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CONSTRUCTIVE APPROXIMATION

Classical-Quantum Correspondence and Functional Relations for Painlevé Equations

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Abstract In light of the quantum Painlevé–Calogero correspondence, we investigate the inverse problem. We imply that this type of the correspondence (classical-quantum correspondence) holds true, and we find out what kind of potentials arise from the compatibility conditions of the related linear problems. The latter conditions are written as functional equations for the potentials depending on a choice of a single function—the left-upper element of the Lax connection. The conditions of the correspondence impose restrictions on this function. In particular, it satisfies the heat equation. It is shown that all natural choices of this function (rational, hyperbolic, and elliptic) reproduce exactly the Painlevé list of equations. In this sense, the classical-quantum correspondence can be regarded as an alternative definition of the Painlevé equations.

Keywords Painlevé equations · Painlevé–Calogero correspondence · Functional relations

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1 Introduction

The Painlevé equations (PI–PVI) discovered by P. Painlevé, R. Fuchs and B. Gambier [13,15,43] have been studied extensively during the last century [7,20]. Their applications include self-similar reductions of nonlinear integrable partial differential equations [11], correlation functions of integrable models [3,23], quantum gravity and string theory [5], topological field theories [8], 2D polymers [59], random matrices [12,51] and stochastic growth processes [30], conformal field theories and KZ equations [14,24], the AGT conjecture [10,37] and spectral duality [38–40], to mention only a few applications and references.

As is known from classical works [13,16,44], the Painlevé equations describe the monodromy preserving deformations of a system of linear differential equations with rational coefficients. The monodromy approach was developed by H. Flaschka and A. Newell and by M. Jimbo, T. Miwa, K. Ueno [11,21,22,25], see also [19]. At present, different types of linear problems are known (scalar [13,16], 2×2-matrix [21] (see also [32,60]), 3×3-matrix [9,26], 8×8-matrix [41], and higher-dimensional Lax pairs [35]).

We deal with the linear problems depending on a spectral parameter [7–43]:

$$\begin{cases} \partial_x \boldsymbol{\Psi} = \mathbf{U}(x,t)\boldsymbol{\Psi}, \\ \partial_t \boldsymbol{\Psi} = \mathbf{V}(x,t)\boldsymbol{\Psi}, \end{cases} \qquad \boldsymbol{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{1.1}$$

where $\mathbf{U}, \mathbf{V} \in sl_2$ explicitly depend on the spectral parameter x and on the deformation parameter t (time variable) and contain an unknown function $u(t)$ to be constrained by the condition that the two equations have a family of common solutions.[1] In fact, the latter is equivalent to the compatibility of the linear problems expressed as the zero curvature equation (integrability condition):

$$\partial_x \mathbf{V} - \partial_t \mathbf{U} + [\mathbf{V}, \mathbf{U}] = 0. \tag{1.2}$$

Set

$$\mathbf{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The matrices $\mathbf{U}, \mathbf{V}$ are traceless, i.e., $a + d = 0$, $A + D = 0$. Then the zero curvature equation gives:

$$\begin{cases} a_t - A_x + bC - cB = 0, \\ b_t - B_x + 2aB - 2bA = 0, \\ c_t - C_x + 2cA - 2aC = 0. \end{cases} \tag{1.3}$$

In [57,58], by applying the diagonal gauge transformation $\Omega = \mathrm{diag}(\omega, \omega^{-1})$, we chose the matrices $\mathbf{U}, \mathbf{V}$ such that

$$b_x = 2B. \tag{1.4}$$

[1] This function is going to satisfy one of the six Painlevé equations (in the Calogero form).

Then the linear system (1.1) for the vector function $\mathbf{\Psi} = (\psi_1, \psi_2)^{\mathrm{t}}$ can be reduced to two scalar equations for $\psi := \psi_1$:

$$\begin{cases} \left(\frac{1}{2}\,\partial_x^2 - \frac{1}{2}\,(\partial_x \log b)\,\partial_x \; + W(x,t)\right)\psi = 0, \\ \partial_t \psi = \left(\frac{1}{2}\,\partial_x^2 + U(x,t)\right)\psi, \end{cases} \tag{1.5}$$

where

$$W = U(x,t) - \frac{1}{2}\,\partial_t \log b + \frac{1}{4}\,\partial_x^2 \log b + \frac{1}{4}(\partial_x \log b)^2$$

and

$$U(x,t) = \frac{1}{2}\,(ad - bc - a_x) + A \; = \; \frac{1}{2}\,\det \mathbf{U} - \frac{a_x}{2} + A. \tag{1.6}$$

The second equation in (1.5) has the form of a nonstationary Schrödinger equation in imaginary time with the potential $U(x,t)$. It describes the isomonodromic deformations of the first one, and their compatibility implies the Painlevé equation (in the Calogero form) for the function $u = u(t)$:

$$\ddot{u} = -\partial_u \tilde{V}(u,t), \tag{1.7}$$

generated by the Hamiltonian $H(\dot{u}, u, t) = \frac{1}{2}\dot{u}^2 + \tilde{V}(u,t)$. The function $u = u(t)$ is defined as a (simple) zero of the function $b(x)$:

$$b(u) = 0.$$

The second important condition we are going to use together with (1.4) is

$$U(x,t) = U(x, \dot{u}(t), u(t), t) = V(x,t) - H(\dot{u}, u, t), \tag{1.8}$$

where $H(\dot{u}, u, t)$ is the classical Hamiltonian. The x-dependent part of the potential $V(x,t)$ does not contain the dependent variable u. Therefore, the second equation in (1.5) acquires the form

$$\partial_t \Psi(x,t) = \left(\frac{1}{2}\,\partial_x^2 + V(x,t)\right)\Psi(x,t), \tag{1.9}$$

with

$$\Psi(x,t) = e^{\int^t H(\dot{u},u,t')dt'}\,\psi(x,t). \tag{1.10}$$

Notice that condition (1.4) can be easily satisfied by choosing a suitable gauge. However, together with (1.8), it becomes a nontrivial condition and leads to the *quantum Painlevé–Calogero correspondence* (see below), which relates the potentials of the classical problem $\tilde{V}$ with V in the quantum one. It appears that the potentials differ only by "quantum corrections" of the coupling constants. Therefore, (1.9) is the quantization of (1.7) with the unit Planck constant.

In [57,58], it was shown that there exists a choice of gauge and variables (x, t) such that the nonstationary Schrödinger equation becomes a quantized Painlevé equation. Thus, the linear problem (1.1) leads to both classical and quantum Painlevé equations. The classical one is written in the variable $u(t)$ and follows from the zero-curvature equation (1.1) valid for all x. The quantum one is written in terms of the spectral parameter x for a component of the common solution ψ_1 of the linear problems. We have called this construction the *quantum Painlevé–Calogero correspondence*. It is a quantum version of the classical correspondence introduced by A. Levin and M. Olshanetsky [31] and developed by K. Takasaki [49]. It should be mentioned that a phenomenon similar to the quantum Painlevé–Calogero correspondence was first observed by B. Suleimanov [46,47] in terms of rational linear problems. See also S. Slavyanov's papers [45].

Let us note that the phenomenon of the classical-quantum correspondence is also known in the theory of integrable systems in some other contexts. There are interrelations between classical and quantum problems of a simingly different type [1,54,55], where Bethe vectors of integrable quantum spin chains are related to some data of classical integrable many-body systems. A similarity between quantum transfer matrices and classical τ-functions was pointed out in [27,29,56].

The aim of this paper is to address the inverse problem. We start with the system of scalar equations (1.5) and assume that the quantum Painlevé–Calogero correspondence takes place, i.e., equations (1.7)–(1.8) hold true. (In this paper, we refer to it as *classical-quantum correspondence,* since it is not clear initially which equations satisfy the conditions). Then we derive and solve functional equations[2] for the potential V searching through possible choices of the function b. In other words, we assume that the classical-quantum correspondence holds true and find out what kind of potentials arise from the compatibility conditions.

We prove the following:

Theorem 1.1 *Let the compatibility condition for the system* (1.5) *with*

$$U(x, \dot{u}(t), u(t), t) = V(x, t) - H(\dot{u}, u, t)$$

and

$$H(\dot{u}, u, t) = \frac{1}{2}\dot{u}^2 + \tilde{V}(u, t)$$

be equivalent to

$$\ddot{u} = -\partial_u \tilde{V}(u, t),$$

where u is defined as a simple zero of the function $b(x, t)$: $b(x, t) \mid_{x=u} = 0$. Then there are two possibilities:

[2] It should be mentioned that functional equations play a very important role in the theory of integrable systems; they underlie the Lax equations, the r-matrix, and other structures [6,28].

1.

$$b(x,u,t)=b(x-u,t). \tag{1.11}$$

The function $b(z,t)$ satisfies the heat equation

$$2\partial_t b(z,t)=\partial_z^2 b(z,t);$$

the quantum potential coincides with the classical one,

$$\tilde{V}(u,t)=V(u,t),$$

and satisfies the functional equation

$$\boxed{V_t(x)-V_t(u)-\tfrac{1}{2}f(x-u,t)\big(V'(x)+V'(u)\big)-f_x(x-u,t)\,(V(x)-V(u))=0,} \tag{1.12}$$

where $f(x,t)=b_x(x,t)/b(x,t)$.

2.

$$b(x,u,t)=b(x-u,t)b(x+u,t). \tag{1.13}$$

The function $b(z,t)$ satisfies the heat equation

$$2\partial_t b(z,t)=\partial_z^2 b(z,t);$$

the classical and quantum potentials are related by

$$\tilde{V}(u,t)=V(u,t)+\frac{1}{2}\,\partial_x^2\log b(x,t)\Big|_{x=2u},$$

and $V(x,t)$ satisfies the functional equation

$$\boxed{\begin{aligned}V_t(x)-V_t(u)-\tfrac{1}{2}f(x-u,t)\,\big(V'(x)+V'(u)\big)-\tfrac{1}{2}f(x+u,t)\,\big(V'(x)-V'(u)\big)\\ +\left(f_x(x-u,t)+f_x(x+u,t)\right)(V(u)-V(x))=0,\end{aligned}} \tag{1.14}$$

where $f(x,t)=b_x(x,t)/b(x,t)$.

The proof of the theorem is based on Propositions 3.1, 3.2, and 3.3. For possibilities different from (1.11) and (1.13), see the remark after (3.33) and Appendix A.

Solving equations (1.12) and (1.14), we get the following results: for the rational (in x) function b, we obtain PI and PII from (1.12) and PIV from (1.14); for the hyperbolic, we obtain PIII from (1.12) and PV from (1.14). The most general equation PVI arises for the θ-functional ansatz for b from (1.14), while the equation from (1.12) is shown to have only trivial solutions in this case.

Finally, it is shown that all natural choices of the function b (rational, hyperbolic, and elliptic) reproduce exactly the Painlevé list of equations. In this sense, the classical-quantum correspondence can be viewed as an alternative definition for the Painlevé equations.

The paper is organized as follows. In the next section, we recall the quantum Painlevé–Calogero correspondence. In Section 3, we derive the functional equations from (1.12) and (1.14) and then solve these equations in Sections 4-6. In the appendices, we give the definitions and identities for necessary elliptic functions, discuss some special cases of the b-function, and list the $\mathbf{U}-\mathbf{V}$ pairs for PI–PV which are acceptable for the quantum Painlevé–Calogero correspondence.

2 Quantum Painlevé–Calogero Correspondence

The quantum Painlevé–Calogero correspondence states that for Painlevé equations, the nonstationary Baxter equation at $\hbar = 1$ represents a classical linear problem. Let us start from example.

2.1 Example of Painlevé V

The PV equation is conventionally written as:

$$\partial_T^2 y = \left(\frac{1}{2y}+\frac{1}{y-1}\right)(\partial_T y)^2 - \frac{\partial_T y}{T} + \frac{y(y-1)^2}{T^2}\left(\alpha+\frac{\beta}{y^2}+\frac{\gamma T}{(y-1)^2}+\frac{\delta T^2(y+1)}{(y-1)^3}\right),$$

where α, β, γ, δ are parameters.[3] Making change of variable

$$y = \coth^2 u \tag{2.1}$$

together with

$$T = e^{2t}, \tag{2.2}$$

PV acquires the form

$$\ddot{u} = -\frac{2\alpha\cosh u}{\sinh^3 u} - \frac{2\beta\sinh u}{\cosh^3 u} - \gamma e^{2t}\sinh(2u) - \frac{1}{2}\delta e^{4t}\sinh(4u). \tag{2.3}$$

The later equation is Hamiltonian with $H_{\mathrm{V}}(p,x) = \frac{p^2}{2} + V_{\mathrm{V}}(u,t)$, where

$$V_{\mathrm{V}}(u,t) = -\frac{\alpha}{\sinh^2 u} - \frac{\beta}{\cosh^2 u} + \frac{\gamma e^{2t}}{2}\cosh(2u) + \frac{\delta e^{4t}}{8}\cosh(4u). \tag{2.4}$$

[3] There are in fact three essentially independent parameters.

The zero curvature representation is known from [21]. It is rational in spectral parameter X. As was shown in [57], the change

$$X = \cosh^2 x$$

with (2.1) and (2.2) and some special gauge transformation brings the Jimbo–Miwa $\mathbf{U}-\mathbf{V}$ pair to the one given in (C.1)–(C.4). Then the first component of the linear problem (1.1) ψ satisfies the nonstationary Schrödinger equation

$$\partial_t \psi = \left(H_{\mathrm{V}}^{(\alpha-\frac{1}{8},\, \beta+\frac{1}{8},\, \gamma,\, \frac{1}{2})}(\partial_x, x) - H_{\mathrm{V}}^{(\alpha,\beta,\gamma,\frac{1}{2})}(\dot{u}, u) \right) \psi,$$

and, therefore,

$$\partial_t \Psi = H_{\mathrm{V}}^{(\alpha-\frac{1}{8},\, \beta+\frac{1}{8},\, \gamma,\, \frac{1}{2})}(\partial_x, x)\Psi = \left(\frac{1}{2}\, \partial_x^2 + V_{\mathrm{V}}^{(\alpha-\frac{1}{8},\, \beta+\frac{1}{8},\, \gamma,\, \frac{1}{2})}(x, t) \right) \Psi$$

for $\Psi(x,t) = e^{\int^t H_{\mathrm{V}}^{(\alpha,\, \beta,\, \gamma,\, \frac{1}{2})}(\dot{u},u)dt'} \psi(x,t)$; i.e., the linear problem admits the form of the quantized equation (in spectral parameter). Notice that the parameters α, β are shifted by $\pm\frac{1}{8}$ in the quantum Hamiltonian.

2.2 Summary

The following theorem summarizes the results of [57,58] for all Painlevé equations, see also the table of changes of variables below.

Theorem [57,58] *For any of the six equations from the Painlevé list written in the Calogero form as classical nonautonomous Hamiltonian systems with time-dependent Hamiltonians $H(p, u, t)$, there exists a pair of compatible linear problems*

$$\begin{cases} \partial_x \mathbf{\Psi} = \mathbf{U}(x, t, u, \dot{u}, \{c_k\})\mathbf{\Psi}, \\ \partial_t \mathbf{\Psi} = \mathbf{V}(x, t, u, \dot{u}, \{c_k\})\mathbf{\Psi}, \end{cases} \qquad \mathbf{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

where $\mathbf{U}$ *and* $\mathbf{V}$ *are sl_2-valued functions, x is a spectral parameter, t is the time variable and $\{c_k\} = \{\alpha, \beta, \gamma, \delta\}$ is the set of parameters involved in the Painlevé equation, such that:*

1) The zero curvature condition

$$\partial_t \mathbf{U} - \partial_x \mathbf{V} + [\mathbf{U}, \mathbf{V}] = 0$$

is equivalent to the Painlevé equation for the variable u defined as any (simple) zero of the right-upper element of the matrix $\mathbf{U}(x,t)$ *in the spectral parameter:* $\mathbf{U}_{12}(u, t) = 0$*;*

2) *The function* $\Psi = e^{\int^t H(\dot{u},u,t')dt'}\psi_1$, *where* ψ_1 *is the first component of* $\boldsymbol{\Psi}$, *satisfies the nonstationary Schrödinger equation in imaginary time*

$$\partial_t \Psi = \left(\frac{1}{2}\,\partial_x^2 + \tilde{V}(x,t)\right)\Psi$$

with the potential

$$\tilde{V}(x,t) = V(x,t,\{\tilde{c}_k\}),$$
$$V(x,t,\{\tilde{c}_k\}) - \left(\frac{1}{2}\dot{u}^2 + V(u,t,\{c_k\})\right) = \frac{1}{2}\Big[\det(\mathbf{U}) - \partial_x \mathbf{U}_{11} + 2\mathbf{V}_{11}\Big],$$

which coincides with the classical potential $V(u,t) = V(u,t,\{c_k\})$ *up to possible shifts of the parameters* $\{c_k\}$*:*

$$(\tilde{\alpha},\tilde{\beta}) = (\alpha, \beta + \tfrac{1}{2}) \quad \textit{for PIV},$$
$$(\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta}) = \left(\alpha - \tfrac{1}{8}, \beta + \tfrac{1}{8}, \gamma, \delta\right) \quad \textit{for PV},$$
$$(\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta}) = \left(\alpha - \tfrac{1}{8}, \beta + \tfrac{1}{8}, \gamma - \tfrac{1}{8}, \delta + \tfrac{1}{8}\right) \quad \textit{for PVI}.$$

The list of changes of variables is summarized in the following table:

Equation	$y(u,t)$	$T(t)$	$X(x,t)$	$\mathbf{U}_{12}(x,t)$
PI	u	t	x	$x-u$
PII	u	t	x	$x-u$
PIV	u^2	t	x^2	x^2-u^2
PIII	e^{2u}	e^t	e^{2x}	$2e^{t/2}\sinh(x-u)$
PV	$\coth^2 u$	e^{2t}	$\cosh^2 x$	$2e^t\sinh(x-u)\sinh(x+u)$
PVI	$\frac{\wp(u)-\wp(\omega_1)}{\wp(\omega_2)-\wp(\omega_1)}$	$\frac{\wp(\omega_3)-\wp(\omega_1)}{\wp(\omega_2)-\wp(\omega_1)}$	$\frac{\wp(x)-\wp(\omega_1)}{\wp(\omega_2)-\wp(\omega_1)}$	$\vartheta_1(x-u)\vartheta_1(x+u)h(u,t)$

Function $h(u,t)$ for the PVI case can be found in [58]. Notice that the changes of variables given above can be derived in a general form from (1.4) and the requirement that the potential (1.6) could be presented as a sum of two parts depending on x,t and u,t separately. This calculation was made in [58] for the most general Painlevé VI equation. The appropriate $\mathbf{U}-\mathbf{V}$ pairs for PI–PV are given in Appendix C.

3 The Scalar Linear Problems and Functional Equations

It was shown in [57,58] that each of the six equations from the Painlevé list, hereinafter referred to as PI–PVI, written in the so-called Calogero form, can be obtained as integrability conditions for two Schrödinger-like equations

$$\begin{cases} \left(\frac{1}{2}\,\partial_x^2 - \frac{b_x}{2b}\,\partial_x\, + W(x,t)\right)\Psi = 0, \\ \partial_t \Psi = \left(\frac{1}{2}\,\partial_x^2 + V(x,t)\right)\Psi, \end{cases} \tag{3.1}$$

stationary and nonstationary. The time-dependent potentials W and V are related by

$$W(x,t) = U - \frac{2\dot{b} - b_{xx}}{4b} = V(x,t) - H - \frac{2\dot{b} - b_{xx}}{4b}, \tag{3.2}$$

where H does not depend on x and b is some function of the spectral parameter x and time t to be chosen in such a way that the two linear problems are compatible for some $V(x,t)$. Suppose it has a (simple) zero at the point $x = u = u(t)$: $b(u,t) = 0$, and let $V(x,t)$ be a function that depends on x, t in an explicit way only (i.e., $V(x,t)$ does not contain u). Also let H be a function of u and $\dot{u}$.

Remark Note that function b may depend on t in two ways — explicit and implicit. The latter means the time dependence through the unknown functions of t (dependent variables). Writing $\partial_t b$, we mean the derivative with respect to the explicit dependence only. For example, $\partial_t(z-u) = 0$. The lower index t means the same ($\partial_t b(z,u(t),t) = b_t$), while the dot is the full time derivative: $\dot{b}(z,u(t),t) = \dot{u}\partial_u b + b_t$. The same notation is used for other functions depending on t and $u(t)$ apart from Ψ in the linear problem (where the partial derivative symbols ∂_x, ∂_t are traditionally used, but, in fact, the operator ∂_t acts as the full time derivative).

Combining equations (3.1), one can write another pair of linear problems whose compatibility implies the Painlevé equations:

$$\begin{cases} \left(\frac{1}{2}\,\partial_x^2 - \frac{b_x}{2b}\,\partial_x\, + W(x,t)\right)\Psi = 0, \\ \partial_t \Psi = \left(\frac{b_x}{2b}\,\partial_x + \frac{2\dot{b} - b_{xx}}{4b} + H\right)\Psi, \end{cases} \tag{3.3}$$

(The first equation is the same, while the other one is a first-order equation.) Passing to the function $\tilde{\Psi} = \Psi/\sqrt{b}$, we can write these linear problems in the Fuchs–Garnier form:

$$\begin{cases} \left(\frac{1}{2}\,\partial_x^2 + S(x,t)\right)\tilde{\Psi} = 0, \\ \partial_t \tilde{\Psi} = \left(\frac{1}{2} f \partial_x - \frac{1}{4}\, f_x\right)\tilde{\Psi}, \end{cases} \qquad f = \partial_x \log b, \quad f_x \equiv \partial_x f, \tag{3.4}$$

where we have introduced the function $S = S(x,t)$ by the formula

$$S = U - \frac{\dot{b}}{2b} + \frac{b_{xx}}{2b} - \frac{3}{8}\left(\frac{b_x}{b}\right)^2 = V - H - \frac{\dot{b}}{2b} + \frac{b_{xx}}{2b} - \frac{3}{8}\left(\frac{b_x}{b}\right)^2. \tag{3.5}$$

Their integrability is equivalent to the condition

$$\Big[\frac{1}{2}\,\partial_x^2+S,\ \partial_t-\frac{1}{2}f\,\partial_x+\frac{1}{4}\,f_x\Big]\tilde{\Psi}=0,$$

which implies

$$\dot{S}=Sf_x+\frac{1}{2}fS_x+\frac{1}{8}\,f_{xxx} \tag{3.6}$$

or

$$\boxed{\begin{gathered} bb_xU_x+2bb_{xx}U-2b_x^2U-2\dot{U}b^2 \\ -\tfrac{1}{2}b_xb_{xxx}+b_x\dot{b}_x-\dot{b}^2-b\dot{b}_{xx}+b\ddot{b}+\tfrac{1}{4}bb_{xxxx}+\tfrac{1}{4}b_{xx}^2=0. \end{gathered}} \tag{3.7}$$

This equation is our main interest in this paper. In the next sections, we determine the potential V making one or another ansatz for b.

Notice that the equation (3.7) can be obtained from the compatibility of initial matrix linear problem (1.3) with U defined by (1.6). One can express all elements of $\mathbf{U}$ and $\mathbf{V}$ in terms of three functions $a=\mathbf{U}_{11}$, $b=\mathbf{U}_{12}$, and U:

$$\mathbf{U}:\quad \begin{gathered} \mathbf{U}_{11}=a,\quad \mathbf{U}_{12}=b, \\ \mathbf{U}_{21}=-\tfrac{1}{2b^2}\left(2a^2b+2a_xb-2ab_x+4Ub-2\dot{b}+b_{xx}\right), \end{gathered}$$

$$\mathbf{V}:\quad \begin{gathered} \mathbf{V}_{11}=\tfrac{1}{4b}\left(2ab_x+2\dot{b}-b_{xx}\right),\quad \mathbf{V}_{12}=\tfrac{1}{2}b_x, \\ \mathbf{V}_{21}=-\tfrac{1}{4b^2}\left(4\dot{a}b-2\dot{b}_x+b_{xxx}-2ab_{xx}+2a^2b_x+4b_xU\right). \end{gathered}$$

The function a cancels out from compatibility condition (1.3).

Recall that the dynamical variable u is defined as a zero of the function $b(x,t)=b(x,u(t),t)$: $b(u,u,t)=0$. Suppose b is an analytical function near $x=u$; then in the vicinity of $x=u$,

$$b=b_1(u,t)(x-u)+b_2(u,t)(x-u)^2+b_3(u,t)(x-u)^3+\cdots. \tag{3.8}$$

Consider equation (3.7) at $x=u$:

$$\left(-2b_x^2U-\frac{1}{2}b_xb_{xxx}+b_x\dot{b}_x-\dot{b}^2+\frac{1}{4}b_{xx}^2\right)|_{x=u}=0, \tag{3.9}$$

where we used that $U=V(x,t)-H(\dot{u},u,t)$, and, therefore, it is a regular function at $x=u$ and $[bU]\,(x=u)=0$. From the expansion (3.8), we get

$$b_x\,|_{x=u}=b_1,\quad b_{xxx}\,|_{x=u}=6b_3,\quad \dot{b}\,|_{x=u}=-\dot{u}b_1,\quad \dot{b}_x\,|_{x=u}=\dot{u}(b_1'-2b_2)+\partial_tb_1.$$

Plugging this into (3.9), we obtain:

$$U\,|_{x=u} = -\frac{1}{2}v^2 + \frac{1}{2b_1^2}\left[\left(b_2 - \frac{1}{2}b_1'\right)^2 - 3b_1b_3 + b_1\partial_t b_1 + \frac{1}{4}b_2^2\right],$$

where

$$v = \dot{u} + \frac{b_2}{b_1} - \frac{b_1'}{2b_1}. \tag{3.10}$$

The latter expression is the "momentum." Notice that this local evaluation at $x = u$ fixes the dependence $H(\dot{u})$, since $V(x, t)$ is independent of $\dot{u}$. We consider some nontrivial cases ($v \neq \dot{u}$) in Appendix A.

Let us find out what kind of restriction on the behavior of $b = b(x, u(t), t)$ arises from the classical-quantum correspondence. First, recall that the quantum Hamiltonian which we use in (1.5), (3.1) has the form $\hat{H} = \frac{1}{2}\partial_x^2 + V(x, t)$. Therefore, the classical one is $H(p_x, x, t) = \frac{1}{2}p_x^2 + V(x, t)$. The classical-quantum correspondence implies that the classical equations for $u(t)$ arising from the compatibility condition (1.2) (or (3.6) or (3.7)) are generated by $H(\dot{u}, u, t)$, which differs from $H(p_x, x, t)$ by only possible "quantum corrections" of the potential. Thus, the classical Hamiltonian should have the "Calogero form," i.e., $H(\dot{u}, u, t) = \frac{1}{2}\dot{u}^2 + \tilde{V}(u, t)$. At the moment, we do not assume any relations between $V(x, t)$ and $\tilde{V}(u, t)$. However, the Calogero form of the Hamiltonian provides some special properties of $b(x, u(t), t)$.

Proposition 3.1 *Let the compatibility condition* (3.7) *describe nonautonomous dynamics*

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= \ddot{u} = -\partial_u \tilde{V}(u, t) \end{aligned} \tag{3.11}$$

generated by the Hamiltonian

$$H(\dot{u}, u, t) = \frac{1}{2}v^2 + \tilde{V}(u, t). \tag{3.12}$$

Then $b(x, u(t), t)$ factorizes into the product

$$b(x, u(t), t) = b_1(x - u(t), t)\, b_2(x + u(t), t), \tag{3.13}$$

and each of the factors satisfies the heat equation:

$$2\partial_t b_{1,2}(z, t) = \partial_z^2 b_{1,2}(z, t).$$

Proof Substituting (3.11) and (3.12) into (3.7), we get an equation where the left-hand side. is quadratic in $v = \dot{u}$. Since v is an independent variable, all the coefficients in front of v^k ($k = 2, 1, 0$) vanish. The coefficient in front of v^2 gives (it comes from terms $2bb_{xx}U$, $-2b_x^2U$, $-\dot{b}^2$ and $b\ddot{b}$ in (3.7))

$$b_x^2 - b_u^2 + bb_{uu} - bb_{xx} = 0$$

or

$$\left(\partial_x^2 - \partial_u^2\right) \log b = (\partial_x - \partial_u)\,(\partial_x + \partial_u) \log b = 0,$$

which is equivalent to (3.13). The coefficient in front of v gives

$$b_x b_{xu} - bb_{xxu} + 2bb_{tu} - 2b_u b_t = 0$$

or

$$\left(\frac{b_{xu}}{b}\right)_x = 2\left(\frac{b_t}{b}\right)_u. \tag{3.14}$$

Plugging (3.13) into (3.14), we obtain:

$$2\left(\frac{b_{2t}}{b_2}\right)' - 2\left(\frac{b_{1t}}{b_1}\right)' = \left(\frac{b_2''}{b_2}\right)' - \left(\frac{b_1''}{b_1}\right)'.$$

The variables $x - u$ and $x + u$ are independent. Therefore,

$$2\left(\frac{b_{kt}}{b_k}\right)' = \left(\frac{b_k''}{b_k}\right)', \quad k = 1, 2.$$

Then

$$2b_{kt} = b_k'' + c(t)b_k, \quad k = 1, 2,$$

where $c(t)$ is the integration constant. The term with $c(t)$ can be removed by the substitution $b \to be^{\int_t c(t)}$. □

The coefficient in front of v^0 gives rise to equations for $V(x,t)$ and $\tilde{V}(u(t),t)$. We study these equations in the next sections.

3.1 One Simple Zero

Let us first consider the case when b has only a simple zero at $u(t)$. The reason for this behavior of $b(z,t)$ is partly explained in Section 6.2.

Proposition 3.2 *Let $b(z,t)$ satisfy the heat equation*

$$2\partial_t b(z,t) = \partial_z^2 b(z,t), \tag{3.15}$$

and let u be a simple zero of the function b: $b(x-u,t)\,|_{x=u} = 0$. *Then integrability condition* (3.7) *implies that*

$$H = \frac{1}{2}\dot{u}^2 + V(u), \tag{3.16}$$

$$\ddot{u} = -V'(u), \tag{3.17}$$

and

$$\boxed{V_t(x) - V_t(u) - \tfrac{1}{2}f(x-u)\left(V'(x) + V'(u)\right) - f_x(x-u)\left(V(x) - V(u)\right) = 0,} \tag{3.18}$$

where $f(x) = f(x,t) = b_x(x,t)/b(x,t)$ *(for brevity we do not indicate the* t*-dependence of* f *explicitly). In particular, if* $f(x) = \frac{1}{x} + c_1 x + c_3 x^3 + \cdots$, *then*

$$V_t' = \frac{1}{12}V''' + 2c_1 V', \tag{3.19}$$

$$\frac{1}{120}V^{(5)} = \frac{1}{2}c_1 V''' + 24 c_3 V', \tag{3.20}$$

where $V_t(u) = \partial_t V(u,t)$.

Proof Direct substitution of $b = b(x-u(t),t)$ into (3.6) together with (3.15) yields

$$V_t(x) - \dot{H} - \frac{1}{2}f\left(V'(x) - \ddot{u}\right) - f_x\left(V(x) + \frac{1}{2}\dot{u}^2 - H\right) = 0.$$

Locally, $f = \frac{b_x}{b} \sim \frac{1}{z-u}$. Therefore, the cancellation of the second-order pole leads to (3.16). At this stage, we have

$$V_t(x) - V_t(u) - \dot{u}(\ddot{u} + V'(u)) - \frac{1}{2}f\left(V'(x) - \ddot{u}\right) - f_x\left(V(x) - V(u)\right) = 0.$$

From the last two terms, it is easy to see that the cancellation of the first-order term gives (3.17). Substituting (3.17) into the above equation, we get (3.18). The differential equations (3.19), (3.20) follow from the local expansion of (3.18) near $x = u$. To be exact, (3.20) follows from (3.19) and $V_t''' = \frac{3}{40}V^{(5)} + \frac{5}{2}c_1 V''' + 24 c_3 V'$. □

In this proof, only the heat equation was used. In what follows, we need some more properties that follow from the heat equation.

Lemma 3.1 *Let b satisfy the heat equation* (3.15) *and* $f = \partial_x \log b$. *Then*

$$\partial_t f = \frac{1}{2}\partial_x\left(f^2 + f_x\right) = f_x f + \frac{1}{2}f_{xx}, \tag{3.21}$$

$$\partial_t f_x = f_{xx} f + f^2 + \frac{1}{2} f_{xxx}. \tag{3.22}$$

Suppose also that b is an odd function of x and has a simple zero at $x = 0$. Then

$$\frac{1}{2} f_x(x-w) f_x(x+w) = (f_x(x-w) + f_x(x+w)) f_{xx}(2w) \\ + (f(x+w) - f(x-w)) f_{xx}(2w) - \partial_t f_x(2w). \tag{3.23}$$

Proof The proof of (3.21) and (3.22) is direct. Identity (3.23) is proved via consideration of the local expansion and comparing of the poles taking into account (3.22). □

3.2 Two Simple Zeros

Suppose b has two simple poles. Let us derive an analogue of (3.16)–(3.18) for this case.

Proposition 3.3 *Let $b = b_1(z,t) b_2(z,t)$, and let each factor satisfy the heat equation*

$$2\partial_t b_{1,2}(z,t) = \partial_z^2 b_{1,2}(z,t). \tag{3.24}$$

Suppose that $b_{1,2}$ has a simple zero $u_{1,2}$: $b_{1,2}(x - u_{1,2}, t) \mid_{x=u_{1,2}} = 0$. Then equation (3.6) has the following solution:

$$u_1 = -u_2, \quad V(u) = V(-u), \quad b_1 = b(x - u(t), t), \quad b_2 = b(x + u(t), t), \tag{3.25}$$

$$H = \frac{1}{2}\dot{u}^2 + V(u) + \frac{1}{2} f_x(2u), \tag{3.26}$$

$$\ddot{u} = -V'(u) - f_{xx}(2u), \tag{3.27}$$

where $b(x,t)$ is an odd function of x, $f = \partial_x \log b$, and the potential satisfies

$$\boxed{\begin{gathered} V_t(x) - V_t(u) - \tfrac{1}{2} f(x-u)\left(V'(x) + V'(u)\right) - \tfrac{1}{2} f(x+u)\left(V'(x) - V'(u)\right) \\ + (f_x(x-u) + f_x(x+u))(V(u) - V(x)) = 0. \end{gathered}} \tag{3.28}$$

In particular, if $f = \frac{1}{x} + c_1 x + c_3 x^3 + \cdots$, then

$$V_t' = \frac{1}{12} V''' + \frac{1}{2} f(2x) V'' + (2c_1 + f_x(2x)) V', \tag{3.29}$$

$$V_t''' = \frac{3}{40} V^{(5)} + \frac{1}{2} f(2x) V^{(4)} + \frac{5}{2} (c_1 + f_x(2x)) V'''$$
$$+ \frac{9}{2} f_{xx}(2x) V'' + (24c_3 + 3 f_{xxx}(2x)) V'. \tag{3.30}$$

Proof The direct substitution leads to

$$\begin{aligned} &V_t(x) - \dot{H} - \frac{1}{2} \frac{b_{1,x}}{b_1} \left(V'(x) - \ddot{u}_1\right) - \frac{1}{2} \frac{b_{2,x}}{b_2} \left(V'(x) - \ddot{u}_2\right) \\ &- \left(\frac{b_{1,x}}{b_1}\right)_x \left(V(x) + \frac{1}{2}\dot{u}_1^2 - H + \frac{1}{2}(\dot{u}_1 + \dot{u}_2) \frac{b_{2,x}}{b_2}\right) \\ &- \left(\frac{b_{2,x}}{b_2}\right)_x \left(V(x) + \frac{1}{2}\dot{u}_2^2 - H + \frac{1}{2}(\dot{u}_1 + \dot{u}_2) \frac{b_{1,x}}{b_1}\right) \\ &- \frac{1}{2} \left(\frac{b_{1,x}}{b_1}\right)_x \left(\frac{b_{2,x}}{b_2}\right)_x = 0. \end{aligned} \tag{3.31}$$

From cancellation of the second-order poles, we get

$$\begin{aligned} H &= \frac{1}{2}\dot{u}_1^2 + V(u_1) + \frac{1}{2}(\dot{u}_1 + \dot{u}_2) \frac{b_{2,x}}{b_2}(u_1) + \frac{1}{2} \left(\frac{b_{2,x}}{b_2}\right)_x (u_1), \\ H &= \frac{1}{2}\dot{u}_2^2 + V(u_2) + \frac{1}{2}(\dot{u}_1 + \dot{u}_2) \frac{b_{1,x}}{b_1}(u_2) + \frac{1}{2} \left(\frac{b_{1,x}}{b_1}\right)_x (u_2). \end{aligned} \tag{3.32}$$

Comparing these two expressions, one can see that (3.25) and (3.26) indeed satisfy (3.6). Then vanishing of the first-order poles at $\pm u$ gives (3.27). Substituting (3.27) into (3.31), we get

$$\begin{aligned} &V_t(x) - \partial_t \left(V(z) + \frac{1}{2} f(2z)\right)\Big|_{z=u(t)} \\ &- \frac{1}{2} \frac{b_{1,x}}{b_1} \left(V'(x) + V'(u) + f_{xx}(2u)\right) - \frac{1}{2} \frac{b_{2,x}}{b_2} \left(V'(x) - V'(u) - f_{xx}(2u)\right) \\ &+ \left(\left(\frac{b_{1,x}}{b_1}\right)_x + \left(\frac{b_{2,x}}{b_2}\right)_x\right) \left(V(u) - V(x) + \frac{1}{2} f_{xx}(2u)\right) \\ &- \frac{1}{2} \left(\frac{b_{1,x}}{b_1}\right)_x \left(\frac{b_{2,x}}{b_2}\right)_x = 0. \end{aligned} \tag{3.33}$$

All terms that do not contain V cancel because of (3.23), and we get (3.28). Differential equations (3.29), (3.30) follow from the local expansion of (3.28) near $x = u$. □

Remark To investigate the case more general than (3.25), one should solve the equation emerging from equality of right-hand sides of (3.32) (see Appendix A).

Notice also that the right-hand side of (3.29) and (3.30) are full derivatives:

$$V_t' = \partial_x \left(\frac{1}{12} V'' + 2c_1 V + \frac{1}{2} f(2x) V'\right),$$

$$V_t''' = \partial_x \left(\frac{3}{40} V^{(4)} + \frac{5}{2} c_1 V'' + 24 c_3 V + \frac{3}{2} f_{xx}(2x) V' + \frac{3}{2} f_x(2x) V'' + \frac{1}{2} f(2x) V''' \right).$$

In particular, this leads to the following equation:

$$V^{(4)} - 60 c_1 V'' + 60 f_{xx}(2x) V' + 60 f_x(2x) V'' + 24 c_3 V = \mathrm{const}(t).$$

4 Rational Solutions

4.1 The Simplest Case: $b = x - u(t)$

The simplest possibility is to set

$$b = x - u(t). \tag{4.1}$$

We will see that already this case is meaningful and leads to PI and PII equations.

In this case, integrability condition (3.18) turns into

$$\Big(V_t(x) - V_t(u)\Big) - \frac{1}{2(x-u)} \Big(V'(x) + V'(u)\Big) + \frac{1}{(x-u)^2} \Big(V(x) - V(u)\Big) = 0 \tag{4.2}$$

or

$$2(x-u)^2 \Big(V_t(x) - V_t(u)\Big) - (x-u) \Big(V'(x) + V'(u)\Big) + 2 \Big(V(x) - V(u)\Big) = 0. \tag{4.3}$$

It should be an identity for all x, u which enter here as independent variables on equal footing. The way to proceed is to take the third derivative of (4.3) with respect to x. The result is

$$\begin{aligned} &2u^2 V_t'''(x) + u \left(V^{\mathrm{IV}}(x) - 4x V_t'''(x) - 12 V_t''(x) \right) \\ &\quad + \; 12 V_t'(x) + 12 x V_t''(x) + 2x^2 V_t'''(x) - V'''(x) - x V^{\mathrm{IV}}(x) = 0. \end{aligned}$$

The equality holds identically if the coefficients in front of u^2, u, and the free term in u vanish. This implies the conditions

$$\begin{cases} V_t'''(x) = 0, \\ 12 V_t'(x) = V'''(x). \end{cases}$$

From the first equation, it follows that $V_t(x)$ is a polynomial in x of second degree at most, while from the second one, it then follows that $V(x)$ is a polynomial in x of fourth degree at most. There are three possibilities:

1) $V_t'(x) \equiv 0$; then $V(x)$ is a quadratic polynomial $V(x) = a_2 x^2 + a_1 x + a_0$ with $\dot{a}_2 = \dot{a}_1 = 0$. Plugging it into equation (4.3), we see that the equation holds identically for any constants a_2, a_1, with the irrelevant free term a_0 being an arbitrary function of t. This is the potential for the harmonic oscillator.

2) $V_t''(x) \equiv 0$; then $V(x)$ is a 3-d degree polynomial $V(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ with $\dot{a}_3 = \dot{a}_2 = 0$. By rescaling and shift of the variable x, we can set $a_3 = 1$, $a_2 = 0$. The free term, a_0, is irrelevant since it cancels in equation (4.3). Plugging the potential in the form $V(x) = x^3 + a_1x$ into equation (4.3), we get $(x-u)^2(2\dot{a}_1 - 1) = 0$. Therefore, $a_1 = t/2$, and

$$V(x) = x^3 + \frac{tx}{2}.$$

This is, up to a common factor, the potential for the PI equation.

3) $V_t''(x) \neq 0$; then $V(x)$ is a 4-th degree polynomial $V(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ with $\dot{a}_4 = \dot{a}_3 = 0$. Again, we can set $a_4 = 1$, $a_3 = 0$, and $a_0 = 0$. Plugging the potential in the form $V(x) = x^4 + a_2x^2 + a_1x$ into equation (4.3), we get $(x^2 - u^2)(\dot{a}_2 - 1) + \dot{a}_1 = 0$. Therefore, $a_2 = t$, $a_1 = -2\alpha$, where α is an arbitrary constant. Up to a common factor, we obtain the potential

$$V(x) = x^4 + tx^2 - 2\alpha x$$

for the PII equation with the parameter α.

4.2 The Case $b = (x - u_1(t))(x - u_2(t))$

Let us make the similar calculations for $b = (x - u_1(t))(x - u_2(t))$. Instead of

$$V_t(x) - H_t - \frac{V'(x) - \ddot{u}}{2(x-u)} + \frac{V(x) - H + \dot{u}^2/2}{(x-u)^2} = 0,$$

we get, after cancellation of third- and fourth-order poles:

$$\begin{aligned} &V_t(x) - H_t - \frac{1}{2(x-u_1)}\left(V'(x) - \ddot{u}_1 - \frac{2}{(u_1-u_2)^3}\right) \\ &- \frac{1}{2(x-u_2)}\left(V'(x) - \ddot{u}_2 - \frac{2}{(u_2-u_1)^3}\right) \\ &+ \frac{1}{(x-u_1)^2}\left(V(x) - H + \frac{1}{2}\frac{\dot{u}_1 + \dot{u}_2}{u_1 - u_2} + \frac{1}{2}\dot{u}_1^2 - \frac{1}{2(u_1-u_2)^2}\right) \\ &+ \frac{1}{(x-u_2)^2}\left(V(x) - H + \frac{1}{2}\frac{\dot{u}_1 + \dot{u}_2}{u_2 - u_1} + \frac{1}{2}\dot{u}_2^2 - \frac{1}{2(u_1-u_2)^2}\right) = 0. \end{aligned} \quad (4.4)$$

Cancellation of the second-order poles at $x = u_{1,2}$ yields

$$H = \frac{1}{2}\dot{u}_1^2 + \frac{1}{2}\frac{\dot{u}_1 + \dot{u}_2}{u_1 - u_2} + V(u_1) - \frac{1}{2(u_1-u_2)^2}$$

and

$$H = \frac{1}{2}\dot{u}_2^2 + \frac{1}{2}\frac{\dot{u}_1 + \dot{u}_2}{u_2 - u_1} + V(u_2) - \frac{1}{2(u_1-u_2)^2}.$$

By equating the two "kinetic" terms, we get the following two possibilities:

$$1)\ \dot{u}_1 + \dot{u}_2 = 0, \quad 2)\ \partial_t (u_1 - u_2)^2 = -4.$$

In the first case, $u_1 + u_2 = \text{const}$, and one can shift x in the initial problem to set $u_1 = -u_2 \equiv u$. Therefore, the two possibilities are rewritten as

$$1) \begin{cases} u_1 = -u_2 \equiv u, \\ V(u) = V(-u), \end{cases} \quad 2) \begin{cases} u_1 = u_2 + \sqrt{c - 4t}, \\ V(u) = V(u - \sqrt{c-4t}), \end{cases} \tag{4.5}$$

where c is some constant. The second case is given in Appendix A. Here we consider the first one. In this case, (4.4) leads to integrability condition (3.28):

$$V_t(x) - V_t(u) - \frac{1}{2(x-u)} \left(V'(x) + V'(u) - 2\,\frac{V(x) - V(u)}{x-u} \right)$$
$$- \frac{1}{2(x+u)} \left(V'(x) - V'(u) - 2\,\frac{V(x) - V(u)}{x+u} \right) = 0,$$

or, equivalently,

$$2(x^2-u^2)^2(V_t(x) - V_t(u)) - (x+u)(x^2-u^2)\left(V'(x) + V'(u)\right)$$
$$- (x-u)(x^2-u^2)\left(V'(x) - V'(u)\right) + 4(x^2+u^2)(V(x) - V(u)) = 0. \tag{4.6}$$

Since the maximal degree of x in (4.6) is 4, the differential operator ∂_X^5 applied to this equation kills all terms containing $V(u)$, and we are left with

$$\partial_X^5 \Big[(x^2-u^2)^2 V_t(x) - x(x^2-u^2)V'(x) + 2(x^2+u^2)V(x) \Big] = 0.$$

Equating the coefficients in front of u^4, u^2, and u^0 to zero, we get the following conditions:

$$\begin{cases} \partial_x^5 V_t(x) = 0, \\ \partial_x^5 \Big[-2x^2 V_t(x) + xV'(x) + 2V(x) \Big] = 0, \\ \partial_x^5 \Big[x^4 V_t(x) - x^3 V'(x) + 2x^2 V(x) \Big] = 0. \end{cases}$$

They mean that the expressions in the square brackets are polynomials in x of at most fourth degree:

$$V_t(x) = P_4(x),$$
$$-2x^2 V_t(x) + xV'(x) + 2V(x) = Q_4(x),$$
$$x^4 V_t(x) - x^3 V'(x) + 2x^2 V(x) = R_4(x).$$

Combining these conditions, we find that $x^2V(x)$ must be a polynomial of at most 8-th degree such that its highest and lowest coefficients do not depend on t. We also recall that it must contain only even powers of x. So we can write

$$V(x) = \mu x^6 + a_4x^4 + a_2x^2 + a_0 + \frac{\nu}{x^2}, \qquad \dot{\mu} = \dot{\nu} = 0.$$

Plugging this potential back into equation (4.6), we obtain

$$(x^4 - u^4)(x^2 - u^2)(\dot{a}_4 - 4\mu) + (x^2 - u^2)^2(\dot{a}_2 - 2a_4) = 0.$$

The solution is $a_4 = 4\mu t + \alpha_4$, $a_2 = 4\mu t^2 + 2\alpha_4 t + \alpha_2$, with integration constants α_4, α_2, and a_0 is arbitrary. There are three cases:

1) $\mu \neq 0$ (the case of general position); then one can set it equal to 1 by rescaling and set $\alpha_4 = 0$ by a shift of the t-variable. Then the potential acquires the form

$$V(x,t) = x^6 + 4tx^4 + (4t^2 + \alpha_2)x^2 + a_0(t) + \frac{\nu}{x^2}. \tag{4.7}$$

This is the potential for the PIV equation.

2) $\mu = 0$ but $\alpha_4 \neq 0$; then one can set α_4 equal to 1 by rescaling and set $\alpha_2 = 0$ by a shift of the t-variable. The potential is

$$V(x,t) = x^4 + 2tx^2 + a_0(t) + \frac{\nu}{x^2}.$$

It generates the equation

$$\ddot{u} = -4u^3 - 4tu + \frac{2\nu}{u^3}.$$

The change of the dependent variable $u \to y$ such that $u^2 + y^2 + \frac{1}{2}\dot{y} + t = 0$ (a version of a similar change in [17, section14.331]) brings the equation to the form $\ddot{y} = 8y^3 + 8ty + \sqrt{-32\nu} - 2$, which is equivalent to the PII equation.

3) $\mu = \alpha_4 = 0$; then

$$V(x,t) = \alpha_2 x^2 + \frac{\nu}{x^2} + a_0(t).$$

This gives the exactly solvable rational 2-particle Calogero model in the harmonic potential. The x-independent term $a_0(t)$ is irrelevant.

5 Hyperbolic Solutions

5.1 The Case $b = e^{t/2}\sinh(x - u(t))$

Let us consider the case when b is a trigonometric (hyperbolic, to be exact) function with one simple zero in the strip of periodicity:

$$b = e^{t/2}\sinh(x - u(t)).$$

We will see that it leads to the PIII equation. Since b satisfies the heat equation (3.15), Proposition 3.2 can be applied. The integrability condition (3.18) with $\frac{b_x}{b} = \coth(x)$ becomes

$$2\sinh^2(x-u)\Big(V_t(x) - V_t(u)\Big) - \sinh(x-u)\cosh(x-u)\Big(V'(x) + V'(u)\Big) \\ +2\Big(V(x) - V(u)\Big) = 0. \tag{5.1}$$

Let us make the change of variables $V \to \mathcal{V}$, $x \to X$, $u \to U$ such that

$$V(x) \equiv e^{-4x}\mathcal{V}(e^{2x}), \quad X = e^{2x}, \quad U = e^{2u};$$

then equation (5.1) is rewritten as

$$(X-U)^2\left(U^2\mathcal{V}_t(X) - X^2\mathcal{V}_t(U)\right) - UX(X^2-U^2)\left(U\mathcal{V}'(X) + X\mathcal{V}'(U)\right) \\ +2(X^2-U^2)\left(U^2\mathcal{V}(X) + X^2\mathcal{V}(U)\right) + 4UX\left(U^2\mathcal{V}(X) - X^2\mathcal{V}(U)\right) = 0. \tag{5.2}$$

Since the maximal degree of X here equals 4, the differential operator ∂_X^5 applied to this equation kills all terms containing $\mathcal{V}(U)$, and we are left with

$$\partial_X^5\Big[(X-U)^2\mathcal{V}_t(X) - X(X^2-U^2)\mathcal{V}'(X) + 2(X^2-U^2+2UX)\mathcal{V}(X)\Big] = 0.$$

Equating the coefficients in front of U^2, U^1, and U^0 to zero, we get the following conditions:

$$\begin{cases} \partial_X^5\Big[\mathcal{V}_t(X) + X\mathcal{V}'(X) - 2\mathcal{V}(X)\Big] = 0, \\ \partial_X^5\Big[-X\mathcal{V}_t(X) + 2X\mathcal{V}(X)\Big] = 0, \\ \partial_X^5\Big[X^2\mathcal{V}_t(X) - X^3\mathcal{V}'(X) + 2X^2\mathcal{V}(X)\Big] = 0. \end{cases}$$

They mean that the expressions in the square brackets are polynomials in X of at most fourth degree:

$$\begin{aligned} &\mathcal{V}_t(X) + X\mathcal{V}'(X) - 2\mathcal{V}(X) = P_4(X), \\ &-X\mathcal{V}_t(X) + 2X\mathcal{V}(X) = Q_4(X), \\ &X^2\mathcal{V}_t(X) - X^3\mathcal{V}'(X) + 2X^2\mathcal{V}(X) = R_4(X). \end{aligned} \tag{5.3}$$

Combining these conditions, we obtain that $X^2\mathcal{V}'(X)$ and $X^2\mathcal{V}_t(X)$ are polynomials of at most 5-th and 6-th degrees, respectively. It is easy to see that the former polynomial must be divisible by X^2. Indeed, let it be $X^2\mathcal{V}'(X) = X^2P_3(X) + p_1X + p_0$ with some nonzero $p_{0,1}$. Then the first equation in (5.3) implies $p_0 = 0$ (otherwise the left-hand side contains a nonpolynomial term $\propto X^{-1}$) and the second equation multiplied by X implies $p_1 = 0$ (otherwise the left-hand side contains a nonpolynomial term $\propto X^2 \log X$). Therefore, we conclude that $\mathcal{V}'(X)$ is a polynomial of at most third degree, and, thus, $\mathcal{V}(X)$ itself is a polynomial of at most fourth degree:

$$\mathcal{V}(X) = a_4 X^4 + a_3 X^3 + a_2 X^2 + a_1 X + a_0.$$

Let us plug it in equation (5.2). After simple transformations, we obtain the relation

$$(X-U)(X^2-U^2)(\dot{a}_4 - 2a_4) + (X-U)^2(\dot{a}_3 - a_3) \\ - \frac{(X-U)^2}{UX}(\dot{a}_1 - a_1) - \frac{(X-U)(X^2-U^2)}{U^2X^2}(\dot{a}_0 - 2a_0) = 0.$$

It must be satisfied identically for all X, U. This implies $\dot{a}_4 = 2a_4$, $\dot{a}_3 = a_3$, $\dot{a}_1 = a_1$, $\dot{a}_0 = 2a_0$, and no condition for a_2. Therefore, the potential $V(x,t)$ is fixed to be

$$V(x,t) = \alpha_1 e^{2t+4x} + \alpha_2 e^{2t-4x} + \alpha_3 e^{t+2x} + \alpha_4 e^{t-2x} + a(t),$$

where α_i are arbitrary constants. This is precisely the potential for the PIII equation.

5.2 The Case $b = e^t \sinh(x - u(t)) \sinh(x + u(t))$

In this case, $b = \left(e^{t/2}\sinh(x-u)\right)\left(e^{t/2}\sinh(x+u)\right)$. Each of the multiples satisfies the heat equation (3.24). Therefore, Proposition 3.3 can be applied. Then equation (3.28) assumes the form

$$V_t(x) - V_t(u) - \frac{1}{2}\coth(x-u)\Big(V'(x) + V'(u)\Big) - \frac{1}{2}\coth(x+u)\Big(V'(x) - V'(u)\Big) \\ + \Big(V(x) - V(u)\Big)\left(\frac{1}{\sinh^2(x-u)} + \frac{1}{\sinh^2(x+u)}\right) = 0.$$

Multiplying by $32\sinh^2(x-u)\sinh^2(x+u)$ and making change of variables $X = \cosh^2(x)$, $y = \coth^2(u)$, we get

$$(Xy - X - y)^2(V_t(X) - V_t(y)) - 2X(X-1)(y-1)(Xy - X - y)V'(X) \\ +2y(y-1)(Xy - X - y)V'(y) + 2(y-1)(Xy + X - y)(V(X) - V(y)) = 0.$$

Now one can apply the calculation method similar to the previous cases. That is to take the third derivative with respect to X and analyze the differential equations (the later equations appear as the coefficients behind different powers of y). This analysis gives the potential of the Painlevé V equation after some tedious evaluations. Instead of proceeding in this manner, let us simplify the problem by assuming that the solution is a sum of terms of the form $V(x) = e^{kt} v(X)$. Making this substitution, one gets:

$$k(Xy - X - y)^2(v(X) - v(y)) - 2X(X-1)(y-1)(Xy - X - y)v'(X) \\ +2y(y-1)(Xy - X - y)v'(y) + 2(y-1)(Xy + X - y)(v(X) - v(y)) = 0.$$

We will see that nontrivial solutions exist for $k = 0, 2, 4$. The way to proceed is to take the third derivative of the expression with respect to X. The equality holds identically if the coefficients in front of y^2, y and the free term in y vanish. This implies the following conditions:

$$\begin{cases} X(X-1)v'''(X) + 3(2X-1)v''(X) + 3(2-k)v'(X) = 0, \\ X(X-1)v^{(4)}(X) + 4(2X-1)v'''(X) + 3(4-k)v''(X) = 0, \\ kv'''(X) = 0. \end{cases} \tag{5.4}$$

Consider the last equation. If $k = 0$, one gets

$$v(X) = \frac{c_1}{X} + \frac{c_2}{X-1} + c_3 = \frac{\tilde{c}_1}{\sinh^2 x} + \frac{\tilde{c}_2}{\cosh^2 x} + \tilde{c}_3,$$

else $v'''(X) = 0$. The later case leads to $(k-4)v''(X) = 0$ (from the second equation in (5.4)). Then, $k = 4$ or $v''(X) = 0$. In the latter case, one gets $(k-2)v'(X) = 0$ (from the first equation in (2.4)). In this way, one can easily recover the potential of the Painlevé V equation (2.4):

$$V(x,t) = -\frac{2(\xi+\sigma)^2}{\sinh^2 x} + \frac{2\zeta^2}{\cosh^2 x} + \frac{e^{2t}}{2}(2\sigma-1)\cosh(2x) - \frac{e^{4t}}{16}\cosh(4x).$$

6 Elliptic Solutions

6.1 The Case $b = \vartheta_1(x - u(t), 2\pi i t)$

Consider an elliptic curve with moduli $\tau = 2\pi i t$,

$$\Sigma_\tau : \ \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau,$$

and let $b = \vartheta_1(x - u(t), 2\pi i t)$. Definitions and properties of elliptic functions are given in Appendix B. Then from (3.18), we have

$$V_t(x) - V_t(u) - \frac{1}{2}E_1(x-u)(V'(u) + V'(x)) + E_2(x-u)\,(V(x) - V(u)) = 0.$$

We will show that this equation has only trivial solutions $V(x,t) = f(t)$. For this purpose, consider the same equation at $x + \tau$ and subtract it from the initial one. Then, using the behavior of $E_1(z)$ (B.1) and $E_2(z)$ (B.2) on the torus lattice, we get

$$\begin{aligned} &V_t(x+\tau) - V_t(x) - \frac{1}{2}E_1(x-u)(V'(x+\tau) - V'(x)) \\ &\quad + E_2(x-u)(V(x+\tau) - V(x)) + \pi i(V'(u) + V'(x+\tau)) = 0. \end{aligned} \tag{6.1}$$

Let us now differentiate the obtained equality with respect to x:

$$V_t'(x+\tau) - V_t'(x) - \frac{1}{2}E_1(x-u)(V''(x+\tau) - V''(x)) + \pi i V''(x+\tau)$$
$$+\frac{3}{2}E_2(x-u)(V'(x+\tau) - V'(x)) + E_2'(x-u)(V(x+\tau) - V(x)) = 0. \quad (6.2)$$

Similarly, let us shift the argument $u \to u+\tau$ in the equation (6.2) and subtract it from (6.2) itself (keeping in mind that E_2' is the double-periodic function). This gives

$$V''(x+\tau) - V''(x) = 0$$

or

$$V(x+\tau) - V(x) = a(\tau)x + b(\tau).$$

Plugging this back into (6.1), one can easily get that $a(\tau) = b(\tau) = 0$ by analyzing coefficients behind the poles at $x-u$ of the second and the first orders. Therefore, the potential should be a double-periodic function. If it is, then (6.1) reduces to

$$\partial_t(V(x+\tau) - V(x)) + \pi i(V'(u) - V'(x)) = 0,$$

since $\partial_t(V(x+\tau) - V(x)) = V_t(x+\tau) - V_t(x) + 2\pi i V'(x+\tau)$. The latter equation should hold for all x and u. Then the only solution is

$$V(x,t) = f(t).$$

6.2 The Case $b = \vartheta_1(x-u(t), 2\pi i t)\vartheta_1(x+u(t), 2\pi i t)$

Equation (3.28) in this case has the form

$$V_t(x) - V_t(u) - \frac{1}{2}E_1(x-u)(V'(u) + V'(x)) - \frac{1}{2}E_1(x+u)(-V'(u) + V'(x))$$
$$+(E_2(x-u) + E_2(x+u))(V(x) - V(u)) = 0. \quad (6.3)$$

Let us make a change of variables:

$$X(x,t) = \frac{\wp(x) - e_1}{e_2 - e_1}, \quad Q(u,t) = \frac{\wp(u) - e_1}{e_2 - e_1}, \quad T(t) = \frac{e_3 - e_1}{e_2 - e_1}.$$

Then

$$E_1(x+u) + E_1(x-u) = 2E_1(x) + \frac{\wp'(x)}{\wp(x) - \wp(u)} = 2E_1(x) + \frac{X_x}{X-Q},$$
$$E_1(x+u) - E_1(x-u) = 2E_1(u) + \frac{\wp'(u)}{\wp(u) - \wp(x)} = 2E_1(x) - \frac{Q_u}{X-Q},$$

$$E_2(x+u)+E_2(x-u)=2E_2(u)+\frac{Q_{uu}}{X-Q}+\frac{Q_u^2}{(X-Q)^2}.$$

Therefore, equation (6.3) is written as

$$\begin{aligned}&(V_T(X)-V_T(Q))T_t+V_X(X)X_t-V_Q(Q)Q_t\\&-\frac{1}{2}V_X(X)X_x\left(2E_1(x)+\frac{X_x}{X-Q}\right)+\frac{1}{2}V_Q(Q)Q_u\left(2E_1(u)-\frac{Q_u}{X-Q}\right)\\&+\left(2\left[2\eta_1+e_1+(e_2-e_1)Q\right]+\frac{Q_{uu}}{X-Q}+\frac{Q_u^2}{(X-Q)^2}\right)(V(x)-V(u))=0.\end{aligned}$$

It follows from (B.8) that

$$\begin{aligned}X_t-X_xE_1(x)&=X_x(E_1(x+\omega_3)-E_1(x)-E_1(\omega_3))=\frac{1}{2}X_x\frac{\wp'(x)}{\wp(x)-\wp(\omega_3)}\\&=\frac{1}{2}\frac{X_x^2}{X-T}.\end{aligned}$$

Therefore,

$$\begin{aligned}&(V_T(X)-V_T(Q))T_t+V_X(X)\frac{1}{2}\frac{X_x^2}{X-T}-V_Q(Q)\frac{1}{2}\frac{Q_u^2}{Q-T}\\&-\frac{1}{2}V_X(X)\frac{X_x^2}{X-Q}-\frac{1}{2}V_Q(Q)\frac{Q_u^2}{X-Q}\\&+\left(2\left[2\eta_1+e_1+(e_2-e_1)Q\right]+\frac{Q_{uu}}{X-Q}+\frac{Q_u^2}{(X-Q)^2}\right)(V(x)-V(u))=0.\end{aligned}\tag{6.4}$$

Now let us proceed as in the previous examples. First, multiply (6.4) by $(X-Q)^2$. Secondly, take the third derivative with respect to X. This excludes $V(Q)$. Thirdly, substitute $Q_u^2=4(e_2-e_1)Q(Q-1)(Q-T)$ and $Q_{uu}=2(e_2-e_1)(3Q^2-2Q(T+1)+T)$. Then, the coefficients in front of Q^2, Q^1, and Q^0 should vanish independently:

$$\begin{cases}F+2V(X)(2\eta_1+e_1+X(e_2-e_1))=P_2(X),\\-2XF+\frac{1}{2}V_X(X)X_x^2\\+2V(X)(X^2(e_2-e_1)+4X(e_1-\eta_1)+T(e_2-e_1))=Q_2(X),\\X^2F-\frac{1}{2}V_X(X)X_x^2X+2V(X)((2\eta_1+e_1)X^2+(e_2-e_1)XT)=R_2(X),\end{cases}\tag{6.5}$$

where $P_2(X)$, $Q_2(X)$, $R_2(X)$ are the second-order polynomials in X with time-dependent coefficients and $F=V_T(X)T_t+V_X(X)\frac{1}{2}\frac{X_x^2}{X-T}$.

Excluding F from the two upper equations in (6.5), we obtain the following equality:

$$V_X(X)X(X-1)(X-T)+V(X)\Big(X(X-1)+X(X-T)+(X-1)(X-T)\Big)$$
$$=\frac{1}{e_2-e_1}\Big(\frac{1}{2}Q_2(X)+XP_2(X)\Big).$$

General solution of the latter equation has a form:

$$V(X)=\frac{1}{X(X-1)(X-T)}\int^X dZ\frac{1}{e_2-e_1}\Big(\frac{1}{2}Q_2(Z)+ZP_2(Z)\Big)=\frac{H_4(X)}{X(X-1)(X-T)},$$

where $H_4(X)$ are the forth-order polynomials in X with time-dependent coefficients. Therefore, $V(X)$ can be presented as

$$V(X)=a(T)X+\frac{b(T)}{X}+\frac{c(T)}{X-1}+\frac{d(T)}{X-T}+h(T). \tag{6.6}$$

The last term $h(T)$ is not fixed by (6.3); i.e., $h(T)$ is arbitrary.

Plugging (6.6) into (6.4) and multiplying the result by $(X-Q)X(X-1)(X-T)Q(Q-1)(Q-T)$, we get a polynomial function in X and Q. The coefficients in front of Q^kX^j provides differential equations. It can be verified that all of them are equivalent to the following system:

$$\begin{cases} a_T(T)T(T-1)(e_2-e_1)+a(T)(e_3+2\eta_1)=0,\\ b_T(T)T(T-1)(e_2-e_1)+b(T)(e_2+2\eta_1)=0,\\ c_T(T)T(T-1)(e_2-e_1)+c(T)(e_1+2\eta_1)=0,\\ d_T(T)T(T-1)(e_2-e_1)+d(T)(-2e_3+2\eta_1)=0. \end{cases}$$

Its solutions (see (B.6)–(B.7)) are

$$\begin{cases} a(T)=\alpha(e_2-e_1), \quad \alpha=\text{const},\\ b(T)=\beta(e_2-e_1)T, \quad \beta=\text{const},\\ c(T)=\gamma(e_2-e_1)(T-1), \quad \gamma=\text{const},\\ d(T)=\delta(e_2-e_1)T(T-1), \quad \delta=\text{const}. \end{cases} \tag{6.7}$$

Then, in view of (B.5), we have

$$V(x)=\alpha\wp(x)+\beta\wp(x+\omega_1)+\gamma\wp(x+\omega_2)+\delta\wp(x+\omega_3)+h(t). \tag{6.8}$$

This is the potential of the Painlevé VI equation in the elliptic form [18,36,53,60] (see also [32,50] and [9]). We remark that the nonstationary Lamé equation in connection with the PVI equation (and with the 8-vertex model) was discussed in [4]. Recently, the nonstationary Lamé equation has appeared [10,37–40] in the context of the AGT conjecture. The results of [38–40] allow one in principle to construct higher Painlevé equations[4] in terms of 2x2 linear problems related to spin chains via spectral duality transformation. We are going to study this possibility in our future publications.

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Appendix A: Special Cases

$b=(x-u(t))e^{g(t)x}$ and $b=(x-u(t))e^{g(t)x^2}$

Let $b=(x-u(t))e^{g(t)x}$. The calculation similar to the one leading to (4.2) gives in this case

$$V_t(x)-V_t(u)-\frac{V'(x)+V'(u)}{2(x-u)}+\frac{V(x)-V(u)}{(x-u)^2}-\frac{\ddot g}{2}(x-u)-\frac{g}{2}\Big(V'(x)-V'(u)\Big)=0 \tag{A.1}$$

and

$$H=\frac{1}{2}\left(\dot u+\frac{g}{2}\right)^2+V(u,t)-\frac{1}{2}u\dot g+\frac{1}{8}g^2, \tag{A.2}$$

with equation of motion

$$\ddot u=-V'(u).$$

It is easy to see that equation (A.1) becomes equivalent to (4.2) for the potential $\tilde V(\tilde x)$ after the change of variables

$$x\to\tilde x=x-\frac{1}{2}G(t),\qquad V(x)\to\tilde V(x)=V\left(x-\frac{1}{2}G(t)\right)-\frac{\dot g}{2}x,$$

where $\dot G=g$. Notice also that the dependence $H(\dot u)$ in (A.2) can be obtained from (3.10) via the local expansion (3.8). The later gives $b_1=e^{ug}$ and $b_2=g\,e^{ug}$. Then $v=\dot u+\frac{g}{2}$.

Consider now the case $b=(x-u(t))e^{g(t)x^2}$. Let us perform the calculation similar to the one leading to (4.2) again. In this case, we have:

$$\begin{aligned}V_t(x)-V_t(u)-\frac{V'(x)+V'(u)}{2(x-u)}+\frac{V(x)-V(u)}{(x-u)^2}-2g\,(V(x)-V(u))\\-g\Big(xV'(x)+uV'(u)\Big)+(x^2-u^2)\left[3g\dot g-\frac{1}{2}\ddot g-2g^3\right]=0\end{aligned} \tag{A.3}$$

[4] See also [42,48] and [52].

and

$$H = \frac{1}{2}\,(\dot{u} + gu)^2 + V(u,t) + \frac{1}{2}(g^2 - \dot{g})u^2 + \frac{3}{2}g, \tag{A.4}$$

with equation of motion

$$\ddot{u} = -V'(u).$$

As in the previous example, it can be shown that equation (A.3) becomes equivalent to (4.2) for the potential $\tilde{V}(\tilde{x})$ after the following change of variables:

$$\begin{aligned} x \to \tilde{x} &= \alpha x = x\, e^{\int_t g(t)},\ \alpha = e^{\int_t g(t)}, \\ V(x) \to \tilde{V}(x) &= \alpha^2\Big(V(\alpha x) - x^2\left(g^2 - \int_t \frac{\ddot{g} - 2g\dot{g}}{2\alpha^2}\right)\Big) \\ &= e^{2\int_t g(t)}\left(V(x\, e^{\int_t g(t)}) - x^2\left(g^2 - \int_t \left[e^{-2\int_t g(t)}\left(\frac{1}{2}\ddot{g} - g\dot{g}\right)\right]\right)\right). \end{aligned}$$

Notice also that the dependence $H(\dot{u})$ in (A.4) can be obtained from (3.10) via the local expansion (3.8). The latter gives $b_1 = e^{gu^2}$ and $b_2 = 2gu\, e^{gu^2}$. Then $v = \dot{u} + gu$.

$b = (x - u_1(t))(x - u_2(t))(x - u_3(t))$

When $b = (x - u_1)(x - u_2)(x - u_3)$, the coefficients behind the second-order pole $\frac{1}{(x-u_1)^2}$ in (3.6) have the following form:

$$\begin{aligned} &V(x,t) - H + \frac{1}{2}\dot{u}_1^2 + \frac{1}{2}\frac{\dot{u}_1 + \dot{u}_2}{u_1 - u_2} + \frac{1}{2}\frac{\dot{u}_1 + \dot{u}_3}{u_1 - u_3} - \frac{1}{2}\frac{1}{(u_1 - u_2)^2} - \frac{1}{2}\frac{1}{(u_1 - u_3)^2} \\ &\quad + \frac{1}{2}\frac{1}{(u_1 - u_2)(u_1 - u_3)}, \end{aligned}$$

and two other coefficients can be obtained by the cyclic permutations. All three coefficients cannot vanish simultaneously. Therefore, some other anzats for W (3.2) should be used in this case. This reflects the fact that (3.1)–(3.2) imply the one degree of freedom case.

$b = (x - u_1(t))^\gamma$ and $b = (x - u_1(t))^{\gamma_1}(x - u_2(t))^{\gamma_2}$

Let us study the case $b = (x - u_1(t))^\gamma$, where $\gamma \in \mathbb{C}^*$ (the case $\gamma = 0$ is trivial). Notice that under change $b \to b^\gamma$ the functions f (3.4) and S (3.5) transform as follows:

$$\begin{aligned} f &= \frac{b_x}{b} \longrightarrow \gamma\frac{b_x}{b}, \\ S &\longrightarrow V - H - \frac{1}{2}\gamma\frac{b_t}{b} + \frac{1}{2}\gamma\frac{b_{xx}}{b} + \frac{1}{2}\left(\frac{1}{4}\gamma^2 - \gamma\right)\left(\frac{b_x}{b}\right)^2. \end{aligned}$$

For the case under consideration, we have $f = \gamma \frac{1}{x-u}$ and

$$S = V - H + \frac{\gamma}{2}\dot{u}\frac{1}{x-u} + \frac{1}{2}\left(\frac{1}{4}\gamma^2 - \gamma\right)\frac{1}{(x-u)^2}.$$

Substituting it into (3.6), we obtain the following condition for cancellation of the fourth- and the third-order poles:

$$(x-u)^{-4}: \quad 0 = \frac{1}{4}\gamma(\gamma^2 - 4\gamma + 3),$$
$$(x-u)^{-3}: \quad \dot{u}\left(\frac{1}{4}\gamma^2 - \gamma\right) = -\frac{3}{4}\gamma^2\dot{u}.$$

The first equation gives $\gamma = \{0, 1, 3\}$, while the second one $\gamma = \{0, 1\}$. Therefore, the nontrivial solution is

$$\gamma = 1.$$

Similarly, the case $b = (x - u_1(t))^{\gamma_1}(x - u_2(t))^{\gamma_2}$ leads to the following conditions:

$$(x-u_1)^{-4}: \tfrac{1}{4}\gamma_1(\gamma_1^2 - 4\gamma_1 + 3) = 0,$$
$$(x-u_1)^{-3}: \tfrac{1}{2}\tfrac{\gamma_1(\gamma_1-1)}{u_1-u_2}\left(2\dot{u}_1(u_1-u_2) + \gamma_2\right),$$

$$(x-u_2)^{-4}: \tfrac{1}{4}\gamma_2(\gamma_2^2 - 4\gamma_2 + 3) = 0,$$
$$(x-u_2)^{-3}: \tfrac{1}{2}\tfrac{\gamma_2(\gamma_2-1)}{u_2-u_1}\left(2\dot{u}_2(u_2-u_1) + \gamma_1\right),$$

which give

$$\gamma_1 = \gamma_2 = 1.$$

$\boldsymbol{b = \exp\left((z/u(t))^{\gamma}\right)}$

First, it can be shown that $\gamma = 0, 1, 2, 3...$

Consider $\gamma = 1$. Substituting $b(z, u(t), t) = \exp(z/u(t))$ into (3.6), we get

$$\left(-\frac{\dot{u}^2}{u^3} + \frac{1}{2}\frac{\ddot{u}}{u^2}\right)x + V_t - H_t - \frac{1}{2}\frac{\dot{u}}{u^3} - \frac{1}{2u}V'_x = 0. \tag{A.5}$$

Applying ∂_x^2 gives

$$V_t'' - \frac{1}{2u}V''' = 0.$$

Notice that the function $U(z, \dot{u}, u, t)$ satisfies the same equation even if we do not impose the condition $U = V(x, t) - H(\dot{u}, u, t)$. Under assumption $U = V(x, t) - H(\dot{u}, u, t)$, we have

$$V''' = V_t'' = 0.$$

This leads to

$$V(x,t) = \frac{\alpha}{2}x^2 + b(t)x + c(t), \quad \alpha = \text{const.}$$

Plugging it back into (A.5), we obtain the following two equations (as coefficients behind x^1 and x^0):

$$\begin{cases} \ddot{u} = 2\frac{\dot{u}^2}{u} - 2\dot{b}u^2 + \alpha u, \\ H_t = \frac{1}{2}\frac{\dot{u}^2}{u^3} + \dot{c} - \frac{1}{2u}b. \end{cases}$$

Case 2 in (4.5)

Here it may be useful to use variable $u = u_1 - \frac{1}{2}\sqrt{c-4t}$ (then $\dot{u} = \dot{u}_1 + \frac{1}{\sqrt{c-4t}}$). Then

$$H = \frac{1}{2}\left(\dot{u}_1 + \frac{1}{\sqrt{c-4t}}\right)^2 + V(u_1) = \frac{1}{2}\dot{u}^2 + V\left(u + \frac{1}{2}\sqrt{c-4t}\right), \tag{A.6}$$

and, therefore,

$$V_t(x) - H_t - \frac{1}{2(x-u_1)}\left(V'(x) - \ddot{u}_1 - 2(c-4t)^{-\frac{3}{2}} - 2\frac{V(x)-V(u_1)}{x-u_1}\right)$$
$$-\frac{1}{2(x-u_2)}\left(V'(x) - \ddot{u}_2 + 2(c-4t)^{-\frac{3}{2}} - 2\frac{V(x)-V(u_2)}{x-u_2}\right) = 0$$

Cancellation of the first-order poles at $x = u_{1,2}$ yields $\ddot{u}_1 = -V'(u_1) - 2(c-4t)^{-\frac{3}{2}}$. On this equation, $H_t = V_t(u_1) - V'(u_1)\frac{1}{\sqrt{c-4t}}$. Thus we arrive at

$$V_t(x) - V_t(u_1) + V'(u_1)\frac{1}{\sqrt{c-4t}} - \frac{1}{2(x-u_1)}\left(V'(x) + V'(u_1) - 2\frac{V(x)-V(u_1)}{x-u_1}\right)$$
$$-\frac{1}{2(x-u_2)}\left(V'(x) + V'(u_1) - 2\frac{V(x)-V(u_2)}{x-u_2}\right) = 0. \tag{A.7}$$

By analogy with (4.7), we get

$$\begin{cases} V_t^{\mathrm{V}}(x) = 0, \\ -20V_t^{\mathrm{IV}}(x) + V^{\mathrm{VI}}(x) = 0, \\ -13V^{\mathrm{V}}(x) + 120V_t^{\mathrm{III}}(x) = 0, \\ V^{\mathrm{IV}}(x) - 6V_t^{\mathrm{II}}(x) = 0, \\ 6\partial_z V_t(x) - V^{\mathrm{III}}(x) + \left(-\frac{16}{13}t + \frac{4}{13}\right)V_t^{\mathrm{III}}(x) = 0. \end{cases} \tag{A.8}$$

From the two upper equations, it follows that $V(x)$ is the 6-th degree polynomial. Plugging it into (A.8) drops the degree to 4 (similar to the Painlevé I, II cases). However, after substituting it back into (A.7), we get only the trivial solution

$$V(x,t) = f(t).$$

Appendix B: Elliptic Functions

Here we give a short version of the Appendix in [58].

Theta-functions

The Jacobi's theta-functions $\vartheta_a(z) = \vartheta_a(z|\tau)$, $a = 0, 1, 2, 3$, are defined by the formulas

$$\vartheta_1(z) = -\sum_{k\in\mathbb{Z}} \exp\left(\pi i\tau \left(k+\frac{1}{2}\right)^2 + 2\pi i \left(z+\frac{1}{2}\right)\left(k+\frac{1}{2}\right)\right),$$
$$\vartheta_2(z) = \sum_{k\in\mathbb{Z}} \exp\left(\pi i\tau \left(k+\frac{1}{2}\right)^2 + 2\pi i z \left(k+\frac{1}{2}\right)\right),$$
$$\vartheta_3(z) = \sum_{k\in\mathbb{Z}} \exp\left(\pi i\tau k^2 + 2\pi i z k\right),$$
$$\vartheta_0(z) = \sum_{k\in\mathbb{Z}} \exp\left(\pi i\tau k^2 + 2\pi i \left(z+\frac{1}{2}\right)k\right),$$

where τ is a complex parameter (the modular parameter) such that $\operatorname{Im}\tau > 0$. Set

$$\omega_0 = 0, \quad \omega_1 = \frac{1}{2}, \quad \omega_2 = \frac{1+\tau}{2}, \quad \omega_3 = \frac{\tau}{2};$$

then the function $\vartheta_a(z)$ has simple zeros at the points of the lattice $\omega_{a-1} + \mathbb{Z} + \mathbb{Z}\tau$ (here $\omega_a \equiv \omega_{a+4}$).

Weierstrass ℘-function

The Weierstrass $\wp$-function is defined as

$$\wp(z) = -\partial_z^2 \log \vartheta_1(z) - 2\eta,$$

where

$$\eta = -\frac{1}{6}\frac{\vartheta_1'''(0)}{\vartheta_1'(0)} = -\frac{2\pi i}{3}\,\partial_\tau \log \theta_1'(0|\tau).$$

Its derivative is given by

$$\wp'(z) = -\frac{2\,(\vartheta_1'(0))^3}{\vartheta_2(0)\vartheta_3(0)\vartheta_0(0)}\,\frac{\vartheta_2(z)\vartheta_3(z)\vartheta_0(z)}{\vartheta_1^3(z)}.$$

The values at the half-periods

$$e_1 = \wp(\omega_1), \quad e_2 = \wp(\omega_2), \quad e_3 = \wp(\omega_3)$$

have special properties. For example, $e_1 + e_2 + e_3 = 0$. The differences $e_j - e_k$ can be represented in two different ways:

$$\begin{aligned} e_1 - e_2 = \pi^2\vartheta_0^4(0) \;&=\; 4\pi i\,\partial_\tau \log\frac{\vartheta_3(0)}{\vartheta_2(0)}, \\ e_1 - e_3 = \pi^2\vartheta_3^4(0) \;&=\; 4\pi i\,\partial_\tau \log\frac{\vartheta_0(0)}{\vartheta_2(0)}, \\ e_2 - e_3 = \pi^2\vartheta_2^4(0) \;&=\; 4\pi i\,\partial_\tau \log\frac{\vartheta_0(0)}{\vartheta_3(0)}. \end{aligned}$$

The second representation is a consequence of the heat equation (B.3) (see below):

$$e_k = 4\pi i\,\partial_\tau\Big(\frac{1}{3}\,\log\vartheta_1'(0) - \log\vartheta_{k+1}(0)\Big)$$

or

$$\pi i\,\partial_\tau \log(e_j - e_k) = -e_l - 2\eta,$$

where $\{jkl\}$—any cyclic permutation of $\{123\}$. The $\wp$-function satisfies the differential equation

$$(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

We also mention the formulae

$$\wp(z) - e_k = \frac{(\vartheta_1'(0))^2}{\vartheta_{k+1}^2(0)}\,\frac{\vartheta_{k+1}^2(z)}{\vartheta_1^2(z)}.$$

Eisenstein functions and Φ-function

By definition,

$$E_1(z) = \partial_z \log\vartheta_1(z), \qquad E_2(z) = -\partial_z E_1(z) = -\partial_z^2 \log\vartheta_1(z) = \wp(z) + 2\eta.$$

Behavior on the lattice:

$$E_1(z+1) = E_1(z), \quad E_1(z+\tau) = E_1(z) - 2\pi i, \tag{B.1}$$

$$E_2(z+1) = E_2(z), \quad E_2(z+\tau) = E_2(z). \tag{B.2}$$

The local expansion near $z = 0$:

$$E_1(z) = \frac{1}{z} - 2\eta z + \cdots, \qquad E_2(z) = \frac{1}{z^2} + 2\eta + \cdots$$

Values at half-periods:

$$E_1(\omega_j) = -2\pi i \partial_\tau \omega_j,$$

and, therefore,

$$E_1(\omega_j) + E_1(\omega_k) = E_1(\omega_j + \omega_k)$$

holds true for any different $j, k = 1, 2, 3$.

Another useful function is

$$\Phi(u, z) = \frac{\vartheta_1(u + z)\vartheta_1'(0)}{\vartheta_1(u)\vartheta_1(z)}.$$

It has the following properties:

$$\begin{aligned}
\Phi(u, z) &= \Phi(z, u), \\
\Phi(-u, -z) &= -\Phi(u, z), \\
\Phi(u, z)\Phi(-u, z) &= \wp(z) - \wp(u), \\
\Phi(u, z)\Phi(w, z) &= \Phi(u + w, z)(E_1(z) + E_1(u) + E_1(w) - E_1(z + u + w)), \\
\Phi(u, z) &= \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \wp(u)) + O(z^2), \\
\partial_z \Phi(u, z) &= \Phi(u, z)(E_1(u + z) - E_1(z)).
\end{aligned}$$

Behavior on the lattice:

$$\Phi(u, z + 1) = \Phi(u, z), \quad \Phi(u, z + \tau) = e^{-2\pi i u}\Phi(u, z).$$

Is is also convenient to introduce

$$\varphi_j(z) = e^{2\pi i z \partial_\tau \omega_j}\Phi(z, \omega_j), \quad j = 1, 2, 3,$$

with properties:

$$\begin{aligned}
&\varphi_j^2(z) = \wp(z) - e_j, \quad \varphi_j^2(z) - \varphi_k^2(z) = e_k - e_j, \\
&\varphi_j(z)\varphi_k(z) = \varphi_l(z)(E_1(z) + E_1(\omega_l) - E_1(z + \omega_l)), \\
&\partial_z \varphi_j(z) = \varphi_j(z)\Big[E_1(z + \omega_j) - E_1(\omega_j) - E_1(z)\Big] = -\varphi_k(z)\varphi_l(z),
\end{aligned}$$

where j, k, l is any cyclic permutation of 1, 2, 3.

Heat equation and related formulae

All the theta-functions satisfy the "heat equation"

$$4\pi i\,\partial_\tau \vartheta_a(z|\tau) = \partial_z^2 \vartheta_a(z|\tau) \tag{B.3}$$

or

$$2\partial_t \vartheta_a(z) = \partial_z^2 \vartheta_a(z) \quad t = \frac{\tau}{2\pi i}.$$

One can also introduce the "heat coefficient" $\kappa = \frac{1}{2\pi i}$ and rewrite the heat equation in the form $\partial_\tau \vartheta_a(z|\tau) = \frac{\kappa}{2}\,\partial_z^2 \vartheta_a(z|\tau)$. All formulas for derivatives of elliptic functions with respect to the modular parameter are based on the heat equation.

The τ-derivatives are given by the following:

Proposition B.1 *The identities*

$$\partial_\tau \Phi(z,u) = \kappa\,\partial_z \partial_u \Phi(z,u), \tag{B.4}$$

$$\partial_\tau E_1(z) = \frac{\kappa}{2}\,\partial_z (E_1^2(z) - \wp(z)),$$
$$\partial_\tau E_2(z) = \kappa E_1(z) E_2'(z) - \kappa E_2^2(z) + \frac{\kappa}{2}\,\wp''(z),$$

with the "heat coefficient" $\kappa = \frac{1}{2\pi i}$, *hold true.*[5]

The proof can be found in [58].

Introduce now

$$X(x,t) = \frac{\wp(x) - e_1}{e_2 - e_1}, \qquad T(t) = \frac{e_3 - e_1}{e_2 - e_1} = \left(\frac{\vartheta_3(0|\tau)}{\vartheta_0(0|\tau)}\right)^4.$$

Then we have

$$X = \frac{\wp(x) - e_1}{e_2 - e_1}, \qquad X - 1 = \frac{\wp(x) - e_2}{e_2 - e_1}, \qquad X - T = \frac{\wp(x) - e_3}{e_2 - e_1},$$

and, therefore,

$$\left(\frac{\partial X}{\partial x}\right)^2 = 4(e_2 - e_1)\,X(X-1)(X-T),$$
$$\frac{\partial^2 X}{\partial x^2} = 2(e_2 - e_1)\,X(X-1)(X-T)\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-T}\right).$$

[5] (B.4) was obtained in [31,50].

Let us give some more relations:

$$\begin{aligned}
\frac{(e_2-e_1)T}{X} &= \wp(x+\omega_1)-e_1,\\
-\frac{(e_2-e_1)(T-1)}{X-1} &= \wp(x+\omega_2)-e_2,\\
\frac{(e_2-e_1)T(T-1)}{X-T} &= \wp(x+\omega_3)-e_3, \qquad \text{(B.5)}\\
\frac{\partial T}{\partial t} &= 2(e_2-e_1)T(T-1), \qquad \text{(B.6)}\\
\partial_T(e_2-e_1) &= \partial_t(e_2-e_1)\frac{1}{T_t} = -\frac{e_3+2\eta_1}{T(T-1)}. \qquad \text{(B.7)}
\end{aligned}$$

The following identity holds true[6]:

$$\frac{\partial X}{\partial t} = \frac{\partial X}{\partial x}\,\frac{\vartheta_0'(x)}{\vartheta_0(x)}$$

or

$$\partial_\tau X = \kappa\partial_z X\,(E_1(z+\omega_3)-E_1(\omega_3)) = \kappa\,\partial_z X\,\partial_z\log\theta_0(z). \qquad \text{(B.8)}$$

Appendix C: U−V pairs for PI–PV

Here we list the **U**−**V** pairs for PI–PV satisfying zero curvature equation (1.2) and admitting the quantum Painlevé–Calogero correspondence. The PVI case is too complicated. In principle, it is gauge equivalent to different types of known elliptic 2×2 **U**−**V** pairs (see [32,60]) which are in their turn related by Hecke transformations [33,34].

Painlevé I

$$\begin{aligned}
4\ddot{u} &= 6u^2+t,\\
H_{\mathrm{I}}(p,u) &= \frac{p^2}{2}-\frac{u^3}{2}-\frac{tu}{4},\\
\mathbf{U}(x,t) &= \begin{pmatrix} \dot{u} & x-u\\ x^2+xu+u^2+\frac{1}{2}t & -\dot{u}\end{pmatrix}, \qquad \mathbf{V}(x,t) = \begin{pmatrix} 0 & \frac{1}{2}\\ \frac{1}{2}x+u & 0\end{pmatrix}.
\end{aligned}$$

Painlevé II

$$\begin{aligned}
\ddot{u} &= 2u^3+tu-\alpha,\\
H_{\mathrm{II}}(p,u) &= \frac{p^2}{2}-\frac{1}{2}\left(u^2+\frac{t}{2}\right)^2+\alpha u,
\end{aligned}$$

[6] This formula was proved by K.Takasaki in [49] by comparison of analytic properties of both sides. In [58], the proof was given by a direct computation.

$$\mathbf{U}=\begin{pmatrix} x^2+\dot{u}-u^2 & x-u \\ (x+u)(2u^2-2\dot{u}+t)-2\alpha-1 & -x^2-\dot{u}+u^2 \end{pmatrix},$$
$$\mathbf{V}=\begin{pmatrix} \frac{x+u}{2} & \frac{1}{2} \\ u^2-\dot{u}+\frac{t}{2} & -\frac{x+u}{2} \end{pmatrix}.$$

Painlevé III

$$2\ddot{u}=e^t(\alpha e^{2u}+\beta e^{-2u})+e^{2t}(\gamma e^{4u}+\delta e^{-4u}),$$
$$H_{\mathrm{III}}(p,u)=\frac{p^2}{2}-\nu^2e^t\cosh(2u-2\varrho)-\mu^2e^{2t}\cosh(4u),$$
$$\mathbf{U}_{11}=\dot{u}e^{2u-2x}+\theta\left(1-e^{2u-2x}\right)+\frac{1}{2}\left(e^{2x+t}-e^{2u-2x}-e^{4u+t-2x}+1\right),$$
$$\mathbf{U}_{12}=e^{\frac{t}{2}}\left(e^{-u+x}-e^{u-x}\right),$$
$$\begin{aligned}\mathbf{U}_{21}&=\dot{u}^2e^{u-\frac{t}{2}-3x}\left(e^{2x}+e^{2u}\right)-\dot{u}e^{u-\frac{t}{2}-3x}\left(e^{2x}+e^{2u+t+2x}+(1+2\theta)e^{2u}+e^{4u+t}\right)\\ &+\theta^2\left(-e^{u-\frac{t}{2}-x}+e^{3u-\frac{t}{2}-3x}\right)+\theta\left(e^{3u-\frac{t}{2}-3x}+e^{5u+\frac{t}{2}-3x}\right)+4\lambda e^{-u+\frac{t}{2}-x}\\ &-4\chi\left(e^{-3u+\frac{3t}{2}-x}+e^{-u+\frac{3t}{2}-3x}\right)\\ &+\frac{1}{4}\left(e^{u-\frac{t}{2}-x}+2e^{3u+\frac{t}{2}-x}+e^{5u+\frac{3t}{2}-x}+e^{3u-\frac{t}{2}-3x}+2e^{5u+\frac{t}{2}-3x}+e^{7u+\frac{3t}{2}-3x}\right).\end{aligned}$$
$$\mathbf{V}_{11}=-\frac{1}{2}\dot{u}\left(e^{2u-2x}+1\right)+\frac{\theta}{2}\left(1+e^{2u-2x}\right)+\frac{1}{4}\left(e^{2x+t}+e^{2u-2x}+e^{4u+t-2x}+1+2e^{2u+t}\right),$$
$$\mathbf{V}_{12}=\frac{1}{2}e^{\frac{t}{2}}\left(e^{-u+x}+e^{u-x}\right),$$
$$\begin{aligned}\mathbf{V}_{21}&=\frac{1}{2}\dot{u}^2e^{u-\frac{t}{2}-3x}\left(e^{2x}-e^{2u}\right)-\frac{1}{2}\dot{u}e^{u-\frac{t}{2}-3x}\left(e^{2x}+e^{2u+t+2x}-(1+2\theta)e^{2u}-e^{4u+t}\right)\\ &-\frac{\theta^2}{2}\left(e^{u-\frac{t}{2}-x}+e^{3u-\frac{t}{2}-3x}\right)-\frac{\theta}{2}\left(e^{3u-\frac{t}{2}-3x}+e^{5u+\frac{t}{2}-3x}\right)+2\lambda e^{-u+\frac{t}{2}-x}\\ &-2\chi\left(e^{-3u+\frac{3t}{2}-x}-e^{-u+\frac{3t}{2}-3x}\right)\\ &+\frac{1}{8}\left(e^{u-\frac{t}{2}-x}+2e^{3u+\frac{t}{2}-x}+e^{5u+\frac{3t}{2}-x}-e^{3u-\frac{t}{2}-3x}-2e^{5u+\frac{t}{2}-3x}-e^{7u+\frac{3t}{2}-3x}\right).\end{aligned}$$

Notice that an interesting equation holds:

$$\partial_x\left(\mathbf{U}_{21}e^{2x}\right)=2\left(\mathbf{V}_{21}e^{2x}\right)$$

(in this case $X=e^{2x}$). Therefore, some relation exists between $\mathbf{U}_{21}$ and $\mathbf{V}_{21}$ elements just as for (12)-elements. For example, for PII we have $\partial_x\mathbf{U}_{21}=2\mathbf{V}_{21}$.

Truncated Painlevé III [2]: $\ddot{u}=2\nu^2e^t\sinh(2u)$

$$\mathbf{U}(x,t)=\begin{pmatrix} \dot{u} & 2\nu e^{t/2}\sinh(x-u) \\ 2\nu e^{t/2}\sinh(x+u) & -\dot{u} \end{pmatrix},$$
$$\mathbf{V}(x,t)=\begin{pmatrix} 0 & \nu e^{t/2}\cosh(x-u) \\ \nu e^{t/2}\cosh(x+u) & 0 \end{pmatrix}.$$

Painlevé IV

$$\ddot{u} = \frac{3}{4}u^5 + 2tu^3 + (t^2 - \alpha)u + \frac{\beta}{2u^3},$$

$$H_{\mathrm{IV}}^{(\alpha,\beta)}(p,u) = \frac{p^2}{2} - \frac{u^6}{8} - \frac{tu^4}{2} - \frac{1}{2}\left(t^2 - \alpha\right)u^2 + \frac{\beta}{4u^2}.$$

$$\mathbf{U} = \begin{pmatrix} \frac{x^3}{2} + tx + \frac{Q + \frac{1}{2}}{x} & x^2 - u^2 \\ \frac{Q^2 + \frac{\beta}{2}}{u^2 x^2} - Q - \alpha - 1 & -\frac{x^3}{2} - tx - \frac{Q + \frac{1}{2}}{x} \end{pmatrix},$$

$$\mathbf{V} = \begin{pmatrix} \frac{x^2 + u^2}{2} + t & x \\ -\frac{Q + \alpha + 1}{x} & -\frac{x^2 + u^2}{2} - t \end{pmatrix},$$

where

$$Q = u\dot{u} - \frac{u^4}{2} - tu^2.$$

Painlevé V

$$\ddot{u} = -\frac{2\alpha\cosh u}{\sinh^3 u} - \frac{2\beta\sinh u}{\cosh^3 u} - \gamma e^{2t}\sinh(2u) - \frac{1}{2}\delta e^{4t}\sinh(4u),$$

$$H_{\mathrm{V}}(p,u) = \frac{p^2}{2} - \frac{\alpha}{\sinh^2 x} - \frac{\beta}{\cosh^2 x} + \frac{\gamma e^{2t}}{2}\cosh(2x) + \frac{\delta e^{4t}}{8}\cosh(4x),$$

$$\mathbf{U}_{11} = \dot{u}\frac{\sinh(2u)}{\sinh(2x)} - \frac{2\sigma}{\sinh(2x)}\Big(\cosh(2x) - \cosh(2u)\Big) + \frac{e^{2t}}{4\sinh(2x)}\Big(\cosh(4x) - \cosh(4u)\Big) + \coth(2x). \tag{C.1}$$

$$\mathbf{U}_{12} = e^t\Big(\cosh(2x) - \cosh(2u)\Big). \tag{C.2}$$

$$\begin{aligned}\mathbf{U}_{21} = {} & \dot{u}^2\frac{e^{-t}}{\sinh^2(2x)}\Big(\cosh(2u) + \cosh(2x)\Big) \\ & + \dot{u}\frac{\sinh(2u)}{\sinh^2(2x)}\Big(4\sigma e^{-t} - e^t\Big[\cosh(2u) + \cosh(2x)\Big]\Big) \\ & + 8\sigma^2 e^{-t}\frac{\coth^2(u)}{\sinh^2(2x)}\Big(\sinh^2(u) - \cosh^2(x)\Big) - 2\sigma e^t\frac{\sinh^2(2u)}{\sinh^2(2x)} \\ & - 2e^{-t}\frac{\xi^2 + 2\xi\sigma}{\sinh^2(u)\sinh^2(x)} + 2e^{-t}\frac{\zeta^2}{\cosh^2(u)\cosh^2(x)} \\ & + \frac{e^{3t}\sinh^2(2u)}{4\sinh^2(2x)}\Big(\cosh(2u) + \cosh(2x)\Big).\end{aligned} \tag{C.3}$$

$$
\begin{aligned}
\mathbf{V}_{11} &= \frac{1}{2}e^{2t}\Big(\cosh(2x)+\cosh(2u)\Big) - 2\sigma + \frac{1}{2},\\
\mathbf{V}_{12} &= e^{t}\sinh(2x),\\
\mathbf{V}_{21} &= \frac{e^{-t}}{\sinh(2x)}\bigg(\Big(\dot{u}^2 - \frac{1}{2}\dot{u}e^{2t}\sinh(2u)\Big)^2 + \frac{4\zeta^2}{\cosh^2(u)}\\
&\quad - 4\frac{\xi^2+2\xi\sigma}{\sinh^2(u)} - 4\sigma^2\coth^2(u)\bigg).
\end{aligned} \tag{C.4}
$$

References

1. Alexandrov, A., Kazakov, V., Leurent, S., Tsuboi, Z., Zabrodin, A.: Classical tau-function for quantum spin chains. arXiv:1112.3310 [math-ph]
2. Aminov, G., Arthamonov, S.: New 2 × 2-matrix linear problems for Painlevé equations. arXiv:1112.4688 [nlin.SI]
3. Barouch, E., McCoy, B., Tracy, C., Wu, T.: Zero field susceptibility of the two-dimensional Ising model near T_c. Phys. Rev. Lett. **31**, 1409–1411 (1973)
4. Bazhanov, V., Mangazeev, V.: The eight-vertex model and Painlevé VI. J. Phys. A Math. Gen. **39**, 12235–12243 (2006)
5. Brézin, E., Kazakov, V.: Exactly solvable field theories of closed strings. Phys. Lett. **B236**, 144–150 (1990); Gross, D., Migdal, A.: Nonperturbative two-dimensional quantum gravity. Phys. Rev. Lett. **64**, 127–130 (1990); Douglas, M., Shenker, S.: Strings in less than one dimension. Nucl. Phys. **B335**, 635–654 (1990)
6. Calogero, F.: On a functional equation connected with integrable many-body problems. Lettere Al Nuovo Cimento **16**(3), 77–80 (1976); Buchstaber, V.M., Perelomov, A.M.: On the functional equation related to the quantum three-body problem, Contemporary mathematical physics. Am. Math. Soc. Transl. Ser. **2**(175), 15–34 (1996). arXiv:math-ph/0205032; Braden, H.W., Buchstaber, V.M.: Integrable systems with pairwise interactions and functional equations arXiv:hep-th/9411240
7. Conte, R. (ed.): The Painleve Property. One Century Later, CRM Series in Mathematical Physics, XXVI, 810 p (1999)
8. Dubrovin, B.: Geometry of 2D topological field theories, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math., vol. 1620, Springer, Berlin, pp. 120–348 (1996); Dubrovin, B.: Painlevé equations in 2D topological field theories. In: Painleve Property, One Century Later, Cargése (1996). arXiv:math.AG/9803107
9. Dubrovin, B., Mazzocco, M.: Monodromy of certain Painlevé-VI transcendents and reflection groups. Invent. Math. **141**(1), 55–147 (2000); Guzzetti, D.: The elliptic representation of the general Painlevé VI equation. Commun. Pure Appl. Math. **55**(10), 1280–1363 (2002); Boalch, P.: From Klein to Painlevé via Fourier, Laplace and Jimbo. Proc. Lond. Math. Soc. **90**(01), 167–208 (2005); Guzzetti, D.: A review on the sixth Painlevé equation. arXiv:1210.0311 [math.CA]
10. Fateev, V., Litvinov, I.: On AGT conjecture. JHEP **1002**, 014 (2010). arXiv:0912.0504
11. Flaschka, H., Newell, A.: Monodromy- and spectrum-preserving deformations, I. Commun. Math. Phys. **76**, 65–116 (1980)
12. Forrester, P., Witte, N.: Application of the τ-function theory of Painlevé equations to random matrices: PIV, PII and the GUE, Commun. Math. Phys. **219** 357–398 (2001); Forrester, P., Witte, N.: Random matrix theory and the sixth Painlevé equation. J. Phys. A Math. Gen. **39**, 12211–12233 (2006)
13. Fuchs, R.: Sur quelques équations différentielles lineáres du second ordre. C. R. Acad. Sci. (Paris) **141**, 555–558 (1905)
14. Gamayun, O., Iorgov, N., Lisovyy, O.: Conformal field theory of Painlevé VI. JHEP **2012**(10), 38 (2012). arXiv:1207.0787 [hep-th]
15. Gambier, B.: Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critique fixés. C. R. Acad. Sci. (Paris) **142**, 266–269 (1906)
16. Garnier, R.: Sur des equations différentielles du troisiéme ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critique fixés. Ann. Ecol. Norm. Sup. **29**, 1–126 (1912)
17. Ince, E.: Ordinary Differential Equations. Dover, New York (1956)

18. Inozemtsev, V.I.: Lax representation with spectral parameter on a torus for integrable particle systems. Lett. Math. Phys. **17**, 11–17 (1989)
19. Its, A., Novokshenov, V.: The isomonodromic deformation method in the theory of Painlevé equations, Lecture Notes in Math. **1191**. Springer, Berlin (1986); Fokas, A., Its, A., Kapaev, A., Novokshenov, V.: Painlevé transcendents: the Riemann–Hilbert approach. AMS Mathematical Surveys and Monographs, vol. 128, Providence, RI (2006)
20. Iwasaki, K., Kimura, H., Shimomura, S., Yoshida, M.: From Gauss to Painlevé, a Modern Theory of Special Funtions, Aspects of Mathematics, E16. Friedr. Vieweg & Sohn, Braunschweig (1991)
21. Jimbo, M., Miwa, T.: Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II. Phys. D **2**, 407–448 (1981)
22. Jimbo, M., Miwa, T.: Monodromy preserving deformation of linear ordinary differential equations with rational coefficients III. Phys. D **4**, 26–46 (1981)
23. Jimbo, M., Miwa, T., Mori, Y., Sato, M.: Density matrix of an impenetrable gas and the fifth Painlevé transcendent. Phys. D **1**, 80–158 (1980)
24. Jimbo, M., Nagoya, H., Sun, J.: Remarks on the confluent KZ equation for sl_2 and quantum Painlevé equations. J. Phys. A Math. Theory **41**, 175205 (2008); Nagoya, H.: Hypergeometric solutions to Schrödinger equations for the quantum Painlevé equations. J. Math. Phys. **52**, 083509 (2011)
25. Jimbo, M., Miwa, T., Ueno, K.: Monodromy preserving deformation of linear ordinary differential equations with rational coefficients I. General theory and τ-function. Phys. D **2**, 306–352 (1981)
26. Joshi, N., Kitaev, A., Treharne, P.: On the linearization of the Painlevé III–VI equations and reductions of the three-wave resonant system. J. Math. Phys. **48**, 103512 (2007). (42 pp.), arXiv:0706.1750
27. Kazakov, V., Sorin, A., Zabrodin, A.: Supersymmetric Bethe ansatz and Baxter equations from discrete Hirota dynamics. Nucl. Phys. **B790**, 345–413 (2008); Zabrodin, A.: Bäcklund transformations for difference Hirota equation and supersymmetric Bethe ansatz, Teor. Mat. Fyz. **155**, 74–93 [English translation: Theor. Math. Phys. **155**, 567–584 (2008)]
28. Krichever, I.: Elliptic solutions of the Kadomtsev-Petviashvili equation and integrable systems of particles. Funk. Anal. i ego Pril. **14**(4), 45–54 (1980). [translation: Funct. Anal. Appl. **14**(4), 282–290 (1980)]; Krichever, I.: On rational solutions of the Kadomtsev–Petviashvili equation and on integrable systems of N particles on the line. Funct. Anal. Appl. **12**(1), 76–78 (1978); Krichever, I., Zabrodin, A.: Spin generalization of the Ruijsenaars–Schneider model, non-abelian 2D Toda chain and representations of Sklyanin algebra. Uspekhi Mat. Nauk **50**(6), 3–56 (1995). [translation: Russ. Math. Surv. **50**(6), 1101–1150 (1995)]
29. Krichever, I., Lipan, O., Wiegmann, P., Zabrodin, A.: Quantum integrable systems and elliptic solutions of classical discrete nonlinear equations. Commun. Math. Phys. **188**, 267–304 (1997)
30. Lee, S.-Y., Teodorescu, R., Wiegmann, P.: Viscous shocks in Hele–Shaw flow and Stokes phenomena of the Painleve I transcendent. Phys. D **240**, 1080–1091 (2011)
31. Levin, A., Olshanetsky, M.: Painlevé–Calogero correspondence, Calogero–Moser–Sutherlend models (Montreal, 1997), CRM Ser. Math. Phys., Springer, pp. 313–332 (2000). arXiv: alg-geom/9706010
32. Levin, A., Olshanetsky, M., Zotov, A.: Painlevé VI, Rigid tops and reflection equation. Commun. Math. Phys. **268**(1), 67–103 (2006). arXiv:math/0508058 [math.QA]; Chernyakov, Y., Levin, A., Olshanetsky, M., Zotov, A.: Elliptic Schlesinger system and Painlevé VI. J. Phys. A Math. Gen. **39**, 12083–12101 (2006). arXiv:nlin/0602043 [nlin.SI]; Levin, A., Zotov, A.: On rational and elliptic forms of Painlevé VI equation, Moscow Seminar on Mathematical Physics, II. Am. Math. Soc. Transl. Ser. 2 **221**, 173–184 (2007)
33. Levin, A., Olshanetsky, M., Zotov, A.: Hitchin systems—symplectic hecke correspondence and two-dimensional version. Commun. Math. Phys. **236**(1) 93–133 (2003). arXiv:nlin/0110045 [nlin.SI]; Levin, A.M., Olshanetsky, M.A., Smirnov, A.V., Zotov, A.V.: Characteristic classes and hitchin systems. General construction. Commun. Math. Phys. **316**(1), 1–44 (2012). arXiv:1006.0702 [math-ph]
34. Levin, A.M., Olshanetsky, M.A., Smirnov, A.V., Zotov, A.V.: Calogero–Moser systems for simple Lie groups and characteristic classes of bundles. J. Geometry Phys. **62**, 1810–1850 (2012). arXiv:1007.4127 [math-ph]; Levin, A.M., Olshanetsky, M.A., Smirnov, A.V., Zotov, A.V.: Hecke transformations of conformal blocks in WZW theory. I. KZB equations for non-trivial bundles. SIGMA **8**, 095, 37 pp., (2012). arXiv:1207.4386 [math-ph]; Levin, A.M., Olshanetsky, M.A., Smirnov, A.V., Zotov, A.V.: Characteristic classes of SL(N)-bundles and quantum dynamical elliptic R-matrices. J. Phys. A Math. Theor. to appear. arXiv:1208.5750 [math-ph]
35. Levin, A., Olshanetsky, M., Zotov, A.: Planck constant as spectral parameter in integrable systems and KZB equations. JHEP **10**, 109 (2014). arXiv:1408.6246 [hep-th]

36. Manin, Y.: Sixth Painlevé equation, universal elliptic curve, and mirror of $\mathbb{P}^2$. AMS Transl. (2) **186**, 131–151 (1998)
37. Marshakov, A., Mironov, A., Morozov, A.: On AGT relations with surface operator insertion and stationary limit of beta-ensembles. J. Geom. Phys. **61**, 1203–1222 (2011)
38. Mironov, A., Morozov, A., Runov, B., Zenkevich, Y., Zotov, A.: Spectral duality between Heisenberg chain and Gaudin model. Lett. Math. Phys. (2012), to appear, doi:10.1007/s11005-012-0595-0; arXiv:1206.6349 [hep-th]
39. Mironov, A., Morozov, A., Shakirov, S.: Towards a proof of AGT conjecture by methods of matrix models. Int. J. Mod. Phys. A **27**, 1230001 (2012). arXiv:1011.5629
40. Mironov, A., Morozov, A., Zenkevich, Y., Zotov, A.: Spectral duality in integrable systems from AGT conjecture. JETP Lett. **97**(1) (2013). arXiv:1204.0913 [hep-th]
41. Noumi, M., Yamada, Y.: A New Lax Pair for the Sixth Painlevé Equation Associated with $\widehat{so}(8)$, Microlocal Analysis and Complex Fourier Analysis, 238252. World Sci. Publ, River Edge, NJ (2002)
42. Novikov, D.P.: The 2×2 matrix Schlesinger system and the Belavin–Polyakov–Zamolodchikov system. Theor. Math. Phys. **161**(2), 1485–1496 (2009)
43. Painlevé, P.: Memoire sur les équations différentielles dont l'intégrale générale est uniforme. Bull. Soc. Math. Phys. Fr. **28**, 201–261; Painlevé, P.: Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme. Acta Math. **21**(1902), 1–85 (1900)
44. Schlesinger, L.: Über eine Klasse von Differentialsystemen beliebiger Ordnung mit feten kritischen Punkten. J. Reine u. Angew. Math. **141**, 96–145 (1912)
45. Slavyanov, S.: Painlevé equations as classical analogs of Heun equations. J. Phys. A Math. Gen. **29**, 7329–7335 (1996); Slavyanov, S.: Painlevé equations and isomonodromic deformations of equations of the Heun class. Theor. Math. Phys. **125**(1), 744–753 (2000)
46. Suleimanov, B.: The Hamiltonian property of Painlevé equations and the method of isomonodromic deformations. Differ. Equ. **30**(5), 726–732 (1994). [Translated from Differentsialnie Uravneniya **30**(5), 791–796 (1994)]
47. Suleimanov, B.: "Quantizations" of the second Painlevé equation and the problem of the equivalence of its L-A pairs. Theor. Math. Phys. **156**, 1280–1291 (2008). [Translated from Teor. Mat. Fys. **156**, 364–377 (2008)]
48. Suleimanov, B.: "Quantization" of higher hamiltonian analogues of the Painleve I and Painleve II equations with two degrees of freedom. arXiv:1204.4006 [nlin.SI]
49. Takasaki, K.: Painlevé–Calogero correspondence revisited. J. Math. Phys. **42**, 1443–1473 (2001)
50. Takasaki, K.: Elliptic Calogero–Moser systems and isomonodromic deformations. J. Math. Phys. **40**, 57–87 (1999)
51. Tracy, C., Widom, H.: Fredholm determinants, differential equations and matrix models. Commun. Math. Phys. **163**, 33–72 (1994)
52. Tsuda, T.: Universal characters, integrable chains and the Painlevé equations. Adv. Math. **197**(2), 587–606 (2005)
53. Painlevé, P.: Sur les équations différentielles du second ordre à points critiques fixés. C. R. Acad. Sci. (Paris) **143**, 1111–1117 (1906)
54. Zabrodin, A.: The master T-operator for vertex models with trigonometric R-matrices as classical tau-function. arXiv:1205.4152 [math-ph]
55. Zabrodin, A.: Bethe ansatz and Hirota equation in integrable models. arXiv:1211.4428 [math-ph]
56. Zabrodin, A.: Discrete Hirotas equation in quantum integrable models. Int. J. Mod. Phys. **B11**, 3125–3158 (1997); Zabrodin, A.: Hirota equation and Bethe ansatz. Teor. Mat. Fyz. **116**, 54–100 (1998). [English translation: Theor. Math. Phys. **116**, 782–819 (1998)]
57. Zabrodin, A., Zotov, A.: Quantum Painlevé–Calogero correspondence. J. Math. Phys. **53**, 073507 (2012). arXiv:1107.5672 [math-ph]
58. Zabrodin, A., Zotov, A.: Quantum Painlevé–Calogero correspondence for Painlevé VI. J. Math. Phys. **53**, 073508 (2012). arXiv:1107.5672 [math-ph]
59. Zamolodchikov, Al: Painlevé III and 2D polymers. Nucl. Phys. **B432**, 427–456 (1994)
60. Zotov, A.: Elliptic linear problem for Calogero–Inozemtsev model and Painlevé VI equation. Lett. Math. Phys. **67**, 153–165 (2004). arXiv:hep-th/0310260