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CONSTRUCTIVE APPROXIMATION

The Mahler Measure of the Rudin–Shapiro Polynomials

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Abstract Littlewood polynomials are polynomials with each of their coefficients in $\{-1, 1\}$. A sequence of Littlewood polynomials that satisfies a remarkable flatness property on the unit circle of the complex plane is given by the Rudin–Shapiro polynomials. It is shown in this paper that the Mahler measure and the maximum modulus of the Rudin–Shapiro polynomials on the unit circle of the complex plane have the same size. It is also shown that the Mahler measure and the maximum norm of the Rudin–Shapiro polynomials have the same size even on not too small subarcs of the unit circle of the complex plane. Not even nontrivial lower bounds for the Mahler measure of the Rudin–Shapiro polynomials have been known before.

Keywords Rudin–Shapiro polynomials · Littlewood polynomials · Mahler measure

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1 Introduction

Let $\alpha < \beta$ be real numbers. The Mahler measure $M_0(Q, [\alpha, \beta])$ is defined for bounded measurable functions Q defined on $[\alpha, \beta]$ as

$$M_0(Q, [\alpha, \beta]) := \exp\left(\frac{1}{\beta - \alpha}\int_\alpha^\beta \log\left|Q(e^{it})\right| dt\right).$$

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It is well known, see [20], for instance, that

$$M_0(Q, [\alpha, \beta]) = \lim_{q \to 0+} M_q(Q, [\alpha, \beta]),$$

where

$$M_q(Q, [\alpha, \beta]) := \left(\frac{1}{\beta - \alpha} \int_\alpha^\beta \left| Q(e^{it}) \right|^q dt \right)^{1/q}, \quad q > 0.$$

It is a simple consequence of the Jensen formula that

$$M_0(Q) := M_0(Q, [0, 2\pi]) = |c| \prod_{k=1}^n \max\{1, |z_k|\}$$

for every polynomial of the form

$$Q(z) = c \prod_{k=1}^n (z - z_k), \quad c, z_k \in \mathbb{C}.$$

See [3, p. 271] or [2, p. 3], for instance. Let $D := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk of the complex plane. Let $\partial D := \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle of the complex plane.

Finding polynomials with suitably restricted coefficients and maximal Mahler measure has interested many authors. The classes

$$\mathcal{L}_n := \left\{ p : p(z) = \sum_{k=0}^n a_k z^k, \quad a_k \in \{-1, 1\} \right\}$$

of Littlewood polynomials and the classes

$$\mathcal{K}_n := \left\{ p : p(z) = \sum_{k=0}^n a_k z^k, \quad a_k \in \mathbb{C}, \ |a_k| = 1 \right\}$$

of unimodular polynomials are two of the most important classes considered. Observe that $\mathcal{L}_n \subset \mathcal{K}_n$ and

$$M_0(Q) = M_0(Q, [0, 2\pi]) \le M_2(Q, [0, 2\pi]) = \sqrt{n+1}$$

for every $Q \in \mathcal{K}_n$. Beller and Newman [1] constructed unimodular polynomials $Q_n \in \mathcal{K}_n$ whose Mahler measure $M_0(Q, [0, 2\pi])$ is at least $\sqrt{n} - c/\log n$.

Section 4 of [2] is devoted to the study of Rudin–Shapiro polynomials. Littlewood asked if there were polynomials $p_{n_k} \in \mathcal{L}_{n_k}$ satisfying

$$c_1 \sqrt{n_k + 1} \le |p_{n_k}(z)| \le c_2 \sqrt{n_k + 1}, \quad z \in \partial D,$$

with some absolute constants $c_1 > 0$ and $c_2 > 0$, see [2, p. 27] for a reference to this problem of Littlewood. To satisfy just the lower bound, by itself, seems very hard, and no such sequence (p_{n_k}) of Littlewood polynomials $p_{n_k} \in \mathcal{L}_{n_k}$ is known. A sequence of Littlewood polynomials that satisfies just the upper bound is given by the Rudin–Shapiro polynomials. The Rudin–Shapiro polynomials appear in Harold Shapiro's 1951 thesis [23] at MIT and are sometimes called just Shapiro polynomials. They also arise independently in Golay's paper [19]. They are remarkably simple to construct and are a rich source of counterexamples to possible conjectures.

The Rudin–Shapiro polynomials are defined recursively as follows:

$$P_0(z) := 1, \quad Q_0(z) := 1,$$

and

$$P_{n+1}(z) := P_n(z) + z^{2^n} Q_n(z),$$
$$Q_{n+1}(z) := P_n(z) - z^{2^n} Q_n(z),$$

for $n = 0, 1, 2, \ldots$. Note that both P_n and Q_n are polynomials of degree $N - 1$ with $N := 2^n$ having each of their coefficients in $\{-1, 1\}$. It is well known and easy to check by using the parallelogram law that

$$|P_{n+1}(z)|^2 + |Q_{n+1}(z)|^2 = 2(|P_n(z)|^2 + |Q_n(z)|^2), \quad z \in \partial D.$$

Hence

$$|P_n(z)|^2 + |Q_n(z)|^2 = 2^{n+1} = 2N, \quad z \in \partial D. \tag{1.1}$$

It is also well known (see Section 4 of [2], for instance), that

$$|Q_n(z)| = |P_n(-z)|, \quad z \in \partial D. \tag{1.2}$$

Peter Borwein's book [2] presents a few more basic results on the Rudin–Shapiro polynomials. Cyclotomic properties of the Rudin–Shapiro polynomials are discussed in [8]. Obviously $M_2(P_n, [0, 2\pi]) = 2^{n/2}$ by the Parseval formula. In 1968 Littlewood [21] evaluated $M_4(P_n, [0, 2\pi])$ and found that $M_4(P_n, [0, 2\pi]) \sim (4^{n+1}/3)^{1/4}$. Rudin–Shapiro-like polynomials in L_4 on the unit circle of the complex plane are studied in [6]. In 1980, Saffari conjectured that

$$M_q(P_n, [0, 2\pi]) \sim \frac{2^{(n+1)/2}}{(q/2 + 1)^{1/q}}$$

for all even integers $q > 0$, perhaps for all real $q > 0$. This conjecture was proved for all even values of $q \leq 52$ by Doche [14] and Doche and Habsieger [15].

Despite the simplicity of their definition, not much is known about the Rudin–Shapiro polynomials. It is shown in this paper that the Mahler measure and the maximum modulus of the Rudin–Shapiro polynomials on the unit circle of the complex plane have the same size. A consequence of this result is also proved. It is also shown in this paper that the Mahler measure and the maximum norm of the Rudin–Shapiro polynomials have the same size even on not too small subarcs of the unit circle of the complex plane. Not even nontrivial lower bounds for the Mahler measure of the Rudin–Shapiro polynomials have been known before.

Borwein and Lockhart [7] investigated the asymptotic behavior of the mean value of normalized L_p norms of Littlewood polynomials for arbitrary $p > 0$. They proved that

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{(M_q(f, [0, 2\pi]))^q}{n^{q/2}} = \Gamma\left(1 + \frac{q}{2}\right).$$

An analogue of this result for $q = 0$ (the Mahler measure) has been proved recently in [10]. Similar results on the average Mahler measure and $L_q(\partial D)$ norms of unimodular polynomials $P \in \mathcal{K}_n$ had been established earlier in [12].

For a prime number p, the p-th Fekete polynomial is defined as

$$f_p(z) := \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) z^k,$$

where

$$\left(\frac{k}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv k (\text{mod } p) \text{ has a nonzero solution,} \\ 0 & \text{if } p \text{ divides } k, \\ -1 & \text{otherwise} \end{cases}$$

is the usual Legendre symbol. Since f_p has constant coefficient 0, it is not a Littlewood polynomial, but g_p defined by $g_p(z) := f_p(z)/z$ is a Littlewood polynomial of degree $p - 2$ and has the same Mahler measure as f_p. Fekete polynomials are examined in detail in [2,4,5,13,16–18,22]. In [9,11], the authors examined the maximal size of the Mahler measure of sums on n monomials on the unit circle as well as on subarcs of the unit circles. In the constructions appearing in [9], properties of the Fekete polynomials f_p turned out to be quite useful.

2 Main Theorems

Our first theorem states that the Mahler measure and the maximum norm of the Rudin–Shapiro polynomials on the unit circle of the complex plane have the same size.

Theorem 2.1 *Let P_n and Q_n be the n-th Rudin–Shapiro polynomials defined in Sect. 1. There is an absolute constant $c_1 > 0$ such that*

$$M_0(P_n, [0, 2\pi]) = M_0(Q_n, [0, 2\pi]) \geq c_1\sqrt{N},$$

where

$$N := 2^n = \deg(P_n) + 1 = \deg(Q_n) + 1.$$

By following the line of our proof, it is easy to verify that $c_1 = e^{-227}$ is an appropriate choice in Theorem 2.1. To formulate our next theorem, we define

$$\widetilde{P}_n := 2^{-(n+1)/2} P_n \quad \text{and} \quad \widetilde{Q}_n := 2^{-(n+1)/2} Q_n. \tag{2.1}$$

By using the above normalization, (1.1) can be rewritten as

$$\left|\widetilde{P}_n(z)\right|^2 + \left|\widetilde{Q}_n(z)\right|^2 = 1, \quad z \in \partial D. \tag{2.2}$$

Let

$$I_q\left(\widetilde{P}_n\right) := \left(M_q\left(\widetilde{P}_n, [0, 2\pi]\right)\right)^q := \frac{1}{2\pi}\int_0^{2\pi} \left|\widetilde{P}_n(e^{i\tau})\right|^q d\tau, \quad q > 0.$$

The following result on the moments of the Rudin–Shapiro polynomials is a simple consequence of Theorem 2.1.

Theorem 2.2 *There is a constant $L < \infty$ independent of n such that*

$$\sum_{k=1}^{\infty} \frac{I_k\left(\widetilde{P}_n\right)}{k} < L, \quad n = 0, 1, \ldots.$$

Our final result states that the Mahler measure and the maximum norm of the Rudin–Shapiro polynomials have the same size even on not too small subarcs of the unit circle of the complex plane.

Theorem 2.3 *There is an absolute constant $c_2 > 0$ such that*

$$M_0(P_n, [\alpha, \beta]) \geq c_2\sqrt{N}, \quad N := 2^n = \deg(P_n) + 1,$$

for all $n \in \mathbb{N}$ and for all $\alpha, \beta \in \mathbb{R}$ such that

$$\frac{32\pi}{N} \leq \frac{(\log N)^{3/2}}{N^{1/2}} \leq \beta - \alpha \leq 2\pi.$$

The same lower bound holds for $M_0(Q_n, [\alpha, \beta])$.

It looks plausible that Theorem 2.3 holds whenever $32\pi/N \leq \beta - \alpha \leq 2\pi$, but we do not seem to be able to handle the case $32\pi/N \leq \beta - \alpha \leq (\log N)^{3/2} N^{-1/2}$ in this paper.

3 Lemmas

A key to the Proof of Theorem 2.1 is the following observation which is a straightforward consequence of the definition of the Rudin–Shapiro polynomials P_n and Q_n.

Lemma 3.1 *Let $n \geq 2$ be an integer, $N := 2^n$, and let*

$$z_j := \mathrm{e}^{it_j}, \quad t_j := \frac{2\pi j}{N}, \quad j \in \mathbb{Z}.$$

We have

$$P_n(z_j) = 2P_{n-2}(z_j), \quad j = 2u,\ u \in \mathbb{Z},$$
$$P_n(z_j) = (-1)^{(j-1)/2} 2i\, Q_{n-2}(z_j), \quad j = 2u + 1,\ u \in \mathbb{Z},$$

where i is the imaginary unit.

Another key to the Proof of Theorem 2.1 is Theorem 1.3 from [16]. Let $\mathcal{P}_N$ be the set of all polynomials of degree at most N with real coefficients.

Lemma 3.2 *Assume that $n, m \geq 1$,*

$$0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m \leq 2\pi, \quad \tau_0 := \tau_m - 2\pi, \quad \tau_{m+1} := \tau_1 + 2\pi.$$

Let

$$\delta := \max\{\tau_1 - \tau_0, \tau_2 - \tau_1, \ldots, \tau_m - \tau_{m-1}\}.$$

For every $A > 0$, there is a $B > 0$ depending only on A such that

$$\sum_{j=1}^{m} \frac{\tau_{j+1} - \tau_{j-1}}{2} \log|P(\mathrm{e}^{i\tau_j})| \leq \int_0^{2\pi} \log|P(\mathrm{e}^{i\tau})| \mathrm{d}\tau + B$$

for all $P \in \mathcal{P}_N$ and $\delta \leq AN^{-1}$. Moreover, the choice $B = 9A^2$ is appropriate.

Our next lemma can be proved by a routine zero counting argument. Let $\mathcal{T}_k$ be the set of all real trigonometric polynomials of degree at most k.

Lemma 3.3 *For $k \in \mathbb{N}$, $M > 0$, and $\alpha \in \mathbb{R}$, let $T \in \mathcal{T}_k$ be defined by*

$$T(t) = \frac{M}{2}(1 - \cos(k(t - t_0))) = M \sin^2\left(\frac{k(t - t_0)}{2}\right). \tag{3.1}$$

Let $a \in \mathbb{R}$ be fixed. Assume that $S \in \mathcal{T}_k$ satisfies $S(a) = T(a) > 0$ and $0 \leq S(t) \leq M$ holds for all $t \in \mathbb{R}$. Then:

(i) $S(t) > T(t)$ holds for all $t \in (y, a)$ if T is increasing on (y, a).

(ii) $S(t) > T(t)$ *holds for all* $t \in (a, y)$ *if* T *is decreasing on* (a, y).

Proof of Lemma 3.3 If the lemma were false, then $S - T \in \mathcal{T}_k$ would have at least $2k + 1$ zeros in a period, by counting multiplicities. □

A straightforward consequence of Lemma 3.3 is the following. For the sake of brevity, let

$$\gamma := \sin^2(\pi/8) = \frac{1}{2}(1 - \cos(\pi/4)).$$

Lemma 3.4 *Assume that* $S \in \mathcal{T}_k$ *satisfies* $0 \le S(t) \le M$ *for all* $t \in \mathbb{R}$. *Let* $a \in \mathbb{R}$, *and assume that* $S(a) \ge (1 - \gamma)M$. *Then*

$$S(t) \ge \gamma M, \quad t \in [a - \delta, a + \delta], \quad \delta := \frac{\pi}{2k}.$$

Proof of Lemma 3.4 Let $T \in \mathcal{T}_k$ be defined by (3.1). Pick a $t_0 \in \mathbb{R}$ such that $S(a) = T(a)$ and $T'(a) \ge 0$. Now observe that T is increasing on $[a - \delta, a]$ and $T(a - \delta) \ge \gamma M$, and Lemma 3.3 (i) gives the lower bound of the lemma for all $t \in [a - \delta, a]$. Now pick a $t_0 \in \mathbb{R}$ such that $S(a) = T(a)$ and $T'(a) \le 0$. Now observe that T is decreasing on $[a, a + \delta]$ and $T(a + \delta) \ge \gamma M$, and hence Lemma 3.3 (ii) gives the lower bound of the lemma for all $t \in [a, a + \delta]$. □

Combining Lemmas 3.1 and 3.4 we easily obtain the following.

Lemma 3.5 *Let* P_n *and* Q_n *be the* n*-th Rudin–Shapiro polynomials. Let* $N := 2^n$ *and* $\gamma := \sin^2(\pi/8)$. *Let*

$$z_j := e^{it_j}, \quad t_j := \frac{2\pi j}{N}, \quad j \in \mathbb{Z}.$$

We have

$$\max\left\{|P_n(z_j)|^2, |P_n(z_{j+r})|^2\right\} \ge \gamma 2^{n+1} = 2\gamma N, \quad r \in \{-1, 1\},$$

for every $j = 2u$, $u \in \mathbb{Z}$.

Lemma 3.5 tells us that the modulus of the Rudin–Shapiro polynomials P_n is certainly larger than $\sqrt{2\gamma N}$ at least at one of any two consecutive N-th roots of unity, where $N := 2^n$. This is a crucial observation of this paper, and despite its simplicity it does not seem to have been observed before in the literature or elsewhere. Moreover, note that while our Theorems 2.1 and 2.3 are proved with rather small multiplicative positive absolute constants, Lemma 3.5 is stated with a quite decent explicit constant.

Proof of Lemma 3.5 Let $k := 2^{n-2}$, $j = 2u$, $u \in \mathbb{Z}$. We introduce the trigonometric polynomial $S \in \mathcal{T}_k$ by

$$S(t) := |Q_{n-2}(e^{it})|^2, \quad t \in \mathbb{R}. \tag{3.2}$$

Assume that

$$|P_n(z_j)|^2 < \gamma 2^{n+1}.$$

Then (1.1) implies that

$$|Q_n(z_j)|^2 > (1-\gamma)2^{n+1}. \tag{3.3}$$

By Lemma 3.1, we have

$$|P_{n-2}(z_j)|^2 = \frac{1}{4}\,|P_n(z_j)|^2,$$

and hence (1.1) implies that

$$|Q_{n-2}(z_j)|^2 = \frac{1}{4}\,|Q_n(z_j)|^2.$$

Combining this with (3.3), we obtain

$$|Q_{n-2}(z_j)|^2 = \frac{1}{4}\,|Q_n(z_j)|^2 > (1-\gamma)2^{n-1}.$$

Hence, using Lemma 3.4 with $S \in \mathcal{T}_k$ and $M := 2^{n-1}$ [recall that (3.2) and (1.1) imply that $0 \le S(t) \le 2^{n-1}$ for all $t \in \mathbb{R}$], we can deduce that

$$|Q_{n-2}(z_{j+r})|^2 \ge \gamma 2^{n-1}, \quad r \in \{-1, 1\}.$$

Finally we use Lemma 3.1 again to conclude that

$$|P_n(z_{j+r})|^2 = 4|Q_{n-2}(z_{j+r})|^2 \ge \gamma 2^{n+1}, \quad r \in \{-1, 1\}.$$

□

To prove Theorem 2.3, we need Theorem 2.1 from [16]. We state it as our next lemma by using a slightly modified notation.

Lemma 3.6 *Let* $\omega_1 < \omega_2 \le \omega_1 + 2\pi$,

$$\begin{aligned}
&\omega_1 \ \le \ \theta_1 < \theta_2 < \cdots < \theta_\mu \le \omega_2, \\
&\theta_0 := \omega_1 - (\theta_1 - \omega_1), \quad \theta_{\mu+1} := \omega_2 + (\omega_2 - \theta_\mu), \\
&\delta := \max\left\{\theta_1 - \theta_0, \theta_2 - \theta_1, \ldots, \theta_{\mu+1} - \theta_\mu\right\} \le \frac{1}{2}\,\sin\frac{\omega_2 - \omega_1}{2}.
\end{aligned}$$

There is an absolute constant $c_3 > 0$ *such that*

$$\sum_{j=1}^{\mu} \frac{\theta_{j+1} - \theta_{j-1}}{2} \log|P(\mathrm{e}^{i\theta_j})| \le \int_{\omega_1}^{\omega_2} \log|P(\mathrm{e}^{i\theta})|\mathrm{d}\theta + c_3 E(N, \delta, \omega_1, \omega_2)$$

for every polynomial P of the form

$$P(z) = \sum_{j=0}^{N} b_j z^j, \quad b_j \in \mathbb{C},\ b_0 b_N \neq 0,$$

where

$$E(N, \delta, \omega_1, \omega_2) := (\omega_2 - \omega_1) N\delta + N\delta^2 \log(1/\delta) + \sqrt{N \log R}\left(\delta \log(1/\delta) + \frac{\delta^2}{\omega_2 - \omega_1}\right)$$

and $R := |b_0 b_N|^{-1/2} \|P\|_{\partial D}$.

Observe that R appearing in the above lemma can be easily estimated by

$$R \leq |b_0 b_N|^{-1/2}(|b_0| + |b_1| + \cdots + |b_N|).$$

4 Proof of Theorems 2.1, 2.2, and 2.3

Proof of Theorem 2.1 Let, as before, $\gamma = \sin^2(\pi/8)$, $N := 2^n$, and

$$z_j := e^{it_j}, \quad t_j := \frac{2\pi j}{N}, \quad j \in \mathbb{Z}.$$

By Lemma 3.5, we can choose

$$0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m \leq 2\pi, \quad \tau_0 := \tau_m - 2\pi, \quad \tau_{m+1} := \tau_1 + 2\pi,$$

so that

$$\{\tau_1, \tau_2, \ldots, \tau_m\} \subset \{t_1, t_2, \ldots, t_N\}, \tag{4.1}$$

$$\tau_{j+1} - \tau_j \leq \frac{4\pi}{N}, \quad j = 0, 1, \ldots, m-1, \tag{4.2}$$

and

$$|P_n(e^{i\tau_j})|^2 \geq \gamma 2^{n+1}, \quad j = 1, 2, \ldots, m. \tag{4.3}$$

Then the value

$$\delta := \max\{\tau_1 - \tau_0, \tau_2 - \tau_1, \ldots, \tau_m - \tau_{m-1}\}$$

appearing in Lemma 3.2 satisfies $\delta \leq AN^{-1}$ with $A = 4\pi$. Let $B > 0$ be chosen for $A := 4\pi$ according to Lemma 3.2. Combining $P_n \in \mathcal{P}_N$, (4.1), and Lemma 3.2, we conclude that

$$2\pi\left(\frac{1}{2}\log 2^{n+1}+\frac{1}{2}\log\gamma\right)\le\sum_{j=1}^{m}\frac{\tau_{j+1}-\tau_{j-1}}{2}\log|P_n(\mathrm{e}^{i\tau_j})|$$
$$\le\int_0^{2\pi}\log|P_n(\mathrm{e}^{i\tau})|\mathrm{d}\tau+B,$$

and hence

$$M_0(P_n,[0,2\pi])\ge\exp(-B/(2\pi))\sqrt{2\gamma}\,2^{n/2}=c\sqrt{N}$$

follows with the absolute constant $c:=\exp(-B/(2\pi))\sqrt{2\gamma}>0$. To complete the proof, observe that

$$M_0(P_n,[0,2\pi])=M_0(Q_n,[0,2\pi])$$

is an immediate consequence of (1.2). □

Proof of Theorem 2.2 Using (2.2), the power series expansion of the function $f(z):=\log(1-z)$ on $(-1,1)$, (1.2), and the monotone convergence theorem, we deduce that

$$\begin{aligned}\int_0^{2\pi}\log\left|\widetilde{P}_n(\mathrm{e}^{i\tau})\right|^2\mathrm{d}\tau&=\int_0^{2\pi}\log\left(1-\left|\widetilde{Q}_n(\mathrm{e}^{i\tau})\right|^2\right)\mathrm{d}\tau\\&=\int_0^{2\pi}-\sum_{k=1}^{\infty}\frac{\left|\widetilde{Q}_n(\mathrm{e}^{i\tau})\right|^{2k}}{k}\mathrm{d}\tau\\&=\int_0^{2\pi}-\sum_{k=1}^{\infty}\frac{\left|\widetilde{P}_n(\mathrm{e}^{i\tau})\right|^{2k}}{k}\mathrm{d}\tau=-\sum_{k=1}^{\infty}\int_0^{2\pi}\frac{\left|\widetilde{P}_n(\mathrm{e}^{i\tau})\right|^{2k}}{k}\mathrm{d}\tau\\&=-2\pi\sum_{k=1}^{\infty}\frac{I_{2k}\left(\widetilde{P}_n\right)}{k}.\end{aligned}$$

Combining this with Theorem 2.1 gives that there is an $L<\infty$ independent of n such that

$$\sum_{k=1}^{\infty}\frac{I_{2k}\left(\widetilde{P}_n\right)}{k}<L.$$

As $I_k\left(\widetilde{P}_n\right)$ is a decreasing function of $k\in\mathbb{N}$, the theorem follows. □

Proof of Theorem 2.3 The theorem follows from Lemmas 3.5 and 3.6 in a straightforward fashion. Note that

$$(M_0(f,[\alpha,\beta]))^{\beta-\alpha}=(M_0(f,[\alpha,\gamma]))^{\gamma-\alpha}(M_0(f,[\gamma,\beta]))^{\beta-\gamma}$$

for all $\alpha < \gamma < \beta \le \alpha + 2\pi$ and for all functions f continuous on $[\alpha, \beta]$. Hence, to prove the theorem, without loss of generality, we may assume that $\beta - \alpha \le \pi$. Let, as before, $\gamma = \sin^2(\pi/8)$, $N := 2^n$, and

$$z_j := \mathrm{e}^{it_j}, \quad t_j := \frac{2\pi j}{N}, \quad j \in \mathbb{Z}.$$

By Lemma 3.5, we can choose

$$0 < \tau_1 \le \tau_2 \le \cdots \le \tau_m \le 2\pi, \quad \tau_0 := \tau_m - 2\pi, \quad \tau_{m+1} := \tau_1 + 2\pi,$$

so that (4.1), (4.2), and (4.3) hold. Let

$$\{\theta_1 < \theta_2 < \cdots < \theta_\mu\} := \{\tau_j \in [\alpha, \beta] : j = 1, 2, \ldots, m\}. \tag{4.4}$$

The assumption on N guarantees that the value of δ defined in Lemma 3.6 is at most $4\pi/N$ and

$$\frac{4\pi}{N} \le \frac{\beta - \alpha}{8} \le \frac{1}{2} \sin \frac{\beta - \alpha}{2}.$$

Observe also that $P_n \in \mathcal{L}_{N-1}$, and hence when we apply Lemma 3.6 to P_n we have $R \le N$. By (4.3), we have

$$|P_n(\mathrm{e}^{i\theta_j})|^2 \ge \gamma 2^{n+1}, \quad j = 1, 2, \ldots, \mu.$$

Applying Lemma 3.6 with $P := P_n$, $N := 2^n$, and $\{\theta_1 < \theta_2 < \cdots < \theta_\mu\}$ defined by (4.4), we obtain

$$\begin{aligned}(\beta - \alpha)\left(\frac{1}{2}\log 2^{n+1} + \frac{1}{2}\log \gamma\right) &\le \sum_{j=1}^{\mu} \frac{\theta_{j+1} - \theta_{j-1}}{2} \log |P_n(\mathrm{e}^{i\theta_j})| \\ &\le \int_\alpha^\beta \log |P_n(\mathrm{e}^{i\theta})| \mathrm{d}\theta + c_3 E(N, 4\pi/N, \alpha, \beta),\end{aligned}$$

where the assumption

$$\frac{(\log N)^{3/2}}{N^{1/2}} \le \beta - \alpha \le 2\pi$$

implies that

$$\begin{aligned}E(N, 4\pi/N, \alpha, \beta) &\le c_4 \left(\frac{(\beta - \alpha)N}{N} + \frac{\log N}{N}\right. \\ &\quad \left. + \sqrt{N \log N}\left(\frac{\log N}{N} + \frac{1}{N^2(\beta - \alpha)}\right)\right) \\ &\le c_5(\beta - \alpha)\end{aligned}$$

with absolute constants $c_4 > 0$ and $c_5 > 0$. Hence

$$M_0(P_n, [\alpha, \beta]) \geq \exp(-c_3 c_5) \sqrt{2\gamma}\, 2^{n/2} = c\sqrt{N}$$

with the absolute constant $c := \exp(-c_3 c_5) \sqrt{2\gamma} > 0$. Combining this with (1.2), we obtain

$$M_0(Q_n, [\alpha, \beta]) \geq c\sqrt{N}$$

with the same absolute constant $c > 0$. □

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