}^{k} V_n(p_i)(\mathbf{t}) \frac{\partial f}{\partial t_i}(0) - \frac{1}{2}\sum_{i,j=1}^{k} V_n\left(p_i p_j\right)(\mathbf{t}) \frac{\partial^2 f}{\partial t_i \partial t_j}(0)}{\sum_{i=1}^{k} \left\| V_n\left(p_i^2\right)\right\|} = 0$$

uniformly with respect to $\mathbf{t} = (t_1, \ldots, t_k) \in \Lambda_k$.

Proof The function $s : \Lambda_k \to [0, \infty)$, $s(t_1, \ldots, t_k) = t_1 + \cdots + t_k$, is continuous and $s(t_1, \ldots, t_k) > 0, \forall (t_1, \ldots, t_k) \in \Lambda_k - \{(0, \ldots, 0)\}$. Since $s = p_1 + \cdots + p_k$, for every $i = 1, \ldots, k$, $V_n\left(p_i^2 s\right) = \sum_{j=1}^{n} V_n\left(p_i^2 p_j\right)$, $\forall n \in \mathbb{N}$, and from (ii), $\lim_{n\to\infty} \frac{V_n\left(p_i^2 s\right)}{\left\|V_n\left(p_i^2\right)\right\|} = 0$ uniformly. We apply Theorem 4. □

7 The First Asymptotic Evaluation for Multivariate Cesàro and Volterra Type Operators

To avoid repetition in this section, we consider $k \geq 2$ a natural number. A typical element in $\mathbb{R}^k$ will be denoted either by $(t_1, \ldots, t_k)$, or $\mathbf{t}$; if $\mathbf{s}, \mathbf{t} \in \mathbb{R}^k$, we define $\mathbf{st} = (s_1 t_1, \ldots, s_k t_k)$. In the study of the Volterra type operators will appear the function $P : \mathbb{R}^k \to \mathbb{R}$, $P(\mathbf{t}) = t_1 \cdots t_k$; we need the relations $P(\mathbf{st}) = P(\mathbf{s}) P(\mathbf{t})$, $p_i(\mathbf{st}) = p_i(\mathbf{s}) p_i(\mathbf{t})$, $\forall \mathbf{s}, \mathbf{t} \in \mathbb{R}^k$, $i = 1, \ldots, k$. Also X is a real Banach space. $\Lambda_k \subset [0, 1]^k$ is a compact Jordan measurable set such that $\lambda_k(\Lambda_k) > 0$, λ_k is the Lebesgue k-dimensional measure, $0 \in \Lambda_k$, 0 is an accumulation point of Λ_k, and $D \subset \mathbb{R}^k$ is an open set such that $\Lambda_k \subset D$. We suppose moreover that $\forall \mathbf{s}, \mathbf{t} \in \Lambda_k$, we have $\mathbf{st} \in \Lambda_k$. For $\varphi : \Lambda_k \to [0, \infty)$ a continuous function such that $\int_{\Lambda_k} \varphi(\mathbf{s})\, \mathbf{ds} > 0$, we define

$$\alpha = \int_{\Lambda_k} \varphi(\mathbf{s})\, \mathbf{ds};\ \alpha_i = \int_{\Lambda_k} s_i \varphi(\mathbf{s})\, \mathbf{ds},\ i = 1, \ldots, k;$$

$$\beta_{ij} = \int_{\Lambda_k} s_i s_j \varphi(\mathbf{s})\, \mathbf{ds},\ i, j = 1, \ldots, k. \mathbf{ds} = \mathrm{d}s_1 \cdots \mathrm{d}s_k.$$

Proposition 1 *Let* $\varphi : \Lambda_k \to [0, \infty)$ *be a continuous function such that* $\int_{\Lambda_k} \varphi(\mathbf{s})\, \mathbf{ds} > 0$. *Then* $0 < \int_{\Lambda_k} s_i s_j \varphi(\mathbf{s})\, \mathbf{ds} < \int_{\Lambda_k} s_i \varphi(\mathbf{s})\, \mathbf{ds}$ *for all* $i, j = 1, \ldots, k$.

Proof Let us suppose, for example, that $\int_{\Lambda_k} s_1 s_2 \varphi(\mathbf{s})\, \mathbf{ds} \geq \int_{\Lambda_k} s_1 \varphi(\mathbf{s})\, \mathbf{ds}$, or $\int_{\Lambda_k} s_1(1 - s_2) \varphi(\mathbf{s})\, \mathbf{ds} \leq \mathbf{0}$. Since $\Lambda_k \subset [0, 1]^k$, we have $s_1(1 - s_2) \geq 0$, $\forall \mathbf{s} \in \Lambda_k$, and from $s_1(1 - s_2) \varphi(\mathbf{s}) \geq 0$, we get $\int_{\Lambda_k} s_1(1 - s_2) \varphi(\mathbf{s})\, \mathbf{ds} = 0$. Then it follows that $s_1(1 - s_2) \varphi(\mathbf{s}) = 0$, for λ_k-almost all $\mathbf{s} \in \Lambda_k$; i.e., $\varphi(\mathbf{s}) = 0$ for λ_k-almost all $\mathbf{s} \in \Lambda_k$. Then $(L) \int_{\Lambda_k} \varphi(\mathbf{s})\, \mathbf{ds} = 0$ (the Lebesgue integral). Since φ is continuous, as is well known, $(L) \int_{\Lambda_k} \varphi(\mathbf{s})\, \mathbf{ds} = \int_{\Lambda_k} \varphi(\mathbf{s})\, \mathbf{ds}$ and thus $\int_{\Lambda_k} \varphi(\mathbf{s})\, \mathbf{ds} = 0$, which is impossible. □

Corollary 11 *Let $\mathcal{C}_{X,\varphi} : C(\Lambda_k, X) \to C(\Lambda_k, X)$ be the multivariate Cesàro type operator defined by*

$$\mathcal{C}_{X,\varphi}(f)(\mathbf{t}) = \int_{\Lambda_k} \varphi(\mathbf{s}) f(\mathbf{st}) \, \mathbf{ds}.$$

Then for every function $f : D \to X$ differentiable at 0*, we have*

$$\lim_{n\to\infty} \frac{\mathcal{C}^n_{X,\varphi}(f)(\mathbf{t}) - \alpha^n f(0)}{L^n} = \sum_{i\in A} t_i \frac{\partial f}{\partial t_i}(0)$$

uniformly with respect to $\mathbf{t} \in \Lambda_k$, where $L = \max_{1\le i\le k} \alpha_i$, $A = \{1 \le i \le k \mid \alpha_i = L\}$.

Proof Let $\beta \ge 0$. For every $i = 1, \dots, k$, by induction on n, we can prove that $\mathcal{C}^n_{\varphi}\left(p_i^{\beta}\right) = \lambda^n_{i\beta} p_i^{\beta}$, $\forall n \in \mathbb{N}$, where $\lambda_{i\beta} = \int_{\Lambda_k} s_i^{\beta} \varphi(\mathbf{s}) \, \mathbf{ds}$. In particular,

$$\mathcal{C}^n_{\varphi}(\mathbf{1}) = \alpha^n, \mathcal{C}^n_{\varphi}(p_i) = \alpha_i^n p_i, \mathcal{C}^n_{\varphi}\left(p_i^2\right) = \beta^n_{ii} p_i^2, \forall n \in \mathbb{N}.$$

Let $i, j = 1, \dots, k$. By induction on n, we can prove that $\mathcal{C}^n_{\varphi}\left(p_i p_j\right) = \beta^n_{ij} \cdot p_i p_j$, $\forall n \in \mathbb{N}$. For all $n \in \mathbb{N}$, we have $\frac{\left\|\mathcal{C}^n_{\varphi}(p_i \cdot p_j)\right\|}{\left\|\mathcal{C}^n_{\varphi}(p_i)\right\|} = \left(\frac{\beta_{ij}}{\alpha_i}\right)^n$, and since by Proposition 1, $0 < \frac{\beta_{ij}}{\alpha_i} < 1$, we obtain $\lim_{n\to\infty} \frac{\left\|\mathcal{C}^n_{\varphi}(p_i \cdot p_j)\right\|}{\left\|\mathcal{C}^n_{\varphi}(p_i)\right\|} = 0$. By Corollary 4, we have

$$\lim_{n\to\infty} \frac{\mathcal{C}^n_{X,\varphi}(f)(\mathbf{t}) - \mathcal{C}^n_{\varphi}(\mathbf{1})(\mathbf{t}) f(0) - \sum_{i=1}^{k} \mathcal{C}^n_{\varphi}(p_i)(\mathbf{t}) \frac{\partial f}{\partial t_i}(0)}{\sum_{i=1}^{k} \left\|\mathcal{C}^n_{\varphi}(p_i)\right\|} = 0$$

uniformly with respect to $\mathbf{t} \in \Lambda_k$; that is,

$$\lim_{n\to\infty} \frac{\mathcal{C}^n_{X,\varphi}(f)(\mathbf{t}) - \alpha^n f(0) - \sum_{i=1}^{k} \alpha_i^n t_i \frac{\partial f}{\partial t_i}(0)}{\sum_{i=1}^{k} \alpha_i^n} = 0$$

uniformly with respect to $\mathbf{t} \in \Lambda_k$. Let us observe that for every $i \notin A$, $0 < \alpha_i < L$, which gives us that $\lim_{n\to\infty} \frac{\alpha_i^n}{L^n} = \lim_{n\to\infty} \left(\frac{\alpha_i}{L}\right)^n = 0$. Then $\lim_{n\to\infty} \frac{\sum_{i=1}^{k} \alpha_i^n}{L^n} = card(A) + \lim_{n\to\infty} \sum_{i\notin A} \left(\frac{\alpha_i}{L}\right)^n = card(A)$. We deduce that

$$\lim_{n\to\infty}\frac{\mathcal{C}^n_{X,\varphi}(f)(\mathbf{t})-\alpha^n f(0)-\sum_{i=1}^{k}\alpha_i^n t_i\frac{\partial f}{\partial t_i}(0)}{L^n}=0,$$

uniformly with respect to $\mathbf{t}\in\Lambda_k$. Since

$$\sum_{i=1}^{k}\frac{\partial f}{\partial t_i}(0)\,\alpha_i^n p_i=\sum_{i\in A}\frac{\partial f}{\partial t_i}(0)\,\alpha_i^n p_i+\sum_{i\notin A}\frac{\partial f}{\partial t_i}(0)\,\alpha_i^n p_i$$
$$=L^n\sum_{i\in A}\frac{\partial f}{\partial t_i}(0)\,p_i+\sum_{i\notin A}\frac{\partial f}{\partial t_i}(0)\,\alpha_i^n p_i,$$

and, as we already observed, $\lim_{n\to\infty}\frac{\alpha_i^n}{L^n}=0$, $\forall i\notin A$, we obtain the evaluation from the statement. □

Corollary 12 *Let* $\mathcal{C}_X : C\left([0,1]^k, X\right)\to C\left([0,1]^k, X\right)$ *be the multivariate Cesàro operator defined by*

$$\mathcal{C}_X(f)(\mathbf{t})=\int_{[0,1]^k} f(\mathbf{st})\,\mathbf{ds}.$$

Then for every function $f : D\to X$ *differentiable at* 0*, we have*

$$\lim_{n\to\infty}2^n\left(\mathcal{C}_X^n(f)(\mathbf{t})-f(0)\right)=\sum_{i=1}^{k}t_i\frac{\partial f}{\partial t_i}(0)$$

uniformly with respect to $\mathbf{t}\in[0,1]^k$.

Proof With the same notation as in Corollary 11, $\alpha=\int_{[0,1]^k}1\mathbf{ds}=1$, $\alpha_i=\int_{[0,1]^k}s_i\mathbf{ds}=\frac{1}{2}=L$ and $A=\{1\le i\le k\mid \alpha_i=L\}=\{1,\ldots,k\}$. □

Corollary 13 *Let* $T_k=\left\{(s_1,\ldots,s_k)\in\mathbb{R}^k\mid s_1\ge 0,\ldots,s_k\ge 0, s_1+\cdots+s_k\le 1\right\}$ *and* $\mathcal{C}_X : C(T_k, X)\to C(T_k, X)$ *be the multivariate Cesàro operator defined by*

$$\mathcal{C}_X(f)(\mathbf{t})=\int_{T_k} f(\mathbf{st})\,\mathbf{ds}.$$

Then for every function $f : D\to X$ *differentiable at* 0*, we have*

$$\lim_{n\to\infty}(k+1)^n\left((k!)^n\,\mathcal{C}_X^n(f)(\mathbf{t})-f(0)\right)=\sum_{i=1}^{k}t_i\frac{\partial f}{\partial t_i}(0)$$

uniformly with respect to $\mathbf{t}\in T_k$.

Proof We will use that if $T_{a,k} = \left\{(s_1, \ldots, s_k) \in \mathbb{R}^k \mid s_1 \geq 0, \ldots, s_k \geq 0, s_1 + \cdots + s_k \leq a\right\}$, then $\lambda_k\left(T_{a,k}\right) = \frac{a^k}{k!}$, $a > 0$. With the same notation as in Corollary 11, $\alpha = \frac{1}{k!}$, $\alpha_i = \alpha_1 = \int_{T_k} s_1 \mathbf{ds} = \int_0^1 s_1 \mathrm{d}s_1 \int_{T_{1-s_1,k-1}} \mathrm{d}s_2 \cdots \mathrm{d}s_k = \frac{1}{(k-1)!} \int_0^1 s_1 (1-s_1)^{k-1} \mathrm{d}s_1 = \frac{1}{(k+1)!} = L$ and $A = \{i \mid \alpha_i = L\} = \{1, \ldots, k\}$. □

Corollary 14 *Let* $S_k^+ = \left\{(s_1, \ldots, s_k) \in \mathbb{R}^k \mid s_1 \geq 0, \ldots, s_k \geq 0, s_1^2 + \cdots + s_k^2 \leq 1\right\}$ *and* $\mathcal{C}_X : C(S_k, X) \to C(S_k, X)$ *be the multivariate Cesàro operator defined by*

$$\mathcal{C}_X(f)(\mathbf{t}) = \int_{S_k^+} f(\mathbf{st})\, \mathbf{ds}.$$

Then for every function $f : D \to X$ *differentiable at* 0*, we have*

$$\lim_{n\to\infty} \left(\frac{2^{k-1}(k+1)\Gamma\left(\frac{k+1}{2}\right)}{\pi^{\frac{k-1}{2}}}\right)^n \left(\mathcal{C}_X^n(f)(\mathbf{t}) - \left(\frac{\pi^{\frac{k}{2}}}{2^k \Gamma\left(\frac{k}{2}+1\right)}\right)^n f(0)\right)$$
$$= \sum_{i=1}^k t_i \frac{\partial f}{\partial t_i}(0) \text{ uniformly with respect to } \mathbf{t} \in S_k^+.$$

Γ *is the gamma function of Euler.*

Proof We will use that if $a > 0$, $S_{a,k} = \left\{(s_1, \ldots, s_k) \in \mathbb{R}^k \mid s_1^2 + \cdots + s_k^2 \leq a^2\right\}$, then $\lambda_k\left(S_{a,k}\right) = \frac{\pi^{\frac{k}{2}} a^k}{\Gamma\left(\frac{k}{2}+1\right)}$ and thus $\lambda_k\left(S_k^+\right) = \frac{1}{2^k}\lambda_k\left(S_{1,k}\right) = \frac{\pi^{\frac{k}{2}}}{2^k\Gamma\left(\frac{k}{2}+1\right)}$. With the same notation as in Corollary 11, $\alpha = \frac{\pi^{\frac{k}{2}}}{2^k\Gamma\left(\frac{k}{2}+1\right)}$, $\alpha_i = \alpha_1 = \int_{S_k^+} s_1 \mathbf{ds} = \int_0^1 s_1 \mathrm{d}s_1 \int_{s_2 \geq 0, \ldots, s_k \geq 0, s_2^2 + \cdots + s_k^2 \leq 1 - s_1^2} \mathrm{d}s_2 \cdots \mathrm{d}s_k = \frac{\pi^{\frac{k-1}{2}}}{2^{k-1}\Gamma\left(\frac{k-1}{2}+1\right)} \int_0^1 s_1 \left(1 - s_1^2\right)^{\frac{k-1}{2}} \mathrm{d}s_1 = \frac{\pi^{\frac{k-1}{2}}}{2^{k-1}(k+1)\Gamma\left(\frac{k-1}{2}+1\right)} = L$. Thus $A = \{i \mid \alpha_i = L\} = \{1, \ldots, k\}$. Now apply Corollary 11. □

Corollary 15 *Let* $Pir = \left\{(x, y, z) \in \mathbb{R}^3 \mid x + y \leq 1, x \geq 0, y \geq 0, 0 \leq z \leq 1\right\}$ *and* $\mathcal{C}_X : C(Pir, X) \to C(Pir, X)$ *be the trivariate Cesàro operator defined by*

$$\mathcal{C}_X(f)(t_1, t_2, t_3) = \iiint_{x+y\leq 1, x\geq 0, y\geq 0, 0\leq z\leq 1} f(t_1 x, t_2 y, t_3 z)\, \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Then for every function $f : D \to X$ *differentiable at* $(0, 0, 0)$*, we have*

$$\lim_{n\to\infty} 2^n \left(2^n \mathcal{C}_X^n(f)(t_1, t_2, t_3) - f(0, 0, 0)\right) = t_3 \frac{\partial f}{\partial t_3}(0, 0, 0)$$

uniformly with respect to $(t_1, t_2, t_3) \in Pir$.

Proof By taking $\varphi = 1$ in Corollary 11, we have

$$\alpha = \iiint_{Pir} 1\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{2}, \alpha_1 = \iiint_{Pir} x\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 x(1-x)\,\mathrm{d}x = \frac{1}{6},$$
$$\alpha_2 = \iiint_{Pir} y\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{6}, \alpha_3 = \iiint_{Pir} z\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{4}.$$

In this case $\max\limits_{1\le i\le 3} \alpha_i = \frac{1}{4} = L$, $A = \left\{1 \le i \le 3 \mid \alpha_i = \frac{1}{4}\right\} = \{3\}$. We apply Corollary 11. □

Corollary 16 *Let* $\Sigma = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \le 1, x \ge 0, y \ge 0, 0 \le z \le 1\right\}$ *and* $\mathcal{C}_X : C(\Sigma, X) \to C(\Sigma, X)$ *be the trivariate Cesàro operator defined by*

$$\mathcal{C}_X(f)(t_1, t_2, t_3) = \iiint_{x^2+y^2\le 1, x\ge 0, y\ge 0, 0\le z\le 1} f(t_1x, t_2y, t_3z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Then for every function $f : D \to X$ *differentiable at* $(0, 0, 0)$*, we have*

$$\lim_{n\to\infty} 2^n \left(\left(\frac{4}{\pi}\right)^n \mathcal{C}_X^n(f)(t_1, t_2, t_3) - f(0, 0, 0)\right) = t_3 \frac{\partial f}{\partial t_3}(0, 0, 0)$$

uniformly with respect to $(t_1, t_2, t_3) \in \Sigma$.

Proof By taking $\varphi = 1$ in Corollary 11, we have $\alpha = \iiint_\Sigma 1\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{\pi}{4}$,

$$\alpha_1 = \iiint_\Sigma x\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{x^2+y^2\le 1, x\ge 0, y\ge 0} x\mathrm{d}x\mathrm{d}y = \iint_{[0,1]\times\left[0,\frac{\pi}{2}\right]} \rho^2 \cos\theta\,\mathrm{d}\rho\mathrm{d}\theta = \frac{1}{3},$$
$$\alpha_2 = \iiint_\Sigma y\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{3}, \alpha_3 = \iiint_\Sigma z\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{2}\iint_{x^2+y^2\le 1, x\ge 0, y\ge 0} \mathrm{d}x\mathrm{d}y = \frac{\pi}{8}.$$

In this case $\max\limits_{1\le i\le 3} \alpha_i = \frac{\pi}{8} = L$, $A = \left\{i \mid \alpha_i = \frac{\pi}{8}\right\} = \{3\}$. We apply Corollary 11. □

Corollary 17 *Let* $\mathcal{V}_{X,\varphi} : C(\Lambda_k, X) \to C(\Lambda_k, X)$ *be the multivariate Volterra type operator defined by*

$$\mathcal{V}_{X,\varphi}(f)(\mathbf{t}) = P(\mathbf{t}) \int_{\Lambda_k} \varphi(\mathbf{s})\, f(\mathbf{st})\,\mathrm{d}\mathbf{s}.$$

Suppose that for every $i, j = 1, \ldots, k$, we have $\lim_{n\to\infty} \frac{\prod_{m=0}^{n-1} \int_{\Lambda_k} s_i s_j \varphi(\mathbf{s}) P^m(\mathbf{s}) \mathrm{d}s}{\prod_{m=0}^{n-1} \int_{\Lambda_k} s_i \varphi(\mathbf{s}) P^m(\mathbf{s}) \mathrm{d}\mathbf{s}} = 0$. *Then for every function $f : D \to X$ differentiable at* 0, *we have*

$$\lim_{n\to\infty} \frac{\mathcal{V}^n_{X,\varphi}(f)(\mathbf{t}) - P^n(\mathbf{t}) a_n f(0) - P^n(\mathbf{t}) \sum_{i=1}^k b_{ni} t_i \frac{\partial f}{\partial t_i}(0)}{\sum_{i=1}^k b_{ni}} = 0$$

uniformly with respect to $\mathbf{t} \in \Lambda_k$, *where* $a_n = \prod_{m=0}^{n-1} \int_{\Lambda_k} \varphi(\mathbf{s}) P^m(\mathbf{s}) \mathrm{d}\mathbf{s}$, $b_{ni} = \prod_{m=0}^{n-1} \int_{\Lambda_k} s_i \varphi(\mathbf{s}) P^m(\mathbf{s}) \mathrm{d}\mathbf{s}$, $i = 1, \ldots, k$.

Proof By induction (on n), we can prove that for every $n \in \mathbb{N}$, we have

$$\mathcal{V}^n_{\varphi}(\mathbf{1}) = a_n P^n, \mathcal{V}^n_{\varphi}(p_i) = \prod_{m=0}^{n-1} \int_{\Lambda_k} s_i \varphi(\mathbf{s}) P^m(\mathbf{s}) \mathrm{d}\mathbf{s} = b_{ni} p_i P^n,$$

$$\mathcal{V}^n_{\varphi}\left(p_i p_j\right) = \left(\prod_{m=0}^{n-1} \int_{\Lambda_k} s_i s_j \varphi(\mathbf{s}) P^m(\mathbf{s}) \mathrm{d}s\right) p_i p_j P^n, i, j = 1, \ldots, k.$$

Let $i, j = 1, \ldots, k$. By the hypothesis, $\lim_{n\to\infty} \frac{\left\|\mathcal{V}^n_{\varphi}(p_i \cdot p_j)\right\|}{\left\|\mathcal{V}^n_{\varphi}(p_i)\right\|} = 0$. From Corollary 9 we obtain

$$\lim_{n\to\infty} \frac{\mathcal{V}^n_{X,\varphi}(f)(\mathbf{t}) - \mathcal{V}^n_{\varphi}(\mathbf{1})(\mathbf{t}) f(0) - \sum_{i=1}^k \mathcal{V}^n_{\varphi}(p_i)(\mathbf{t}) \frac{\partial f}{\partial t_i}(0)}{\sum_{i=1}^k \left\|\mathcal{V}^n_{\varphi}(p_i)\right\|} = 0$$

uniformly with respect to $\mathbf{t} \in \Lambda_k$. After some simple calculations we get the statement. □

Corollary 18 *Let* $\mathcal{V}_X : C\left([0,1]^k, X\right) \to C\left([0,1]^k, X\right)$ *be the multivariate Volterra operator defined by*

$$\mathcal{V}_X(f)(\mathbf{t}) = P(\mathbf{t}) \int_{[0,1]^k} f(\mathbf{st}) \mathrm{d}\mathbf{s}.$$

Then for every function $f : D \to X$ *differentiable at* 0*, we have*

$$\lim_{n\to\infty}\left[(n!)^k (n+1)\left(\mathcal{V}_X^n(f)(\mathbf{t}) - \frac{P^n(\mathbf{t})}{(n!)^k} f(0)\right) - P^n(\mathbf{t})\sum_{i=1}^k t_i \frac{\partial f}{\partial t_i}(0)\right] = 0$$

uniformly with respect to $\mathbf{t} \in [0,1]^k$ *and thus*

$$\lim_{n\to\infty}(n!)^k (n+1)\left(\mathcal{V}_X^n(f)(\mathbf{t}) - \frac{P^n(\mathbf{t})}{(n!)^k} f(0)\right) = \begin{cases} 0 & \text{if } (t_1,\ldots,t_k) \neq (1,\ldots,1), \\ \sum_{i=1}^k \frac{\partial f}{\partial t_i}(0) & \text{if } (t_1,\ldots,t_k) = (1,\ldots,1). \end{cases}$$

Proof With the same notation as in Corollary 17, we have $\int_{[0,1]^k} P^m(\mathbf{s})\,d\mathbf{s} = \frac{1}{(m+1)^k}$, $a_n = \frac{1}{(n!)^k}$ and $\int_{[0,1]^k} s_i P^m(\mathbf{s})\,d\mathbf{s} = \frac{1}{(m+1)^{k-1}(m+2)}$, $b_{ni} = \frac{1}{(n!)^k(n+1)}$. By Corollary 17 we deduce that

$$\lim_{n\to\infty}\frac{\mathcal{V}_X^n(f)(\mathbf{t}) - \frac{P^n(\mathbf{t})}{(n!)^k} f(0) - \frac{P^n(\mathbf{t})}{(n!)^k(n+1)}\sum_{i=1}^k t_i \frac{\partial f}{\partial t_i}(0)}{\frac{k}{(n!)^k(n+1)}} = 0$$

uniformly with respect to $\mathbf{t} \in [0,1]^k$; that is,

$$\lim_{n\to\infty}\left[(n!)^k (n+1)\left(\mathcal{V}_X^n(f)(\mathbf{t}) - \frac{P^n(\mathbf{t})}{(n!)^k} f(0)\right) - P^n(\mathbf{t})\sum_{i=1}^k t_i \frac{\partial f}{\partial t_i}(0)\right] = 0$$

uniformly with respect to $\mathbf{t} \in [0,1]^k$. The second part is obvious. □

8 The Second Asymptotic Evaluation for Multivariate Cesàro and Volterra Type Operators

Corollary 19 *Let* $\mathcal{C}_{X,\varphi} : C(\Lambda_k, X) \to C(\Lambda_k, X)$ *be the multivariate Cesàro type operator defined by*

$$\mathcal{C}_{X,\varphi}(f)(\mathbf{t}) = \int_{\Lambda_k} \varphi(\mathbf{s})\, f(\mathbf{st})\,d\mathbf{s}.$$

Then for every function $f : D \to X$ *twice differentiable at* 0*, we have*

$$\lim_{n\to\infty}\frac{\mathcal{C}_{X,\varphi}^n(f)(\mathbf{t}) - P_{n,2}(f)(\mathbf{t})}{M^n} = 0$$

uniformly with respect to $\mathbf{t} \in \Lambda_k$*, where* $M = \max_{1\leq i\leq k} \beta_{ii}$ *and*

$$P_{n,2}(f)(\mathbf{t}) = \alpha^n f(0) + \sum_{i=1}^{k} \alpha_i^n t_i \frac{\partial f}{\partial t_i}(0) + \frac{1}{2}\sum_{i,j=1}^{k} \beta_{ij}^n t_i t_j \frac{\partial^2 f}{\partial t_i \partial t_j}(0).$$

Proof We know from the proof of Corollary 11 that

$$\mathcal{C}_\varphi^n(\mathbf{1}) = \alpha^n, \mathcal{C}_\varphi^n(p_i) = \alpha_i^n p_i, \mathcal{C}_\varphi^n\left(p_i^2\right) = \beta_{ii}^n p_i^2, \mathcal{C}_\varphi^n\left(p_i^3\right) = \lambda_{i3}^n p_i^3, \forall n \in \mathbb{N},$$

where $\lambda_{i3} = \int_{\Lambda_k} s_i^3 \varphi(\mathbf{s})\, \mathbf{ds}$. For $i, j = 1, \ldots, k$, we have

$$\mathcal{C}_\varphi^n\left(p_i p_j\right) = \beta_{ij}^n \cdot p_i p_j, \mathcal{C}_\varphi^n\left(p_i^2 p_j\right) = \theta_{ij}^n \cdot p_i^2 p_j, \forall n \in \mathbb{N},$$

where $\theta_{ij} = \int_{\Lambda_k} s_i^2 s_j \varphi(\mathbf{s})\, \mathbf{ds}$. For every $i, j = 1, \ldots, k$, by Proposition 1, $0 < \frac{\theta_{ij}}{\beta_{ii}} < 1$, and thus $\lim_{n\to\infty} \frac{\left\|\mathcal{C}_\varphi^n(p_i^2 p_j)\right\|}{\left\|\mathcal{C}_\varphi^n(p_i^2)\right\|} = \lim_{n\to\infty}\left(\frac{\theta_{ij}}{\beta_{ii}}\right)^n = 0$. Then, by Corollary 10, $\lim_{n\to\infty} \frac{\mathcal{C}_{X,\varphi}^n(f)(\mathbf{t}) - P_{n,2} f(\mathbf{t})}{\sum_{i=1}^{k}\left\|\mathcal{C}_\varphi^n(p_i^2)\right\|} = 0$ uniformly with respect to $\mathbf{t} \in \Lambda_k$, where

$$\begin{aligned} P_{n,2} f(\mathbf{t}) &= \mathcal{C}_\varphi^n(\mathbf{1})(\mathbf{t}) f(0) + \sum_{i=1}^{k} \mathcal{C}_\varphi^n(p_i)(\mathbf{t}) \frac{\partial f}{\partial t_i}(0) + \frac{1}{2}\sum_{i,j=1}^{k} \mathcal{C}_\varphi^n\left(p_i p_j\right)(\mathbf{t}) \frac{\partial^2 f}{\partial t_i \partial t_j}(0) \\ &= \alpha^n f(0) + \sum_{i=1}^{k} \alpha_i^n t_i \frac{\partial f}{\partial t_i}(0) + \frac{1}{2}\sum_{i,j=1}^{k} \beta_{ij}^n t_i t_j \frac{\partial^2 f}{\partial t_i \partial t_j}(0). \end{aligned}$$

Thus

$$\lim_{n\to\infty} \frac{\mathcal{C}_{X,\varphi}^n(f)(\mathbf{t}) - P_{n,2} f(\mathbf{t})}{\sum_{i=1}^{k} \beta_{ii}^n} = 0$$

uniformly with respect to $\mathbf{t} \in \Lambda_k$. This gives us the statement, because $\lim_{n\to\infty} \frac{\sum_{i=1}^{k} \beta_{ii}^n}{M^n} = card(I)$, where $I = \{1 \leq i \leq k \mid \beta_{ii} = M\}$. □

Corollary 20 *Let* $\mathcal{C}_X : C\left([0,1]^k, X\right) \to C\left([0,1]^k, X\right)$ *be the multivariate Cesàro operator defined by*

$$\mathcal{C}_X(f)(\mathbf{t}) = \int_{[0,1]^k} f(\mathbf{st})\, \mathbf{ds}.$$

Then for every function $f : D \to X$ twice differentiable at 0*, we have*

$$\lim_{n\to\infty} 3^n \left(\mathcal{C}_X^n (f)(\mathbf{t}) - f(0) - \frac{1}{2^n} \sum_{i=1}^{k} t_i \frac{\partial f}{\partial t_i}(0) \right) = \frac{1}{2} \sum_{i=1}^{k} t_i^2 \frac{\partial^2 f}{\partial t_i^2}(0)$$

uniformly with respect to $\mathbf{t} \in [0,1]^k$.

Proof For $\varphi = \mathbf{1}$ in Corollary 19, we have $\lim\limits_{n\to\infty} \frac{\mathcal{C}_X^n(f)(\mathbf{t}) - P_{n,2}(f)(\mathbf{t})}{M^n} = 0$ uniformly with respect to $\mathbf{t} \in [0,1]^k$, where $M = \max\limits_{1\le i\le k} \beta_{ii}$ and

$$P_{n,2}(f)(\mathbf{t}) = \alpha^n f(0) + \sum_{i=1}^{k} \alpha_i^n t_i \frac{\partial f}{\partial t_i}(0) + \frac{1}{2} \sum_{i,j=1}^{k} \beta_{ij}^n t_i t_j \frac{\partial^2 f}{\partial t_i \partial t_j}(0).$$

In this case $\alpha = \int_{[0,1]^k} \mathbf{1} \mathbf{ds} = 1$,

$$\alpha_i = \int_{[0,1]^k} s_i \mathbf{ds} = \frac{1}{2},\ \beta_{ii} = \int_{[0,1]^k} s_i^2 \mathbf{ds} = \frac{1}{3};\ \beta_{ij} = \int_{[0,1]^k} s_i s_j \mathbf{ds} = \frac{1}{4},\ i \neq j.$$

We obtain

$$P_{n,2}(f)(\mathbf{t}) = f(0) + \frac{1}{2^n} \sum_{i=1}^{k} t_i \frac{\partial f}{\partial t_i}(0) + \frac{1}{2\cdot 3^n} \sum_{i=1}^{k} t_i^2 \frac{\partial^2 f}{\partial t_i^2}(0) + \frac{1}{2\cdot 4^n} \sum_{i,j=1,i\neq j}^{k} t_i t_j \frac{\partial^2 f}{\partial t_i \partial t_j}(0).$$

From these relations we easily obtain the statement. □

Corollary 21 *Let* $T_k = \left\{ (s_1, \ldots, s_k) \in \mathbb{R}^k \mid s_1 \ge 0, \ldots, s_k \ge 0, s_1 + \cdots + s_k \le 1 \right\}$ *and* $\mathcal{C}_X : C(T_k, X) \to C(T_k, X)$ *be the multivariate Cesàro operator defined by*

$$\mathcal{C}_X(f)(\mathbf{t}) = \int_{T_k} f(\mathbf{st})\, \mathbf{ds}.$$

Then for every function $f : D \to X$ twice differentiable at 0*, we have*

$$\lim_{n\to\infty} \frac{(k+1)^n (k+2)^n}{2^n} \left((k!)^n \mathcal{C}_{X,\varphi}^n (f)(\mathbf{t}) - f(0) - \frac{1}{(k+1)^n} \sum_{i=1}^{k} t_i \frac{\partial f}{\partial t_i}(0) \right)$$
$$= \frac{1}{2} \sum_{i=1}^{k} t_i^2 \frac{\partial^2 f}{\partial t_i^2}(0) \text{ uniformly with respect to } \mathbf{t} \in T_k.$$

Proof For $a > 0$ let $T_{a,k} = \left\{ (s_1, \dots, s_k) \in \mathbb{R}^k \mid s_1 \geq 0, \dots, s_k \geq 0, s_1 + \cdots + s_k \leq a \right\}$. We have shown in the proof of Corollary 13 that $\alpha = \frac{1}{k!}, \alpha_i = \frac{1}{(k+1)!}$. In addition,

$$\beta_{ii} = \beta_{11} = \int_{T_k} s_1^2 \mathbf{ds} = \int_0^1 s_1^2 \mathrm{d}s_1 \int_{T_{1-s_1,k-1}} \mathrm{d}s_2 \cdots \mathrm{d}s_k$$
$$= \frac{1}{(k-1)!} \int_0^1 s_1^2 (1-s_1)^{k-1} \mathrm{d}s_1 = \frac{2}{(k+2)!}.$$

Also

$$\beta_{ij} = \beta_{12} = \int_{T_k} s_1 s_2 \mathbf{ds} = \int_0^1 s_1 \mathrm{d}s_1 \int_{T_{1-s_1,k-1}} s_2 \mathrm{d}s_2 \cdots$$
$$\mathrm{d}s_k = \frac{1}{k!} \int_0^1 s_1 (1-s_1)^k \mathrm{d}s_1 = \frac{1}{(k+2)!}.$$

We have used that

$$\int_{T_{a,k-1}} s_2 \mathrm{d}s_2 \cdots \mathrm{d}s_k = \int_0^a s_2 \mathrm{d}s_2 \int_{T_{a-s_2,k-2}} \mathrm{d}s_3 \cdots \mathrm{d}s_k$$
$$= \frac{1}{(k-2)!} \int_0^a s_2 (a-s_2)^{k-2} = \frac{a^k}{k!}.$$

From Corollary 19 we have

$$\lim_{n \to \infty} \frac{C^n_{X,\varphi}(f)(\mathbf{t}) - P_{n,2}(f)(\mathbf{t})}{M^n} = 0$$

uniformly with respect to $\mathbf{t} \in T_k$, where $M = \max\limits_{1 \leq i \leq k} \beta_{ii} = \frac{2}{(k+2)!}$ and

$$P_{n,2}(f)(\mathbf{t}) = \alpha^n f(0) + \sum_{i=1}^k \alpha_i^n t_i \frac{\partial f}{\partial t_i}(0)$$
$$+ \frac{1}{2} \sum_{i=1}^k \beta_{ii}^n t_i^2 \frac{\partial^2 f}{\partial t_i^2}(0) + \frac{1}{2} \sum_{i,j=1, i \neq j}^k \beta_{ij}^n t_i t_j \frac{\partial^2 f}{\partial t_i \partial t_j}(0).$$

To finish the proof, let us note that

$$P_{n,2}(f)(\mathbf{t}) = \frac{1}{(k!)^n} \left(f(0) + \frac{A}{(k+1)^n} + \frac{2^n B}{(k+1)^n (k+2)^n} + \frac{C}{(k+1)^n (k+2)^n} \right),$$

where $A = \sum\limits_{i=1}^k t_i \frac{\partial f}{\partial t_i}(0)$, $B = \frac{1}{2} \sum\limits_{i=1}^k t_i^2 \frac{\partial^2 f}{\partial t_i^2}(0)$, $C = \frac{1}{2} \sum\limits_{i,j=1,i \neq j}^k t_i t_j \frac{\partial^2 f}{\partial t_i \partial t_j}(0)$. □

Corollary 22 *Let $\mathcal{V}_{X,\varphi} : C(\Lambda_k, X) \to C(\Lambda_k, X)$ be the multivariate Volterra type operator defined by*

$$\mathcal{V}_{X,\varphi}(f)(\mathbf{t}) = P(\mathbf{t}) \int_{\Lambda_k} \varphi(\mathbf{s}) f(\mathbf{st}) \, \mathrm{d}\mathbf{s}.$$

Let us suppose that $\lim_{n\to\infty} \frac{\prod_{m=1}^{n-1} \int_{\Lambda_k} s_i^2 s_j \varphi(\mathbf{s}) P^m(\mathbf{s}) \mathrm{d}\mathbf{s}}{\prod_{m=0}^{n-1} \int_{\Lambda_k} s_i^2 \varphi(\mathbf{s}) P^m(\mathbf{s}) \mathrm{d}\mathbf{s}} = 0$ *for every $i, j = 1, \ldots, k$. Then for every function $f : D \to X$ twice differentiable at 0, we have*

$$\lim_{n\to\infty} \frac{\mathcal{V}_{X,\varphi}^n(f)(\mathbf{t}) - P^n(\mathbf{t}) a_n f(0) - P^n(\mathbf{t}) \sum_{i=1}^k b_{ni} t_i \frac{\partial f}{\partial t_i}(0) - \frac{P^n(\mathbf{t})}{2} \sum_{i,j=1}^k c_{nij} t_i t_j \frac{\partial^2 f}{\partial t_i \partial t_j}(0)}{\sum_{i=1}^k \left(\int_{\Lambda_k} s_i^2 \varphi(s) \, \mathrm{d}s\right)\left(\int_{\Lambda_k} s_i^2 \varphi(s) P(s) \, \mathrm{d}s\right) \cdots \left(\int_{\Lambda_k} s_i^2 \varphi(s) P^{n-1}(s) \, \mathrm{d}s\right)} = 0$$

uniformly with respect to $\mathbf{t} \in \Lambda_k$, where

$$a_n = \prod_{m=0}^{n-1} \int_{\Lambda_k} \varphi(\mathbf{s}) P^m(\mathbf{s}) \, \mathrm{d}\mathbf{s},$$

$$b_{ni} = \prod_{m=0}^{n-1} \int_{\Lambda_k} s_i \varphi(\mathbf{s}) P^m(\mathbf{s}) \, \mathrm{d}\mathbf{s},$$

$$c_{nij} = \prod_{m=0}^{n-1} \int_{\Lambda_k} s_i s_j \varphi(\mathbf{s}) P^m(\mathbf{s}) \, \mathrm{d}\mathbf{s}.$$

Proof Let $\beta \geq 0$. For every $i = 1, \ldots, k$, by induction on n, we have

$$\mathcal{V}_\varphi^n\left(p_i^\beta\right) = \left(\prod_{m=0}^{n-1} \int_{\Lambda_k} s_i^\beta \varphi(\mathbf{s}) P^m(\mathbf{s})\right) p_i^\beta P^n,$$

$$\mathcal{V}_\varphi^n\left(p_i^2 p_j\right) = \left(\prod_{m=1}^{n-1} \int_{\Lambda_k} s_i^2 s_j \varphi(\mathbf{s}) P^m(\mathbf{s}) \, \mathrm{d}\mathbf{s}\right) p_i^2 p_j P^n.$$

Let $i, j = 1, \ldots, k$. For every $n \in \mathbb{N}$, we have

$$\frac{\left\|\mathcal{V}_\varphi^n\left(p_i^2 p_j\right)\right\|}{\left\|\mathcal{V}_\varphi^n\left(p_i^2\right)\right\|} = \frac{\prod_{m=1}^{n-1} \int_{\Lambda_k} s_i^2 s_j \varphi(\mathbf{s}) P^m(\mathbf{s}) \, \mathrm{d}\mathbf{s}}{\prod_{m=0}^{n-1} \int_{\Lambda_k} s_i^2 \varphi(\mathbf{s}) P^m(\mathbf{s}) \, \mathrm{d}\mathbf{s}}$$

and by the hypothesis, $\lim_{n\to\infty} \frac{\left\| \mathcal{V}_{\varphi}^{n}(p_i^2 p_j) \right\|}{\left\| \mathcal{V}_{\varphi}^{n}(p_i^2) \right\|} = 0$. From Corollary 10, for every function $f : D \to \mathbb{R}$ twice differentiable at 0, we have

$$\lim_{n\to\infty} \frac{\mathcal{V}_{X,\varphi}^{n}(f)(\mathbf{t}) - S_{n,2}(f)(\mathbf{t})}{\sum_{i=1}^{k} \left\| \mathcal{V}_{\varphi}^{n}\left(p_i^2\right) \right\|} = 0$$

uniformly with respect to $\mathbf{t} \in \Lambda_k$, where

$$S_{n,2}(f)(\mathbf{t}) := \mathcal{V}_{\varphi}^{n}(\mathbf{1})(\mathbf{t}) f(0) + \sum_{i=1}^{k} \mathcal{V}_{\varphi}^{n}(p_i)(\mathbf{t}) \frac{\partial f}{\partial t_i}(0) + \frac{1}{2} \sum_{i,j=1}^{k} \mathcal{V}_{\varphi}^{n}\left(p_i p_j\right)(\mathbf{t}) \frac{\partial^2 f}{\partial t_i \partial t_j}(0).$$

By simple calculation, we deduce that

$$S_{n,2}(f)(\mathbf{t}) = P^n(\mathbf{t}) \left(a_n f(0) + \sum_{i=1}^{k} b_{ni} t_i \frac{\partial f}{\partial t_i}(0) + \frac{1}{2} \sum_{i,j=1}^{k} c_{nij} t_i t_j \frac{\partial^2 f}{\partial t_i \partial t_j}(0) \right),$$

which completes the proof. □

Corollary 23 *Let* $\mathcal{V}_X : C\left([0,1]^k, X\right) \to C\left([0,1]^k, X\right)$ *be the multivariate Cesàro operator defined by*

$$\mathcal{V}_X(f)(\mathbf{t}) = \int_{[0,1]^k} f(\mathbf{st})\, \mathrm{d}\mathbf{s}.$$

Then for every function $f : D \to X$ *twice differentiable at* 0*, we have*

$$\lim_{n\to\infty} \left[(n!)^k (n+1)(n+2) \left(\mathcal{V}_X^n(f)(\mathbf{t}) - \frac{P^n(\mathbf{t}) f(0)}{(n!)^k} - \frac{P^n(\mathbf{t}) A}{(n!)^k (n+1)} \right) - \frac{P^n(\mathbf{t}) B}{2} \right] = 0$$

uniformly with respect to $\mathbf{t} \in [0,1]^k$*, where* $A = \sum_{i=1}^{k} t_i \frac{\partial f}{\partial t_i}(0)$*,* $B = \sum_{i=1}^{k} t_i^2 \frac{\partial^2 f}{\partial t_i^2}(0)$ *and thus* $\lim_{n\to\infty} (n!)^k (n+1)(n+2) \left(\mathcal{V}^n(f)(\mathbf{t}) - f(0) \frac{(t_1 t_2 \cdots t_k)^n}{(n!)^k} - \sum_{i=1}^{k} \frac{\partial f}{\partial t_i}(0) \frac{t_i (t_1 t_2 \cdots t_k)}{(n+1)(n!)^k} \right)$

$$= \begin{cases} 0 & \text{if } (t_1, \ldots, t_k) \neq (1, \ldots, 1), \\ \sum_{i,j=1}^{k} \frac{\partial^2 f}{\partial t_i^2}(0) & \text{if } (t_1, \ldots, t_k) = (1, \ldots, 1). \end{cases}$$

Proof Take $\varphi = 1$ in Corollary 22. With the same notation, we have

$$\int_{[0,1]^k} P^m(\mathbf{s})\, d\mathbf{s} = \frac{1}{(m+1)^k}, a_n = \frac{1}{(n!)^k},$$
$$\int_{[0,1]^k} s_i P^m(\mathbf{s})\, d\mathbf{s} = \frac{1}{(m+1)^{k-1}(m+2)}, b_{ni} = \frac{1}{(n!)^k (n+1)},$$

and similarly,

$$\int_{[0,1]^k} s_i^2 P^m(\mathbf{s})\, d\mathbf{s} = \frac{1}{(m+1)^{k-1}(m+3)}, c_{nii} = \frac{2}{(n!)^k (n+1)(n+2)},$$
$$\left(\int_{[0,1]^k} s_i^2 d\mathbf{s}\right)\left(\int_{[0,1]^k} s_i^2 P(\mathbf{s})\, d\mathbf{s}\right)\cdots\left(\int_{[0,1]^k} s_i^2 P^{n-1}(\mathbf{s})\, d\mathbf{s}\right) = \frac{2}{(n!)^k (n+1)},$$

and

$$\int_{[0,1]^k} s_i s_j P^m(\mathbf{s})\, d\mathbf{s} = \frac{1}{(m+1)^{k-2}(m+2)^2}; c_{nij} = \frac{1}{(n!)^k (n+1)^2}.$$

From Corollary 22, we have

$$\lim_{n\to\infty} \frac{\mathcal{V}_X^n(f)(\mathbf{t}) - S_{n,2}(f)(\mathbf{t})}{\frac{2k}{(n!)^k (n+1)(n+2)}} = 0$$

uniformly with respect to $\mathbf{t} \in [0,1]^k$, where

$$S_{n,2}(f)(\mathbf{t}) = \frac{P^n(\mathbf{t})}{(n!)^k}\left(f(0) + \frac{A}{n+1} + \frac{B}{(n+1)(n+2)} + \frac{C}{(n+1)^2(n+2)^2}\right)$$

and $A = \sum_{i=1}^k t_i \frac{\partial f}{\partial t_i}(0)$, $B = \sum_{i=1}^k t_i^2 \frac{\partial^2 f}{\partial t_i^2}(0)$, $C = \frac{1}{2}\sum_{i,j=1, i\neq j}^k t_i t_j \frac{\partial^2 f}{\partial t_i \partial t_j}(0)$. From these relations, we get the first part of the conclusion. The second part is obvious. □

Acknowledgements We would like to express our gratitude to the two reviewers for their very careful reading of the manuscript and their many valuable and constructive comments that have improved the final version of the paper.

References

1. Altomare, F., Campiti, M.: Korovkin-Type Approximation Theory and its Applications. de Gruyter Studies in Mathematics, vol. 17. Walter de Gruyter & Co., Berlin (1994)
2. Bourbaki, N.: Functions of a real variable: elementary theory, Trans. from the 1976 French original by Philip Spain, Berlin, Springer (1976)
3. Cartan, H.: Calcul Differentiel. Hermann, Paris (1967)
4. Defant, A., Floret, K.: Tensor Norms and Operator Ideals. North-Holland Math. Studies, vol. 176. North-Holland Publishing Co., Amsterdam (1993)

5. Diestel, J., Uhl, J.J.: Vector Measures. Mathematical Surveys, No. 15. American Mathematical Society, Providence (1977)
6. Galaz Fontes, F., Solís, F.J.: Iterating the Ces àro operators. Proc. Am. Math. Soc. **136**, 2147–2153 (2008)
7. Gavrea, I., Ivan, M.: The iterates of positive linear operators preserving constants. Appl. Math. Lett. **24**, 2068–2071 (2011)
8. Karlin, S., Ziegler, Z.: Iteration of positive approximation operators. J. Approx. Theory **3**, 310–339 (1970)
9. Kelisky, R.P., Rivlin, T.J.: Iterates of Bernstein polynomials. Pac. J. Math. **21**, 511–520 (1967)
10. Lomeli, H.E., Garcia, C.L.: Variations on a Theorem of Korovkin. Am. Math. Mon. **113**, 744–750 (2006)
11. Niculescu, C.P.: An overview of absolute continuity and its applications, Inequalities and applications. Internat. Ser. Numer. Math., vol. 157, pp. 201–214. Birkhäuser, Basel (2009)
12. Ryan, R.A.: Introduction to Tensor Products of Banach Spaces. Springer Monographs in Mathematics. Springer, London (2002)