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CONSTRUCTIVE APPROXIMATION

Positive L^p-Bounded Dunkl-Type Generalized Translation Operator and Its Applications

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Abstract We prove that the spherical mean value of the Dunkl-type generalized translation operator τ^y is a positive L^p-bounded generalized translation operator T^t. As applications, we prove the Young inequality for a convolution defined by T^t, the L^p-boundedness of τ^y on radial functions for $p > 2$, the L^p-boundedness of the Riesz potential for the Dunkl transform, and direct and inverse theorems of approximation theory in L^p-spaces with the Dunkl weight.

Keywords Dunkl transform · Generalized translation operator · Convolution · Riesz potential

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1 Introduction

During the last three decades, many important elements of harmonic analysis with Dunkl weight on $\mathbb{R}^d$ and $\mathbb{S}^{d-1}$ were proved; see, e.g., the papers by Dunkl [14–16], Rösler [40–43], de Jeu [24,25], Trimèche [52,53], Xu [54,55], and the recent works [1,11,12,19,20].

Yet there are still several gaps in our knowledge of Dunkl harmonic analysis. In particular, Young's convolution inequality, several important polynomial inequalities, and basic approximation estimates are not established in the general case. One of the main reasons is the lack of tools related to the translation operator. Needless to say, the standard translation operator $f \mapsto f(\cdot + y)$ plays a crucial role both in classical approximation theory and harmonic analysis, in particular, by introducing several smoothness characteristics of f. In Dunkl analysis, its analogue is the generalized translation operator τ^y defined by Rösler [40]. Unfortunately, the L^p-boundedness of τ^y is not obtained in general.

To overcome this difficulty, the spherical mean value of the translation operator τ^y was introduced in [28] and was studied in [42], where, in particular, its positivity was shown. Our main goal in this paper is to prove that this operator is a positive L^p-bounded operator T^t, which may be considered as a generalized translation operator. It is worth mentioning that this operator can be applied to problems where it is essential to deal with radial multipliers. This is because by virtue of T^t we can define the convolution operator that coincides with the known convolution introduced by Thangavelu and Xu in [48] using the operator τ^y.

For this convolution, we prove the Young inequality and, subsequently, an L^p-boundedness of the operator τ^y on radial functions for $p > 2$. For $1 \leq p \leq 2$ it was proved in [48].

Let us mention here two applications of the operator T^t. The first one is the Riesz potential defined in [49], where its boundedness properties were obtained for the reflection group $\mathbb{Z}_2^d$. For the general case see [21]. Using the L^p-boundedness of the operator T^t allows us to give a different simple proof, which follows ideas of Thangavelu and Xu [49]. Another application is basic inequalities of approximation theory in the weighted L^p spaces. With the help of the operator T^t one can define moduli of smoothness, which are equivalent to the K-functionals, and prove the direct and inverse approximation theorems. For the reflection group $\mathbb{Z}_2^d$, basic approximation inequalities were studied in [11,12].

The paper is organized as follows. In the next section, we give some basic notation and facts of Dunkl harmonic analysis. In Sect. 3, we study the operator T^t, define a convolution operator, and prove the Young inequality. As a consequence, we obtain an L^p-boundedness of the operator τ^y on radial functions. The weighted Riesz potential is studied in Sect. 4. Section 5 consists of a study of interrelation between several classes of entire functions. We also obtain multidimensional weighted analogues of Plancherel–Polya–Boas inequalities, which are of their own interest. In Sect. 6, we introduce moduli of smoothness and the K-functional, associated to the Dunkl weight, and prove equivalence between them as well as the Jackson inequality. Section 7 consists of weighted analogues of Nikol'skiǐ, Bernstein, and Boas inequalities for entire functions of exponential type. In Sect. 8, we obtain that moduli of smoothness are equivalent to the realization of the K-functional. We conclude with Sect. 9, where we prove the inverse theorems in L^p-spaces with the Dunkl weight.

2 Notation

In this section, we recall the basic notation and results of Dunkl harmonic analysis, see, e.g., [43].

Throughout the paper, $\langle x, y\rangle$ denotes the standard Euclidean scalar product in d-dimensional Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$, equipped with a norm $|x| = \sqrt{\langle x, x\rangle}$. For $r > 0$ we write $B_r = \{x \in \mathbb{R}^d : |x| \le r\}$. Define the following function spaces:

- $C(\mathbb{R}^d)$ the space of continuous functions,
- $C_b(\mathbb{R}^d)$ the space of bounded continuous functions with the norm $\|f\|_\infty = \sup_{\mathbb{R}^d} |f|$,
- $C_0(\mathbb{R}^d)$ the space of continuous functions that vanish at infinity,
- $C^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions,
- $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space,
- $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions,
- $X(\mathbb{R}_+)$ the space of even functions from $X(\mathbb{R})$, where X is one of the spaces above,
- $X_{\mathrm{rad}}(\mathbb{R}^d)$ the subspace of $X(\mathbb{R}^d)$ consisting of radial functions $f(x) = f_0(|x|)$.

Let a finite subset $R \subset \mathbb{R}^d \setminus \{0\}$ be a root system; R_+ a positive subsystem of R; $G(R) \subset O(d)$ the finite reflection group, generated by reflections $\{\sigma_a : a \in R\}$, where σ_a is a reflection with respect to hyperplane $\langle a, x\rangle = 0$; $k : R \to \mathbb{R}_+$ a G-invariant multiplicity function. Recall that a finite subset $R \subset \mathbb{R}^d \setminus \{0\}$ is called a root system if

$$R \cap \mathbb{R}a = \{a, -a\} \quad \text{and} \quad \sigma_a R = R \quad \text{for all } a \in R.$$

Let

$$v_k(x) = \prod_{a \in R_+} |\langle a, x\rangle|^{2k(a)}$$

be the Dunkl weight,

$$c_k^{-1} = \int_{\mathbb{R}^d} e^{-|x|^2/2} v_k(x)\,\mathrm{d}x, \quad \mathrm{d}\mu_k(x) = c_k v_k(x)\,\mathrm{d}x,$$

and $L^p(\mathbb{R}^d, \mathrm{d}\mu_k), 0 < p < \infty$, be the space of complex-valued Lebesgue measurable functions f for which

$$\|f\|_{p,\mathrm{d}\mu_k} = \left(\int_{\mathbb{R}^d} |f|^p \,\mathrm{d}\mu_k\right)^{1/p} < \infty.$$

We also assume that $L^\infty \equiv C_b$ and $\|f\|_{\infty,\mathrm{d}\mu_k} = \|f\|_\infty$.

Example If the root system R is $\{\pm e_1, \ldots, \pm e_d\}$, where $\{e_1, \ldots, e_d\}$ is an orthonormal basis of $\mathbb{R}^d$, then $v_k(x) = \prod_{j=1}^d |x_j|^{2k_j}$, $k_j \geq 0$, $G = \mathbb{Z}_2^d$.

Let

$$D_j f(x) = \frac{\partial f(x)}{\partial x_j} + \sum_{a \in R_+} k(a)\langle a, e_j\rangle \frac{f(x) - f(\sigma_a x)}{\langle a, x\rangle}, \quad j = 1, \ldots, d,$$

be differential-differences Dunkl operators and $\Delta_k = \sum_{j=1}^d D_j^2$ be the Dunkl Laplacian. The Dunkl kernel $e_k(x, y) = E_k(x, iy)$ is a unique solution of the system

$$D_j f(x) = iy_j f(x), \quad j = 1, \ldots, d, \quad f(0) = 1,$$

and it plays the role of a generalized exponential function. Its properties are similar to those of the classical exponential function $e^{i\langle x,y\rangle}$. Several basic properties follow from an integral representation [41]:

$$e_k(x, y) = \int_{\mathbb{R}^d} e^{i\langle \xi,y\rangle} \,\mathrm{d}\mu_x^k(\xi),$$

where μ_x^k is a probability Borel measure, whose support is contained in

$$\mathrm{co}(\{gx \colon g \in G(R)\}),$$

the convex hull of the G-orbit of x in $\mathbb{R}^d$. In particular, $|e_k(x, y)| \leq 1$.

For $f \in L^1(\mathbb{R}^d, \mathrm{d}\mu_k)$, the Dunkl transform is defined by the equality

$$\mathcal{F}_k(f)(y) = \int_{\mathbb{R}^d} f(x)\overline{e_k(x, y)} \,\mathrm{d}\mu_k(x).$$

For $k \equiv 0$, $\mathcal{F}_0$ is the classical Fourier transform $\mathcal{F}$. We also note that $\mathcal{F}_k(e^{-|\cdot|^2/2})(y) = e^{-|y|^2/2}$ and $\mathcal{F}_k^{-1}(f)(x) = \mathcal{F}_k(f)(-x)$. Let

$$\mathcal{A}_k = \left\{f \in L^1(\mathbb{R}^d, \mathrm{d}\mu_k) \cap C_0(\mathbb{R}^d) \colon \mathcal{F}_k(f) \in L^1(\mathbb{R}^d, \mathrm{d}\mu_k)\right\}. \tag{2.1}$$

Let us now list several basic properties of the Dunkl transform.

Proposition 2.1 (1) *For* $f \in L^1(\mathbb{R}^d, \mathrm{d}\mu_k)$, $\mathcal{F}_k(f) \in C_0(\mathbb{R}^d)$.
(2) *If* $f \in \mathcal{A}_k$, *we have the pointwise inversion formula*

$$f(x) = \int_{\mathbb{R}^d} \mathcal{F}_k(f)(y) e_k(x, y)\, \mathrm{d}\mu_k(y).$$

(3) *The Dunkl transform leaves the Schwartz space* $\mathcal{S}(\mathbb{R}^d)$ *invariant.*
(4) *The Dunkl transform extends to a unitary operator in* $L^2(\mathbb{R}^d, \mathrm{d}\mu_k)$.

Let $\lambda \geq -1/2$ and $J_\lambda(t)$ be the classical Bessel function of degree λ and

$$j_\lambda(t) = 2^\lambda \Gamma(\lambda + 1) t^{-\lambda} J_\lambda(t)$$

be the normalized Bessel function. Set

$$b_\lambda^{-1} = \int_0^\infty e^{-t^2/2} t^{2\lambda+1}\, \mathrm{d}t = 2^\lambda \Gamma(\lambda + 1), \quad \mathrm{d}\nu_\lambda(t) = b_\lambda t^{2\lambda+1}\, \mathrm{d}t, \quad t \in \mathbb{R}_+.$$

The norm in $L^p(\mathbb{R}_+, \mathrm{d}\nu_\lambda)$, $1 \leq p < \infty$, is given by

$$\|f\|_{p,\mathrm{d}\nu_\lambda} = \left(\int_{\mathbb{R}_+} |f(t)|^p\, \mathrm{d}\nu_\lambda(t) \right)^{1/p}.$$

Define $\|f\|_\infty = \operatorname{ess\,sup}_{t \in \mathbb{R}_+} |f(t)|$.

The Hankel transform is defined as follows:

$$\mathcal{H}_\lambda(f)(r) = \int_{\mathbb{R}_+} f(t) j_\lambda(rt)\, \mathrm{d}\nu_\lambda(t), \quad r \in \mathbb{R}_+.$$

It is a unitary operator in $L^2(\mathbb{R}_+, \mathrm{d}\nu_\lambda)$ and $\mathcal{H}_\lambda^{-1} = \mathcal{H}_\lambda$ [2, Chap. 7].

Note that if $\lambda = d/2 - 1$, the Hankel transform is a restriction of the Fourier transform on radial functions, and if $\lambda = \lambda_k = d/2 - 1 + \sum_{a \in R_+} k(a)$, of the Dunkl transform.

Let $\mathbb{S}^{d-1} = \{x' \in \mathbb{R}^d : |x'| = 1\}$ be the Euclidean sphere and $\mathrm{d}\sigma_k(x') = a_k v_k(x')\, \mathrm{d}x'$ be the probability measure on $\mathbb{S}^{d-1}$. We have

$$\int_{\mathbb{R}^d} f(x)\, \mathrm{d}\mu_k(x) = \int_0^\infty \int_{\mathbb{S}^{d-1}} f(tx')\, \mathrm{d}\sigma_k(x')\, \mathrm{d}\nu_{\lambda_k}(t). \tag{2.2}$$

We need the following partial case of the Funk–Hecke formula [55]

$$\int_{\mathbb{S}^{d-1}} e_k(x, ty')\, \mathrm{d}\sigma_k(y') = j_{\lambda_k}(t|x|). \tag{2.3}$$

Throughout the paper, we will assume that $A \lesssim B$ means that $A \leq CB$ with a constant C depending only on nonessential parameters.

3 Generalized Translation Operators and Convolutions

Let $y \in \mathbb{R}^d$ be given. Rösler [40] defined a generalized translation operator τ^y in $L^2(\mathbb{R}^d, \mathrm{d}\mu_k)$ by the equation

$$\mathcal{F}_k(\tau^y f)(z) = e_k(y, z)\mathcal{F}_k(f)(z).$$

Since $|e_k(y, z)| \le 1$, we have $\|\tau^y\|_{2\to 2} \le 1$. If $f \in \mathcal{A}_k$ (recall that $\mathcal{A}_k$ is given by (2.1)), then, for any $x, y \in \mathbb{R}^d$,

$$\tau^y f(x) = \int_{\mathbb{R}^d} e_k(y, z)e_k(x, z)\mathcal{F}_k(f)(z)\,\mathrm{d}\mu_k(z). \tag{3.1}$$

Note that $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{A}_k \subset L^2(\mathbb{R}^d, \mathrm{d}\mu_k)$. Trimèche [53] extended the operator τ^y on $C^\infty(\mathbb{R}^d)$.

The explicit expression of $\tau^y f$ is known only in the case of the reflection group $\mathbb{Z}_2^d$. In particular, in this case $\tau^y f$ is not a positive operator [39]. Note that in the case of symmetric group S_d, the operator $\tau^y f$ is also not positive [48].

It remains an open question whether $\tau^y f$ is an L^p bounded operator on $\mathcal{S}(\mathbb{R}^d)$ for $p \ne 2$. It is known [39,48] only for $G = \mathbb{Z}_2^d$. Note that a positive answer would follow from the L^1-boundedness.

Let

$$\lambda_k = d/2 - 1 + \sum_{a\in R_+} k(a).$$

We have $\lambda_k \ge -1/2$ and, moreover, $\lambda_k = -1/2$ only if $d = 1$ and $k \equiv 0$. In what follows, we assume that $\lambda_k > -1/2$.

Define another generalized translation operator $T^t\colon L^2(\mathbb{R}^d, \mathrm{d}\mu_k) \to L^2(\mathbb{R}^d, \mathrm{d}\mu_k)$, $t \in \mathbb{R}$, by the relation

$$\mathcal{F}_k(T^t f)(y) = j_{\lambda_k}(t|y|)\mathcal{F}_k(f)(y).$$

Since $|j_{\lambda_k}(t)| \le 1$, it is a bounded operator such that $\|T^t\|_{2\to 2} \le 1$ and

$$T^t f(x) = \int_{\mathbb{R}^d} j_{\lambda_k}(t|y|)e_k(x, y)\mathcal{F}_k(f)(y)\,\mathrm{d}\mu_k(y).$$

This gives $T^t = T^{-t}$. If $f \in \mathcal{A}_k$, then from (2.3) and (3.1) we have (pointwise)

$$T^t f(x) = \int_{\mathbb{R}^d} j_{\lambda_k}(t|y|)e_k(x, y)\mathcal{F}_k(f)(y)\,\mathrm{d}\mu_k(y) = \int_{\mathbb{S}^{d-1}} \tau^{ty'} f(x)\,\mathrm{d}\sigma_k(y'). \tag{3.2}$$

Note that the operator T^t is self-adjoint. Indeed, if $f, g \in \mathcal{A}_k$, then

$$\int_{\mathbb{R}^d} T^t f(x)\, g(x)\,\mathrm{d}\mu_k(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} j_{\lambda_k}(t|y|)e_k(x, y)\mathcal{F}_k(f)(y)\,\mathrm{d}\mu_k(y)\, g(x)\,\mathrm{d}\mu_k(x)$$

$$= \int_{\mathbb{R}^d} j_{\lambda_k}(t|y|)\mathcal{F}_k(f)(y)\mathcal{F}_k(g)(-y)\,\mathrm{d}\mu_k(y)$$
$$= \int_{\mathbb{R}^d} j_{\lambda_k}(t|y|)\mathcal{F}_k(g)(y)\mathcal{F}_k(f)(-y)\,\mathrm{d}\mu_k(y)$$
$$= \int_{\mathbb{R}^d} f(x)\,T^t g(x)\,\mathrm{d}\mu_k(x).$$

Rösler [42] proved that the spherical mean (with respect to the Dunkl weight) of the operator τ^y, i.e., $\int_{\mathbb{S}^{d-1}} \tau^{ty'} f(x)\,\mathrm{d}\sigma_k(y')$, is a positive operator on $C^\infty(\mathbb{R}^d)$ and obtained its integral representation. This implies that T^t is a positive operator on $C^\infty(\mathbb{R}^d)$ and, moreover, for any $t \in \mathbb{R}$, $x \in \mathbb{R}^d$,

$$T^t f(x) = \int_{\mathbb{R}^d} f(z)\,\mathrm{d}\sigma_{x,t}^k(z), \tag{3.3}$$

where $\sigma_{x,t}^k$ is a probability Borel measure,

$$\operatorname{supp} \sigma_{x,t}^k \subset \bigcup_{g\in G} \{z \in \mathbb{R}^d : |z - gx| \le t\} \tag{3.4}$$

and the mapping $(x, t) \to \sigma_{x,t}^k$ is continuous with respect to the weak topology on probability measures.

The representation (3.3) gives a natural extension of the operator T^t on $C_b(\mathbb{R}^d)$; namely, for $f \in C_b(\mathbb{R}^d)$ we define $T^t f(x) \in C(\mathbb{R} \times \mathbb{R}^d)$ by (3.3), and, moreover, the estimate $\|T^t f\|_\infty \le \|f\|_\infty$ holds.

Note that for $k \equiv 0$, T^t is the usual spherical mean

$$T^t f(x) = S^t f(x) = \int_{\mathbb{S}^{d-1}} f(x + ty')\,\mathrm{d}\sigma_0(y'). \tag{3.5}$$

Theorem 3.1 *If $1 \le p \le \infty$, then, for any $t \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$\|T^t f\|_{p,\mathrm{d}\mu_k} \le \|f\|_{p,\mathrm{d}\mu_k}. \tag{3.6}$$

Remark 3.2 (i) The inequality $\|T^t f\|_{p,\mathrm{d}\mu_k} \le c\|f\|_{p,\mathrm{d}\mu_k}$ was proved in [48] for $G = \mathbb{Z}_2^d$.

(ii) $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$, $1 \le p < \infty$, so for any $t \in \mathbb{R}_+$ the operator T^t can be defined on $L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$ and estimate (3.6) holds.

(iii) If $d = 1$, $v_k(x) = |x|^{2\lambda+1}$, $\lambda > -1/2$, inequality (3.6) was proved in [7]. In this case the integral representation of T^t is of the form

$$T^t f(x) = \frac{c_\lambda}{2} \int_0^\pi \{f(A)(1 + B) + f(-A)(1 - B)\} \sin^{2\lambda} \varphi\,\mathrm{d}\varphi,$$

where, for $(x, t) \neq (0, 0)$,

$$c_\lambda = \frac{\Gamma(\lambda+1)}{\sqrt{\pi}\,\Gamma(\lambda+1/2)}, \quad A = \sqrt{x^2+t^2-2xt\cos\varphi}, \quad B = \frac{x-t\cos\varphi}{A}. \tag{3.7}$$

If $\lambda = -1/2$, i.e., $k \equiv 0$, then $T^t f(x) = \frac{1}{2}(f(x+t) + f(x-t))$.

Proof Let $t \in \mathbb{R}_+$ be given and the operator T^t be defined on $\mathcal{S}(\mathbb{R}^d)$ by (3.3). Using (3.2), we have

$$\sup\{\|T^t f\|_2 \colon f \in \mathcal{S}(\mathbb{R}^d),\ \|f\|_2 \le 1\} \le 1,$$

and T^t can be extended to the space $L^2(\mathbb{R}^d, \mathrm{d}\mu_k)$ with preservation of norm; moreover, this extension coincides with (3.2). Furthermore, (3.3) yields

$$\sup\{\|T^t f\|_\infty \colon f \in \mathcal{S}(\mathbb{R}^d),\ \|f\|_\infty \le 1\} \le 1. \tag{3.8}$$

Since the operator T^t is self-adjoint, by (3.8),

$$\begin{aligned}
&\sup\{\|T^t f\|_{1,\mathrm{d}\mu_k} \colon f \in \mathcal{S}(\mathbb{R}^d),\ \|f\|_{1,\mathrm{d}\mu_k} \le 1\} \\
&\quad = \sup\left\{\int_{\mathbb{R}^d} T^t f\, g\, \mathrm{d}\mu_k \colon f, g \in \mathcal{S}(\mathbb{R}^d),\ \|f\|_{1,\mathrm{d}\mu_k} \le 1,\ \|g\|_\infty \le 1\right\} \\
&\quad = \sup\left\{\int_{\mathbb{R}^d} f\, T^t g\, \mathrm{d}\mu_k \colon f, g \in \mathcal{S}(\mathbb{R}^d),\ \|f\|_{1,\mathrm{d}\mu_k} \le 1,\ \|g\|_\infty \le 1\right\} \\
&\quad = \sup\{\|T^t g\|_\infty \colon g \in \mathcal{S}(\mathbb{R}^d),\ \|g\|_\infty \le 1\} \le 1.
\end{aligned}$$

Hence, T^t can be extended to $L^1(\mathbb{R}^d, \mathrm{d}\mu_k)$ with preservation of the norm such that this extension coincides with (3.2) on $L^1(\mathbb{R}^d, \mathrm{d}\mu_k) \cap L^2(\mathbb{R}^d, \mathrm{d}\mu_k)$.

By the Riesz–Thorin interpolation theorem we obtain

$$\sup\{\|T^t f\|_{p,\mathrm{d}\mu_k} \colon f \in \mathcal{S}(\mathbb{R}^d),\ \|f\|_{p,\mathrm{d}\mu_k} \le 1\} \le 1, \quad 1 \le p \le 2.$$

Let $2 < p < \infty$, $1/p + 1/p' = 1$. As for $p = 1$, we get

$$\begin{aligned}
&\sup\{\|T^t f\|_{p,\mathrm{d}\mu_k} \colon f \in \mathcal{S}(\mathbb{R}^d),\ \|f\|_{p,\mathrm{d}\mu_k} \le 1\} \\
&\quad = \sup\{\|T^t g\|_{p',\mathrm{d}\mu_k} \colon g \in \mathcal{S}(\mathbb{R}^d),\ \|g\|_{p',\mathrm{d}\mu_k} \le 1\} \le 1.
\end{aligned}$$

□

For any $f_0 \in L^p(\mathbb{R}_+, \mathrm{d}\nu_\lambda)$, $1 \le p \le \infty$, $\lambda > -1/2$, let us define the Gegenbauer-type translation operator (see, e.g., [34,35])

$$R^t f_0(r) = c_\lambda \int_0^\pi f_0\left(\sqrt{r^2+t^2-2rt\cos\varphi}\right) \sin^{2\lambda}\varphi\, \mathrm{d}\varphi,$$

where c_λ is defined by (3.7). We have that $\|R^t\|_{p\to p} \le 1$ and $\mathcal{H}_\lambda(R^t f_0)(r) = j_\lambda(tr)\mathcal{H}_\lambda(f_0)(r)$, where $f_0 \in \mathcal{S}(\mathbb{R}_+)$. Taking into account (2.3) and (3.2), we note that for $\lambda = \lambda_k$ the operator R^t is a restriction of T^t on radial functions; that is, for $f_0 \in L^p(\mathbb{R}_+, \mathrm{d}\nu_{\lambda_k})$,

$$T^t f_0(|x|) = R^t f_0(r), \quad r = |x|.$$

We also mention the following useful properties of the generalized translation operator T^t.

Lemma 3.3 *Let* $t \in \mathbb{R}$.

(1) *If* $f \in L^1(\mathbb{R}^d, \mathrm{d}\mu_k)$, *then* $\int_{\mathbb{R}^d} T^t f \,\mathrm{d}\mu_k = \int_{\mathbb{R}^d} f \,\mathrm{d}\mu_k$.
(2) *Let* $r > 0$, $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$, $1 \le p < \infty$. *If* supp $f \subset B_r$, *then* supp $T^t f \subset B_{r+|t|}$. *If* supp $f \subset \mathbb{R}^d \setminus B_r$, $r > |t|$, *then* supp $T^t f \subset \mathbb{R}^d \setminus B_{r-|t|}$.

Proof Due to the L^p-boundedness of T^t and the density of $\mathcal{S}(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$, we can assume that $f \in \mathcal{S}(\mathbb{R}^d)$.

(1) Let $s > 0$. By integral representation of $j_{\lambda_k}(z)$ (see, e.g., [2, Sect. 7.12]) we have

$$\begin{aligned}T^t(e^{-s|\cdot|^2})(x) &= R^t(e^{-s(\cdot)^2})(|x|) = c_{\lambda_k}\int_0^\pi e^{-s(|x|^2+t^2-2|x|t\cos\varphi)}\sin^{2\lambda_k}\varphi\,\mathrm{d}\varphi\\ &= e^{-s(|x|^2+t^2)}c_{\lambda_k}\int_0^\pi e^{2s|x|t\cos\varphi}\sin^{2\lambda_k}\varphi\,\mathrm{d}\varphi\\ &= e^{-s(|x|^2+t^2)}j_{\lambda_k}(2is|x|t),\end{aligned}$$

and, in particular,

$$T^t(e^{-s|\cdot|^2})(x) \le e^{-s(|x|^2+t^2)}e^{2s|x|t} = e^{-s(|x|-t)^2} \le 1.$$

Using the self-adjointness of T^t, we obtain

$$\int_{\mathbb{R}^d} T^t f(x)\, e^{-s|x|^2}\,\mathrm{d}\mu_k(x) = \int_{\mathbb{R}^d} f(x) T^t(e^{-s|\cdot|^2})(x)\,\mathrm{d}\mu_k(x).$$

Since for any $t \in \mathbb{R}$, $x \in \mathbb{R}^d$,

$$\lim_{s\to 0} e^{-s|x|^2} = \lim_{s\to 0} T^t(e^{-s|\cdot|^2})(x) = 1,$$

by Lebesgue's dominated convergence theorem we derive (1).
(2) If supp $f \subset B_r$ and $|x| > r+|t|$, then, in light of (3.4) and (3.3), for $z \in \operatorname{supp}\sigma_{x,t}^k$ and $g \in G$, we have that

$$|z| \ge |gx| - |z - gx| = |x| - |z - gx| > r$$

and $f(z) = 0$, which yields $T^t f(x) = 0$.

If $\operatorname{supp} f \subset \mathbb{R}^d \setminus B_r$, $|x| < r - |t|$, then, for $z \in \operatorname{supp} \sigma_{x,t}^k$ and $g \in G$, we similarly obtain $|z| \le |gx| + |z - gx| = |x| + |z - gx| < r$, $f(z) = 0$, and $T^t f(x) = 0$. □

Let g be a radial function, $g(y) = g_0(|y|)$, where $g_0(t)$ is defined on $\mathbb{R}_+$. Note that by virtue of (2.2),

$$\|g\|_{p,\mathrm{d}\mu_k} = \|g_0\|_{p,\mathrm{d}\nu_{\lambda_k}}, \quad \mathcal{F}_k(g)(y) = \mathcal{H}_{\lambda_k}(g_0)(|y|). \tag{3.9}$$

By means of operators T^t and τ^y, define two convolution operators:

$$(f *_{\lambda_k} g_0)(x) = \int_0^\infty T^t f(x) g_0(t)\, \mathrm{d}\nu_{\lambda_k}(t), \tag{3.10}$$

$$(f *_k g)(x) = \int_{\mathbb{R}^d} f(y) \tau^x g(-y)\, \mathrm{d}\mu_k(y). \tag{3.11}$$

Note that operator (3.10) was defined in [48], while (3.11) was investigated in [48,53].

Thangavelu and Xu [48] proved that if $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$, $1 \le p \le \infty$, and $g \in L^1_{\mathrm{rad}}(\mathbb{R}^d, \mathrm{d}\mu_k)$, then

$$\|(f *_k g)\|_{p,\mathrm{d}\mu_k} \le \|f\|_{p,\mathrm{d}\mu_k} \|g\|_{1,\mathrm{d}\mu_k}, \tag{3.12}$$

and if $1 \le p \le 2$, $g \in L^p_{\mathrm{rad}}(\mathbb{R}^d, \mathrm{d}\mu_k)$, then, for any $y \in \mathbb{R}^d$,

$$\|\tau^y g\|_{p,\mathrm{d}\mu_k} \le \|g\|_{p,\mathrm{d}\mu_k}. \tag{3.13}$$

Lemma 3.4 *If $f \in \mathcal{A}_k$, $g_0 \in L^1(\mathbb{R}_+, \mathrm{d}\nu_{\lambda_k})$, $g(y) = g_0(|y|)$, then, for any $x, y \in \mathbb{R}^d$,*

$$(f *_{\lambda_k} g_0)(x) = (f *_k g)(x) = \int_{\mathbb{R}^d} \tau^{-y} f(x) g(y)\, \mathrm{d}\mu_k(y), \tag{3.14}$$

$$\mathcal{F}_k(f *_{\lambda_k} g_0)(y) = \mathcal{F}_k(f *_k g)(y) = \mathcal{F}_k(f)(y)\mathcal{F}_k(g)(y). \tag{3.15}$$

Proof Using (3.2) and (3.9), we get

$$\begin{aligned}(f *_{\lambda_k} g_0)(x) &= \int_0^\infty T^t f(x) g_0(t)\, \mathrm{d}\nu_{\lambda_k}(t)\\ &= \int_0^\infty \int_{\mathbb{R}^d} j_{\lambda_k}(t|y|) e_k(x, y) \mathcal{F}_k(f)(y)\, \mathrm{d}\mu_k(y) g_0(t)\, \mathrm{d}\nu_{\lambda_k}(t)\\ &= \int_{\mathbb{R}^d} e_k(x, y) \mathcal{F}_k(f)(y) \mathcal{F}_k(g)(y)\, \mathrm{d}\mu_k(y),\end{aligned}$$

which gives

$$\mathcal{F}_k(f *_{\lambda_k} g_0)(y) = \mathcal{F}_k(f)(y)\mathcal{F}_k(g)(y).$$

If $g \in \mathcal{A}_k$, then, by (3.1),

$$
\begin{aligned}
(f *_k g)(x) &= \int_{\mathbb{R}^d} f(y)\tau^x g(-y)\,\mathrm{d}\mu_k(y) \\
&= \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} e_k(-y,z)e_k(x,z)\mathcal{F}_k(g)(z)\,\mathrm{d}\mu_k(z)\,\mathrm{d}\mu_k(y) \\
&= \int_{\mathbb{R}^d} e_k(x,z)\mathcal{F}_k(f)(z)\mathcal{F}_k(g)(z)\,\mathrm{d}\mu_k(z),
\end{aligned}
$$

and hence the first equality in (3.14) and the second equality in (3.15) are valid for $g \in \mathcal{A}_k$.

Assuming that $g_0 \in L^1(\mathbb{R}_+, \mathrm{d}\nu_\lambda)$, $(g_n)_0 \in \mathcal{S}(\mathbb{R}_+)$, $g_n \to g$ in $L^1(\mathbb{R}^d, \mathrm{d}\mu_k)$, and taking into account (3.8)–(3.11) and (3.13), we arrive at

$$
\begin{aligned}
\left|(f *_{\lambda_k} g_0)(x) - (f *_k g)(x)\right| &\le \left|(f *_{\lambda_k} (g_0 - (g_n)_0))(x)\right| + |(f *_k (g - g_n))(x)| \\
&\le 2\|f\|_\infty \, \|g - g_n\|_{1,\mathrm{d}\mu_k}.
\end{aligned}
$$

Thus, the first equality in (3.14) holds.

Finally, using (3.1), we get

$$
\begin{aligned}
\int_{\mathbb{R}^d} \tau^{-y} f(x)g(y)\,\mathrm{d}\mu_k(y) &= \int_{\mathbb{R}^d} g(y) \int_{\mathbb{R}^d} e_k(-y,z)e_k(x,z)\mathcal{F}_k(f)(z)\,\mathrm{d}\mu_k(z)\,\mathrm{d}\mu_k(y) \\
&= \int_{\mathbb{R}^d} e_k(x,z)\mathcal{F}_k(f)(z)\mathcal{F}_k(g)(z)\,\mathrm{d}\mu_k(z),
\end{aligned}
$$

and the second part in (3.14) is valid. □

Let $y \in \mathbb{R}^d$ be given. Rösler [42] proved that the operator τ^y is positive on $C_{\mathrm{rad}}^\infty(\mathbb{R}^d)$, i.e., $\tau^y \ge 0$, and moreover, for any $x \in \mathbb{R}^d$,

$$
\tau^y f(x) = \int_{\mathbb{R}^d} f(z)\,\mathrm{d}\rho_{x,y}^k(z), \tag{3.16}
$$

where $\rho_{x,y}^k$ is a radial probability Borel measure such that $\operatorname{supp} \rho_{x,y}^k \subset B_{|x|+|y|}$.

Theorem 3.5 *If $1 \le p \le \infty$, then, for any $x \in \mathbb{R}^d$ and $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$
\|T^t f(x)\|_{p,\mathrm{d}\nu_{\lambda_k}} = \left(\int_{\mathbb{R}_+} |T^t f(x)|^p\,\mathrm{d}\nu_\lambda(t)\right)^{1/p} \le \|f\|_{p,\mathrm{d}\mu_k}. \tag{3.17}
$$

Proof Let $x \in \mathbb{R}^d$ be given. Let an operator B^x be defined on $\mathcal{S}(\mathbb{R}^d)$ as follows (cf. (3.2) and (3.3)): for $f \in \mathcal{S}(\mathbb{R}^d)$,

$$
B^x f(t) = T^t f(x) = \int_{\mathbb{R}^d} j_{\lambda_k}(t|y|)e_k(x,y)\mathcal{F}_k(f)(y)\,\mathrm{d}\mu_k(y) = \int_{\mathbb{R}^d} f(z)\,\mathrm{d}\sigma_{x,t}^k(z).
$$

Let $p=2$. We have

$$T^t f(x)=\int_0^\infty j_{\lambda_k}(tr)\int_{\mathbb{S}^{d-1}} e_k(x,ry')\mathcal{F}_k(f)(ry')\,\mathrm{d}\sigma_k(y')\,\mathrm{d}\nu_{\lambda_k}(r)$$

and

$$\mathcal{H}_{\lambda_k}(T^t f(x))(r)=\int_{\mathbb{S}^{d-1}} e_k(x,ry')\mathcal{F}_k(f)(ry')\,\mathrm{d}\sigma_k(y').$$

This, Hölder's inequality, and the fact that the operators $\mathcal{H}_{\lambda_k}$ and $\mathcal{F}_k$ are unitary imply

$$\begin{aligned}\|T^t f(x)\|_{2,\mathrm{d}\nu_{\lambda_k}}^2&=\|\mathcal{H}_{\lambda_k}(T^t f(x))(r)\|_{2,\mathrm{d}\nu_{\lambda_k}}^2\\&=\int_0^\infty\left|\int_{\mathbb{S}^{d-1}} e_k(x,ry')\mathcal{F}_k(f)(ry')\,\mathrm{d}\sigma_k(y')\right|^2\mathrm{d}\nu_{\lambda_k}(r)\\&\le\int_0^\infty\int_{\mathbb{S}^{d-1}}|\mathcal{F}_k(f)(ry')|^2\,\mathrm{d}\sigma_k(y')\,\mathrm{d}\nu_{\lambda_k}(r)\\&=\|\mathcal{F}_k(f)\|_{2,\mathrm{d}\mu_k}^2=\|f\|_{2,\mathrm{d}\mu_k}^2,\end{aligned}$$

which yields inequality (3.17) for $p=2$. Moreover, B^x can be extended to the space $L^2(\mathbb{R}_+,\mathrm{d}\nu_{\lambda_k})$ with preservation of norm, and, moreover, this extension coincides with (3.2).

Let $p=1$. By (3.14) and (3.16), we obtain

$$\begin{aligned}\|T^t f(x)\|_{1,\mathrm{d}\nu_{\lambda_k}}&=\sup\left\{\int_0^\infty T^t f(x)g_0(t)\,\mathrm{d}\nu_{\lambda_k}(t)\colon g_0\in\mathcal{S}(\mathbb{R}_+),\ \|g_0\|_\infty\le 1\right\}\\&=\sup\left\{\int_{\mathbb{R}^d} f(y)\tau^x g(-y)\,\mathrm{d}\mu_k(y)\colon g\in\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d),\ \|g\|_\infty\le 1\right\}\\&\le\|f\|_{1,\mathrm{d}\mu_k}\sup\left\{\|\tau^x g(-y)\|_\infty\colon g\in\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d),\ \|g\|_\infty\le 1\right\}\\&\le\|f\|_{1,\mathrm{d}\mu_k},\end{aligned}$$

which is the desired inequality (3.17) for $p=1$. Moreover, B^x can be extended to $L^1(\mathbb{R}_+,\mathrm{d}\nu_{\lambda_k})$ with preservation of norm such that the extension coincides with (3.2) on $L^1(\mathbb{R}_+,\mathrm{d}\nu_{\lambda_k})\cap L^2(\mathbb{R}_+,\mathrm{d}\nu_{\lambda_k})$.

By the Riesz–Thorin interpolation theorem we obtain (3.17) for $1<p<2$.

If $2<p<\infty$, $1/p+1/p'=1$, then by (3.14) and (3.13),

$$\begin{aligned}\|T^t f(x)\|_{p,\mathrm{d}\nu_{\lambda_k}}&=\sup\left\{\int_0^\infty T^t f(x)g_0(t)\,\mathrm{d}\nu_{\lambda_k}(t)\colon g_0\in\mathcal{S}(\mathbb{R}_+),\ \|g_0\|_{p',\mathrm{d}\nu_{\lambda_k}}\le 1\right\}\\&=\sup\left\{\int_{\mathbb{R}^d} f(y)\tau^x g(-y)\,\mathrm{d}\mu_k(y)\colon g\in\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d),\ \|g\|_{p',\mathrm{d}\mu_k}\le 1\right\}\\&\le\|f\|_{p,\mathrm{d}\mu_k}\sup\{\|\tau^x g(-y)\|_{p',\mathrm{d}\mu_k}\colon g\in\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d),\ \|g\|_{p',\mathrm{d}\mu_k}\le 1\}\\&\le\|f\|_{p,\mathrm{d}\mu_k}.\end{aligned}$$

Finally, for $p=\infty$, (3.17) follows from representation (3.3). □

We are now in a position to prove the Young inequality for the convolutions (3.10) and (3.11).

Theorem 3.6 *Let* $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} \ge 1$, *and* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. *We have that, for any* $f \in \mathcal{S}(\mathbb{R}^d)$, $g_0 \in \mathcal{S}(\mathbb{R}_+)$, *and* $g \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$,

$$\|(f *_{\lambda_k} g_0)\|_{r,\mathrm{d}\nu_{\lambda_k}} \le \|f\|_{p,\mathrm{d}\mu_k} \|g_0\|_{q,\mathrm{d}\nu_{\lambda_k}}, \tag{3.18}$$

$$\|(f *_k g)\|_{r,\mathrm{d}\mu_k} \le \|f\|_{p,\mathrm{d}\mu_k} \|g\|_{q,\mathrm{d}\mu_k}. \tag{3.19}$$

Proof Since for $g(y) = g_0(|y|)$ we have

$$\|(f *_{\lambda_k} g_0)\|_{r,\mathrm{d}\nu_{\lambda_k}} = \|(f *_k g)\|_{r,\mathrm{d}\mu_k}, \quad \|g_0\|_{q,\mathrm{d}\nu_{\lambda_k}} = \|g\|_{q,\mathrm{d}\mu_k},$$

it is enough to show inequality (3.18). The proof is straightforward using Hölder's inequality and estimates (3.6) and (3.17). For the sake of completeness, we give it here. Let $\frac{1}{\mu} = \frac{1}{p} - \frac{1}{r}$ and $\frac{1}{\nu} = \frac{1}{q} - \frac{1}{r}$, then $\frac{1}{\mu} \ge 0$, $\frac{1}{\nu} \ge 0$, and $\frac{1}{r} + \frac{1}{\mu} + \frac{1}{\nu} = 1$. In virtue of (3.17), we have

$$\begin{aligned}
\left|\int_0^\infty T^t f(x) g_0(t)\, \mathrm{d}\nu_{\lambda_k}(t)\right| &\le \left(\int_0^\infty |T^t f(x)|^p |g_0(t)|^q\, \mathrm{d}\nu_{\lambda_k}(t)\right)^{1/r} \\
&\quad \times \left(\int_0^\infty |T^t f(x)|^p\, \mathrm{d}\nu_{\lambda_k}(t)\right)^{1/\mu} \left(\int_0^\infty |g_0(t)|^q\, \mathrm{d}\nu_{\lambda_k}(t)\right)^{1/\nu} \\
&\le \left(\int_0^\infty |T^t f(x)|^p |g_0(t)|^q\, \mathrm{d}\nu_{\lambda_k}(t)\right)^{1/r} \|f\|_{p,\mathrm{d}\mu_k}^{p/\mu} \|g_0\|_{q,\mathrm{d}\nu_{\lambda_k}}^{q/\nu}.
\end{aligned}$$

Using (3.6), this gives

$$\begin{aligned}
\|(f *_{\lambda_k} g_0)\|_{r,\mathrm{d}\nu_{\lambda_k}} &\le \left(\int_{\mathbb{R}^d} \int_0^\infty |T^t f(x)|^p |g_0(t)|^q\, \mathrm{d}\nu_{\lambda_k}(t)\, \mathrm{d}\mu_k(x)\right)^{1/r} \\
&\quad \times \|f\|_{p,\mathrm{d}\mu_k}^{p/\mu} \|g_0\|_{q,\mathrm{d}\nu_{\lambda_k}}^{q/\nu} \le \|f\|_{p,\mathrm{d}\mu_k} \|g_0\|_{q,\mathrm{d}\nu_{\lambda_k}}.
\end{aligned}$$

□

Theorem 3.7 *Let* $1 \le p \le \infty$ *and* $g \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$. *We have that, for any* $y \in \mathbb{R}^d$,

$$\|\tau^y g\|_{p,\mathrm{d}\mu_k} \le \|g\|_{p,\mathrm{d}\mu_k}. \tag{3.20}$$

Remark 3.8 Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$, $1 \le p < \infty$, the operator τ^y can be defined on $L^p_{\mathrm{rad}}(\mathbb{R}^d, \mathrm{d}\mu_k)$ so that (3.20) holds.

Proof In the case $1 \le p \le 2$, this result was proved in [48]. The case $p = \infty$ follows from (3.16).

Let $2 < p < \infty$. Since $\mathcal{F}_k(g)$ is a radial function and

$$\tau^y g(-x) = \int_{\mathbb{R}^d} e_k(y, z) e_k(-x, z) \mathcal{F}_k(g)(z)\, \mathrm{d}\mu_k(z)$$

$$= \int_{\mathbb{R}^d} e_k(-y, z) e_k(x, z) \mathcal{F}_k(g)(z) \, \mathrm{d}\mu_k(z) = \tau^{-y} g(x),$$

using (3.19) for $r = \infty$, $q = p$, we obtain

$$\begin{aligned} \|\tau^{-y} g\|_{p, \mathrm{d}\mu_k} &= \sup \left\{ \int_{\mathbb{R}^d} \tau^{-y} g(x) f(x) \, \mathrm{d}\mu_k(x) \colon f \in \mathcal{S}(\mathbb{R}^d), \ \|f\|_{p', \mathrm{d}\mu_k} \le 1 \right\} \\ &\le \sup\{\|(f *_k g)(y)\|_{\infty, \mathrm{d}\mu_k} \colon f \in \mathcal{S}(\mathbb{R}^d), \ \|f\|_{p', \mathrm{d}\mu_k} \le 1\} \le \|g\|_{p, \mathrm{d}\mu_k}. \end{aligned}$$

□

Now we give an analogue of Lemma 3.4 for the case when $f \in L^p$.

Lemma 3.9 *Let* $1 \le p \le \infty$, $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k) \cap C_b(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$, $g_0 \in \mathcal{S}(\mathbb{R}_+)$, *and* $g(y) = g_0(|y|)$. *Then, for any* $x \in \mathbb{R}^d$,

$$(f *_{\lambda_k} g_0)(x) = (f *_k g)(x) \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k) \cap C_b(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d), \tag{3.21}$$

and, in the sense of tempered distributions,

$$\mathcal{F}_k(f *_{\lambda_k} g_0) = \mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g). \tag{3.22}$$

Proof First, in light of (3.6) and (3.18), we note that the convolution (3.10) belongs to $L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$. Moreover, (3.3) implies that it is in $C_b(\mathbb{R}^d)$.

Taking into account that $g \in \mathcal{S}(\mathbb{R}^d)$ and $(-\Delta_k)^r e_k(\cdot, z) = |z|^{2r} e_k(\cdot, z)$, we have

$$(-\Delta_k)^r (f *_k g)(x) = \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} e_k(x, z) e_k(-y, z) |z|^{2r} \mathcal{F}_k(g)(z) \, \mathrm{d}\mu_k(z) \, \mathrm{d}\mu_k(y).$$

Let us show that the integral converges uniformly in x. We have

$$\int_{\mathbb{R}^d} e_k(x, z) e_k(-y, z) |z|^{2r} \mathcal{F}_k(g)(z) \, \mathrm{d}\mu_k(z) = \tau^x G(-y),$$

where $G \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ is such that $\mathcal{F}_k(G)(z) = |z|^{2r} \mathcal{F}_k(g)(z)$. Using Hölder's inequality and (3.20), we get

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} e_k(x, z) e_k(-y, z) |z|^{2r} \mathcal{F}_k(g)(z) \, \mathrm{d}\mu_k(z) \, \mathrm{d}\mu_k(y) \right| \\ &\quad = \left| \int_{\mathbb{R}^d} f(y) \tau^x G(-y) \, \mathrm{d}\mu_k(y) \right| \le \|f\|_{p, \mathrm{d}\mu_k} \|\tau^x G\|_{p', \mathrm{d}\mu_k} \le \|f\|_{p, \mathrm{d}\mu_k} \|G\|_{p', \mathrm{d}\mu_k}. \end{aligned}$$

Thus, convolution (3.11) belongs to $C^\infty(\mathbb{R}^d)$.

By Lemma 3.4, the equality in (3.21) holds for any function $f \in \mathcal{S}(\mathbb{R}^d)$. If $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$, $f_n \in \mathcal{S}(\mathbb{R}^d)$, and $f_n \to f$ in $L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$, then Minkowski's inequality and (3.6) give

$$\|((f - f_n) *_{\lambda_k} g_0)\|_{p, \mathrm{d}\mu_k} \le \|f - f_n\|_{p, \mathrm{d}\mu_k} \, \|g_0\|_{1, \mathrm{d}\nu_{\lambda_k}}, \tag{3.23}$$

while Hölder's inequality and (3.20) imply

$$|((f - f_n) *_k g)(x)| \le \|f - f_n\|_{p,\mathrm{d}\mu_k} \|g\|_{p',\mathrm{d}\mu_k}.$$

By (3.23), there is a subsequence $\{n_k\}$ such that $(f_{n_k} *_{\lambda_k} g_0)(x) \to (f *_{\lambda_k} g_0)(x)$ a.e., therefore the relation $(f *_{\lambda_k} g_0)(x) = (f *_k g)(x)$ holds almost everywhere. Since both convolutions are continuous, it holds everywhere.

To prove the second equation of the lemma, we first remark that Lemma 3.4 implies that (3.22) holds pointwise for any $f \in \mathcal{S}(\mathbb{R}^d)$. In the general case, since $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$, $(f *_{\lambda_k} g_0) \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$, and $\mathcal{F}_k(g) \in \mathcal{S}(\mathbb{R}^d)$, the left- and right-hand sides of (3.22) are tempered distributions. Recall that the Dunkl transform of tempered distribution is defined by

$$\langle \mathcal{F}_k(f), \varphi\rangle = \langle f, \mathcal{F}_k(\varphi)\rangle, \quad f \in \mathcal{S}'(\mathbb{R}^d), \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Let $f_n \in \mathcal{S}(\mathbb{R}^d)$ and $f_n \to f$ in $L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then

$$\langle \mathcal{F}_k((f - f_n) *_{\lambda_k} g_0), \varphi\rangle = \langle ((f - f_n) *_{\lambda_k} g_0), \mathcal{F}_k(\varphi)\rangle,$$
$$\langle \mathcal{F}_k(g)\mathcal{F}_k(f - f_n), \varphi\rangle = \langle (f - f_n), \mathcal{F}_k(\mathcal{F}_k(g)\varphi)\rangle$$

and

$$|\langle \mathcal{F}_k((f - f_n) *_{\lambda_k} g_0), \varphi\rangle| \le \|f - f_n\|_{p,\mathrm{d}\mu_k} \|g_0\|_{1,\mathrm{d}\nu_{\lambda_k}} \|\mathcal{F}_k(\varphi)\|_{p',\mathrm{d}\mu_k},$$
$$|\langle \mathcal{F}_k(g)\mathcal{F}_k(f - f_n), \varphi\rangle| \le \|f - f_n\|_{p,\mathrm{d}\mu_k} \|\mathcal{F}_k(\mathcal{F}_k(g)\varphi)\|_{p',\mathrm{d}\mu_k}.$$

Thus, the proof of (3.22) is now complete. □

4 Boundedness of the Riesz Potential

Recall that $\lambda_k = d/2 - 1 + \sum_{a\in R_+} k(a)$. For $0 < \alpha < 2\lambda_k + 2$, the weighted Riesz potential $I_\alpha^k f$ is defined on $\mathcal{S}(\mathbb{R}^d)$ (see [49]) by

$$I_\alpha^k f(x) = \left(d_k^\alpha\right)^{-1} \int_{\mathbb{R}^d} \tau^{-y} f(x) \frac{1}{|y|^{2\lambda_k+2-\alpha}}\, \mathrm{d}\mu_k(y),$$

where $d_k^\alpha = 2^{-\lambda_k-1+\alpha}\Gamma(\alpha/2)/\Gamma(\lambda_k + 1 - \alpha/2)$. We have, in the sense of tempered distributions,

$$\mathcal{F}_k(I_\alpha^k f)(y) = |y|^{-\alpha}\mathcal{F}_k(f)(y).$$

Using (2.2) and (3.2), we obtain

$$I_\alpha^k f(x) = \left(d_k^\alpha\right)^{-1} \int_0^\infty T^t f(x) \frac{1}{t^{2\lambda_k+2-\alpha}}\, \mathrm{d}\nu_{\lambda_k}(t). \tag{4.1}$$

To estimate the L^p-norm of this operator, we use the maximal function defined for $f \in \mathcal{S}(\mathbb{R}^d)$ as follows [48]:

$$M_k f(x) = \sup_{r>0} \frac{|(f *_k \chi_{B_r})(x)|}{\int_{B_r} \mathrm{d}\mu_k},$$

where χ_{B_r} is the characteristic function of the Euclidean ball B_r of radius r centered at 0.

Using (2.2), (3.2), and (3.14), we get

$$M_k f(x) = \sup_{r>0} \frac{\left|\int_0^r T^t f(x)\,\mathrm{d}\nu_{\lambda_k}(t)\right|}{\int_0^r \mathrm{d}\nu_{\lambda_k}}.$$

It is proved in [48] that the maximal function is bounded on $L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$, $1 < p \le \infty$,

$$\|M_k f\|_{p,\mathrm{d}\mu_k} \lesssim \|f\|_{p,\mathrm{d}\mu_k}, \tag{4.2}$$

and it is of weak type $(1, 1)$; that is,

$$\int_{\{x:\, M_k f(x)>a\}} \mathrm{d}\mu_k \lesssim \frac{\|f\|_{1,\mathrm{d}\mu_k}}{a}, \quad a > 0. \tag{4.3}$$

Theorem 4.1 *If* $1 < p < q < \infty$, $0 < \alpha < 2\lambda_k + 2$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda_k+2}$, *then*

$$\|I_\alpha^k f\|_{q,\mathrm{d}\mu_k} \lesssim \|f\|_{p,\mathrm{d}\mu_k}, \quad f \in \mathcal{S}(\mathbb{R}^d). \tag{4.4}$$

The mapping $f \mapsto I_\alpha^k f$ *is of weak type* $(1, q)$*; that is,*

$$\int_{\{x:\, |I_\alpha^k f(x)|>a\}} \mathrm{d}\mu_k \lesssim \left(\frac{\|f\|_{1,\mathrm{d}\mu_k}}{a}\right)^q. \tag{4.5}$$

Remark 4.2 In the case $k \equiv 0$, inequality (4.4) was proved by Soboleff [44] and Thorin [50] and the weighted inequality was studied by Stein and Weiss [46]. For the reflection group $G = \mathbb{Z}_2^d$, Theorem 4.1 was proved in [49]. The general case was obtained in [21]. We give another simple proof based on the L^p-boundedness of T^t given in Theorem 3.5 and follow the proof given in [49] for $G = \mathbb{Z}_2^d$.

Remark 4.3 In Theorem 4.1, dealing with (4.4), we may assume that $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$, $1 < p < \infty$, while proving (4.5), we may assume that $f \in L^1(\mathbb{R}^d, \mathrm{d}\mu_k)$.

Proof Let $R > 0$ be fixed. We write (4.1) as sum of two terms,

$$I_\alpha^k f(x) = \left(d_k^\alpha\right)^{-1} \int_0^R T^t f(x) \frac{1}{t^{2\lambda_k+2-\alpha}}\,\mathrm{d}\nu_{\lambda_k}(t)$$

$$+ \left(d_k^\alpha\right)^{-1} \int_R^\infty T^t f(x) \frac{1}{t^{2\lambda_k+2-\alpha}} \, \mathrm{d}\nu_{\lambda_k}(t) = J_1 + J_2. \tag{4.6}$$

Integrating J_1 by parts, we obtain

$$\begin{aligned} d_k^\alpha J_1 &= \int_0^R t^{-(2\lambda_k+2-\alpha)} \, d\left(\int_0^t T^s f(x) \, \mathrm{d}\nu_{\lambda_k}(s)\right) \\ &= R^\alpha \cdot R^{-(2\lambda_k+2)} \int_0^R T^s f(x) \, \mathrm{d}\nu_{\lambda_k}(s) \\ &\quad + (2\lambda_k + 2 - \alpha) \int_0^R t^{-(2\lambda_k+2)} \int_0^t T^s f(x) \, \mathrm{d}\nu_{\lambda_k}(s) \, t^{\alpha-1} \, \mathrm{d}t. \end{aligned} \tag{4.7}$$

Here we have used that

$$\lim_{\varepsilon \to 0+0} \varepsilon^\alpha \cdot \varepsilon^{-(2\lambda_k+2)} \int_0^\varepsilon T^s f(x) \, \mathrm{d}\nu_{\lambda_k}(s) = 0,$$

since

$$\varepsilon^\alpha \cdot \varepsilon^{-(2\lambda_k+2)} \left| \int_0^\varepsilon T^s f(x) \, \mathrm{d}\nu_{\lambda_k}(s) \right| \lesssim \varepsilon^\alpha \sup_{\varepsilon>0} \frac{\left|\int_0^\varepsilon T^t f(x) \, \mathrm{d}\nu_{\lambda_k}(t)\right|}{\int_0^\varepsilon \mathrm{d}\nu_{\lambda_k}} = \varepsilon^\alpha M_k f(x).$$

In light of (4.7), we have

$$|J_1| \lesssim R^\alpha M_k f(x) + \int_0^R M_k f(x) t^{\alpha-1} \, \mathrm{d}t \lesssim R^\alpha M_k f(x). \tag{4.8}$$

To estimate J_2, we use Hölder's inequality, the relation $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda_k+2}$, and (3.17):

$$\begin{aligned} |J_2| &\le \left(d_k^\alpha\right)^{-1} \left(\int_R^\infty t^{-(2\lambda_k+2-\alpha)p'} \, \mathrm{d}\nu_{\lambda_k}(t)\right)^{1/p'} \|T^t f(x)\|_{p,\mathrm{d}\nu_{\lambda_k}} \\ &\lesssim R^{-(2\lambda_k+2)q} \|f\|_{p,\mathrm{d}\mu_k}. \end{aligned}$$

This, (4.6) and (4.8) yield

$$|I_\alpha^k f(x)| \lesssim R^\alpha M_k f(x) + R^{-(2\lambda_k+2)q} \|f\|_{p,\mathrm{d}\mu_k},$$

for any $R > 0$. Choosing $R = \left(M_k f(x)/\|f\|_{p,\mathrm{d}\mu_k}\right)^{-q/(2\lambda_k+2)}$ implies the inequality

$$|I_\alpha^k f(x)| \lesssim (M_k f(x))^{p/q} (\|f\|_{p,\mathrm{d}\mu_k})^{1-p/q} \tag{4.9}$$

for any $1 \le p < q$. Integrating (4.9) and using (4.2), we have

$$\|I_\alpha^k f\|_{q,\mathrm{d}\mu_k} \lesssim \|M_k f\|_{p,\mathrm{d}\mu_k}^{p/q} \|f\|_{p,\mathrm{d}\mu_k}^{1-p/q} \lesssim \|f\|_{p,\mathrm{d}\mu_k}, \quad p > 1.$$

Finally, we use inequality (4.3) for the maximal function and inequality (4.9) with $p = 1$ to obtain

$$\int_{\{x:\,|I_\alpha^k f(x)|>a\}} \mathrm{d}\mu_k \le \int_{\{x:\,(M_k f(x))^{1/q}(\|f\|_{1,\mathrm{d}\mu_k})^{1-1/q}\gtrsim a\}} \mathrm{d}\mu_k \lesssim \left(\frac{\|f\|_{1,\mathrm{d}\mu_k}}{a}\right)^q.$$

□

5 Entire Functions of Exponential Type and Plancherel–Polya–Boas-Type Inequalities

Let $\mathbb{C}^d$ be the complex Euclidean space of d dimensions. Let also $z = (z_1, \dots, z_d) \in \mathbb{C}^d$, $\operatorname{Im} z = (\operatorname{Im} z_1, \dots, \operatorname{Im} z_d)$, and $\sigma > 0$.

In this section, we define several classes of entire functions of exponential type and study their interrelations. Moreover, we prove the Plancherel–Polya–Boas-type estimates and the Paley–Wiener-type theorems. These classes will be used later to study the approximation of functions on $\mathbb{R}^d$ by entire functions of exponential type.

First, we define two classes of entire functions: $B_{p,k}^\sigma$ and $\widetilde{B}_{p,k}^\sigma$. We say that a function $f \in B_{p,k}^\sigma$ if $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$ is such that its analytic continuation to $\mathbb{C}^d$ satisfies

$$|f(z)| \le c_\varepsilon e^{(\sigma+\varepsilon)|z|}, \quad \forall\varepsilon > 0,\ \forall z \in \mathbb{C}.$$

The smallest $\sigma = \sigma_f$ in this inequality is called a spherical type of f. In other words, the class $B_{p,k}^\sigma$ is the collection of all entire functions of spherical type at most σ.

We say that a function $f \in \widetilde{B}_{p,k}^\sigma$ if $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$ is such that its analytic continuation to $\mathbb{C}^d$ satisfies

$$|f(z)| \le c_f e^{\sigma|\operatorname{Im} z|}, \quad \forall z \in \mathbb{C}^d.$$

Historically, functions from $\widetilde{B}_{p,k}^\sigma$ were basic objects in the Dunkl harmonic analysis. It is clear that $\widetilde{B}_{p,k}^\sigma \subset B_{p,k}^\sigma$. Moreover, if $k \equiv 0$, then both classes coincide (see, e.g., [29]). Indeed, if $f \in B_{p,0}^\sigma$, $1 \le p < \infty$, then Nikol'skii's inequality [31, 3.3.5]

$$\|f\|_\infty \le 2^{\mathrm{d}} \sigma^{d/p} \|f\|_{p,\mathrm{d}\mu_0}$$

and the inequality [31, 3.2.6]

$$\|f(\cdot + iy)\|_\infty \le e^{\sigma|y|}\|f\|_\infty, \quad y \in \mathbb{R}^d,$$

imply that, for $z = x + iy \in \mathbb{C}^d$,

$$|f(z)| \le 2^{\mathrm{d}} \sigma^{d/p} \|f\|_{p,\mathrm{d}\mu_0} e^{\sigma|\operatorname{Im} z|},$$

i.e., $f \in \widetilde{B}_{p,0}^\sigma$.

In fact, the classes $B^\sigma_{p,k}$ and $\widetilde{B}^\sigma_{p,k}$ coincide in the weighted case ($k \neq 0$) as well. To see that, it is enough to show that functions from $B^\sigma_{p,k}$ are bounded on $\mathbb{R}^d$.

Theorem 5.1 *If* $0 < p < \infty$, *then* $B^\sigma_{p,k} = \widetilde{B}^\sigma_{p,k}$.

We will actually prove the more general statement. Let $m \in \mathbb{Z}_+$, $\alpha^1, \dots, \alpha^m \in \mathbb{R}^d \setminus \{0\}$, $k_0 \geq 0$, $k_1, \dots, k_m > 0$, and

$$v(x) = |x|^{k_0} \prod_{j=1}^{m} |\langle \alpha^j, x\rangle|^{k_j} \tag{5.1}$$

be the power weight. The Dunkl weight is a particular case of such weighted functions. The weighted function (5.1) arises in the study of the generalized Fourier transform (see, e.g., [3]).

Let $L^{p,v}(\mathbb{R}^d)$, $0 < p < \infty$, be the space of complex-valued Lebesgue measurable functions f for which

$$\|f\|_{p,v} = \left(\int_{\mathbb{R}^d} |f(x)|^p v(x)\, \mathrm{d}x \right)^{1/p} < \infty.$$

Let $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_d)$, $\sigma_1, \dots, \sigma_d > 0$.

Again, let us define three *anisotropic* classes of entire functions: $B^{\boldsymbol{\sigma}}$, $B^{\boldsymbol{\sigma}}_{p,v}$, and $\widetilde{B}^{\boldsymbol{\sigma}}_{p,v}$.

We say that a function f defined on $\mathbb{R}^d$ belongs to $B^{\boldsymbol{\sigma}}$ if its analytic continuation to $\mathbb{C}^d$ satisfies

$$|f(z)| \leq c_\varepsilon e^{(\sigma_1+\varepsilon)|z_1|+\cdots+(\sigma_d+\varepsilon)|z_d|}, \quad \forall \varepsilon > 0,\ \forall z \in \mathbb{C}^d.$$

We say that a function $f \in B^{\boldsymbol{\sigma}}_{p,v}$ if $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$ is such that its analytic continuation to $\mathbb{C}^d$ belongs to $B^{\boldsymbol{\sigma}}$.

We say that a function $f \in \widetilde{B}^{\boldsymbol{\sigma}}_{p,v}$ if $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$ is such that its analytic continuation to $\mathbb{C}^d$ satisfies

$$|f(z)| \leq c_f e^{\sigma_1|\mathrm{Im}\, z_1|+\cdots+\sigma_d|\mathrm{Im}\, z_d|}, \quad \forall z \in \mathbb{C}^d.$$

We will use the notation $L^p(\mathbb{R}^d)$, $\|\cdot\|_p$, $B^{\boldsymbol{\sigma}}_p$, and $\widetilde{B}^{\boldsymbol{\sigma}}_p$ in the case of the unit weight, i.e., $v \equiv 1$.

Theorem 5.2 *If* $0 < p < \infty$, *then*

(1) $B^{\boldsymbol{\sigma}}_{p,v} \subset B^{\boldsymbol{\sigma}}_p$,
(2) $B^{\boldsymbol{\sigma}}_{p,v} = \widetilde{B}^{\boldsymbol{\sigma}}_{p,v}$,
(3) $B^{\sigma}_{p,v} = \widetilde{B}^{\sigma}_{p,v}$.

Remark 5.3 (i) Part (3) of Theorem 5.2 implies Theorem 5.1.

(ii) Note that in some particular cases ($k_0 = 0$ and $p \geq 1$) a similar result was discussed in [23].

Parts (2) and (3) of Theorem 5.2 follow from (1). Indeed, the embedding in (1) implies that $B^{\boldsymbol{\sigma}}_{p,v} \subset B^{\boldsymbol{\sigma}}_{p} \subset B^{\boldsymbol{\sigma}}_{\infty}$. Hence, a function $f \in B^{\boldsymbol{\sigma}}_{p,v}$ is bounded on $\mathbb{R}^d$ and then $f \in \widetilde{B}^{\boldsymbol{\sigma}}_{p,v}$, which gives (2). Further, $B^{\sigma}_{p,v} \subset B^{\boldsymbol{\sigma}}_{p,v}$ holds, where $\boldsymbol{\sigma} = (\sigma, \dots, \sigma) \in \mathbb{R}^d_+$ since $|z| \leq |z_1| + \cdots + |z_d|$. Hence, similar to the above, we have $B^{\sigma}_{p,v} \subset B^{\boldsymbol{\sigma}}_{\infty}$ and (3) follows. Thus, to prove Theorem 5.2, it is sufficient to verify part (1).

The main difficulty to prove Theorem 5.2 is that the weight $v(x)$ vanishes. In order to overcome this problem, we will first prove two-sided estimates of the L^p norm of entire functions in terms of the weighted l_p norm, $\left(\sum_n v(\lambda^{(n)})|f(\lambda^{(n)})|^p\right)^{1/p}$, $0 < p < \infty$, where v does not vanish at $\{\lambda^{(n)}\} \subset \mathbb{R}^d$.

Such estimates are of their own interest. They generalize the Plancherel–Polya inequality [33], [6, Chapt. 6, 6.7.15]

$$\sum_{k\in\mathbb{Z}} |f(\lambda_k)|^p \leq c(\delta, \sigma, p) \int_{-\infty}^{\infty} |f(x)|^p \,\mathrm{d}x, \quad 0 < p < \infty,$$

where λ_k is an increasing sequence such that $\lambda_{k+1} - \lambda_k \geq \delta > 0$, and f is an entire function of exponential type at most σ; and the Boas inequality [5], [6, Chapt. 10, 10.6.8],

$$\int_{-\infty}^{\infty} |f(x)|^p \,\mathrm{d}x \leq C(\delta, L, \sigma, p) \sum_{k\in\mathbb{Z}} |f(\lambda_k)|^p, \quad 0 < p < \infty, \tag{5.2}$$

where, additionally, $\left|\lambda_k - \frac{\pi}{\sigma} k\right| \leq L$ and the type of f is $< \sigma$.

We write $\boldsymbol{\sigma}' = (\sigma_1', \dots, \sigma_d') < \boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_d)$ if $\sigma_1' < \sigma_1, \dots, \sigma_d' < \sigma_d$. Let $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$ and $\lambda^{(n)} \colon \mathbb{Z}^d \to \mathbb{R}^d$. In what follows, we consider the sequences of the following type:

$$\lambda^{(n)} = (\lambda_1(n_1), \lambda_2(n_1, n_2), \dots, \lambda_d(n_1, \dots, n_d)), \tag{5.3}$$

where $\lambda_i^{(n)} = \lambda_i(n_1, \dots, n_i)$ are sequences increasing with respect to n_i, $i = 1, \dots, d$ for fixed $n_1, \dots, n_{i-1}$.

Definition 5.4 We say that the sequence $\lambda^{(n)}$ satisfies *the separation condition* $\Omega_{\mathrm{sep}}[\delta]$, $\delta > 0$, if, for any $n \in \mathbb{Z}^d$,

$$\lambda_i(n_1, \dots, n_{i-1}, n_i + 1) - \lambda_i(n_1, \dots, n_{i-1}, n_i) \geq \delta, \quad i = 1, \dots, d.$$

Note that if the sequence $\lambda^{(n)}$ satisfies the separation condition $\Omega_{\mathrm{sep}}[\delta]$, then it also satisfies the condition $\inf_{n\neq m} |\lambda^{(n)} - \lambda^{(m)}| > 0$.

Definition 5.5 We say that the sequence $\lambda^{(n)}$ satisfies *the close-lattice condition* $\Omega_{\rm lat}[\mathbf{a}, L]$, $\mathbf{a} = (a_1, \ldots, a_d) > 0$, $L > 0$, if, for any $n \in \mathbb{Z}^d$,

$$\left|\lambda_i(n_1, \ldots, n_i) - \frac{\pi n_i}{a_i}\right| \le L, \quad i = 1, \ldots, d.$$

We start with the Plancherel–Polya-type inequality.

Theorem 5.6 *Assume that* $\lambda^{(n)}$ *satisfies the condition* $\inf_{n\neq m} |\lambda^{(n)} - \lambda^{(m)}| > 0$. *Then for* $f \in B_p^{\boldsymbol{\sigma}}$, $0 < p < \infty$, *we have*

$$\sum_{n\in\mathbb{Z}^d} |f(\lambda^{(n)})|^p \lesssim \int_{\mathbb{R}^d} |f(x)|^p \, \mathrm{d}x.$$

Proof For simplicity, we prove this result for $d = 2$. The proof in the general case is similar.

The function $|f(z)|^p$ is plurisubharmonic, and therefore for any $x = (x_1, x_2) \in \mathbb{R}^2$, one has [38]

$$|f(x_1, x_2)|^p \le \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} |f(x_1 + \rho_1 e^{i\theta_1},\ x_2 + \rho_2 e^{i\theta_2}|^p \, \mathrm{d}\theta_1 \mathrm{d}\theta_2,$$

where $\rho_1,\ \rho_2 > 0$. Following [31, 3.2.5], for $\delta > 0$ and $\xi + i\eta = (\xi_1 + i\eta_1, \xi_2 + i\eta_2)$, we obtain that

$$|f(x_1, x_2)|^p \le \frac{1}{(\pi\delta^2)^2} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \int_{x_1-\delta}^{x_1+\delta} \int_{x_2-\delta}^{x_2+\delta} |f(\xi + i\eta)|^p \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \, \mathrm{d}\eta_1 \, \mathrm{d}\eta_2. \quad (5.4)$$

The separation condition implies that for some $\delta > 0$, the boxes $[\lambda_1^{(n)} - \delta, \lambda_1^{(n)} + \delta] \times [\lambda_2^{(n)} - \delta, \lambda_2^{(n)} + \delta]$ do not overlap for any n.

Since

$$f(x + iy) = \sum_{k\in\mathbb{Z}_+^2} \frac{f^{(k)}(x)}{k!} (iy)^k,$$

where $f^{(k)}$ is a partial derivative f of order $k = (k_1, k_2)$, $k! = k_1!k_2!$, and $(iy)^k = (iy_1)^{k_1}(iy_2)^{k_2}$, by applying Bernstein's inequality (see [31, 3.2.2 and 3.3.5] and [37]), we derive that

$$\|f(\cdot + iy)\|_p \lesssim e^{\sigma_1|y_1|+\sigma_2|y_2|}\|f\|_p.$$

Using this and (5.4), we derive that

$$\sum_{n\in\mathbb{Z}^2} |f(\lambda^{(n)})|^p \le \frac{1}{(\pi\delta^2)^2} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(\xi + i\eta)|^p \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \, \mathrm{d}\eta_1 \, \mathrm{d}\eta_2$$

$$\lesssim \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} e^{p(\sigma_1|\eta_1|+\sigma_2|\eta_d|)}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(\xi)|^p\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2$$
$$\lesssim \int_{\mathbb{R}^2} |f(x)|^p\,\mathrm{d}x.$$

□

Theorem 5.7 *Let the sequence* $\lambda^{(n)}$ *of form* (5.3) *satisfy the conditions* $\Omega_{\mathrm{sep}}[\delta]$ *and* $\Omega_{\mathrm{lat}}[\boldsymbol{\sigma}, L]$. *Assume that* $f \in B^{\boldsymbol{\sigma}'}$, $\boldsymbol{\sigma}' < \boldsymbol{\sigma}$, *is such that* $\sum_{n\in\mathbb{Z}^d} |f(\lambda^{(n)})|^p < \infty$, $0 < p < \infty$. *Then* $f \in L^p(\mathbb{R}^d)$ *and*

$$\int_{\mathbb{R}^d} |f(x)|^p\,\mathrm{d}x \lesssim \sum_{n\in\mathbb{Z}^d} |f(\lambda^{(n)})|^p.$$

Remark 5.8 For $p \geq 1$, a similar two-sided Plancherel–Polya–Boas-type inequality was obtained from [32].

Proof For simplicity, we consider the case $d = 2$. Integrating $|f(x_1, x_2)|^p$ at x_1 and applying inequality (5.2), we get, for any x_2,

$$\int_{-\infty}^{\infty} |f(x_1, x_2)|^p\,\mathrm{d}x_1 \lesssim \sum_{n_1\in\mathbb{Z}} |f(\beta_1(n_1), x_2)|^p.$$

Since by (5.2), for any n_1,

$$\int_{-\infty}^{\infty} |f(\beta_1(n_1), x_2)|^p\,\mathrm{d}x_2 \lesssim \sum_{n_2\in\mathbb{Z}} |f(\beta_1(n_1), \beta_2(n_1, n_2))|^p,$$

we then have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x_1, x_2)|^p\,\mathrm{d}x_1\,\mathrm{d}x_2 \lesssim \sum_{n_1\in\mathbb{Z}} \int_{-\infty}^{\infty} |f(\beta_1(n_1), x_2)|^p\,\mathrm{d}x_2$$
$$\lesssim \sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}} |f(\beta_1(n_1), \beta_2(n_1, n_2))|^p < \infty.$$

□

Using Theorems 5.6 and 5.7 we arrive at the following statement:

Theorem 5.9 *Let the sequence* $\{\lambda^{(n)}\}$ *of form* (5.3) *satisfy the conditions* $\Omega_{\mathrm{sep}}[\delta]$ *and* $\Omega_{\mathrm{lat}}[\boldsymbol{\sigma}, L]$. *If* $f \in B^{\boldsymbol{\sigma}'}$, $\boldsymbol{\sigma}' < \boldsymbol{\sigma}$, *then, for* $0 < p < \infty$,

$$\sum_{n\in\mathbb{Z}^d} |f(\lambda^{(n)})|^p \lesssim \int_{\mathbb{R}^d} |f(x)|^p \, dx \lesssim \sum_{n\in\mathbb{Z}^d} |f(\lambda^{(n)})|^p.$$

We will need the weighted version of the Plancherel–Polya–Boas equivalence. We start with three auxiliary lemmas.

Lemma 5.10 [18] *If $\gamma \geq -1/2$, then there exists an even entire function $\omega_\gamma(z)$, $z \in \mathbb{C}$, of exponential type 2 such that, uniformly in $x \in \mathbb{R}_+$,*

$$\omega_\gamma(x) \asymp \begin{cases} x^{2k+2}, & 0 \leq x \leq 1, \\ x^{2\gamma+1}, & x \geq 1, \end{cases}$$

where $k = [\gamma + 1/2]$ and $[a]$ is the integral part of a. In particular, we can take

$$\omega(z) = z^{2k+2} j_{k-\gamma}(z + i) j_{k-\gamma}(z - i).$$

Lemma 5.11 *Let $m \in \mathbb{N}$, $j = 1, \dots, m$, $b^j = (b_1^j, \dots, b_d^j) \in \mathbb{R}^d \setminus \{0\}$, and either $|b_i^j| \geq 1$, or $b_i^j = 0$, $i = 1, \dots, d$. Then there exists a sequence $\{\rho^{(n)}\} \subset \mathbb{Z}^d \setminus \{0\}$ of the form (5.3) such that, for any $j = 1, \dots, m$ and $i = 1, \dots, d$,*

$$|\rho_i(n_1, \dots, n_i) - n_i| \leq m, \tag{5.5}$$
$$|\langle b^j, \rho^{(n)} \rangle| \geq 1/2. \tag{5.6}$$

Proof To construct a desired sequence

$$\rho^{(n)} = (\rho_1(n_1), \rho_2(n_1, n_2), \dots, \rho_d(n_1, \dots, n_d)) \in \mathbb{Z}^d,$$

we will use the following simple remark. If we throw out m points from $\mathbb{Z}$, then the rest can be numbered such that the obtained sequence will be increasing and (5.5) holds.

Let $J_1 = \{j : b_1^j \neq 0,\ b_2^j = \dots = b_d^j = 0\}$. If $J_1 = \varnothing$, then we set $\rho_1(n_1) = n_1$. If $J_1 \neq \varnothing$, then $\rho_1(n_1)$ is an increasing sequence formed from $\mathbb{Z} \setminus \{0\}$. In both cases (5.5) is valid, and, moreover, for $j \in J_1$ and any $\rho_2(n_1, n_2), \dots, \rho_d(n_1, \dots, n_d)$, one has (5.6) since $|\langle b^j, \rho^{(n)} \rangle| = |b_1^j \rho_1(n_1)| \geq 1$.

Let $J_2 = \{j : b_2^j \neq 0,\ b_3^j = \dots = b_d^j = 0\}$, $n_1 \in \mathbb{Z}$. If $J_2 = \varnothing$, then we set $\rho_2(n_1, n_2) = n_2$. Let $J_2 \neq \varnothing$. If $j \in J_2$ and $b_1^j \rho_1(n_1) + b_2^j t_j = 0$, then $t_j = l_j + \varepsilon_j$, $l_j \in \mathbb{Z}$, $|\varepsilon_j| \leq 1/2$. Here l_j is the nearest integer to t_j. Note that if $\rho_2 \neq l_j$, then $|b_1^j \rho_1(n_1) + b_2^j \rho_2| = |b_2^j(\rho_2 - l_j - \varepsilon_j)| \geq 1/2$.

Let $\rho_2(n_1, n_2)$ be an increasing sequence at n_2 formed from $\mathbb{Z} \backslash \{l_j : j \in J_2\}$. For this sequence (5.5) holds and, for $j \in J_2$ and any $\rho_3(n_1, n_2, n_3), \dots, \rho_d(n_1, \dots, n_d)$, one has

$$|\langle b^j, \rho^{(n)} \rangle| = |b_1^j \rho_1(n_1) + b_2^j \rho_2(n_1, n_2)| \geq 1/2;$$

that is, (5.6) holds as well.

Assume that we have constructed the sets $J_1, \dots, J_{d-1}$, and the sequence $(\rho_1(n_1), \rho_2(n_1, n_2), \dots, \rho_{d-1}(n_1, \dots, n_{d-1})) \in \mathbb{Z}^{d-1}$.

Let $J_d = \{j : b_d^j \neq 0\}$, $(n_1, \dots, n_{d-1}) \in \mathbb{Z}^{d-1}$. If $J_d = \varnothing$, then we set $\rho_d(n_1, \dots, n_{d-1}, n_d) = n_d$. Assume now that $J_d \neq \varnothing$. If $j \in J_d$ and

$$b_1^j \rho_1(n_1) + \cdots + b_{d-1}^j \rho_{d-1}(n_1, \dots, n_{d-1}) + b_d^j t_j = 0,$$

then $t_j = l_j + \varepsilon_j$, $|\varepsilon_j| \le 1/2$. Note that if $\rho_d \neq l_j$, then

$$|b_1^j \rho_1(n_1) + \cdots + b_{d-1}^j \rho_{d-1}(n_1, \dots, n_{d-1}) + b_d^j \rho_d| = |b_d^j(\rho_d - l_j - \varepsilon_j)| \ge 1/2.$$

Let $\rho_d(n_1, \dots, n_d)$ be an increasing sequence in n_d formed from $\mathbb{Z} \setminus \{l_j : j \in J_d\}$, $\rho^{(n)} = (\rho_1(n_1), \rho_2(n_1, n_2), \dots, \rho_d(n_1, \dots, n_d))$. For the sequence $\rho_d(n_1, \dots, n_d)$, inequality (5.5) holds, and, for $j \in J_d$, one has $|\langle b^j, \rho^{(n)} \rangle| \ge 1/2$.

Thus, we construct the desired sequence since, for any $j \in \{1, \dots, m\}$ and some $i \in \{1, \dots, d\}$, $b^j \in J_i$ holds. □

An important ingredient of the proof of Theorem 5.2 is the following corollary of Lemma 5.11:

Lemma 5.12 *If* $\mathbf{a} > 0$, $\alpha^1, \dots, \alpha^m \in \mathbb{R}^d \setminus \{0\}$, *then there exists a sequence* $\lambda^{(n)}$ *of the form* (5.3) *such that for some* δ, $L > 0$ *the conditions* $\Omega_{\text{sep}}[\delta]$, $\Omega_{\text{lat}}[\mathbf{a}, L]$, *and* $\xi_j(\lambda^{(n)}) \ge \delta$, $j = 0, 1, \dots, m$, $n \in \mathbb{Z}^d$, *hold, where*

$$\xi_0(x) = |x|, \quad \xi_j(x) = |\langle \alpha^j, x \rangle|, \; j = 1, \dots, m. \tag{5.7}$$

Indeed, for $m \ge 1$, it is enough to define

$$\begin{aligned} \lambda^{(n)} &= (\lambda_1(n_1), \lambda_2(n_1, n_2), \dots, \lambda_d(n_1, \dots, n_d)) \\ &:= \left(\frac{\pi \rho_1(n_1)}{a_1}, \frac{\pi \rho_2(n_1, n_2)}{a_2}, \dots, \frac{\pi \rho_d(n_1, \dots, n_d)}{a_d} \right), \end{aligned} \tag{5.8}$$

where $\rho^{(n)}$ is the sequence defined in Lemma 5.11. For $m = 0$ in (5.8), we can take $\{\rho^{(n)}\} = \mathbb{Z}^d \setminus \{0\}$.

We are now in a position to state the Plancherel–Polya–Boas inequalities with weights.

Theorem 5.13 *Let* $f \in B^{\boldsymbol{\sigma}}$ *and* $\lambda^{(n)}$ *be the sequence satisfying all conditions of Lemma* 5.12 *with some* $\mathbf{a} > \boldsymbol{\sigma}$. *Then, for* $0 < p < \infty$,

$$\sum_{n \in \mathbb{Z}^d} v(\lambda^{(n)}) |f(\lambda^{(n)})|^p \lesssim \int_{\mathbb{R}^d} |f(x)|^p v(x) \, \mathrm{d}x \lesssim \sum_{n \in \mathbb{Z}^d} v(\lambda^{(n)}) |f(\lambda^{(n)})|^p.$$

Proof Recall that $v(x) = \prod_{j=0}^m v_j(x)$, where $v_j(x) = \xi_j^{k_j}(x)$, $j = 0, 1, \dots, m$ (see (5.1) and (5.7)).

By Lemma 5.10, we construct an entire function of exponential type

$$w(z)=\prod_{j=0}^{m} w_j(z),$$

where $w_0(z)=\omega_{\gamma_0}(|z|)$, $w_j(z)=\omega_{\gamma_j}(\langle\alpha^j,z\rangle)$, $j=1,\dots,m$, and

$$\gamma_j=\frac{k_j}{2p}-\frac{1}{2},\quad j=0,1,\dots,m.$$

For $j=0,1,\dots,m$, we have $w_j\in B^{2\boldsymbol{\mu}^j}$, where

$$\boldsymbol{\mu}^0=(1,\dots,1)\in\mathbb{R}^d,\quad \boldsymbol{\mu}^j=\left(\left|\alpha_1^j\right|,\dots,\left|\alpha_d^j\right|\right),\ j=1,\dots,m,$$

and $w\in B^{2\boldsymbol{\mu}}$, $\boldsymbol{\mu}=\sum_{j=0}^m\boldsymbol{\mu}^j$. Moreover, for any $j=0,1,\dots,m$,

$$\begin{aligned}&w_j^p(x)\lesssim v_j(x),\quad x\in\mathbb{R}^d,\\&w_j^p(x)\gtrsim v_j(x)\gtrsim 1,\quad \text{for}\quad \xi_j(x)\ge\delta>0.\end{aligned}\tag{5.9}$$

Let $f\in B_{p,v}^{\boldsymbol{\sigma}}$, $0<p<\infty$, $\boldsymbol{\sigma}<\mathbf{a}$, and $\lambda^{(n)}$ be the sequence satisfying all conditions of Lemma 5.12. Then, for some $s>0$ such that $\boldsymbol{\sigma}+2s\boldsymbol{\mu}<\mathbf{a}$, we have that $f(x)w(sx)\in B^{\boldsymbol{\sigma}+2s\boldsymbol{\mu}}$.

Using Theorem 5.6 and properties (5.9), we derive

$$\begin{aligned}\sum_{n\in\mathbb{Z}^d}v(\lambda^{(n)})|f(\lambda^{(n)})|^p&\lesssim\sum_{n\in\mathbb{Z}^d}|f(\lambda^{(n)})w(\lambda^{(n)})|^p\\&\lesssim\int_{\mathbb{R}^d}|f(x)w(x)|^p\,\mathrm{d}x\lesssim\int_{\mathbb{R}^d}|f(x)|^pv(x)\,\mathrm{d}x.\end{aligned}$$

Let $\delta>0$, $J\subset J_m:=\{0,1,\dots,m\}$ or $J=\varnothing$,

$$E_\delta(J)=\{x\in\mathbb{R}^d:\xi_j(x)\ge\delta,\ j\in J\text{ and }\xi_j(x)\le\delta,\ j\in J_m\setminus J\}.$$

Since $f(x)\prod_{j\in J}w_j(sx)\in B^{\boldsymbol{\sigma}+2s\boldsymbol{\mu}}$, using Theorems 5.6 and 5.7 and properties (5.9) for δ from Lemma 5.12, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}|f(x)|^pv(x)\,\mathrm{d}x&=\sum_J\int_{E_\delta(J)}|f(x)|^pv(x)\,\mathrm{d}x\lesssim\sum_J\int_{E_\delta(J)}|f(x)|^p\prod_{j\in J}v_j(sx)\,\mathrm{d}x\\&\lesssim\sum_J\int_{E_\delta(J)}\left|f(x)\prod_{j\in J}w_j(sx)\right|^p\mathrm{d}x\lesssim\sum_J\int_{\mathbb{R}^d}\left|f(x)\prod_{j\in J}w_j(sx)\right|^p\mathrm{d}x\\&\lesssim\sum_n|f(\lambda^{(n)})|^p\sum_J\prod_{j\in J}w_j^p(s\lambda^{(n)})\lesssim\sum_n|f(\lambda^{(n)})w(s\lambda^{(n)})|^p\\&\lesssim\sum_n|f(\lambda^{(n)})|^pv(s\lambda^{(n)})\lesssim\sum_n|f(\lambda^{(n)})|^pv(\lambda^{(n)}),\end{aligned}$$

where we have assumed that $\prod_{j\in\varnothing}=1$. □

Proof of Theorem 5.2 Recall that it is enough to show that $B^{\boldsymbol\sigma}_{p,v} \subset B^{\boldsymbol\sigma}_{p}$, and the latter follows from $B^{\boldsymbol\sigma}_{p,v} \subset L^p(\mathbb{R}^d)$.

Let $f \in B^{\boldsymbol\sigma}_{p,v}, 0 < p < \infty, \mathbf{a} > \boldsymbol\sigma$, and $\lambda^{(n)}$ be the sequence satisfying all conditions of Lemma 5.12. Using Theorem 5.6 and properties (5.9) as in Theorem 5.13, we have

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x)|^p \,\mathrm{d}x &\lesssim \sum_{n\in\mathbb{Z}^d} |f(\lambda^{(n)})|^p \lesssim \sum_{n\in\mathbb{Z}^d} |w(\lambda^{(n)})f(\lambda^{(n)})|^p \\ &\lesssim \int_{\mathbb{R}^d} |f(x)w(x)|^p \,\mathrm{d}x \lesssim \int_{\mathbb{R}^d} |f(x)|^p v(x)\,\mathrm{d}x.\end{aligned}$$

□

By the Paley–Wiener theorem for tempered distributions (see [25,53]) and Theorem 5.1, we arrive at the following result.

Theorem 5.14 *A function* $f \in B^{\sigma}_{p,k}$, $1 \le p < \infty$, *if and only if* $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k) \cap C_b(\mathbb{R}^d)$ *and* $\operatorname{supp} \mathcal{F}_k(f) \subset B_\sigma$.

The Dunkl transform $\mathcal{F}_k(f)$ in Theorem 5.14 is understood as a function for $1 \le p \le 2$ and as a tempered distribution for $p > 2$.

We conclude this section by presenting the concept of the best approximation. Let

$$E_\sigma(f)_{p,\mathrm{d}\mu_k} = \inf\{\|f-g\|_{p,\mathrm{d}\mu_k} : g \in B^{\sigma}_{p,k}\}$$

be the best approximation of a function $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$ by entire functions of spherical exponential type σ. We show that the best approximation is achieved.

Theorem 5.15 *For any* $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$, $1 \le p \le \infty$, *there exists a function* $g^* \in B^{\sigma}_{p,k}$ *such that* $E_\sigma(f)_{p,\mathrm{d}\mu_k} = \|f - g^*\|_{p,\mathrm{d}\mu_k}$.

Proof The proof is standard. Let g_n be a sequence from $B^{\sigma}_{p,k}$ such that $\|f - g_n\|_{p,\mathrm{d}\mu_k} \to E_\sigma(f)_{p,\mathrm{d}\mu_k}$. Since it is bounded in $L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$, it is also bounded in $C_b(\mathbb{R}^d)$. A compactness theorem for entire functions [31, 3.3.6] implies that there exist a subsequence g_{n_k} and an entire function g^* of exponential type at most σ such that

$$\lim_{k\to\infty} g_{n_k}(x) = g^*(x), \quad x \in \mathbb{R}^d,$$

and, moreover, convergence is uniform on compact sets. Therefore, for any $R > 0$,

$$\|g^*\chi_{B_R}\|_{p,\mathrm{d}\mu_k} = \lim_{k\to\infty} \|g_{n_k}\chi_{B_R}\|_{p,\mathrm{d}\mu_k} \le M.$$

Letting $R \to \infty$, we have that $g^* \in B^{\sigma}_{p,k}$. In light of

$$\|(f-g^*)\chi_{B_R}\|_{p,\mathrm{d}\mu_k} = \lim_{k\to\infty} \|(f-g_{n_k})\chi_{B_R}\|_{p,\mathrm{d}\mu_k}$$
$$\le \lim_{k\to\infty} \|f-g_{n_k}\|_{p,\mathrm{d}\mu_k} = E_\sigma(f)_{p,\mathrm{d}\mu_k},$$

we have $\|f-g^*\|_{p,\mathrm{d}\mu_k} \le E_\sigma(f)_{p,\mathrm{d}\mu_k}$. □

6 Jackson's Inequality and Equivalence of Modulus of Smoothness and *K*-Functional

6.1 Smoothness Characteristics and *K*-Functional

We define the r-th power of the Dunkl Laplacian as a tempered distribution:

$$\langle(-\Delta_k)^r f, \varphi\rangle = \langle f, (-\Delta_k)^r\varphi\rangle, \quad f\in\mathcal{S}'(\mathbb{R}^d), \quad \varphi\in\mathcal{S}(\mathbb{R}^d), \quad r\in\mathbb{N}.$$

The Dunkl Laplacian can also be written in terms of the Dunkl transform

$$(-\Delta_k)^r f = \mathcal{F}_k^{-1}(|\cdot|^{2r}\mathcal{F}_k(f)). \tag{6.1}$$

Let $W^{2r}_{p,k}$ be the Sobolev space, that is,

$$W^{2r}_{p,k} = \{f\in L^p(\mathbb{R}^d,\mathrm{d}\mu_k)\colon (-\Delta_k)^r f\in L^p(\mathbb{R}^d,\mathrm{d}\mu_k)\}$$

equipped with the Banach norm

$$\|f\|_{W^{2r}_{p,k}} = \|f\|_{p,\mathrm{d}\mu_k} + \|(-\Delta_k)^r f\|_{p,\mathrm{d}\mu_k}.$$

Note that $(-\Delta_k)^r f \in \mathcal{S}(\mathbb{R}^d)$ whenever $f\in\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ is dense in $W^{2r}_{p,k}$. Indeed, if $f\in W^{2r}_{p,k}$, defining

$$\Phi(x) = e^{-|x|^2/2}, \quad \Phi_\varepsilon(x) = \frac{1}{\varepsilon^{2\lambda_k+2}}\,\Phi\left(\frac{x}{\varepsilon}\right),$$

we obtain that $(f *_k \Phi_\varepsilon)\in\mathcal{S}(\mathbb{R}^d)$ and (see [48])

$$\lim_{\varepsilon\to 0}\|f-(f*_k\Phi_\varepsilon)\|_{p,\mathrm{d}\mu_k} = \lim_{\varepsilon\to 0}\|(-\Delta_k)^r f - ((-\Delta_k)^r f *_k \Phi_\varepsilon)\|_{p,\mathrm{d}\mu_k} = 0.$$

Define the K-functional for the couple $(L^p(\mathbb{R}^d,\mathrm{d}\mu_k), W^{2r}_{p,k})$ as follows:

$$K_{2r}(t,f)_{p,\mathrm{d}\mu_k} = \inf\{\|f-g\|_{p,\mathrm{d}\mu_k} + t^{2r}\|(-\Delta_k)^r g\|_{p,\mathrm{d}\mu_k}\colon g\in W^{2r}_{p,k}\}.$$

Note that for any $f_1, f_2\in L^p(\mathbb{R}^d,\mathrm{d}\mu_k)$ and $g\in W^{2r}_{p,k}$, we have

$$\begin{aligned}&\|f_1-g\|_{p,\mathrm{d}\mu_k}+t^{2r}\|(-\Delta_k)^r g\|_{p,\mathrm{d}\mu_k}\\&\quad\le\|f_2-g\|_{p,\mathrm{d}\mu_k}+t^{2r}\|(-\Delta_k)^r g\|_{p,\mathrm{d}\mu_k}+\|f_1-f_2\|_{p,\mathrm{d}\mu_k},\end{aligned}$$

and hence,

$$|K_{2r}(t,f_1)_{p,\mathrm{d}\mu_k}-K_{2r}(t,f_2)_{p,\mathrm{d}\mu_k}|\le\|f_1-f_2\|_{p,\mathrm{d}\mu_k}. \tag{6.2}$$

If $f\in W^{2r}_{p,k}$, then $K_{2r}(t,f)_{p,\mathrm{d}\mu_k}\le t^{2r}\|(-\Delta_k)^r f\|_{p,\mathrm{d}\mu_k}$ and $\lim_{t\to 0}K_{2r}(t,f)_{p,\mathrm{d}\mu_k}=0$. This and (6.2) imply that, for any $f\in L^p(\mathbb{R}^d,\mathrm{d}\mu_k)$,

$$\lim_{t\to 0}K_{2r}(t,f)_{p,\mathrm{d}\mu_k}=0. \tag{6.3}$$

Another important property of the K-functional is

$$K_{2r}(\lambda t,f)_{p,\mathrm{d}\mu_k}\le\max\{1,\lambda^{2r}\}K_{2r}(t,f)_{p,\mathrm{d}\mu_k}. \tag{6.4}$$

Let I be an identical operator and $m\in\mathbb{N}$. Consider the following three differences:

$$\varDelta_t^m f(x)=(I-T^t)^m f(x)=\sum_{s=0}^{m}(-1)^s\binom{m}{s}(T^t)^s f(x), \tag{6.5}$$

$$^*\!\varDelta_t^m f(x)=\sum_{s=0}^{m}(-1)^s\binom{m}{s}T^{st}f(x), \tag{6.6}$$

$$^{**}\!\varDelta_t^m f(x)=\binom{2m}{m}^{-1}\sum_{s=-m}^{m}(-1)^s\binom{2m}{m-s}T^{st}f(x). \tag{6.7}$$

Differences (6.5) and (6.6) coincide with the classical difference for the translation operator $T^t f(x)=f(x+t)$ and correspond to the usual definition of the modulus of smoothness of order m. Difference (6.7) can be seen as follows. Define $\mu_s=(-1)^s\binom{m}{s}$, $s\in\mathbb{Z}$. Then the convolution $\mu*\mu$ is given by

$$\nu_s:=(\mu*\mu)_s=\sum_{l\in\mathbb{Z}}\mu_l\mu_{s+l}=(-1)^s\binom{2m}{m-s}.$$

Note that $\nu_s\ne 0$ if $|s|\le m$. Moreover, if $k\equiv 0$, then

$$\frac{1}{\nu_0}\sum_{s=-m}^{m}\nu_s T^{st}f(x)=f(x)+\frac{2}{\nu_0}\sum_{s=1}^{m}\nu_s S^{st}f(x)=f(x)-V_{m,t}f(x),$$

where the operator S^t was given in (3.5) and the averages

$$V_{m,t}f(x)=\frac{-2}{\nu_0}\sum_{s=1}^{m}\nu_s S^{st}f(x)$$

were defined by Dai and Ditzian in [9].

Definition 6.1 The moduli of smoothness of a function $f \in L^p(\mathbb{R}^d, d\mu_k)$ are defined by

$$\omega_m(\delta, f)_{p,d\mu_k} = \sup_{0<t\le\delta} \|\Delta_t^m f(x)\|_{p,d\mu_k}, \tag{6.8}$$

$$^*\omega_m(\delta, f)_{p,d\mu_k} = \sup_{0<t\le\delta} \|^*\Delta_t^m f(x)\|_{p,d\mu_k}, \tag{6.9}$$

$$^{**}\omega_m(\delta, f)_{p,d\mu_k} = \sup_{0<t\le\delta} \|^{**}\Delta_t^m f(x)\|_{p,d\mu_k}. \tag{6.10}$$

Let us mention some basic properties of these moduli of smoothness. Define by $\Omega_m(\delta, f)_{p,d\mu_k}$ any of the three moduli in Definition 6.1. Using the triangle inequality, estimate (3.6) reveals

$$\Omega_m(\delta, f_1 + f_2)_{p,d\mu_k} \le \Omega_m(\delta, f_1)_{p,d\mu_k} + \Omega_m(\delta, f_2)_{p,d\mu_k}$$

and

$$\begin{aligned} \Omega_m(\delta, f)_{p,d\mu_k} &\lesssim \|f\|_{p,d\mu_k}, \\ |\Omega_m(\delta, f_1)_{p,d\mu_k} - \Omega_m(\delta, f_2)_{p,d\mu_k}| &\lesssim \|f_1 - f_2\|_{p,d\mu_k}. \end{aligned} \tag{6.11}$$

If $f \in \mathcal{S}(\mathbb{R}^d)$, then, by (3.2),

$$\begin{aligned} \mathcal{F}_k(\Delta_t^m f)(y) &= j_{\lambda_k,m}(t|y|)\mathcal{F}_k(f)(y), \\ \mathcal{F}_k(^*\Delta_t^r f)(y) &= j^*_{\lambda_k,m}(t|y|)\mathcal{F}_k(f)(y), \\ \mathcal{F}_k(^{**}\Delta_t^m f)(y) &= j^{**}_{\lambda_k,m}(t|y|)\mathcal{F}_k(f)(y), \end{aligned} \tag{6.12}$$

where $\lambda_k = d/2 - 1 + \sum_{a\in R_+} k(a) > -1/2$,

$$j_{\lambda_k,m}(t) = \sum_{s=0}^{m}(-1)^s \binom{m}{s} \left(j_{\lambda_k}(t)\right)^s = (1 - j_{\lambda_k}(t))^m,$$

$$j^*_{\lambda_k,m}(t) = \sum_{s=0}^{m}(-1)^s \binom{m}{s} j_{\lambda_k}(st),$$

and

$$\begin{aligned} j^{**}_{\lambda_k,m}(t) &= \binom{2m}{m}^{-1} \sum_{s=-m}^{m} (-1)^s \binom{2m}{m-s} j_{\lambda_k}(st) \\ &= 1 + 2\binom{2m}{m}^{-1} \sum_{s=1}^{m}(-1)^s \binom{2m}{m-s} j_{\lambda_k}(st). \end{aligned} \tag{6.13}$$

These formulas alow us to prove the following remark, which will be important further in Theorem 6.6.

Remark 6.2 The functions $j_{\lambda_k,m}(t)$ and $j^{**}_{\lambda_k,m}(t)$ have zero of order $2m$ at the origin, while the function $j^*_{\lambda_k,m}(t)$ has zero of order $m + 1$ if m is odd and of order m if m is even.

Indeed, first we study $j_{\lambda_k,m}(t)=(1-j_{\lambda_k}(t))^m$. Since, for any t,

$$j_\lambda(t)=\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\lambda+1)(t/2)^{2k}}{k!\,\Gamma(k+\lambda+1)},\tag{6.14}$$

we get $j_{\lambda_k,m}(t)\asymp t^{2m}$ as $t\to 0$. Second, since

$$\sum_{s=0}^{m}(-1)^s\binom{m}{s}s^{2k}=0,\quad 0\le 2k\le m-1,$$

(see [36, Sect. 4.2]), using (6.14), we obtain that $j^*_{\lambda_k,m}(t)\asymp t^{2[(m+1)/2]}$. Finally, taking into account

$$\sum_{s=1}^{m}(-1)^s\binom{2m}{m-s}=-\frac{1}{2}\binom{2m}{m},$$
$$\sum_{s=1}^{m}(-1)^s\binom{2m}{m-s}s^{2k}=0,\quad k=1,\dots,m-1,$$

(see [36, Sect. 4.2]) and using again (6.14), we arrive at $j^{**}_{\lambda_k,m}(t)\asymp t^{2m}$. Some of these properties were known (see [9,34,35]).

Remark 6.3 In the paper [9], the authors obtained that $j^{**}_{\lambda_k,m}(t)>0$ for $t>0$.

6.2 Main Results

First we state the Jackson-type inequality.

Theorem 6.4 *Let $\sigma>0$, $1\le p\le\infty$, $r\in\mathbb{Z}_+$, $m\in\mathbb{N}$. We have, for any $f\in W^{2r}_{p,k}$,*

$$E_\sigma(f)_{p,d\mu_k}\lesssim\frac{1}{\sigma^{2r}}\,\Omega_m\left(\frac{1}{\sigma},(-\Delta_k)^r f\right)_{p,d\mu_k},\tag{6.15}$$

where Ω_m is any of the three moduli of smoothness (6.8)–(6.10).

Remark 6.5 (i) For radial functions, inequality (6.15) is the Jackson inequality in $L^p(\mathbb{R}_+,d\nu_{\lambda_k})$. In this case it was obtained in [34,35] for moduli (6.8) and (6.9). For $k\equiv 0$ and the modulus of smoothness (6.10), inequality (6.15) was obtained by Dai and Ditzian [9], see also the paper [10].

(ii) From the proof of Theorem 6.4, we will see that inequality (6.15) for moduli (6.8) and (6.10) can be equivalently written as

$$E_\sigma(f)_{p,d\mu_k}\lesssim\frac{1}{\sigma^{2r}}\left\|\Delta^m_{1/\sigma}((-\Delta_k)^r f)\right\|_{p,d\mu_k},$$
$$E_\sigma(f)_{p,d\mu_k}\lesssim\frac{1}{\sigma^{2r}}\left\|{}^{**}\Delta^m_{1/\sigma}((-\Delta_k)^r f)\right\|_{p,d\mu_k}.$$

The next theorem provides an equivalence between moduli of smoothness and the K-functional.

Theorem 6.6 *If $\delta > 0$, $1 \le p \le \infty$, $r \in \mathbb{N}$, then for any $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$,*

$$\begin{aligned} K_{2r}(\delta, f)_{p,\mathrm{d}\mu_k} &\asymp \omega_r(\delta, f)_{p,\mathrm{d}\mu_k} \asymp {}^{**}\omega_r(\delta, f)_{p,\mathrm{d}\mu_k} \\ &\asymp {}^{*}\omega_{2r-1}(\delta, f)_{p,\mathrm{d}\mu_k} \asymp {}^{*}\omega_{2r}(\delta, f)_{p,\mathrm{d}\mu_k}. \end{aligned} \tag{6.16}$$

Remark 6.7 If $k \equiv 0$, the equivalence between the classical modulus of smoothness and the K-functional is well known [8,26], while the equivalence between modulus (6.10) and the K-functional was shown in [9]. For radial functions, a partial result of (6.16), more precisely, an equivalence of the K-functional and moduli of smoothness (6.8) and (6.9), was proved in [34,35].

Remark 6.8 One can continue equivalence (6.16) as follows (see also Remark 6.16):

$$\ldots \asymp \|\varDelta_\delta^r f\|_{p,\mathrm{d}\mu_k} \asymp \|{}^{**}\varDelta_\delta^r f\|_{p,\mathrm{d}\mu_k}.$$

We give the proof for the difference (6.7) and the modulus of smoothness (6.10). We partially follow the proofs in [27,34,35], which are different from those given in [9]. For moduli of smoothness (6.8) and (6.9), the proofs are similar and will be omitted here (see also [34,35]). The proof makes use of radial multipliers and is based on boundedness of the translation operator T^t. Note that by (6.2) and (6.11), the K-functional and moduli of smoothness depend continuously on a function. Moreover, the best approximations also depend continuously on a function, and therefore one can assume that functions belong to the Schwartz space.

6.3 Properties of the de la Vallée Poussin Type Operators

Let $\eta \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ be such that $\eta(x) = 1$ if $|x| \le 1$, $\eta(x) > 0$ if $|x| < 2$, and $\eta(x) = 0$ if $|x| \ge 2$. We write

$$\eta_r(x) = \frac{1 - \eta(x)}{|x|^{2r}}, \quad \widehat{\eta}_{k,r}(y) = \mathcal{F}_k(\eta_r)(y),$$

where $\mathcal{F}_k(\eta_r)$ is a tempered distribution. If $t = |x|$, $\eta_0(t) = \eta(x)$, and $\eta_{r0}(t) = \eta_r(x)$, then $\mathcal{F}_k(\eta_r)(y) = \mathcal{H}_{\lambda_k}(\eta_{r0})(|y|)$.

Lemma 6.9 *We have $\widehat{\eta}_{k,r} \in L^1(\mathbb{R}^d, \mathrm{d}\mu_k)$, where $r > 0$.*

Proof It is sufficient to prove that $\mathcal{H}_{\lambda_k}(\eta_{r0}) \in L^1(\mathbb{R}_+, \mathrm{d}\nu_{\lambda_k})$. In the case $r \ge 1$, this was proved in [35, (4.25)]. We give the proof for any $r > 0$.

Letting $u_j(t) = (1 + t^2)^{-j}$ and taking into account that

$$\frac{1}{t^{2r}} = \frac{1}{(1+t^2)^r}\left(1 - \frac{1}{1+t^2}\right)^{-r} = \sum_{j=0}^{\infty} \binom{j+r-1}{j} \frac{1}{(1+t^2)^{j+r}}, \quad t \ne 0,$$

we obtain, for any $M \in \mathbb{N}$ and $t \geq 0$,

$$\eta_{r0}(t) = \sum_{j=0}^{\infty} \binom{j+r-1}{j}(1-\eta_0(t))u_{j+r}(t)$$
$$= \sum_{j=0}^{M-1} \binom{j+r-1}{j} u_{j+r}(t) - \eta_0(t) \sum_{j=0}^{M-1} \binom{j+r-1}{j} u_{j+r}(t)$$
$$+ \sum_{j=M}^{\infty} \binom{j+r-1}{j}(1-\eta_0(t))u_{j+r}(t) =: \psi_1(t) + \psi_2(t) + \psi_3(t).$$

For any $r > 0$, we have $\mathcal{H}_{\lambda_k}(u_r) \in L^1(\mathbb{R}_+, \mathrm{d}\nu_{\lambda_k})$ (see [35, Lemma 3.2], [47, Chapt 5, 5.3.1], [31, Chapt 8, 8.1]); therefore $\mathcal{H}_{\lambda_k}(\psi_1) \in L^1(\mathbb{R}_+, \mathrm{d}\nu_{\lambda_k})$. Because of $\psi_2 \in \mathcal{S}(\mathbb{R}_+), \mathcal{H}_{\lambda_k}(\psi_2) \in L^1(\mathbb{R}_+, \mathrm{d}\nu_{\lambda_k})$. Thus, we are left to show that, for sufficiently large M, $\mathcal{H}_{\lambda_k}(\psi_3) \in L^1(\mathbb{R}_+, \mathrm{d}\nu_{\lambda_k})$.

Let $M + r > \lambda_k + 1$, $t \geq 1$. Since $\frac{\Gamma(j+r)}{\Gamma(j+1)} \lesssim j^{r-1}$, we have

$$|\psi_3(t)| \leq \frac{M^{r-1}}{(1+t^2)^{M+r}} \sum_{j=0}^{\infty} \frac{(1+j)^{r-1}}{2^j} \lesssim \frac{1}{(1+t^2)^{M+r}} \lesssim \frac{1}{t^{2M+2r}},$$

and

$$\int_0^{\infty} |\psi_3(t)| \, \mathrm{d}\nu_{\lambda_k}(t) \lesssim \int_1^{\infty} t^{-(2M+2r-2\lambda_k-1)} \, \mathrm{d}t < \infty.$$

Thus, $\psi_3 \in L^1(\mathbb{R}_+, \mathrm{d}\nu_{\lambda_k})$, $\mathcal{H}_{\lambda_k}(\psi_3) \in C(\mathbb{R}_+)$, and $\mathcal{H}_{\lambda_k}(\psi_3) \in L^1([0, 2], \mathrm{d}\nu_{\lambda_k})$.

Recall that the Bessel differential operator is defined by

$$\mathcal{B}_{\lambda_k} = \frac{d^2}{\mathrm{d}t^2} + \frac{(2\lambda_k + 1)}{t} \frac{\mathrm{d}}{\mathrm{d}t}.$$

Using $\psi_3 \in C^{\infty}(\mathbb{R}_+)$, we have, for any $s \in \mathbb{N}$, $\mathcal{B}_{\lambda_k}^s \psi_3 \in L^1([0, 2], \mathrm{d}\nu_{\lambda_k})$.

If $t \geq 2$, then $(1 - \eta_0(t))u_{j+r}(t) = u_{j+r}(t)$ and

$$\mathcal{B}_{\lambda_k} u_{j+r}(t) = 4(j+r)(j+r-\lambda_k)u_{j+r+1}(t) - 4(j+r)(j+r+1)u_{j+r+2}(t).$$

This gives

$$|\mathcal{B}_{\lambda_k} u_{j+r}(t)| \leq 2^3 (j+r+\lambda_k+1)^2 u_{j+r+1}(t).$$

By induction on s,

$$|\mathcal{B}_{\lambda_k}^s u_{j+r}(t)| \leq 2^{3s} (j+r+2s+\lambda_k-1)^{2s} u_{j+r+s}(t),$$

and then, for $t \ge 2$,

$$|\mathcal{B}_{\lambda_k}^s \psi_3(t)| \lesssim \frac{1}{(1+t^2)^{M+r+s}} \sum_{j=0}^{\infty} \frac{(1+j)^{r+2s-1}}{5^j} \lesssim \frac{1}{(1+t^2)^{M+r+s}} \lesssim \frac{1}{t^{2M+2r+2s}},$$

and $\mathcal{B}_{\lambda_k}^s \psi_3 \in L^1([2,\infty), \mathrm{d}\nu_{\lambda_k})$. Thus, we have $\mathcal{B}_{\lambda_k}^s \psi_3 \in L^1(\mathbb{R}_+, \mathrm{d}\nu_{\lambda_k})$ for any s. Choosing $s > \lambda_k + 1$ and using the inequality

$$|\mathcal{H}_{\lambda_k}(\psi_3)(\tau)| \le \frac{1}{\tau^{2s}} \int_0^\infty |\mathcal{B}_{\lambda_k}^s \psi_3(t)| \,\mathrm{d}\nu_{\lambda_k}(t) \lesssim \frac{1}{\tau^{2s}},$$

we arrive at $\mathcal{H}_{\lambda_k}(\psi_3) \in L^1([2,\infty), \mathrm{d}\nu_{\lambda_k})$. Finally, we obtain that $\mathcal{H}_{\lambda_k}(\psi_3) \in L^1(\mathbb{R}_+, \mathrm{d}\nu_{\lambda_k})$. □

For $m, r \in \mathbb{N}$ and $m \ge r$, we set

$$g_{m,r}^*(y) := |y|^{-2r} j_{\lambda_k,m}^{**}(|y|), \quad g_{m,r}(x) := \mathcal{F}_k(g_{m,r}^*)(x),$$
$$g_{m,r}^t(x) := t^{2r-2\lambda_k-2} g_{m,r}\left(\frac{x}{t}\right).$$

Since

$$g_{m,r}^*(y) = j_{\lambda_k,m}^{**}(|y|)\eta_r(y) + \frac{j_{\lambda_k,m}^{**}(|y|)}{|y|^{2r}} \eta(y),$$
$$\frac{j_{\lambda_k,m}^{**}(|y|)}{|y|^{2r}} \in C^\infty(\mathbb{R}^d), \quad \frac{j_{\lambda_k,m}^{**}(|y|)}{|y|^{2r}} \eta(y) \in \mathcal{S}(\mathbb{R}^d),$$

and

$$\mathcal{F}_k(j_{\lambda_k,m}^{**}\eta_r)(x) = \binom{2m}{m}^{-1} \sum_{s=-m}^{m} (-1)^s \binom{2m}{m-s} T^s \widehat{\eta}_{\lambda_k,r}(x),$$

boundedness of the operator T^s in $L^1(\mathbb{R}^d, \mathrm{d}\mu_k)$ and Lemma 6.9 imply that

$$g_{m,r},\ g_{m,r}^t \in L^1(\mathbb{R}^d, \mathrm{d}\mu_k), \quad \|g_{m,r}^t\|_{1,\mathrm{d}\mu_k} = t^{2r} \|g_{m,r}\|_{1,\mathrm{d}\mu_k},$$
$$\mathcal{F}_k^{-1}(g_{m,r}^t)(y) = \mathcal{F}_k(g_{m,r}^t)(y) = t^{2r} g_{m,r}^*(ty) = |y|^{-2r} j_{\lambda_k,m}^{**}(t|y|). \tag{6.17}$$

Lemma 6.10 *Let $m, r \in \mathbb{N}$, $m \ge r$, $1 \le p \le \infty$, and $f \in \mathcal{S}(\mathbb{R}^d)$. We have*

$$^{**}\Delta_t^m f = (-\Delta_k)^r f *_k g_{m,r}^t \tag{6.18}$$

and

$$\|^{**}\Delta_t^m f\|_{p,\mathrm{d}\mu_k} \lesssim t^{2r} \|(-\Delta_k)^r f\|_{p,\mathrm{d}\mu_k}. \tag{6.19}$$

Proof Combining (3.15), (6.1), (6.12), and (6.17), we obtain that

$$\mathcal{F}_k({}^{**}\Delta_t^m f)(y) = j^{**}_{\lambda_k,m}(t|y|)\mathcal{F}_k(f)(y) = |y|^{2r}\mathcal{F}_k(f)(y)\frac{j^{**}_{\lambda_k,m}(t|y|)}{|y|^{2r}}$$
$$= \mathcal{F}_k((-\Delta_k)^r f)(y)\mathcal{F}_k(g^t_{m,r})(y).$$

Then (6.18) follows from (3.11) and Lemma 3.4. Inequality (6.19) follows from (6.17), (6.18), Lemma 3.4, and (3.12). Note that a constant in (6.19) can be taken as $\|g_{m,r}\|_{1,\mathrm{d}\mu_k}$. □

Remark 6.11 Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $W^{2r}_{p,k}$, in light of (6.11), inequality (6.19) holds for any function from $W^{2r}_{p,k}$.

Let $f \in \mathcal{S}(\mathbb{R}^d)$. We set $\theta(x) = \mathcal{F}_k(\eta)(x)$ and $\theta_\sigma(x) = \theta(x/\sigma)$. Then θ, $\theta_\sigma \in \mathcal{S}(\mathbb{R}^d)$. The de la Vallée Poussin type operator is given by $P_\sigma(f) = f *_k \theta_\sigma$. By Lemma 3.4,

$$\mathcal{F}_k(P_\sigma(f))(y) = \eta(y/\sigma)\mathcal{F}_k(f)(y).$$

Lemma 6.12 *If* $\sigma > 0$, $1 \le p \le \infty$, $f \in \mathcal{S}(\mathbb{R}^d)$, *then*

(1) $P_\sigma(f) \in B^{2\sigma}_{p,k}$ *and* $P_\sigma(g) = g$ *for any* $g \in B^{\sigma}_{p,k}$;
(2) $\|P_\sigma(f)\|_{p,\mathrm{d}\mu_k} \lesssim \|f\|_{p,\mathrm{d}\mu_k}$;
(3) $\|f - P_\sigma(f)\|_{p,\mathrm{d}\mu_k} \lesssim E_\sigma(f)_{p,\mathrm{d}\mu_k}$.

Remark 6.13 Property (3) in this lemma means that $P_\sigma(f)$ is *the near best approximant* of f in $L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$.

Proof (1) We observe that $\operatorname{supp}\eta(\cdot/\sigma) \subset B_{2\sigma}$ and then $\operatorname{supp}\mathcal{F}_k(P_\sigma(f)) \subset B_{2\sigma}$. Theorem 5.14 yields $P_\sigma(f) \in B^{2\sigma}_{p,k}$. If $g \in B^{\sigma}_{p,k}$, then by Theorem 5.14, $\operatorname{supp}\mathcal{F}_k(g) \subset B_\sigma$ and $\mathcal{F}_k(P_\sigma(g))(y) = \eta(y/\sigma)\mathcal{F}_k(g)(y) = \mathcal{F}_k(g)(y)$. Hence, $P_\sigma(g) = g$.

(2) In light of (3.12),

$$\|P_\sigma(f)\|_{p,\mathrm{d}\mu_k} = \|f *_k \theta_\sigma\|_{p,\mathrm{d}\mu_k} \le \|\theta_\sigma\|_{1,\mathrm{d}\mu_k}\|f\|_{p,\mathrm{d}\mu_k}$$
$$= \|\theta\|_{1,\mathrm{d}\mu_k}\|f\|_{p,\mathrm{d}\mu_k} \lesssim \|f\|_{p,\mathrm{d}\mu_k}.$$

(3) Using Theorem 5.15, there exists an entire function $g^* \in B^{\sigma}_{p,k}$ such that $\|f - g^*\|_{p,\mathrm{d}\mu_k} = E_\sigma(f)_{p,\mathrm{d}\mu_k}$. Then using $P_\sigma(g^*) = g^*$ implies

$$\|f - P_\sigma(f)\|_{p,\mathrm{d}\mu_k} = \|f - g^* + P_\sigma(g^* - f)\|_{p,\mathrm{d}\mu_k}$$
$$\le \|f - g^*\|_{p,\mathrm{d}\mu_k} + \|P_\sigma(f - g^*)\|_{p,\mathrm{d}\mu_k} \lesssim E_\sigma(f)_{p,\mathrm{d}\mu_k}.$$

□

In the proof of the next lemma we will use the estimate

$$|j^{(n)}_\lambda(t)| \lesssim (|t|+1)^{-(\lambda+1/2)}, \quad t \in \mathbb{R},\ \lambda \ge -1/2,\ n \in \mathbb{Z}_+, \tag{6.20}$$

which follows, by induction on n, from the known properties of the Bessel function [2, Chap. 7]

$$|j_\lambda(t)| \lesssim (|t|+1)^{-(\lambda+1/2)}, \quad j_\lambda'(t) = -\frac{t}{2(\lambda+1)} j_{\lambda+1}(t).$$

Lemma 6.14 *If $\sigma > 0$, $1 \le p \le \infty$, $m \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $f \in \mathcal{S}(\mathbb{R}^d)$, then*

$$\|f - P_{\sigma/2}(f)\|_{p,\mathrm{d}\mu_k} \lesssim \sigma^{-2r} \|{}^{**}\Delta_{a/\sigma}^m((-\Delta_k)^r f)\|_{p,\mathrm{d}\mu_k} \tag{6.21}$$

for some $a = a(\lambda_k, m) > 0$.

Proof We have

$$\begin{aligned}\mathcal{F}_k(f - P_{\sigma/2}(f))(y) &= (1 - \eta(2y/\sigma))\mathcal{F}_k f(y) \\ &= \sigma^{-2r} \frac{1 - \eta(2y/\sigma)}{(|y|/\sigma)^{2r} j_{\lambda_k,m}^{**}(a|y|/\sigma)} \mathcal{F}_k({}^{**}\Delta_{a/\sigma}^m((-\Delta_k)^r f))(y) \\ &= \sigma^{-2r} \varphi(y/\sigma) \mathcal{F}_k({}^{**}\Delta_{a/\sigma}^m((-\Delta_k)^r f))(y), \end{aligned} \tag{6.22}$$

where

$$\varphi(y) = \frac{1 - \eta(2y)}{|y|^{2r} j_{\lambda_k,m}^{**}(a|y|)}, \quad {}^{**}\Delta_{a/\sigma}^m((-\Delta_k)^r f) \in \mathcal{S}(\mathbb{R}^d). \tag{6.23}$$

Setting $j_{\lambda_k,m}^{**}(t) = 1 - \tau_0(t)$, in light of (6.13) and (6.20), we observe that $j_{\lambda_k,m}^{**}(t) \to 1$ as $t \to \infty$. Then we can choose $a > 0$ such that $|\tau_0(t)| \le 1/2$ for $|t| \ge a/2$. For such $a = a(\lambda_k, m)$, we have that $\varphi(y) = 0$ for $|y| \le 1/2$, $\varphi(y) > 0$ for $|y| > 1/2$, and $\varphi \in C^\infty(\mathbb{R}^d)$. Moreover, the derivatives $\varphi^{(k)}(y)$ grow at infinity not faster than $|y|^{a_k}$, which yields $\varphi \in \mathcal{S}'(\mathbb{R}^d)$.

We will use the following decomposition:

$$\varphi(y) = \varphi_1(|y|) + \varphi_2(|y|),$$

where

$$\varphi_1(|y|) = 2^{2r} \eta_r(2y) \left(\frac{1}{1 - \tau_0(a|y|)} - S_N(\tau_0(a|y|) \right)$$

and

$$\varphi_2(|y|) = 2^{2r} \eta_r(2y) S_N(\tau_0(a|y|)), \quad \eta_r(y) = \frac{1 - \eta(y)}{|y|^{2r}}, \quad S_N(t) = \sum_{j=0}^{N-1} t^j.$$

First, we show that $\mathcal{F}_k(\varphi_1(|\cdot|)) \in L^1(\mathbb{R}^d, \mathrm{d}\mu_k)$. Since for a radial function we have

$$\Delta_k \varphi_1(|y|) = \varphi_1''(|y|) + \frac{2\lambda_k + 1}{|y|} \varphi_1'(|y|)$$

and, for $|t| \le 1/2$,

$$(1-t)^{-1} - \sum_{j=0}^{N-1} t^j = (1-t)^{-1} - S_N(t) = \frac{t^N}{1-t},$$

then, by (6.13) and (6.20), we obtain

$$\Delta_k^s \varphi_1(|y|) = O(|y|^{-2r-N(\lambda_k+1/2)}), \quad |y| \ge 1/2, \ s \in \mathbb{Z}_+.$$

Hence, for a fixed $N \ge 2 + 2/(2\lambda_k + 1)$, we have $\Delta_k^s \varphi_1(|y|) \in L^1(\mathbb{R}^d, \mathrm{d}\mu_k)$, where $s \in \mathbb{Z}_+$. Applying (6.1) we derive that

$$|\mathcal{F}_k(\varphi_1(|\cdot|))(x)| = \frac{|\mathcal{F}_k((-\Delta_k)^s \varphi_1(|\cdot|))(x)|}{|x|^{2s}} \le \frac{\|(-\Delta_k)^s \varphi_1(|\cdot|)\|_{1,\mathrm{d}\mu_k}}{|x|^{2s}}.$$

Setting $s > \lambda_k + 1$ yields $\mathcal{F}_k(\varphi_1(|\cdot|)) \in L^1(\mathbb{R}^d, \mathrm{d}\mu_k)$.

Second, let us show that $\mathcal{F}_k(\varphi_2(|\cdot|)) \in L^1(\mathbb{R}^d, \mathrm{d}\mu_k)$ for $r \in \mathbb{N}$. Let

$$\tau_0(t) = \sum_{s=1}^{m} \nu_s j_{\lambda_k}(st), \quad \psi_r(x) = 2^{2r} \mathcal{F}_k(\eta_r(2\,\cdot))(x),$$

$$A^a f(x) = \sum_{s=1}^{m} \nu_s T^{as} f(x), \quad B^a f(x) = \sum_{j=0}^{N-1} (A^a)^j f(x).$$

Boundedness of the operator T^t in $L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$ implies

$$\|A^a\|_{p\to p} = \sup\{\|Af\|_{p,\mathrm{d}\mu_k} : \|f\|_{p,\mathrm{d}\mu_k} \le 1\} \le \sum_{s=1}^{m} |\nu_s|$$

and

$$\|B^a\|_{p\to p} \le \sum_{j=0}^{N-1} (\|A\|_{p\to p})^j \le N\left(1 + \sum_{s=1}^{m} |\nu_s|\right)^{N-1}, \quad 1 \le p < \infty. \tag{6.24}$$

Then for $p = 1$, taking into account Lemma 6.9, we have

$$\|\mathcal{F}_k(\varphi_2(|\cdot|))\|_{1,\mathrm{d}\mu_k} = \left\|B^a \psi_r\right\|_{1,\mathrm{d}\mu_k} \le N\left(1 + \sum_{s=1}^{m} |\nu_s|\right)^{N-1} \|\psi_r\|_{1,\mathrm{d}\mu_k} < \infty.$$

Thus, $\mathcal{F}_k(\varphi) \in L^1(\mathbb{R}^d, \mathrm{d}\mu_k)$. Combining Lemma 3.4, relations (3.12), (6.22), (6.23), and the formula $\|\mathcal{F}_k(\varphi(\cdot/\sigma))\|_{1,\mathrm{d}\mu_k} = \|\mathcal{F}_k(\varphi)\|_{1,\mathrm{d}\mu_k}$, we obtain inequality (6.21) for $r \in \mathbb{N}$.

Now let $r = 0$. Define the operators A_1 and A_2 as follows:

$$\mathcal{F}_k(A_1 g)(y) = \varphi_1(|y|/\sigma)\mathcal{F}_k(g)(y)$$

and

$$\mathcal{F}_k(A_2 g)(y) = \varphi_2(|y|/\sigma)\mathcal{F}_k(g)(y), \quad \varphi_2(|y|) = (1 - \eta(2y))S_N(\tau_0(a|y|)).$$

Since $\mathcal{F}_k(\varphi_1(|\cdot|)) \in L^1(\mathbb{R}^d, \mathrm{d}\mu_k)$,

$$\|A_1 g\|_{p,\mathrm{d}\mu_k} \le \|\mathcal{F}_k(\varphi_1(|y|))\|_{1,\mathrm{d}\mu_k} \|g\|_{p,\mathrm{d}\mu_k} \lesssim \|g\|_{p,\mathrm{d}\mu_k}, \\ 1 \le p \le \infty, \ g \in \mathcal{S}(\mathbb{R}^d). \tag{6.25}$$

We are left to show that

$$\|A_2 g\|_{p,\mathrm{d}\mu_k} \lesssim \|g\|_{p,\mathrm{d}\mu_k}, \quad 1 \le p \le \infty, \ g \in \mathcal{S}(\mathbb{R}^d).$$

We have

$$\begin{aligned}\mathcal{F}_k(A_2 g)(y) &= (1 - \eta(2y/\sigma))S_N(\tau_0(a|y|/\sigma))\mathcal{F}_k(g)(y) \\ &= (1 - \eta(2y/\sigma))\mathcal{F}_k(B^{a/\sigma} g)(y) \\ &= \mathcal{F}_k(B^{a/\sigma} g - P_{\sigma/2}(B^{a/\sigma} g))(y).\end{aligned}$$

Since $B^{a/\sigma} g \in \mathcal{S}(\mathbb{R}^d)$, using Lemma 6.12 and inequality (6.24), we get

$$\|A_2 g\|_{p,\mathrm{d}\mu_k} \le \|B^{a/\sigma} g\|_{p,\mathrm{d}\mu_k} \le N\left(1 + \sum_{s=1}^{m} |\nu_s|\right)^{N-1} \|g\|_{p,\mathrm{d}\mu_k} \lesssim \|g\|_{p,\mathrm{d}\mu_k}. \tag{6.26}$$

Using (6.25) and (6.26) with $g = {}^{**}\!\Delta^m_{a/\sigma} f$, we finally obtain (6.21) for $r = 0$. □

Lemma 6.15 *If $\sigma > 0$, $1 \le p \le \infty$, $m \in \mathbb{N}$, $f \in \mathcal{S}(\mathbb{R}^d)$, then*

$$\|((-\Delta_k)^m P_\sigma(f)\|_{p,\mathrm{d}\mu_k} \lesssim \sigma^{2m} \|{}^{**}\!\Delta^m_{a/(2\sigma)} f\|_{p,\mathrm{d}\mu_k}, \tag{6.27}$$

where $a = a(\lambda_k, m) > 0$ is given in Lemma 6.14.

Proof We have

$$\begin{aligned}\mathcal{F}_k(((-\Delta_k)^m P_\sigma(f))(y) &= |y|^{2m}\eta(y/\sigma)\mathcal{F}_k(f)(y) \\ &= \sigma^{2m}\varphi(y/\sigma) j^{**}_{\lambda_k,m}(a/(2\sigma))\mathcal{F}_k(f)(y) \\ &= \sigma^{2m}\varphi(y/\sigma)\mathcal{F}_k({}^{**}\!\Delta^m_{a/(2\sigma)} f)(y),\end{aligned}$$

where

$$\varphi(y) = \frac{|y|^{2m}\eta(y)}{j^{**}_{\lambda_k,m}(a|y|/2)}.$$

Since $j^{**}_{\lambda_k,m}(a|y|/2)/|y|^{2m} > 0$ for $|y| > 0$, we observe that $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\mathcal{F}_k(\varphi) \in L^1(\mathbb{R}^d, \mathrm{d}\mu_k)$. Then estimate (6.27) follows from Lemma 3.4, Young's inequality (3.12), and $\|\mathcal{F}_k(\varphi(\cdot/\sigma))\|_{1,\mathrm{d}\mu_k} = \|\mathcal{F}_k(\varphi)\|_{1,\mathrm{d}\mu_k}$. □

6.4 Proofs of Theorems 6.4 and 6.6

Proof of Theorem 6.6 In connection with Lemma 6.10 and Remark 6.11, observe that, for $f \in \mathcal{S}(\mathbb{R}^d)$ and $g \in W^{2r}_{p,k}$,

$$\begin{aligned}\|{}^{**}\Delta^r_\delta f\|_{p,\mathrm{d}\mu_k} &\le {}^{**}\omega_r(\delta, f)_{p,\mathrm{d}\mu_k} \le {}^{**}\omega_r(\delta, f-g)_{p,\mathrm{d}\mu_k} + {}^{**}\omega_r(\delta, g)_{p,\mathrm{d}\mu_k}\\ &\lesssim \|f-g\|_{p,\mathrm{d}\mu_k} + \delta^{2r}\|(-\Delta_k)^r g\|_{p,\mathrm{d}\mu_k}.\end{aligned}$$

Then

$$\|{}^{**}\Delta^r_\delta f\|_{p,\mathrm{d}\mu_k} \le {}^{**}\omega_r(\delta, f)_{p,\mathrm{d}\mu_k} \lesssim K_{2r}(\delta, f)_{p,\mathrm{d}\mu_k}. \tag{6.28}$$

On the other hand, $P_\sigma(f) \in W^{2r}_{p,k}$ and

$$K_{2r}(\delta, f)_{p,\mathrm{d}\mu_k} \le \|f - P_\sigma(f)\|_{p,\mathrm{d}\mu_k} + \delta^{2r}\|(-\Delta_k)^r P_\sigma(f)\|_{p,\mathrm{d}\mu_k}. \tag{6.29}$$

In light of Lemma 6.14,

$$\|f - P_\sigma(f)\|_{p,\mathrm{d}\mu_k} \lesssim \|{}^{**}\Delta^r_{a/(2\sigma)} f\|_{p,\mathrm{d}\mu_k}.$$

Further, Lemma 6.15 yields

$$\|((-\Delta_k)^r P_\sigma(f)\|_{p,\mathrm{d}\mu_k} \lesssim \sigma^{2r}\|{}^{**}\Delta^r_{a/(2\sigma)} f\|_{p,\mathrm{d}\mu_k}. \tag{6.30}$$

Setting $\sigma = a/(2\delta)$, from (6.29)–(6.30) we arrive at

$$K_{2r}(\delta, f)_{p,\mathrm{d}\mu_k} \lesssim \|{}^{**}\Delta^r_\delta f\|_{p,\mathrm{d}\mu_k} \lesssim {}^{**}\omega_r(\delta, f)_{p,\mathrm{d}\mu_k}. \tag{6.31}$$

Proof of Theorem 6.4 Using property (6.4) and inequalities (6.28) and (6.31), we obtain

$$\begin{aligned}E_\sigma(f)_{p,\mathrm{d}\mu_k} &\le \|f - P_{\sigma/2}(f)\|_{p,\mathrm{d}\mu_k} \lesssim \sigma^{-2r}\|{}^{**}\Delta^m_{a/\sigma}((-\Delta_k)^r f)\|_{p,\mathrm{d}\mu_k}\\ &\lesssim \frac{1}{\sigma^{2r}} K_{2m}\left(\frac{a}{\sigma}, (-\Delta_k)^r f\right)_{p,\mathrm{d}\mu_k} \lesssim \frac{1}{\sigma^{2r}} K_{2m}\left(\frac{1}{\sigma}, (-\Delta_k)^r f\right)_{p,\mathrm{d}\mu_k}\end{aligned}$$

$$\lesssim \frac{1}{\sigma^{2r}} \|^{**}\Delta^{m}_{1/\sigma}((-\Delta_k)^r f)\|_{p,\mathrm{d}\mu_k} \lesssim \frac{1}{\sigma^{2r}} {}^{**}\omega_m\left(\frac{1}{\sigma}, (-\Delta_k)^r f\right)_{p,\mathrm{d}\mu_k}. \tag{6.32}$$

Remark 6.16 The proofs of estimates (6.31) and (6.32) for the difference (6.7) are based on the fact that the parameter a in Lemmas 6.14 and 6.15 is the same. It is possible due to the fact that $j^{**}_{\lambda_k,m}(t) > 0$ for $t > 0$, see Remark 6.3. This estimate is valid for the difference (6.5) as well, since $j_{\lambda_k,m}(t) = (1 - j_{\lambda_k}(t))^m > 0$ for $t > 0$.

Therefore, the moduli of smoothness (6.8) and (6.10) in inequalities (6.15) and (6.16) can be replaced by the norms of the corresponding differences (6.5) and (6.7). For the modulus of smoothness (6.9), this observation is not valid since $j^{*}_{\lambda_k,m}(t)$ does not keep its sign.

Remark 6.17 Properties (6.3) and (6.4) of the K-functional and the equivalence (6.16) imply the following properties of moduli of smoothness:

(1) $\lim\limits_{\delta\to 0+0} \omega_m(\delta, f)_{p,\mathrm{d}\mu_k} = \lim\limits_{\delta\to 0+0} {}^{*}\omega_m(\delta, f)_{p,\mathrm{d}\mu_k} = \lim\limits_{\delta\to 0+0} {}^{**}\omega_m(\delta, f)_{p,\mathrm{d}\mu_k} = 0$;

(2) $\omega_m(\lambda\delta, f)_{p,\mathrm{d}\mu_k} \lesssim \max\{1, \lambda^{2m}\}\, \omega_m(\delta, f)_{p,\mathrm{d}\mu_k}$;

(3) ${}^{*}\omega_l(\lambda\delta, f)_{p,\mathrm{d}\mu_k} \lesssim \max\{1, \lambda^{2m}\}\, {}^{*}\omega_l(\delta, f)_{p,\mathrm{d}\mu_k}, \quad l = 2m-1,\ 2m$;

(4) ${}^{**}\omega_m(\lambda\delta, f)_{p,\mathrm{d}\mu_k} \lesssim \max\{1, \lambda^{2m}\}\, {}^{**}\omega_m(\delta, f)_{p,\mathrm{d}\mu_k}$.

7 Some Inequalities for Entire Functions

In this section, we study weighted analogues of the inequalities for entire functions. In particular, we obtain Nikolskii's inequality ([31], see Theorem 7.1 below), Bernstein's inequality ([31], Theorem 7.3), Nikolskii–Stechkin's inequality ([30,45], Theorem 7.5), and Boas-type inequality ([4], Theorem 7.7).

Theorem 7.1 *If* $\sigma > 0$, $0 < p \le q \le \infty$, $f \in B^{\sigma}_{p,k}$, *then*

$$\|f\|_{q,\mathrm{d}\mu_k} \lesssim \sigma^{(2\lambda_k+2)(1/p-1/q)} \|f\|_{p,\mathrm{d}\mu_k}. \tag{7.1}$$

Remark 7.2 Observe that the obtained Nikolskii inequality is sharp, i.e., we actually have

$$\sup_{f\in B^{\sigma}_{p,k},\, f\neq 0} \frac{\|f\|_{q,\mathrm{d}\mu_k}}{\|f\|_{p,\mathrm{d}\mu_k}} \asymp \sigma^{(2\lambda_k+2)(1/p-1/q)},$$

and an extremizer can be taken as

$$f_{\sigma,m}(x) = \frac{\sin^{2m}(\theta|x|)}{|x|^{2m}}, \quad \theta = \frac{\sigma}{2m},$$

for sufficiently large $m \in \mathbb{N}$.

Proof Let $f \in B^{\sigma}_{p,k}$, $p \geq 1$, $q = \infty$. By Theorem 5.14, we have $\operatorname{supp} \mathcal{F}_k(f) \subset B_{\sigma}$, and then

$$\mathcal{F}_k(f)(y) = \eta(y/\sigma)\mathcal{F}_k(f)(y), \quad \eta(y) = \eta_0(|y|). \tag{7.2}$$

Lemma 3.9 implies

$$f(x) = (f *_{\lambda_k} \mathcal{H}_{\lambda_k}(\eta_0(\cdot/\sigma)))(x) = \int_0^{\infty} T^t f(x)\mathcal{H}_{\lambda_k}(\eta_0(\cdot/\sigma))(t)\, \mathrm{d}\nu_{\lambda_k}(t).$$

Taking into account that

$$\mathcal{H}_{\lambda_k}(\eta_0(\cdot/\sigma))(t) = \sigma^{2\lambda_k+2}\mathcal{H}_{\lambda_k}(\eta_0)(\sigma t),$$
$$\|\mathcal{H}_{\lambda_k}(\eta_0)(\sigma t)\|_{p',\mathrm{d}\mu_k} = \sigma^{-\frac{2\lambda_k+2}{p'}} \|\mathcal{H}_{\lambda_k}(\eta_0)(t)\|_{p',\mathrm{d}\mu_k},$$

Hölder's inequality and Theorem 3.5 yield

$$\begin{aligned}|f(x)| &\leq \sigma^{2\lambda_k+2}\|T^t f(x)\|_{p,\mathrm{d}\nu_{\lambda_k}} \|\mathcal{H}_{\lambda_k}(\eta_0)(\sigma t)\|_{p',\mathrm{d}\mu_k} \\ &\leq \sigma^{(2\lambda_k+2)/p}\|\mathcal{H}_{\lambda_k}(\eta_0)(t)\|_{p',\mathrm{d}\mu_k}\|f\|_{p,\mathrm{d}\mu_k} \lesssim \sigma^{(2\lambda_k+2)/p}\|f\|_{p,\mathrm{d}\mu_k},\end{aligned}$$

i.e., (7.1) holds.

Let $f \in B^{\sigma}_{p,k}$, $0 < p < 1$, $q = \infty$. By Theorem 5.1, f is bounded and $f \in B^{\sigma}_{1,k}$. We have

$$\|f\|_{1,\mathrm{d}\mu_k} = \||f|^{1-p}|f|^p\|_{1,\mathrm{d}\mu_k} \leq \||f|^{1-p}\|_{\infty}\||f|^p\|_{1,\mathrm{d}\mu_k} = \|f\|_{\infty}^{1-p}\|f\|^p_{p,\mathrm{d}\mu_k}.$$

Using (7.1) with $p = 1$ and $q = \infty$,

$$\|f\|_{1,\mathrm{d}\mu_k} \lesssim \sigma^{2\lambda_k+2}\|f\|_{1,\mathrm{d}\mu_k}\|f\|_{\infty}^{-p}\|f\|^p_{p,\mathrm{d}\mu_k},$$

which gives

$$\|f\|_{\infty} \lesssim \sigma^{(2\lambda_k+2)/p}\|f\|_{p,\mathrm{d}\mu_k}.$$

Thus, the proof of (7.1) for $q = \infty$ is complete.

If $0 < p \leq q < \infty$, we obtain

$$\begin{aligned}\|f\|_{q,\mathrm{d}\mu_k} &= \||f|^{1-p/q}|f|^{p/q}\|_{q,\mathrm{d}\mu_k} \leq \|f\|_{\infty}^{1-p/q}\|f\|^{p/q}_{p,\mathrm{d}\mu_k} \\ &\leq \sigma^{(2\lambda_k+2)(1-p/q)/p}\|f\|^{1-p/q}_{p,\mathrm{d}\mu_k}\|f\|^{p/q}_{p,\mathrm{d}\mu_k} = \sigma^{(2\lambda_k+2)(1/p-1/q)}\|f\|_{p,\mathrm{d}\mu_k}.\end{aligned}$$

□

Theorem 7.3 *If $\sigma > 0$, $r \in \mathbb{N}$, $1 \leq p \leq \infty$, $f \in B^{\sigma}_{p,k}$, then*

$$\|(-\Delta_k)^r f\|_{p,\mathrm{d}\mu_k} \lesssim \sigma^{2r}\|f\|_{p,\mathrm{d}\mu_k}. \tag{7.3}$$

Proof It is enough to consider the case $r = 1$. As in the previous theorem, we use (7.2) to obtain

$$\mathcal{F}_k((-\Delta_k)f)(y) = |y|^2\eta(y/\sigma)\mathcal{F}_k(f)(y) = \sigma^2\varphi_0(|y|/\sigma)\mathcal{F}_k(f)(y),$$

where $\varphi_0(t) = t^2\eta_0(t) \in \mathcal{S}(\mathbb{R}_+)$. Combining Lemma 3.9, inequality (3.12), and $\|\mathcal{F}_k(\varphi_0(|\cdot|/\sigma))\|_{1,d\mu_k} = \|\mathcal{F}_k(\varphi_0(|\cdot|))\|_{1,d\mu_k}$, we arrive at

$$\|(-\Delta_k)f\|_{p,d\mu_k} \le \sigma^2\|\mathcal{F}_k(\varphi_0(|\cdot|))\|_{1,d\mu_k}\|f\|_{p,d\mu_k} \lesssim \sigma^2\|f\|_{p,d\mu_k}.$$

□

The next result follows from Lemma 6.10, Remark 6.11, and Theorem 7.3.

Corollary 7.4 *If* $\sigma,\ \delta > 0$, $m \in \mathbb{N}$, $1 \le p \le \infty$, $f \in B_{p,k}^{\sigma}$, *then*

$$\begin{aligned}
\omega_m(\delta, f)_{p,d\mu_k} &\lesssim (\sigma\delta)^{2m}\|f\|_{p,d\mu_k},\\
{}^{*}\omega_l(\delta, f)_{p,d\mu_k} &\lesssim (\sigma\delta)^{2m}\|f\|_{p,d\mu_k}, \quad l = 2m-1,\ 2m,\\
{}^{**}\omega_m(\delta, f)_{p,d\mu_k} &\lesssim (\sigma\delta)^{2m}\|f\|_{p,d\mu_k},
\end{aligned}$$

where constants do not depend on σ, δ, *and* f.

Theorem 7.5 *If* $\sigma > 0$, $m \in \mathbb{N}$, $1 \le p \le \infty$, $0 < t \le 1/(2\sigma)$, $f \in B_{p,k}^{\sigma}$, *then*

$$\|(-\Delta_k)^m f\|_{p,d\mu_k} \lesssim t^{-2m}\|{}^{**}\Delta_t^m f\|_{p,d\mu_k}. \tag{7.4}$$

Remark 7.6 By Remark 6.8, this inequality can be equivalently written as

$$\|(-\Delta_k)^m f\|_{p,d\mu_k} \lesssim t^{-2m}K_{2m}(t, f)_{p,d\mu_k}.$$

Proof We have

$$\mathcal{F}_k((-\Delta_k)^m f)(y) = \frac{|y|^{2m}\eta(y/\sigma)}{j_{\lambda_k,m}^{**}(t|y|)}\, j_{\lambda_k,m}^{**}(t|y|)\mathcal{F}_k(f)(y).$$

Since for $0 < t \le 1/(2\sigma)$,

$$\eta(y/\sigma) = \eta(y/\sigma)\eta(ty),$$

we obtain that

$$\mathcal{F}_k((-\Delta_k)^m f)(y) = t^{-2m}\eta(y/\sigma)\varphi(ty)\,j_{\lambda_k,m}^{**}(t|y|)\mathcal{F}_k(f)(y),$$

where

$$\varphi(y) = \frac{|y|^{2m}\eta(y)}{j_{\lambda_k,m}^{**}(|y|)} \in \mathcal{S}(\mathbb{R}^d).$$

Using

$$j^{**}_{\lambda_k,m}(t|\cdot|)\mathcal{F}_k(f)=\mathcal{F}_k({}^{**}\Delta_t^m f),\quad {}^{**}\Delta_t^m f\in L^p(\mathbb{R}^d,\mathrm{d}\mu_k),$$

and

$$\|\mathcal{F}_k(\eta(\cdot/\sigma))\|_{1,\mathrm{d}\mu_k}=\|\mathcal{F}_k(\eta)\|_{1,\mathrm{d}\mu_k},\quad \|\mathcal{F}_k(\varphi(t\,\cdot))\|_{1,\mathrm{d}\mu_k}=\|\mathcal{F}_k(\varphi)\|_{1,\mathrm{d}\mu_k},$$

and combining Lemma 3.9 and inequality (3.12), we have

$$\begin{aligned}\|(-\Delta_k)^m f\|_{p,\mathrm{d}\mu_k}&\le t^{-2m}\|\mathcal{F}_k(\eta(\cdot/\sigma))\|_{1,\mathrm{d}\mu_k}\|\mathcal{F}_k(\varphi(t\,\cdot))\|_{1,\mathrm{d}\mu_k}\|{}^{**}\Delta_t^m f\|_{p,\mathrm{d}\mu_k}\\&=t^{-2m}\|\mathcal{F}_k(\eta)\|_{1,\mathrm{d}\mu_k}\|\mathcal{F}_k(\varphi)\|_{1,\mathrm{d}\mu_k}\|{}^{**}\Delta_t^m f\|_{p,\mathrm{d}\mu_k}\\&\lesssim t^{-2m}\|{}^{**}\Delta_t^m f\|_{p,\mathrm{d}\mu_k}.\end{aligned}$$

□

Theorem 7.7 *If $\sigma>0$, $m\in\mathbb{N}$, $1\le p\le\infty$, $0<\delta\le t\le 1/(2\sigma)$, $f\in B^{\sigma}_{p,k}$, then*

$$\delta^{-2m}\|{}^{**}\Delta_\delta^m f\|_{p,\mathrm{d}\mu_k}\lesssim t^{-2m}\|{}^{**}\Delta_t^m f\|_{p,\mathrm{d}\mu_k}. \tag{7.5}$$

Remark 7.8 Using Remark 6.8, Theorem 7.5, and taking into account that $\delta^{-2m}K_{2m}(\delta,f)_{p,\mathrm{d}\mu_k}$ is decreasing in δ (see (6.4)), inequality (7.5) can be equivalently written as

$$\begin{aligned}\|(-\Delta_k)^m f\|_{p,\mathrm{d}\mu_k}&\asymp\delta^{-2m}\|{}^{**}\Delta_\delta^m f\|_{p,\mathrm{d}\mu_k}\asymp t^{-2m}\|{}^{**}\Delta_t^m f\|_{p,\mathrm{d}\mu_k},\\\|(-\Delta_k)^m f\|_{p,\mathrm{d}\mu_k}&\asymp\delta^{-2m}K_{2m}(\delta,f)_{p,\mathrm{d}\mu_k}\asymp t^{-2m}K_{2m}(t,f)_{p,\mathrm{d}\mu_k}.\end{aligned}$$

Proof We have

$$\begin{aligned}\mathcal{F}_k({}^{**}\Delta_\delta^m f)(y)&=j^{**}_{\lambda_k,m}(\delta|y|)\mathcal{F}_k(f)(y)\\&=\eta(y/\sigma)\frac{j^{**}_{\lambda_k,m}(\delta|y|)\eta(ty)}{j^{**}_{\lambda_k,m}(t|y|)}\mathcal{F}_k({}^{**}\Delta_t^m f)(y)\\&=\theta^{2m}\eta(y/\sigma)\varphi_\theta(ty)\mathcal{F}_k({}^{**}\Delta_t^m f)(y),\end{aligned}$$

where $\theta=(\delta/t)^{2m}\in(0,1]$,

$$\varphi_\theta(y)=\frac{\psi(\theta y)\eta(y)}{\psi(y)}\in\mathcal{S}(\mathbb{R}^d),\quad \psi(y)=\frac{j^{**}_{\lambda_k,m}(|y|)}{|y|^{2m}}\in C^\infty(\mathbb{R}^d).$$

Using Lemma 3.9 and estimate (3.12), we arrive at inequality (7.5):

$$\|{}^{**}\Delta_\delta^m f\|_{p,\mathrm{d}\mu_k}\le\theta^{2m}\|\mathcal{F}_k(\eta)\|_{1,\mathrm{d}\mu_k}\max_{0\le\theta\le1}\|\mathcal{F}_k(\varphi_\theta)\|_{1,\mathrm{d}\mu_k}\|{}^{**}\Delta_t^m f\|_{p,\mathrm{d}\mu_k}$$

$$\lesssim \left(\frac{\delta}{t}\right)^{2m} \|^{**}\Delta_t^m f\|_{p,\mathrm{d}\mu_k},$$

provided that the function $n(\theta) = \|\mathcal{F}_k(\varphi_\theta)\|_{1,\mathrm{d}\mu_k}$ is continuous on [0, 1]. Let us prove this.

Set $\varphi_\theta(y) = \varphi_{\theta 0}(|y|)$, $r = |y|$, $\rho = |x|$. Then

$$\begin{aligned} n(\theta) &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \varphi_\theta(y) e_k(x, y)\, \mathrm{d}\mu_k(y) \right| \mathrm{d}\mu_k(x) \\ &= \int_0^\infty \left| \int_0^2 \varphi_{\theta 0}(r) j_{\lambda_k}(\rho r)\, \mathrm{d}\nu_{\lambda_k}(r) \right| \mathrm{d}\nu_{\lambda_k}(\rho) \\ &= b_{\lambda_k}^2 \int_0^\infty \left| \int_0^2 \varphi_{\theta 0}(r) j_{\lambda_k}(\rho r) r^{2\lambda_k+1}\, \mathrm{d}r \right| \rho^{2\lambda_k+1}\, \mathrm{d}\rho. \end{aligned}$$

The inner integral continuously depends on θ. Let us show that the outer integral converges uniformly in $\theta \in [0, 1]$. Since [2, Sect. 7.2]

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(j_{\lambda_k+1}(\rho r) r^{2\lambda_k+2}\right) = (2\lambda_k + 2) j_{\lambda_k}(\rho r) r^{2\lambda_k+1},$$

integrating by parts implies

$$\begin{aligned} &\int_0^2 \varphi_{\theta 0}(r) j_{\lambda_k}(\rho r) r^{2\lambda_k+1}\, \mathrm{d}r = \int_0^2 \varphi_{\theta 0}(r)\, d\left(\int_0^r j_{\lambda_k}(\rho \tau) \tau^{2\lambda_k+1}\right) \\ &= \frac{1}{2\lambda_k + 2} \int_0^2 \varphi_{\theta 0}(r)\, d\left(j_{\lambda_k+1}(\rho r) r^{2\lambda_k+2}\right) \\ &= -\frac{1}{2\lambda_k + 2} \int_0^2 \frac{\varphi_{\theta 0}(r)}{r} j_{\lambda_k+1}(\rho r) r^{2\lambda_k+3}\, \mathrm{d}r = \ldots \\ &= (-1)^s \left(\prod_{j=1}^s (2\lambda_k + 2s)\right)^{-1} \int_0^2 \varphi_{\theta 0}^{[s]}(r) j_{\lambda_k+s}(\rho r) r^{2\lambda_k+2s+1}\, \mathrm{d}r, \end{aligned}$$

where

$$\varphi_{\theta 0}^{[s]}(r) := \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\varphi_{\theta 0}^{[s-1]}(r)}{r}\right).$$

This and (6.20) give

$$\left| \int_0^2 \varphi_{\theta 0}(r) j_{\lambda_k}(\rho r) r^{2\lambda_k+1}\, \mathrm{d}r \right| \le \frac{c_1(\lambda_k, m, s)}{(\rho + 1)^{\lambda_k+s+1/2}}$$

and, for $s > \lambda_k + 3/2$,

$$n(\theta) \le c_2(\lambda_k, m, s) \int_0^\infty (1+\rho)^{-(s-\lambda_k-1/2)}\,\mathrm{d}\rho \le c_3(\lambda_k, m, s),$$

completing the proof. □

Remark 7.9 Combining (7.1) and (7.3), the following Bernstein–Nikolskii inequality is valid:

$$\|(-\Delta_k)^r f\|_{q,\mathrm{d}\mu_k} \lesssim \sigma^{2r+(2\lambda_k+2)(1/p-1/q)} \|f\|_{p,\mathrm{d}\mu_k}, \quad 1 \le p \le q \le \infty.$$

Remark 7.10 For radial functions, Nikolskii inequality (7.1), Bernstein (7.3), Nikolskii–Stechkin (7.4), and Boas inequality (7.5) follow from corresponding estimates in the space $L^p(\mathbb{R}_+, \mathrm{d}\nu_\lambda)$ proved in [34].

8 Realization of *K*-Functionals and Moduli of Smoothness

In the nonweighted case ($k \equiv 0$) the equivalence between the classical modulus of smoothness and the K-functional between L^p and the Sobolev space W_p^r is well known [8,26]: $1 \le p \le \infty$, *for any integer r one has*

$$\omega_r(t, f)_{L^p(\mathbb{R})} \asymp K_r(f, t)_p, \quad 1 \le p \le \infty,$$

where

$$K_r(f, t)_p := \inf_{g \in \dot{W}_p^r} \left(\|f - g\|_p + t^r \|g\|_{\dot{W}_p^r}\right).$$

Starting from the paper [13] (see also [17, Lemma 1.1] for the fractional case), the following equivalence between the modulus of smoothness and the realization of the K-functional is widely used in approximation theory:

$$\omega_r(t, f)_{L^p(\mathbb{R})} \asymp \mathcal{R}_r(t, f)_p = \inf_g \left\{\|f - g\|_p + t^r \|g^{(r)}\|_p\right\},$$

where g is an entire function of exponential type $1/t$.

Let the realization of the K-functional $K_{2r}(t, f)_{p,\mathrm{d}\mu_k}$ be given as follows:

$$\mathcal{R}_{2r}(t, f)_{p,\mathrm{d}\mu_k} = \inf \left\{\|f - g\|_{p,\mathrm{d}\mu_k} + t^{2r} \|(-\Delta_k)^r g\|_{p,\mathrm{d}\mu_k} : g \in B_{p,k}^{1/t}\right\}$$

and

$$\mathcal{R}^*_{2r}(t, f)_{p,\mathrm{d}\mu_k} = \|f - g^*\|_{p,\mathrm{d}\mu_k} + t^{2r} \|(-\Delta_k)^r g^*\|_{p,\mathrm{d}\mu_k},$$

where $g^* \in B_{p,k}^{1/t}$ is a near best approximant.

Theorem 8.1 *If $t > 0$, $1 \le p \le \infty$, $r \in \mathbb{N}$, then for any $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$,*

$$\mathcal{R}_{2r}(t, f)_{p,\mathrm{d}\mu_k} \asymp \mathcal{R}^*_{2r}(t, f)_{p,\mathrm{d}\mu_k} \asymp K_{2r}(t, f)_{p,\mathrm{d}\mu_k} \asymp \omega_r(t, f)_{p,\mathrm{d}\mu_k} \\ \asymp {}^{**}\omega_r(t, f)_{p,\mathrm{d}\mu_k} \asymp {}^{*}\omega_{2r-1}(t, f)_{p,\mathrm{d}\mu_k} \asymp {}^{*}\omega_{2r}(t, f)_{p,\mathrm{d}\mu_k}.$$

Proof By Theorem 6.6,

$$\omega_r(t, f)_{p,\mathrm{d}\mu_k} \asymp {}^{**}\omega_r(t, f)_{p,\mathrm{d}\mu_k} \asymp {}^{*}\omega_{2r-1}(t, f)_{p,\mathrm{d}\mu_k} \asymp {}^{*}\omega_{2r}(t, f)_{p,\mathrm{d}\mu_k} \\ \asymp K_{2r}(t, f)_{p,\mathrm{d}\mu_k} \le \mathcal{R}_{2r}(t, f)_{p,\mathrm{d}\mu_k} \le \mathcal{R}^*_{2r}(t, f)_{p,\mathrm{d}\mu_k},$$

where we have used the fact that $B^{1/t}_{p,k} \subset W^{2r}_{p,k}$, which follows from Theorem 7.3.

Therefore, it is enough to show that

$$\mathcal{R}^*_{2r}(t, f)_{p,\mathrm{d}\mu_k} \le C\omega_r(t, f)_{p,\mathrm{d}\mu_k}.$$

Indeed, for g^* being the best approximant (or near best approximant), the Jackson inequality given in Theorem 6.4 implies that

$$\|f - g^*\|_{p,\mathrm{d}\mu_k} \lesssim E_{1/t}(f)_{p,\mathrm{d}\mu_k} \lesssim \omega_r(t, f)_{p,\mathrm{d}\mu_k}. \tag{8.1}$$

Using the first inequality in Theorem 7.5 and taking into account (8.1), we have

$$\|(-\Delta_k)^r g^*\|_{p,\mathrm{d}\mu_k} \lesssim t^{-2r}\|\varDelta^r_{t/2} g^*\|_{p,\mathrm{d}\mu_k} \\ \lesssim t^{-2r}\|\varDelta^r_{t/2}(g^* - f)\|_{p,\mathrm{d}\mu_k} + t^{-2r}\|\varDelta^r_{t/2} f\|_{p,\mathrm{d}\mu_k} \\ \lesssim t^{-2r}\|g^* - f\|_{p,\mathrm{d}\mu_k} + t^{-2r}\omega_r(t/2, f)_{p,\mathrm{d}\mu_k}.$$

Using again (8.1), we arrive at

$$\|f - g^*\|_{p,\mathrm{d}\mu_k} + t^{2r}\|(-\Delta_k)^r g^*\|_{p,\mathrm{d}\mu_k} \lesssim \omega_r(t, f)_{p,\mathrm{d}\mu_k},$$

completing the proof.

The next result answers the following important question (see, e.g., [22,51]): when does the relation

$$\omega_m\left(\frac{1}{n}, f\right)_{p,\mathrm{d}\mu_k} \asymp E_n(f)_{p,\mathrm{d}\mu_k} \tag{8.2}$$

(or similar relations with concepts in Theorem 8.2) hold?

Theorem 8.2 *Let $1 \le p \le \infty$ and $m \in \mathbb{N}$. We have that (8.2) is valid if and only if*

$$\omega_m\left(\frac{1}{n}, f\right)_{p,\mathrm{d}\mu_k} \asymp \omega_{m+1}\left(\frac{1}{n}, f\right)_{p,\mathrm{d}\mu_k}. \tag{8.3}$$

Proof We prove only the nontrivial part that (8.3) implies (8.2). Since, by (6.4), we have $\omega_m(nt, f)_{p,\mathrm{d}\mu_k} \lesssim n^{2m}\omega_m(t, f)_{p,\mathrm{d}\mu_k}$, relation (8.3) implies that

$$\omega_{m+1}(nt, f)_{p,\mathrm{d}\mu_k} \lesssim n^{2m}\omega_{m+1}(t, f)_{p,\mathrm{d}\mu_k}. \tag{8.4}$$

This and Jackson's inequality give

$$\begin{aligned}
&\frac{1}{n^{2(m+1)}}\sum_{j=0}^{n}(j+1)^{2(m+1)-1}E_j(f)_{p,\mathrm{d}\mu_k}\\
&\lesssim \frac{1}{n^{2(m+1)}}\sum_{j=0}^{n}(j+1)^{2(m+1)-1}\omega_{m+1}\left(\frac{1}{j+1}, f\right)_{p,\mathrm{d}\mu_k}\\
&\lesssim \omega_{m+1}\left(\frac{1}{n}, f\right)_{p,\mathrm{d}\mu_k}.
\end{aligned}$$

Moreover, Theorem 9.1 below implies

$$\begin{aligned}
\omega_{m+1}\left(\frac{1}{ln}, f\right)_{p,\mathrm{d}\mu_k} &\lesssim \frac{1}{(ln)^{2(m+1)}}\sum_{j=0}^{ln}(j+1)^{2(m+1)-1}E_j(f)_{p,\mathrm{d}\mu_k}\\
&\lesssim \frac{1}{l^{2(m+1)}}\,\omega_{m+1}\left(\frac{1}{n}, f\right)_{p,\mathrm{d}\mu_k}\\
&\quad + \frac{1}{(ln)^{2(m+1)}}\sum_{j=n+1}^{ln}(j+1)^{2(m+1)-1}E_j(f)_{p,\mathrm{d}\mu_k},
\end{aligned}$$

or, in other words,

$$\begin{aligned}
&\frac{1}{n^{2(m+1)}}\sum_{j=n+1}^{ln}(j+1)^{2(m+1)-1}E_j(f)_{p,\mathrm{d}\mu_k}\\
&\gtrsim Cl^{2(m+1)}\omega_{m+1}\left(\frac{1}{ln}, f\right)_{p,\mathrm{d}\mu_k} - \omega_{m+1}\left(\frac{1}{n}, f\right)_{p,\mathrm{d}\mu_k}.
\end{aligned}$$

Using again (8.4), we obtain

$$\frac{1}{n^{2(m+1)}}\sum_{j=n+1}^{ln}(j+1)^{2(m+1)-1}E_j(f)_{p,\mathrm{d}\mu_k} \gtrsim (Cl^2-1)\omega_{m+1}\left(\frac{1}{n}, f\right)_{p,\mathrm{d}\mu_k}.$$

Taking into account monotonicity of $E_j(f)_{p,\mathrm{d}\mu_k}$ and choosing l sufficiently large, we arrive at (8.2). □

9 Inverse Theorems of Approximation Theory

Theorem 9.1 *Let* $m, n \in \mathbb{N}$, $1 \le p \le \infty$, $f \in L^p(\mathbb{R}^d, d\mu_k)$. *We have*

$$K_{2m}\left(\frac{1}{n}, f\right)_{p,d\mu_k} \lesssim \frac{1}{n^{2m}} \sum_{j=0}^{n} (j+1)^{2m-1} E_j(f)_{p,d\mu_k}. \tag{9.1}$$

Remark 9.2 By Remark 6.8, $K_{2m}\left(\frac{1}{n}, f\right)_{p,d\mu_k}$ in this inequality can be equivalently replaced by $\omega_m\left(\frac{1}{n}, f\right)_{p,d\mu_k}$, ${}^{**}\omega_m\left(\frac{1}{n}, f\right)_{p,d\mu_k}$, and ${}^{*}\omega_l\left(\frac{1}{n}, f\right)_{p,d\mu_k}$, $l = 2m-1, 2m$.

Proof Let us prove (9.1) for $\omega_m\left(\frac{1}{n}, f\right)_{p,d\mu_k}$. By Theorem 5.15, for any $\sigma > 0$, there exists $f_\sigma \in B_{p,k}^{\sigma}$ such that

$$\|f - f_\sigma\|_{p,d\mu_k} = E_\sigma(f)_{p,d\mu_k}, \quad E_0(f)_{p,d\mu_k} = \|f\|_{p,d\mu_k}.$$

For any $s \in \mathbb{Z}_+$,

$$\begin{aligned}\omega_m(1/n, f)_{p,d\mu_k} &\le \omega_m(1/n, f - f_{2^{s+1}})_{p,d\mu_k} + \omega_m(1/n, f_{2^{s+1}})_{p,d\mu_k}\\ &\lesssim E_{2^{s+1}}(f)_{p,d\mu_k} + \omega_m(1/n, f_{2^{s+1}})_{p,d\mu_k}.\end{aligned}$$

Using Lemma 6.10,

$$\begin{aligned}&\omega_m(1/n, f_{2^{s+1}})_{p,d\mu_k} \lesssim n^{-2m}\|(-\Delta_k)^m f_{2^{s+1}}\|_{p,d\mu_k}\\ &\lesssim \frac{1}{n^{2m}}\left(\|(-\Delta_k)^m f_1\|_{p,d\mu_k} + \sum_{j=0}^{s}\|(-\Delta_k)^m f_{2^{j+1}} - (-\Delta_k)^m f_{2^j}\|_{p,d\mu_k}\right).\end{aligned}$$

Then Bernstein inequality (7.3) implies that

$$\begin{aligned}\|(-\Delta_k)^m f_{2^{j+1}} - (-\Delta_k)^m f_{2^j}\|_{p,d\mu_k} &\lesssim 2^{2m(j+1)}\|f_{2^{j+1}} - f_{2^j}\|_{p,d\mu_k}\\ &\lesssim 2^{2m(j+1)} E_{2^j}(f)_{p,d\mu_k},\\ \|(-\Delta_k)^m f_1\|_{p,d\mu_k} &\lesssim E_0(f)_{p,d\mu_k}.\end{aligned}$$

Thus,

$$\omega_m(1/n, f_{2^{s+1}})_{p,d\mu_k} \lesssim \frac{1}{n^{2m}}\left(E_0(f)_{p,d\mu_k} + \sum_{j=0}^{s} 2^{2m(j+1)} E_{2^j}(f)_{p,d\mu_k}\right).$$

Taking into account that

$$\sum_{l=2^{j-1}+1}^{2^j} l^{2m-1} E_l(f)_{p,d\mu_k} \ge 2^{2m(j-1)} E_{2^j}(f)_{p,d\mu_k}, \tag{9.2}$$

we have

$$\omega_m(1/n, f_{2^{s+1}})_{p,\mathrm{d}\mu_k} \lesssim \frac{1}{n^{2m}} \Bigg(E_0(f)_{p,\mathrm{d}\mu_k} + 2^{2m} E_1(f)_{p,\mathrm{d}\mu_k} + \sum_{j=1}^{s} 2^{4m} \sum_{l=2^{j-1}+1}^{2^j} l^{2m-1} E_l(f)_{p,\mathrm{d}\mu_k} \Bigg) \lesssim \frac{1}{n^{2m}} \sum_{j=0}^{2^s} (j+1)^{2m-1} E_j(f)_{p,\mathrm{d}\mu_k}.$$

Choosing s such that $2^s \le n < 2^{s+1}$ implies (9.1). □

Theorem 9.1 and Jackson's inequality imply the following Marchaud inequality.

Corollary 9.3 *Let $m \in \mathbb{N}$, $1 \le p \le \infty$, $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$. We have*

$$K_{2m}(\delta, f)_{p,\mathrm{d}\mu_k} \lesssim \delta^{2m} \left(\|f\|_{p,\mathrm{d}\mu_k} + \int_\delta^1 t^{-2m} K_{2m+2}(t, f)_{p,\mathrm{d}\mu_k} \frac{\mathrm{d}t}{t} \right).$$

Theorem 9.4 *Let $1 \le p \le \infty$, $f \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$, and $r \in \mathbb{N}$ be such that $\sum_{j=1}^\infty j^{2r-1} E_j(f)_{p,\mathrm{d}\mu_k} < \infty$. Then $f \in W^{2r}_{p,k}$, and, for any $m, n \in \mathbb{N}$, we have*

$$K_{2m}\left(\frac{1}{n}, (-\Delta_k)^r f\right)_{p,\mathrm{d}\mu_k} \lesssim \frac{1}{n^{2r}} \sum_{j=0}^{n} (j+1)^{2k+2r-1} E_j(f)_{p,\mathrm{d}\mu_k} + \sum_{j=n+1}^{\infty} j^{2r-1} E_j(f)_{p,\mathrm{d}\mu_k}. \tag{9.3}$$

Remark 9.5 We can replace $K_{2m}\left(\frac{1}{n}, (-\Delta_k)^r f\right)_{p,\mathrm{d}\mu_k}$ by any of moduli $\omega_m\left(\frac{1}{n}, (-\Delta_k)^r f\right)_{p,\mathrm{d}\mu_k}$, ${}^*\omega_l\left(\frac{1}{n}, (-\Delta_k)^r f\right)_{p,\mathrm{d}\mu_k}$, and ${}^{**}\omega_m\left(\frac{1}{n}, (-\Delta_k)^r f\right)_{p,\mathrm{d}\mu_k}$, $l = 2m-1, 2m$.

Proof Let us prove (9.3) for $\omega_m\left(\frac{1}{n}, (-\Delta_k)^r f\right)_{p,\mathrm{d}\mu_k}$. Consider

$$(-\Delta_k)^r f_1 + \sum_{j=0}^{\infty} \left((-\Delta_k)^r f_{2^{j+1}} - (-\Delta_k)^r f_{2^j} \right). \tag{9.4}$$

By Bernstein's inequality (7.3),

$$\|(-\Delta_k)^r f_{2^{j+1}} - (-\Delta_k)^r f_{2^j}\|_{p,\mathrm{d}\mu_k} \lesssim 2^{(j+1)r} E_{2^j}(f)_{p,\mathrm{d}\mu_k} \lesssim \sum_{l=2^{j-1}+1}^{2^j} l^{r-1} E_l(f)_{p,\mathrm{d}\mu_k}.$$

Therefore, series (9.4) converges to a function $g \in L^p(\mathbb{R}^d, \mathrm{d}\mu_k)$. Let us show that $g = (-\Delta_k)^r f$, i.e., $f \in W^{2r}_{p,k}$. Set

$$S_N = (-\Delta_k)^r f_1 + \sum_{j=0}^{N} \left((-\Delta_k)^r f_{2^{j+1}} - (-\Delta_k)^r f_{2^j} \right).$$

Then

$$\begin{aligned}\langle \mathcal{F}_k(g), \varphi \rangle &= \langle g, \mathcal{F}_k(\varphi) \rangle = \lim_{N\to\infty} \langle S_N, \mathcal{F}_k(\varphi) \rangle \\ &= \lim_{N\to\infty} \langle \mathcal{F}_k(S_N), \varphi \rangle = \lim_{N\to\infty} \langle |y|^{2r} \mathcal{F}_k(f_{2^{N+1}}), \varphi \rangle = \langle |y|^{2r} \mathcal{F}_k(f), \varphi \rangle,\end{aligned}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Hence, $\mathcal{F}_k(g)(y) = |y|^{2r} \mathcal{F}_k(f)(y)$ and $g = (-\Delta_k)^r f$.

To obtain (9.3), we write

$$\omega_m(1/n, (-\Delta_k)^r f)_{p,\mathrm{d}\mu_k} \le \omega_m(1/n, (-\Delta_k)^r f - S_N)_{p,\mathrm{d}\mu_k} + \omega_m(1/n, S_N)_{p,\mathrm{d}\mu_k}.$$

The first term is estimated as follows

$$\begin{aligned}\omega_m(1/n, (-\Delta_k)^r f - S_N)_{p,\mathrm{d}\mu_k} &\lesssim \|(-\Delta_k)^r f - S_N\|_{p,\mathrm{d}\mu_k} \\ &\lesssim \sum_{j=N+1}^{\infty} 2^{2r(j+1)} E_{2^j}(f)_{p,\mathrm{d}\mu_k} \lesssim \sum_{l=2^N+1}^{\infty} l^{2r-1} E_l(f)_{p,\mathrm{d}\mu_k}.\end{aligned}$$

Moreover, by Corollary 7.4,

$$\begin{aligned}\omega_m(1/n, S_N)_{p,\mathrm{d}\mu_k} &\le \omega_m(1/n, (-\Delta_k)^r f_1)_{p,\mathrm{d}\mu_k} \\ &\quad + \sum_{j=0}^{N} \omega_m \left(1/n, (-\Delta_k)^r f_{2^{j+1}} - (-\Delta_k)^r f_{2^j}\right)_{p,\mathrm{d}\mu_k} \\ &\lesssim \frac{1}{n^{2r}} \left(E_0(f)_{p,\mathrm{d}\mu_k} + \sum_{j=0}^{N} 2^{2(m+r)(j+1)} E_{2^j}(f)_{p,\mathrm{d}\mu_k} \right).\end{aligned}$$

Using (9.2) and choosing N such that $2^N \le n < 2^{N+1}$ completes the proof of (9.3). □

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