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CONSTRUCTIVE APPROXIMATION

New 2 × 2-Matrix Linear Problems for the Painlevé Equations III, V

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Abstract We construct 2 × 2-matrix linear problems with a spectral parameter for the Painlevé equations I–V by means of the degeneration processes from the elliptic linear problem for the Painlevé VI equation. These processes supplement the known degeneration relations between the Painlevé equations with the degeneration scheme for the associated linear problems. The degeneration relations constructed in this paper are based on the trigonometric, rational, and Inozemtsev limits. The obtained 2 × 2-matrix linear problems for the Painlevé equations III and V are new.

Keywords Integrable systems · Painlevé equations · Isomonodromic deformations · Painlevé–Calogero correspondence · Inozemtsev limit

Mathematics Subject Classification 34M55 · 34M56 · 70H06

1 Introduction

We study the Painlevé equations and the associated 2 × 2-matrix linear problems. The Painlevé equations are six nonlinear ordinary second-order differential equations

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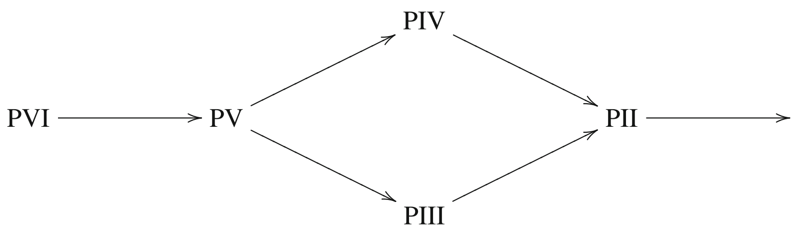
discovered by Fuchs [9], Gambier [10], and Painlevé [29,30], at the beginning of the twentieth century. The approach to the Painlevé equations from the point of view of the monodromy preserving deformations of linear ordinary differential equations was established by Fuchs in the work [9] and generalized in the works by Schlesinger [33] and Garnier [11,12]. After a long break, this approach was further developed in the works [7,16–18,20], see also the books [8,14,32].

Another important approach to the Painlevé equations was established in the work [23]—the Hamiltonian approach. It turns out that each Painlevé equation is equivalent to the equations of motion of some nonautonomous Hamiltonian system. Such Hamiltonian systems were first introduced by Okamoto in the works [26–28]. The next step in the development of the Hamiltonian approach was the representation of the Painlevé equations as nonautonomous Hamiltonian systems describing the motion of a particle in a nonstationary potential. The Hamiltonian of this type was constructed by Manin [24]. Soon after, Levin and Olshanetsky discovered [21] that the Lax pair of the elliptic Calogero system forms the linear problem for the equation of isomonodromic deformations on a torus and, particularly, for the Painlevé VI equation with specific choice of arbitrary constants. This connection between the Painlevé VI equation and the integrable model of the Calogero type [6] was called the Painlevé–Calogero correspondence. Later, Takasaki [34] derived Hamiltonians for the Painlevé equations I–V from the Manin's Hamiltonian using degeneration relations (1.1) between the Painlevé equations [28]. It is worth noting that the similar diagram of degeneration (1.1) was known (without any connection to the Painlevé equations) for the autonomous Inozemtsev systems [35].



(1.1)

Thus, Takasaki extended the Painlevé–Calogero correspondence to the whole set of the Painlevé equations. Recently, this result was further developed in [37,38], where a "quantized" version of the Painlevé–Calogero correspondence was suggested.

The goal of this paper is to construct 2×2-matrix linear problems with a spectral parameter for the Painlevé equations I–V by means of degeneration processes. In papers [1,2,4], we proposed limit relations between the elliptic $\mathrm{SL}(N, \mathbb{C})$ top and Toda systems in the autonomous case for $N \geqslant 2$ and in the nonautonomous case for $N = 2$. These relations are based on the Inozemtsev limit [13] and allow one to obtain the Lax pair of a Toda system from the Lax pair of the elliptic $\mathrm{SL}(N, \mathbb{C})$ top. It is known that there is a connection between the systems discussed above and the Painlevé equations. The equations of motion of a nonautonomous elliptic $\mathrm{SL}(2, \mathbb{C})$ top are equivalent to a particular case of the Painlevé VI equation [22], and the equations of motion of nonautonomous Toda systems are equivalent to the Painlevé III equation with a definite choice of arbitrary constants. Also, in [1], it was shown that the linear problem for the particular case of the Painlevé III equation can be obtained from the particular case of the Painlevé VI equation. Thus, we can apply the procedure from [1] to the linear problem for the general case of the Painlevé VI equation in the elliptic form [5,24,31]. Zotov constructed the 2×2 Lax pair with spectral parameter z for the Calogero–Inozemtsev system with one degree of freedom [39]. This Lax pair also provides the linear problem for the Painlevé VI equation in the elliptic form. The Calogero–Inozemtsev system considered in [39] is described by the Hamiltonian on an elliptic curve $\langle 1, \tau \rangle$, where the parameter τ stands for the time in the nonautonomous version of this system. So, using the linear problem for the Painlevé VI equation from [39], we obtain linear problems for other Painlevé equations by means of the degeneration processes.

In Sects. 2 and 4, new linear problems for the Painlevé equations V and III are constructed. They are obtained as limits of the linear problem for the Painlevé VI equation. The common component of these limits is the following decomposition of the parameter τ of an elliptic curve:

$$\tau = \tau_1 + \tau_2,$$

where τ_1 stands for the time in the limiting system and τ_2 gives the trigonometric limit $\operatorname{Im} \tau_2 \to +\infty$. In other words, we introduce the infinite shift of the parameter of an elliptic curve. Thus, we have the linear problems for the Painlevé equations V and III defined on an infinite complex cylinder $\mathbb{C}/\mathbb{Z}$. The difference between the limits from Sects. 2 and 4 is due to the infinite shifts of the Calogero–Inozemtsev system coordinate u and spectral parameter z. The limits also differ in the scalings of constants of the linear problem for the Painlevé VI equation [39].

In Sect. 3, we construct a linear problem for the Painlevé IV equation using the result of Sect. 2. In Sects. 5 and 6, using the linear problem from Sect. 4, we obtain linear problems for the Painlevé equations II and I, respectively. The limiting procedures used in Sects. 3, 5, and 6 are based on the transformation of an infinite complex cylinder $\mathbb{C}/\mathbb{Z}$ to a complex plane $\mathbb{C}$. Thus, the degeneration relations between the linear problems obtained in this paper can be described by the following diagram:

$$\begin{array}{ccccc} & \nearrow & \mathrm{PV}(2.7, 2.15) & \longrightarrow & \mathrm{PIV}(3.3) \\ \mathrm{PVI} & \longrightarrow & \mathrm{PIII}(4.9, 4.17) & \longrightarrow & \mathrm{PII}(5.3) \\ & & & \searrow & \mathrm{PI}(6.3) \end{array} \tag{1.2}$$

The diagram (1.2) differs from (1.1) due to the fact that the known degeneration procedures [15] for the Painlevé equations themselves cannot be directly applied to the corresponding linear problems.

1.1 Painlevé Equations

We will now review general facts and notation about the equations under consideration. The six Painlevé equations [9,10,29,30] in the rational form [15] are

$$\begin{aligned}
\text{PVI:}\quad \frac{d^2\lambda}{dt^2} &= \frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}\\
&\quad+\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[\alpha-\beta\frac{t}{\lambda^2}+\gamma\frac{t-1}{(\lambda-1)^2}+\left(\frac{1}{2}-\delta\right)\frac{t(t-1)}{(\lambda-t)^2}\right],\\
\text{PV:}\quad \frac{d^2\lambda}{dt^2} &= \left(\frac{1}{2\lambda}+\frac{1}{\lambda-1}\right)\left(\frac{d\lambda}{dt}\right)^2-\frac{1}{t}\frac{d\lambda}{dt}+\frac{(\lambda-1)^2}{t^2}\left(\alpha\lambda+\frac{\beta}{\lambda}\right)+\gamma\frac{\lambda}{t}\\
&\quad+\delta\frac{\lambda(\lambda+1)}{(\lambda-1)},\\
\text{PIV:}\quad \frac{d^2\lambda}{dt^2} &= \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2+\frac{3}{2}\lambda^3+4t\lambda^2+2\left(t^2-\alpha\right)\lambda+\frac{\beta}{\lambda},\\
\text{PIII:}\quad \frac{d^2\lambda}{dt^2} &= \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2-\frac{1}{t}\frac{d\lambda}{dt}+\frac{1}{t}\left(\alpha\lambda^2+\beta\right)+\gamma\lambda^3+\frac{\delta}{\lambda},\\
\text{PII:}\quad \frac{d^2\lambda}{dt^2} &= 2\lambda^3+t\lambda+\alpha,\\
\text{PI:}\quad \frac{d^2\lambda}{dt^2} &= 6\lambda^2+t,
\end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary complex constants. The Painlevé equations I–V can be derived from the Painlevé VI equation by means of the degeneration processes (1.1) [28].

1.2 Elliptic Linear Problem for the Painlevé VI Equation

The sixth Painlevé equation in the elliptic form [5,24,31] is

$$\frac{d^2u}{d\tau^2} = -2\sum_{\alpha=0}^{3}\nu_\alpha^2 E_2'\left(u+\omega_\alpha,\tau\right),$$

where $\omega_\alpha=\left\{0,\frac{1}{2},\frac{\tau}{2},\frac{1+\tau}{2}\right\}$ and the second Eisenstein function $E_2(z)$ (9.1) is defined on a complex torus $\langle 1,\tau\rangle$ (see "Appendix 2"). In this form, the Painlevé VI equation is equivalent to the equation of motion of a nonautonomous Calogero–Inozemtsev system with one degree of freedom, which is described by the Hamiltonian

$$H^{\mathrm{VI}} = v^2-\sum_{\alpha=0}^{3}\nu_\alpha^2 E_2\left(u+\omega_\alpha\right). \tag{1.3}$$

We will also need the following equivalent form to calculate the limits in Sects. 3 and 5:

$$\widetilde{H}^{\mathrm{VI}} = v^2 - \sum_{\alpha=0}^{3} \nu_\alpha^2 \left(E_2\left(u+\omega_\alpha\right) - E_2\left(z+\omega_\alpha\right)\right). \tag{1.4}$$

The nonautonomous version of the Calogero–Inozemtsev system with one degree of freedom has the following 2×2 Lax representation constructed in [39]:

$$\partial_\tau L^{\mathrm{VI}} - \frac{1}{2\pi \mathrm{i}} \partial_z M^{\mathrm{VI}} = \left[L^{\mathrm{VI}}, M^{\mathrm{VI}}\right]. \tag{1.5}$$

The Lax pair L^{VI}, M^{VI} is of the form

$$L^{\mathrm{VI}} = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} + \sum_{\alpha=0}^{3} L_\alpha^{\mathrm{VI}}, \quad L_\alpha^{\mathrm{VI}} = \begin{pmatrix} 0 & \nu_\alpha \varphi_\alpha\left(u+\omega_\alpha, z\right) \\ \nu_\alpha \varphi_\alpha\left(-u+\omega_\alpha, z\right) & 0 \end{pmatrix}, \tag{1.6a}$$

$$M^{\mathrm{VI}} = \sum_{\alpha=0}^{3} M_\alpha^{\mathrm{V}I}, \quad M_\alpha^{\mathrm{V}I} = \begin{pmatrix} 0 & \nu_\alpha f_\alpha\left(u+\omega_\alpha, z\right) \\ \nu_\alpha f_\alpha\left(-u+\omega_\alpha, z\right) & 0 \end{pmatrix}, \tag{1.6b}$$

where functions φ_α, f_α [19] are defined in the following way (see "Appendix 2"):

$$\begin{aligned} \varphi_\alpha\left(u+\omega_\beta, z\right) &= \mathbf{e}\left(z \partial_\tau \omega_\alpha\right) \phi\left(u+\omega_\beta, z\right), \\ f_\alpha\left(u+\omega_\beta, z\right) &= \mathbf{e}\left(z \partial_\tau \omega_\alpha\right) \partial_w \phi\left(w, z\right)|_{w=u+\omega_\beta}, \\ \phi(u, z) &= \frac{\theta_{11}(u+z)\theta_{11}'(0)}{\theta_{11}(u)\theta_{11}(z)}. \end{aligned}$$

2 Linear Problem for the Painlevé V Equation

In order to obtain a relation between linear problems for the Painlevé equations VI and V, we consider two different degeneration procedures. These procedures give linear problems for the following ordinary differential equations:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = C_0^2 \frac{\cos u}{\sin^3 u} + C_1^2 \frac{\sin u}{\cos^3 u} + C_2^2 \mathrm{e}^{2t} \sin(4u) + C_2 C_3 \mathrm{e}^t \sin(2u), \tag{2.1}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = C_0^2 \frac{\cos u}{\sin^3 u} + C_1^2 \frac{\sin u}{\cos^3 u} + C_4^2 \mathrm{e}^t \sin(2u), \tag{2.2}$$

which are the particular cases of the Painlevé V equation (8.2). Equations (2.1) and (2.2) describe the Painlevé V equation with any choice of arbitrary constants. Even though these equations are connected by the following limit:

$$C_2 \to 0, \qquad C_3 \to \infty, \qquad C_2 C_3 \to \mathrm{const} = C_4^2, \tag{2.3}$$

the Lax pair of the linear problem for Eq. (2.1) (obtained in Sect. 2.1) diverges upon taking (2.3). Thus, we construct linear problems for Eqs. (2.1) and (2.2) separately.

In both degeneration procedures, we use the following decomposition of the parameter τ of an elliptic curve:

$$\tau = \tau_1 + \tau_2, \tag{2.4}$$

where τ_1 stands for the time in the limiting system and τ_2 gives the trigonometric limit $\operatorname{Im}\tau_2 \to +\infty$. In other words, we introduce the infinite shift of the parameter of an elliptic curve. Thus, we have the linear problem for the Painlevé V equation defined on an infinite complex cylinder $\mathbb{C}/\mathbb{Z}$. The difference between the degeneration procedures is due to the infinite shifts of coordinate u and spectral parameter z. The degeneration procedures also differ in the scalings of constants of the linear problem for the Painlevé VI equation.

We will start with the degeneration procedure giving the linear problem for Eq. (2.1).

2.1 Linear Problem for Equation (2.1)

We decompose the parameter τ of an elliptic curve as it was described earlier, $\tau = \tau_1 + \tau_2$, which implies

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \frac{\mathrm{d}u}{\mathrm{d}\tau_1}.$$

The scalings of coupling constants are defined by the limiting behavior of Lax matrices (1.6a), (1.6b) as follows:

$$\nu_0 = \frac{\widetilde{\nu}_0}{\pi}, \qquad \nu_1 = \frac{\widetilde{\nu}_1}{\pi}, \qquad \nu_2 = \frac{-\widetilde{\nu}_2 q_2^{-\frac{1}{2}} + \widetilde{\nu}_3}{2\pi}, \qquad \nu_3 = \frac{\widetilde{\nu}_2 q_2^{-\frac{1}{2}} + \widetilde{\nu}_3}{2\pi},$$

where $q_2 \equiv \mathbf{e}(\tau_2)$. Thus, we obtain the limiting Hamiltonian and the linear problem of the following form:

$$H^{\mathrm{V}} = v^2 - \frac{\widetilde{\nu}_0^2}{\sin^2(\pi u)} - \frac{\widetilde{\nu}_1^2}{\cos^2(\pi u)} - 8 q_1^{\frac{1}{2}} \widetilde{\nu}_2 \widetilde{\nu}_3 \cos(2\pi u) + 8 q_1 \widetilde{\nu}_2^2 \cos(4\pi u), \tag{2.5}$$

$$\partial_{\tau_1} L^{\mathrm{V}} - \frac{1}{2\pi \mathrm{i}} \partial_z M^{\mathrm{V}} = \left[L^{\mathrm{V}}, M^{\mathrm{V}}\right] = \left\{H^{\mathrm{V}}, L^{\mathrm{V}}\right\}, \tag{2.6}$$

where

$$H^{\mathrm{V}} = \lim_{\operatorname{Im}\tau_2 \to +\infty} H^{\mathrm{VI}}, \qquad q_1 \equiv \mathbf{e}(\tau_1),$$

$$L^{\mathrm{V}} = \lim_{\operatorname{Im}\tau_2 \to +\infty} L^{\mathrm{VI}} = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} + \sum_{\alpha=0}^{3} \widetilde{\nu}_\alpha L_\alpha^{\mathrm{V}},$$

$$L_0^{\mathrm{V}} = \begin{pmatrix} 0 & \cot(\pi u) + \cot(\pi z) \\ -\cot(\pi u) + \cot(\pi z) & 0 \end{pmatrix},$$

$$L_1^{\mathrm{V}} = \begin{pmatrix} 0 & \cot(\pi z) - \tan(\pi u) \\ \cot(\pi z) + \tan(\pi u) & 0 \end{pmatrix},$$

$$L_2^{\mathrm{V}} = 4q_1^{\frac{1}{2}} \begin{pmatrix} 0 & -\sin(\pi(2u+z)) \\ \sin(\pi(2u-z)) & 0 \end{pmatrix}, \qquad L_3^{\mathrm{V}} = \frac{1}{\sin(\pi z)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.7a}$$

$$M^{\mathrm{V}} = \lim_{\operatorname{Im}\tau_2 \to +\infty} M^{\mathrm{VI}} = \sum_{\alpha=0}^{3} \widetilde{\nu}_\alpha M_\alpha^{\mathrm{V}},$$

$$M_0^{\mathrm{V}} = -\frac{\pi}{\sin^2(\pi u)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_1^{\mathrm{V}} = -\frac{\pi}{\cos^2(\pi u)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$M_2^{\mathrm{V}} = -8\pi q_1^{\frac{1}{2}} \begin{pmatrix} 0 & \cos(\pi(2u+z)) \\ \cos(\pi(2u-z)) & 0 \end{pmatrix}, \quad M_3^{\mathrm{V}} = 0. \tag{2.7b}$$

We will also need an equivalent form of the Hamiltonian to calculate the limit in Sect. 3. This form can be derived from expression (1.4):

$$\begin{aligned} \widetilde{H}^{\mathrm{V}} = \lim_{\operatorname{Im}\tau_2 \to +\infty} \widetilde{H}^{\mathrm{VI}} &= v^2 - \widetilde{\nu}_0^2 \left(\frac{1}{\sin^2(\pi u)} - \frac{1}{\sin^2(\pi z)} \right) \\ &- \widetilde{\nu}_1^2 \left(\frac{1}{\cos^2(\pi u)} - \frac{1}{\cos^2(\pi z)} \right) - 8q_1^{\frac{1}{2}} \widetilde{\nu}_2 \widetilde{\nu}_3 \left(\cos(2\pi u) - \cos(2\pi z)\right) \\ &+ 8q_1 \widetilde{\nu}_2^2 \left(\cos(4\pi u) - \cos(4\pi z)\right). \end{aligned} \tag{2.8}$$

Limiting Hamiltonian (2.5) coincides with the one known for the fifth Painlevé equation [34]. It is useful to note that (2.6) is equivalent to (2.1), which is a particular case of the Painlevé V equation. Indeed, we rewrite (2.6) as a system of two first-order differential equations

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}\tau_1} = 2v, \\ \dfrac{\mathrm{d}v}{\mathrm{d}\tau_1} = -2\pi \widetilde{\nu}_0^2 \dfrac{\cos(\pi u)}{\sin^3(\pi u)} + 2\pi \widetilde{\nu}_1^2 \dfrac{\sin(\pi u)}{\cos^3(\pi u)} - 16\pi q_1^{\frac{1}{2}} \widetilde{\nu}_2 \widetilde{\nu}_3 \sin(2\pi u) \\ \qquad + 32\pi q_1 \widetilde{\nu}_2^2 \sin(4\pi u), \end{cases}$$

which gives

$$\begin{aligned} \frac{\mathrm{d}^2 u}{\mathrm{d}\tau_1^2} = &- 4\pi \widetilde{\nu}_0^2 \frac{\cos(\pi u)}{\sin^3(\pi u)} + 4\pi \widetilde{\nu}_1^2 \frac{\sin(\pi u)}{\cos^3(\pi u)} - 32\pi q_1^{\frac{1}{2}} \widetilde{\nu}_2 \widetilde{\nu}_3 \sin(2\pi u) \\ &+ 64\pi q_1 \widetilde{\nu}_2^2 \sin(4\pi u). \end{aligned}$$

2.2 Linear Problem for Equation (2.2)

In this subsection, besides decomposition (2.4) of the parameter τ, the following shifts are used:

$$u = \widetilde{u} - \frac{\tau}{2}, \qquad z = \widetilde{z} - \frac{\tau}{2}. \tag{2.9}$$

The scalings of coupling constants are defined by the limiting behavior of Hamiltonian (1.3) as follows:

$$\nu_0 = \frac{\mathrm{i}\widetilde{\nu}_2 q_2^{-1/4}}{\sqrt{2}\pi}, \qquad \nu_1 = \frac{\widetilde{\nu}_3 q_2^{-1/4}}{\sqrt{2}\pi}, \qquad \nu_2 = \frac{\widetilde{\nu}_0}{\pi}, \qquad \nu_3 = \frac{\widetilde{\nu}_1}{\pi}.$$

Since τ_1 stands for the time in the limiting system, the shifts (2.9) are time-dependent, hence,

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \frac{\mathrm{d}u}{\mathrm{d}\tau_1} = \frac{\mathrm{d}\widetilde{u}}{\mathrm{d}\tau_1} - \frac{1}{2},$$

and the Hamiltonian defining the equations of motion of the limiting system is given by

$$H^{\mathrm{V}} = \lim_{\mathrm{Im}\,\tau_2 \to +\infty} H^{\mathrm{VI}} + \frac{1}{2}v + \frac{1}{16}.$$

Using the Hamiltonian (1.3) of the Painlevé VI equation, we get

$$H^{\mathrm{V}} = \left(v + \frac{1}{4}\right)^2 - \frac{\widetilde{\nu}_0^2}{\sin^2(\pi\widetilde{u})} - \frac{\widetilde{\nu}_1^2}{\cos^2(\pi\widetilde{u})} - 4\left(\widetilde{\nu}_2^2 + \widetilde{\nu}_3^2\right) q_1^{1/2} \cos(2\pi\widetilde{u}). \tag{2.10}$$

In order to obtain convergent Lax matrices, it is necessary to perform the gauge transformation of the form

$$L^{\mathrm{VI}} \to gL^{\mathrm{VI}}g^{-1}, \qquad M^{\mathrm{VI}} \to gM^{\mathrm{VI}}g^{-1},$$
$$g = \begin{pmatrix} 1 & 0 \\ 0 & q_2^{-1/4} \end{pmatrix}.$$

Since the shift of the spectral parameter in the degeneration procedure under consideration is time-dependent, Eq. (1.5) turns into

$$\partial_{\tau_1} L^{\mathrm{VI}} - \partial_{\widetilde{z}} \left(\frac{1}{2\pi\mathrm{i}} M^{\mathrm{VI}} - \frac{1}{2} L^{\mathrm{VI}} \right) = \left[L^{\mathrm{VI}}, M^{\mathrm{VI}} \right],$$

where $L^{\mathrm{VI}} = L^{\mathrm{VI}}(\widetilde{u} - \tau/2, v, \widetilde{z} - \tau/2, \tau)$, $M^{\mathrm{VI}} = M^{\mathrm{VI}}(\widetilde{u} - \tau/2, \widetilde{z} - \tau/2, \tau)$. Thus, the Lax pair of the linear problem for Eq. (2.2) is defined via

$$L^{\mathrm{V}} = 2\pi\mathrm{i} \lim_{\mathrm{Im}\tau_2 \to +\infty} gL^{\mathrm{VI}}g^{-1}, \qquad M^{\mathrm{V}} = \lim_{\mathrm{Im}\tau_2 \to +\infty} g\left(M^{\mathrm{VI}} - \pi\mathrm{i}L^{\mathrm{VI}}\right)g^{-1}.$$

Equation of zero curvature (1.5) takes the following form in the limit:

$$\partial_{\tau_1} L^{\mathrm{V}} - \partial_{\widetilde{z}} M^{\mathrm{V}} = \left[L^{\mathrm{V}}, M^{\mathrm{V}} \right], \tag{2.11}$$

where

$$
\begin{aligned}
L^{\mathrm{V}} &= 2\pi \mathrm{i} \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} + \sum_{\alpha=0}^{3} \widetilde{\nu}_\alpha L_\alpha^{\mathrm{V}}, \\
L_0^{\mathrm{V}} &= \frac{2\pi \mathrm{i}}{\sin(\pi \widetilde{u})} \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\pi(\widetilde{u}+\widetilde{z})} q_1^{-1/4} \\ -\mathrm{e}^{-\mathrm{i}\pi(\widetilde{u}+\widetilde{z})} q_1^{1/4} & 0 \end{pmatrix}, \\
L_1^{\mathrm{V}} &= \frac{2\pi \mathrm{i}}{\cos(\pi \widetilde{u})} \begin{pmatrix} 0 & \mathrm{i}\mathrm{e}^{\mathrm{i}\pi(\widetilde{u}+\widetilde{z})} q_1^{-1/4} \\ \mathrm{i}\mathrm{e}^{-\mathrm{i}\pi(\widetilde{u}+\widetilde{z})} q_1^{1/4} & 0 \end{pmatrix}, \\
L_2^{\mathrm{V}} &= 2\sqrt{2}\pi \mathrm{i} \begin{pmatrix} 0 & \mathrm{e}^{2\pi \mathrm{i}(\widetilde{u}+\widetilde{z})} - 1 \\ -2\mathrm{i}\mathrm{e}^{-\mathrm{i}\pi(\widetilde{u}+\widetilde{z})} q_1^{1/2} \sin(\pi(\widetilde{u}-\widetilde{z})) & 0 \end{pmatrix}, \\
L_3^{\mathrm{V}} &= -2\sqrt{2}\pi \begin{pmatrix} 0 & \mathrm{e}^{2\pi \mathrm{i}(\widetilde{u}+\widetilde{z})} + 1 \\ 2\mathrm{e}^{-\mathrm{i}\pi(\widetilde{u}+\widetilde{z})} q_1^{1/2} \cos(\pi(\widetilde{u}-\widetilde{z})) & 0 \end{pmatrix},
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
M^{\mathrm{V}} &= \mathrm{i}\pi \begin{pmatrix} -v & 0 \\ 0 & v \end{pmatrix} + \sum_{\alpha=0}^{3} \widetilde{\nu}_\alpha M_\alpha^{\mathrm{V}}, \\
M_0^{\mathrm{V}} &= -\pi \frac{\cos(\pi \widetilde{u})}{\sin^2(\pi \widetilde{u})} \begin{pmatrix} 0 & q_1^{-1/4} \mathrm{e}^{\mathrm{i}\pi(\widetilde{u}+\widetilde{z})} \\ q_1^{1/4} \mathrm{e}^{-\mathrm{i}\pi(\widetilde{u}+\widetilde{z})} & 0 \end{pmatrix}, \\
M_1^{\mathrm{V}} &= \mathrm{i}\pi \frac{\sin(\pi \widetilde{u})}{\cos^2(\pi \widetilde{u})} \begin{pmatrix} 0 & q_1^{-1/4} \mathrm{e}^{\mathrm{i}\pi(\widetilde{u}+\widetilde{z})} \\ -q_1^{1/4} \mathrm{e}^{-\mathrm{i}\pi(\widetilde{u}+\widetilde{z})} & 0 \end{pmatrix}, \\
M_2^{\mathrm{V}} &= \sqrt{2}\pi \mathrm{i} \begin{pmatrix} 0 & 1 + \mathrm{e}^{2\mathrm{i}\pi(\widetilde{u}+\widetilde{z})} \\ q_1^{1/2} \left(\mathrm{e}^{-2\mathrm{i}\pi \widetilde{z}} + \mathrm{e}^{-2\mathrm{i}\pi \widetilde{u}}\right) & 0 \end{pmatrix}, \\
M_3^{\mathrm{V}} &= \sqrt{2}\pi \begin{pmatrix} 0 & 1 - \mathrm{e}^{2\mathrm{i}\pi(\widetilde{u}+\widetilde{z})} \\ q_1^{1/2} \left(\mathrm{e}^{-2\mathrm{i}\pi \widetilde{z}} - \mathrm{e}^{-2\mathrm{i}\pi \widetilde{u}}\right) & 0 \end{pmatrix}.
\end{aligned} \tag{2.13}
$$

Lax pair (2.12), (2.13) can be simplified by means of the following gauge transformation:

$$
\begin{aligned}
&\widetilde{L}^{\mathrm{V}} = \widetilde{g} L^{\mathrm{V}} \widetilde{g}^{-1} - (\partial_{\widetilde{z}} \widetilde{g})\, \widetilde{g}^{-1}, \qquad \widetilde{M}^{\mathrm{V}} = \widetilde{g} M^{\mathrm{V}} \widetilde{g}^{-1} - \left(\partial_{\tau_1} \widetilde{g}\right) \widetilde{g}^{-1}, \\
&\widetilde{g} = \begin{pmatrix} q_1^{1/8} \mathrm{e}^{-\mathrm{i}\pi(\widetilde{u}+\widetilde{z})/2} & 0 \\ 0 & q_1^{-1/8} \mathrm{e}^{\mathrm{i}\pi(\widetilde{u}+\widetilde{z})/2} \end{pmatrix}.
\end{aligned} \tag{2.14}
$$

After this transformation, the coordinate velocity v enters into the Lax matrix $\widetilde{L}^{\mathrm{V}}$ with the same shift $v + \frac{1}{4}$ as in the Hamiltonian (2.10), i.e.,

$$
\widetilde{L}^{\mathrm{V}} = 2\pi \mathrm{i} \begin{pmatrix} v + \frac{1}{4} & 0 \\ 0 & -v - \frac{1}{4} \end{pmatrix} + \sum_{\alpha=0}^{3} \widetilde{\nu}_\alpha \widetilde{L}_\alpha^{\mathrm{V}},
$$

$$\widetilde{L}_0^{\mathrm{V}} = \frac{2\pi\mathrm{i}}{\sin(\pi\widetilde{u})}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad \widetilde{L}_1^{\mathrm{V}} = -\frac{2\pi}{\cos(\pi\widetilde{u})}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$
$$\widetilde{L}_2^{\mathrm{V}} = 4\sqrt{2}\pi q_1^{1/4}\begin{pmatrix}0 & -\sin(\pi(\widetilde{u}+\widetilde{z}))\\ \sin(\pi(\widetilde{u}-\widetilde{z})) & 0\end{pmatrix},$$
$$\widetilde{L}_3^{\mathrm{V}} = -4\sqrt{2}\pi q_1^{1/4}\begin{pmatrix}0 & \cos(\pi(\widetilde{u}+\widetilde{z}))\\ \cos(\pi(\widetilde{u}-\widetilde{z})) & 0\end{pmatrix}. \tag{2.15a}$$

Using the Hamilton equation of motion for the coordinate $\widetilde{u}$

$$\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}\tau_1} = \left\{H^{\mathrm{V}}, \widetilde{u}\right\} = 2v + \frac{1}{2},$$

one can ensure that transformation (2.14) removes v from the second Lax matrix (2.13), namely,

$$\widetilde{M}^{\mathrm{V}} = \sum_{\alpha=0}^{3}\widetilde{\nu}_\alpha \widetilde{M}_\alpha^{\mathrm{V}},$$
$$\widetilde{M}_0^{\mathrm{V}} = -\pi\frac{\cos(\pi\widetilde{u})}{\sin^2(\pi\widetilde{u})}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \widetilde{M}_1^{\mathrm{V}} = \mathrm{i}\pi\frac{\sin(\pi\widetilde{u})}{\cos^2(\pi\widetilde{u})}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$
$$\widetilde{M}_2^{\mathrm{V}} = 2\sqrt{2}\pi\mathrm{i}q_1^{1/4}\begin{pmatrix}0 & \cos(\pi(\widetilde{u}+\widetilde{z}))\\ \cos(\pi(\widetilde{u}-\widetilde{z})) & 0\end{pmatrix},$$
$$\widetilde{M}_3^{\mathrm{V}} = 2\sqrt{2}\pi\mathrm{i}q_1^{1/4}\begin{pmatrix}0 & -\sin(\pi(\widetilde{u}+\widetilde{z}))\\ \sin(\pi(\widetilde{u}-\widetilde{z})) & 0\end{pmatrix}. \tag{2.15b}$$

To show that Eqs. (2.11) and (2.2) are equivalent, we rewrite (2.11) in the form of a system of two first-order differential equations

$$\begin{cases} \dfrac{\mathrm{d}\widetilde{u}}{\mathrm{d}\tau_1} = 2v + \dfrac{1}{2}, \\ \dfrac{\mathrm{d}v}{\mathrm{d}\tau_1} = -2\pi\widetilde{\nu}_0^2\dfrac{\cos(\pi\widetilde{u})}{\sin^3(\pi\widetilde{u})} + 2\pi\widetilde{\nu}_1^2\dfrac{\sin(\pi\widetilde{u})}{\cos^3(\pi\widetilde{u})} - 8\pi q_1^{\frac{1}{2}}\left(\widetilde{\nu}_2^2 + \widetilde{\nu}_3^2\right)\sin(2\pi\widetilde{u}). \end{cases}$$

Eliminating v, we obtain

$$\frac{\mathrm{d}^2\widetilde{u}}{\mathrm{d}\tau_1^2} = -4\pi\widetilde{\nu}_0^2\frac{\cos(\pi\widetilde{u})}{\sin^3(\pi\widetilde{u})} + 4\pi\widetilde{\nu}_1^2\frac{\sin(\pi\widetilde{u})}{\cos^3(\pi\widetilde{u})} - 16\pi q_1^{\frac{1}{2}}\left(\widetilde{\nu}_2^2 + \widetilde{\nu}_3^2\right)\sin(2\pi\widetilde{u}).$$

3 Linear Problem for the Painlevé IV Equation

We construct a linear problem for the Painlevé IV equation as the limit of the linear problem for the Painlevé V equation obtained in Sect. 2.1. We make the substitutions

$$\tau_1 = \frac{tw^2}{2\pi\mathrm{i}}, \qquad u = \widetilde{u}\frac{w}{2\pi\mathrm{i}}, \qquad z = \widetilde{z}\frac{w}{2\pi\mathrm{i}}, \qquad v = \frac{\widetilde{v}}{w}, \tag{3.1}$$

the scalings of coupling constants and the limit $w \to 0$. Note that the limit $w \to 0$ describes the transformation of an infinite complex cylinder from Sect. 2 to a complex plane. After applying (3.1), the canonical Poisson bracket acquires the following form:

$$\{\widetilde{v}, \widetilde{u}\} = 2\pi \mathrm{i}.$$

We define scalings of coupling constants by the limiting behavior of equations of motion (2.6) and the Lax matrices (2.7a), (2.7b) via

$$\widetilde{\nu}_0 = \frac{\beta}{4\sqrt{2}}, \qquad \widetilde{\nu}_1 = -2\mathrm{i}w^{-4}, \qquad \widetilde{\nu}_2 = \frac{\mathrm{i}}{4w^4}, \qquad \widetilde{\nu}_3 = \frac{8\mathrm{i} + w^4 \mathrm{i}\alpha}{4w^4}.$$

To obtain the Hamiltonian of the limiting system, we use Hamiltonian (2.8) for the Painlevé V equation, because the other Hamiltonian (2.5) diverges as $w \to 0$. Thus, the Hamiltonian and the Lax matrices of the limiting system are defined as

$$H^{\mathrm{IV}} = \lim_{w\to 0} \frac{w^2}{2\pi\mathrm{i}} \widetilde{H}^{\mathrm{V}}, \qquad L^{\mathrm{IV}} = \lim_{w\to 0} w L^{\mathrm{V}}, \qquad M^{\mathrm{IV}} = \lim_{w\to 0} \frac{w^2}{2\pi\mathrm{i}} M^{\mathrm{V}}.$$

Finally, we get the following limiting Hamiltonian and the equation of zero curvature:

$$\begin{aligned} H^{\mathrm{IV}} = &-\frac{\mathrm{i}\widetilde{v}^2}{2\pi} + \frac{\mathrm{i}\left(\widetilde{u}^6 - \widetilde{z}^6\right)}{32\pi} + \frac{\mathrm{i}t\left(\widetilde{u}^4 - \widetilde{z}^4\right)}{8\pi} + \frac{\mathrm{i}\left(t^2-\alpha\right)\left(\widetilde{u}^2 - \widetilde{z}^2\right)}{8\pi} \\ &- \frac{\mathrm{i}\beta^2}{16\pi}\left(\frac{1}{\widetilde{u}^2} - \frac{1}{\widetilde{z}^2}\right), \\ &\partial_t L^{\mathrm{IV}} - \partial_{\widetilde{z}} M^{\mathrm{IV}} = \left[L^{\mathrm{IV}}, M^{\mathrm{IV}}\right] = \left\{H^{\mathrm{IV}}, L^{\mathrm{IV}}\right\}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} L^{\mathrm{IV}} = &\begin{pmatrix} \widetilde{v} & -\frac{\widetilde{u}^3}{4} - \frac{\widetilde{u}^2\widetilde{z}}{4} - \widetilde{u}\frac{\widetilde{z}^2+4t}{8} - \frac{\widetilde{z}^3}{32} - \frac{t\widetilde{z}}{4} \\ \frac{\widetilde{u}^3}{4} - \frac{\widetilde{u}^2\widetilde{z}}{4} + \widetilde{u}\frac{\widetilde{z}^2+4t}{8} - \frac{\widetilde{z}^3}{32} - \frac{t\widetilde{z}}{4} & -\widetilde{v} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & -\frac{\alpha}{2\widetilde{z}} + \frac{\mathrm{i}\beta}{2\sqrt{2}}\left(\frac{1}{\widetilde{u}} + \frac{1}{\widetilde{z}}\right) \\ -\frac{\alpha}{2\widetilde{z}} + \frac{\mathrm{i}\beta}{2\sqrt{2}}\left(\frac{1}{\widetilde{z}} - \frac{1}{\widetilde{u}}\right) & 0 \end{pmatrix}, \end{aligned} \tag{3.3a}$$

$$M^{\mathrm{IV}} = \begin{pmatrix} 0 & -\frac{3\widetilde{u}^2}{4} - \frac{\widetilde{u}\widetilde{z}}{2} - \frac{t}{2} - \frac{\widetilde{z}^2}{8} - \frac{\mathrm{i}\beta}{2\sqrt{2}\widetilde{u}^2} \\ -\frac{3\widetilde{u}^2}{4} + \frac{\widetilde{u}\widetilde{z}}{2} - \frac{t}{2} - \frac{\widetilde{z}^2}{8} - \frac{\mathrm{i}\beta}{2\sqrt{2}\widetilde{u}^2} & 0 \end{pmatrix}. \tag{3.3b}$$

The equivalence of Eq. (3.2) to the Painlevé IV equation in the form (8.4) can be shown in two steps. First, we rewrite (3.2) as a system of two differential equations

$$\begin{cases} \dfrac{d\widetilde{u}}{dt} = 2\widetilde{v}, \\ \dfrac{d\widetilde{v}}{dt} = \dfrac{\beta^2}{4\widetilde{u}^3} + \dfrac{1}{2}\left(t^2 - \alpha\right)\widetilde{u} + t\widetilde{u}^3 + \dfrac{3\widetilde{u}^5}{8}. \end{cases}$$

Second, after eliminating v from the system we get the following second-order differential equation:

$$\frac{d^2\widetilde{u}}{dt^2} = \frac{\beta^2}{2\widetilde{u}^3} + \left(t^2 - \alpha\right)\widetilde{u} + 2t\widetilde{u}^3 + \frac{3\widetilde{u}^5}{4}.$$

4 Linear Problem for the Painlevé III Equation

As in Sect. 2 we construct a limiting procedure which transforms the linear problem (1.6) for the Painlevé VI equation into linear problems for the following two equations:

$$\frac{d^2u}{dt^2} = C_0^2 e^{2t+2u} + C_2^2 e^{2t-2u} + C_0 C_1 e^{t+u} + C_2 C_3 e^{t-u}, \qquad (4.1)$$

$$\frac{d^2u}{dt^2} = C_2^2 e^{2t-2u} + C_0^2 e^{t+u} + C_2 C_3 e^{t-u}. \qquad (4.2)$$

Equations (4.1) and (4.2) describe the Painlevé III equation (8.6) with any choice of arbitrary constants. In Sects. 4.1 and 4.2, we construct two distinct degeneration procedures which give different linear problems for Eq. (4.1). A linear problem for Eq. (4.2) is constructed in Sect. 4.3.

Degeneration procedures under consideration are based on a generalization of the Inozemtsev limit and differ in shifts of the spectral parameter and scalings of coupling constants. The generalization of the Inozemtsev limit consists of the decomposition of the parameter τ,

$$\tau = \tau_1 + \tau_2, \qquad (4.3)$$

with τ_1 denoting the time of the system, the shift of the coordinate

$$u = \widetilde{u} + \frac{\tau}{4}, \qquad (4.4)$$

and the trigonometric limit $\operatorname{Im}\tau_2 \to +\infty$. In other words, we introduce the infinite shift of the parameter of an elliptic curve. Thus, we have the linear problem for the Painlevé III equation defined on an infinite complex cylinder $\mathbb{C}/\mathbb{Z}$.

4.1 First Linear Problem for Equation (4.1)

To construct a linear problem associated with Eq. (4.1), we use decomposition (4.3), shift of the coordinate u (4.4), and the trigonometric limit $\operatorname{Im}\tau_2 \to +\infty$. From the decomposition of the Lax matrices (1.6a), (1.6b) as a series in q, one can determine the scalings of coupling constants

$$\nu_0 = \frac{\widetilde{\nu}_0 q_2^{-\frac{1}{4}} + \widetilde{\nu}_1}{2\pi}, \qquad \nu_1 = \frac{-\widetilde{\nu}_0 q_2^{-\frac{1}{4}} + \widetilde{\nu}_1}{2\pi}, \qquad \nu_2 = \frac{\widetilde{\nu}_2 q_2^{-\frac{1}{4}} + \widetilde{\nu}_3}{2\pi},$$
$$\nu_3 = \frac{-\widetilde{\nu}_2 q_2^{-\frac{1}{4}} + \widetilde{\nu}_3}{2\pi}.$$

Since τ_1 is the time of the system, shift (4.4) of the coordinate u is time-dependent, which implies

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \frac{\mathrm{d}u}{\mathrm{d}\tau_1} = \frac{\mathrm{d}\widetilde{u}}{\mathrm{d}\tau_1} + \frac{1}{4}. \tag{4.5}$$

Thus, the Hamiltonian of the limiting system has the following form:

$$H^{\mathrm{III}} = \lim_{\mathrm{Im}\,\tau_2 \to +\infty} H^{\mathrm{VI}} - \frac{1}{4}v + \frac{1}{64}.$$

Using the Hamiltonian for the Painlevé VI equation in the form (1.3), we get

$$H^{\mathrm{III}} = \left(v - \frac{1}{8}\right)^2 + 4q_1^{\frac{1}{2}} \widetilde{\nu}_0^2 \mathrm{e}^{4\pi \mathrm{i}\widetilde{u}} + 4q_1^{\frac{1}{2}} \widetilde{\nu}_2^2 \mathrm{e}^{-4\pi \mathrm{i}\widetilde{u}} + 4q_1^{\frac{1}{4}} \widetilde{\nu}_0 \widetilde{\nu}_1 \mathrm{e}^{2\pi \mathrm{i}\widetilde{u}} + 4q_1^{\frac{1}{4}} \widetilde{\nu}_2 \widetilde{\nu}_3 \mathrm{e}^{-2\pi \mathrm{i}\widetilde{u}}. \tag{4.6}$$

Expression (1.4) gives an equivalent Hamiltonian

$$\begin{aligned} \widetilde{H}^{\mathrm{III}} &= \lim_{\mathrm{Im}\,\tau_2 \to +\infty} \widetilde{H}^{\mathrm{VI}} - \frac{1}{4}v + \frac{1}{64} = \left(v - \frac{1}{8}\right)^2 + 4q_1^{\frac{1}{2}} \widetilde{\nu}_0^2 \left(\mathrm{e}^{4\pi \mathrm{i}\widetilde{u}} - \mathrm{e}^{4\pi \mathrm{i}z}\right) \\ &\quad + 4q_1^{\frac{1}{2}} \widetilde{\nu}_2^2 \left(\mathrm{e}^{-4\pi \mathrm{i}\widetilde{u}} - \mathrm{e}^{-4\pi \mathrm{i}z}\right) + 4q_1^{\frac{1}{4}} \widetilde{\nu}_0 \widetilde{\nu}_1 \left(\mathrm{e}^{2\pi \mathrm{i}\widetilde{u}} - \mathrm{e}^{2\pi \mathrm{i}z}\right) \\ &\quad + 4q_1^{\frac{1}{4}} \widetilde{\nu}_2 \widetilde{\nu}_3 \left(\mathrm{e}^{-2\pi \mathrm{i}\widetilde{u}} - \mathrm{e}^{-2\pi \mathrm{i}z}\right), \end{aligned} \tag{4.7}$$

which will be used to calculate a limit in Sect. 5.

The equation of zero curvature (1.5) preserves the form in the limit

$$\partial_{\tau_1} L^{\mathrm{III}} - \frac{1}{2\pi \mathrm{i}} \partial_z M^{\mathrm{III}} = \left[L^{\mathrm{III}}, M^{\mathrm{III}}\right], \tag{4.8}$$

where

$$L^{\mathrm{III}} = \lim_{\mathrm{Im}\,\tau_2 \to +\infty} L^{\mathrm{VI}} = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} + \sum_{\alpha=0}^{3} \widetilde{\nu}_\alpha L_\alpha^{\mathrm{III}},$$

$$L_0^{\mathrm{III}} = 2\mathrm{i}q_1^{\frac{1}{4}}\mathrm{e}^{2\pi\mathrm{i}\widetilde{u}}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad L_1^{\mathrm{III}} = \begin{pmatrix}0 & -\mathrm{i}+\cot(\pi z)\\ \mathrm{i}+\cot(\pi z) & 0\end{pmatrix},$$

$$L_2^{\mathrm{III}} = 2\mathrm{i}q_1^{\frac{1}{4}}\begin{pmatrix}0 & \mathrm{e}^{-\mathrm{i}\pi(2\widetilde{u}+z)}\\ -\mathrm{e}^{-\mathrm{i}\pi(2\widetilde{u}-z)} & 0\end{pmatrix}, \quad L_3^{\mathrm{III}} = \frac{1}{\sin(\pi z)}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{4.9a}$$

$$M^{\mathrm{III}} = \lim_{\mathrm{Im}\,\tau_2\to+\infty} L^{\mathrm{VI}} = \sum_{\alpha=0}^{3}\widetilde{\nu}_\alpha M_\alpha^{\mathrm{III}},$$

$$M_0^{\mathrm{III}} = 4\pi q_1^{\frac{1}{4}}\mathrm{e}^{2\pi\mathrm{i}\widetilde{u}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad M_1^{\mathrm{III}} = 0,$$

$$M_2^{\mathrm{III}} = 4\pi q_1^{\frac{1}{4}}\begin{pmatrix}0 & \mathrm{e}^{-\mathrm{i}\pi(2\widetilde{u}+z)}\\ \mathrm{e}^{-\mathrm{i}\pi(2\widetilde{u}-z)} & 0\end{pmatrix}, \qquad M_3^{\mathrm{III}} = 0. \tag{4.9b}$$

Equation (4.8) describes the Hamilton equations of motion of the limiting system

$$\begin{cases} \dfrac{\mathrm{d}\widetilde{u}}{\mathrm{d}\tau_1} = \left\{H^{\mathrm{III}}, \widetilde{u}\right\} = 2v - \dfrac{1}{4}, \\ \dfrac{\mathrm{d}v}{\mathrm{d}\tau_1} = \left\{H^{\mathrm{III}}, v\right\} = -16\mathrm{i}\pi q_1^{\frac{1}{2}}\widetilde{\nu}_0^2\mathrm{e}^{4\pi\mathrm{i}\widetilde{u}} + 16\mathrm{i}\pi q_1^{\frac{1}{2}}\widetilde{\nu}_2^2\mathrm{e}^{-4\pi\mathrm{i}\widetilde{u}} - 8\mathrm{i}\pi q_1^{\frac{1}{4}}\widetilde{\nu}_0\widetilde{\nu}_1\mathrm{e}^{2\pi\mathrm{i}\widetilde{u}} \\ \qquad + 8\mathrm{i}\pi q_1^{\frac{1}{4}}\widetilde{\nu}_2\widetilde{\nu}_3\mathrm{e}^{-2\pi\mathrm{i}\widetilde{u}}, \end{cases}$$

and is equivalent to the following second-order differential equation:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\tau_1^2} = -32\mathrm{i}\pi q_1^{\frac{1}{2}}\widetilde{\nu}_0^2\mathrm{e}^{4\pi\mathrm{i}\widetilde{u}} + 32\mathrm{i}\pi q_1^{\frac{1}{2}}\widetilde{\nu}_2^2\mathrm{e}^{-4\pi\mathrm{i}\widetilde{u}} - 16\mathrm{i}\pi q_1^{\frac{1}{4}}\widetilde{\nu}_0\widetilde{\nu}_1\mathrm{e}^{2\pi\mathrm{i}\widetilde{u}} + 16\mathrm{i}\pi q_1^{\frac{1}{4}}\widetilde{\nu}_2\widetilde{\nu}_3\mathrm{e}^{-2\pi\mathrm{i}\widetilde{u}},$$

which in turn coincides with (4.1) up to a change of arbitrary constants.

4.2 Second Linear Problem for Equation (4.1)

Adding to the degeneration procedure described in Sect. 4.1 the shift of the spectral parameter

$$z = \widetilde{z} + \frac{\tau}{2}, \tag{4.10}$$

we get another linear problem associated with Eq. (4.1). The gauge equivalence of the second linear problem to the first linear problem 4.9 has not been established.

Thus, we use decomposition (4.3) of the parameter τ, shifts of the coordinate (4.4) and the spectral parameter (4.10), and the trigonometric limit $Im\,\tau_2 \to +\infty$. Scalings of coupling constants are determined from the decomposition of the Hamiltonian (1.3) as a series in q,

$$\nu_0 = \frac{\widetilde{\nu}_0 q_2^{-\frac{1}{4}} + \widetilde{\nu}_1}{2\pi}, \qquad \nu_1 = \frac{-\widetilde{\nu}_0 q_2^{-\frac{1}{4}} + \widetilde{\nu}_1}{2\pi}, \qquad \nu_2 = \frac{\widetilde{\nu}_2 q_2^{-\frac{1}{4}} + \widetilde{\nu}_3}{2\pi},$$
$$\nu_3 = \frac{\widetilde{\nu}_2 q_2^{-\frac{1}{4}} - \widetilde{\nu}_3}{2\pi}.$$

Since substitution (4.4) is time-dependent, the time derivative of the coordinate u of the Calogero–Inozemtsev system and the time derivative of the coordinate $\widetilde{u}$ of the limiting system are connected via (4.5) as follows:

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \frac{\mathrm{d}u}{\mathrm{d}\tau_1} = \frac{\mathrm{d}\widetilde{u}}{\mathrm{d}\tau_1} + \frac{1}{4}.$$

Thus, the Hamiltonian of the limiting system is of the form

$$H^{\mathrm{III}} = \lim_{\mathrm{Im}\,\tau_2 \to +\infty} H^{\mathrm{VI}} - \frac{1}{4}v + \frac{1}{64}.$$

Using the Hamiltonian (1.3) associated with the Painlevé VI equation, one can derive the following explicit formula for H^{III}:

$$H^{\mathrm{III}} = \left(v - \frac{1}{8}\right)^2 + 4\widetilde{\nu}_0^2 q_1^{1/2} \mathrm{e}^{4\pi \mathrm{i}\widetilde{u}} + 4\widetilde{\nu}_0\widetilde{\nu}_1 q_1^{1/4} \mathrm{e}^{2\pi \mathrm{i}\widetilde{u}} + 4\widetilde{\nu}_2^2 q_1^{1/2} \mathrm{e}^{-4\pi \mathrm{i}\widetilde{u}}$$
$$+ 4\widetilde{\nu}_2\widetilde{\nu}_3 q_1^{1/4} \mathrm{e}^{-2\pi \mathrm{i}\widetilde{u}}. \tag{4.11}$$

After substitutions (4.3), (4.4), and (4.10), the equation of zero curvature (1.5) transforms into

$$\partial_{\tau_1} L^{\mathrm{VI}} - \partial_{\widetilde{z}} \left(\frac{1}{2\pi \mathrm{i}} M^{\mathrm{VI}} + \frac{1}{2} L^{\mathrm{VI}}\right) = \left[L^{\mathrm{VI}}, M^{\mathrm{VI}}\right],$$

where $L^{\mathrm{VI}} = L^{\mathrm{VI}}(\widetilde{u} + \tau/4, v, \widetilde{z} + \tau/2, \tau)$ and $M^{\mathrm{VI}} = M^{\mathrm{VI}}(\widetilde{u} + \tau/4, \widetilde{z} + \tau/2, \tau)$. This implies the following definitions of the Lax matrices:

$$L^{\mathrm{III}} = 2\pi \mathrm{i} \lim_{\mathrm{Im}\tau_2 \to +\infty} L^{\mathrm{VI}}, \qquad M^{\mathrm{III}} = \lim_{\mathrm{Im}\tau_2 \to +\infty} \left(M^{\mathrm{VI}} + \pi \mathrm{i} L^{\mathrm{VI}}\right).$$

Finally, the equation of zero curvature acquires the form

$$\partial_{\tau_1} L^{\mathrm{III}} - \partial_{\widetilde{z}} M^{\mathrm{III}} = \left[L^{\mathrm{III}}, M^{\mathrm{III}}\right], \tag{4.12}$$

where

$$L^{\mathrm{III}} = \lim_{\operatorname{Im}\tau_2\to+\infty} L^{\mathrm{VI}} = 2\pi\mathrm{i}\begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} + \sum_{\alpha=0}^{3} \widetilde{\nu}_\alpha L_\alpha^{\mathrm{III}},$$

$$L_0^{\mathrm{III}} = 4\pi q_1^{\frac{1}{4}} \begin{pmatrix} 0 & \mathrm{e}^{2\pi\mathrm{i}\widetilde{u}} - \mathrm{e}^{-2\pi\mathrm{i}(\widetilde{u}+\widetilde{z})} \\ -\mathrm{e}^{2\pi\mathrm{i}\widetilde{u}} & 0 \end{pmatrix}, \quad L_1^{\mathrm{III}} = 4\pi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$L_2^{\mathrm{III}} = 4\pi q_1^{\frac{1}{4}} \begin{pmatrix} 0 & \mathrm{e}^{\pi\mathrm{i}\widetilde{z}} - \mathrm{e}^{-\pi\mathrm{i}(4\widetilde{u}+\widetilde{z})} \\ \mathrm{e}^{\pi\mathrm{i}\widetilde{z}} & 0 \end{pmatrix}, \quad L_3^{\mathrm{III}} = -4\pi\mathrm{e}^{-\pi\mathrm{i}(2\widetilde{u}+\widetilde{z})} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.13}$$

$$M^{\mathrm{III}} = \lim_{\operatorname{Im}\tau_2\to+\infty} M^{\mathrm{VI}} = \mathrm{i}\pi\begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} + \sum_{\alpha=0}^{3} \widetilde{\nu}_\alpha M_\alpha^{\mathrm{III}},$$

$$M_0^{\mathrm{III}} = 2\pi q_1^{1/4} \begin{pmatrix} 0 & \mathrm{e}^{-2\mathrm{i}\pi(\widetilde{u}+\widetilde{z})} + 3\mathrm{e}^{2\mathrm{i}\pi\widetilde{u}} \\ \mathrm{e}^{2\mathrm{i}\pi\widetilde{u}} & 0 \end{pmatrix}, \qquad M_1^{\mathrm{III}} = 2\pi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$M_2^{\mathrm{III}} = 2\pi q_1^{1/4} \begin{pmatrix} 0 & 3\mathrm{e}^{-\mathrm{i}\pi(4\widetilde{u}+\widetilde{z})} + \mathrm{e}^{\mathrm{i}\pi\widetilde{z}} \\ \mathrm{e}^{\mathrm{i}\pi\widetilde{z}} & 0 \end{pmatrix}, \qquad M_3^{\mathrm{III}} = 2\pi\mathrm{e}^{-\mathrm{i}\pi(2\widetilde{u}+\widetilde{z})} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.14}$$

As in Sect. 2.2, we can remove v from the second Lax matrix M^{III} (4.14) by means of the gauge transformation

$$L^{\mathrm{III}} \to gL^{\mathrm{III}}g^{-1} - (\partial_{\widetilde{z}} g)\, g^{-1}, \qquad M^{\mathrm{III}} \to gM^{\mathrm{III}}g^{-1} - \left(\partial_{\tau_1} g\right) g^{-1},$$
$$g = \begin{pmatrix} q_1^{1/16}\mathrm{e}^{\mathrm{i}\pi(2\widetilde{u}+\widetilde{z})/4} & 0 \\ 0 & q_1^{-1/16}\mathrm{e}^{-\mathrm{i}\pi(2\widetilde{u}+\widetilde{z})/4} \end{pmatrix}.$$

Applying the transformation, we get the first Lax matrix L^{III} (4.13) with the same shifted velocity $v - \frac{1}{8}$ as in the Hamiltonian (4.11).

Equation (4.12) is equivalent to the Hamilton equations of motion

$$\begin{cases} \dfrac{\mathrm{d}\widetilde{u}}{\mathrm{d}\tau_1} = 2v - \dfrac{1}{4}, \\ \dfrac{\mathrm{d}v}{\mathrm{d}\tau_1} = -16\pi\mathrm{i}\widetilde{\nu}_0^2 q_1^{1/2}\mathrm{e}^{4\pi\mathrm{i}\widetilde{u}} - 8\pi\mathrm{i}\widetilde{\nu}_0\widetilde{\nu}_1 q_1^{1/4}\mathrm{e}^{2\pi\mathrm{i}\widetilde{u}} + 16\pi\mathrm{i}\widetilde{\nu}_2^2 q_1^{1/2}\mathrm{e}^{-4\pi\mathrm{i}\widetilde{u}} \\ \qquad\quad + 8\pi\mathrm{i}\widetilde{\nu}_2\widetilde{\nu}_3 q_1^{1/4}\mathrm{e}^{-2\pi\mathrm{i}\widetilde{u}}. \end{cases}$$

These equations in turn are equivalent to the following second-order differential equation:

$$\frac{\mathrm{d}^2\widetilde{u}}{\mathrm{d}\tau_1^2} = -\,32\pi\mathrm{i}\widetilde{\nu}_0^2 q_1^{1/2}\mathrm{e}^{4\pi\mathrm{i}\widetilde{u}} - 16\pi\mathrm{i}\widetilde{\nu}_0\widetilde{\nu}_1 q_1^{1/4}\mathrm{e}^{2\pi\mathrm{i}\widetilde{u}} + 32\pi\mathrm{i}\widetilde{\nu}_2^2 q_1^{1/2}\mathrm{e}^{-4\pi\mathrm{i}\widetilde{u}}$$
$$+\,16\pi\mathrm{i}\widetilde{\nu}_2\widetilde{\nu}_3 q_1^{1/4}\mathrm{e}^{-2\pi\mathrm{i}\widetilde{u}},$$

which coincides with (4.1) up to a change of arbitrary constants.

4.3 Linear Problem for Equation (4.2)

In this subsection we use decomposition (4.3) of the parameter τ of an elliptic curve, the substitution of the coordinate (4.4), and the shift of the spectral parameter

$$z = \widetilde{z} + \frac{\tau}{4}.$$

The scalings of coupling constants are determined from the decomposition of the Hamiltonian (1.3) as a series in q in the following way:

$$\nu_0 = \frac{\widetilde{\nu}_0 q_2^{-1/8}}{2\pi}, \quad \nu_1 = \frac{\mathrm{i}\widetilde{\nu}_1 q_2^{-1/8}}{2\pi}, \quad \nu_2 = \frac{\widetilde{\nu}_2 q_2^{-1/4} + \widetilde{\nu}_3}{2\pi}, \quad \nu_3 = \frac{-\widetilde{\nu}_2 q_2^{-1/4} + \widetilde{\nu}_3}{2\pi}.$$

As was mentioned in Sect. 4.1, shift of the coordinate (4.4) is time-dependent. Thus, the Hamiltonians of the limiting system have the following form:

$$H^{\mathrm{III}} = \lim_{\mathrm{Im}\,\tau_2 \to +\infty} H^{\mathrm{VI}} - \frac{1}{4} v + \frac{1}{64}.$$

Using the Hamiltonian (1.3) for the Painlevé VI equation, one can derive

$$H^{\mathrm{III}} = \left(v - \frac{1}{8}\right)^2 + \left(\widetilde{\nu}_0^2 + \widetilde{\nu}_1^2\right) q_1^{1/4} \mathrm{e}^{2\mathrm{i}\pi\widetilde{u}} + 4\widetilde{\nu}_2^2 q_1^{1/2} \mathrm{e}^{-4\mathrm{i}\pi\widetilde{u}} + 4\widetilde{\nu}_2\widetilde{\nu}_3 q_1^{1/4} \mathrm{e}^{-2\mathrm{i}\pi\widetilde{u}}. \tag{4.15}$$

In this case, in order to get convergent Lax matrices it is necessary to make the gauge transformation

$$L^{\mathrm{VI}} \to g L^{\mathrm{VI}} g^{-1}, \qquad M^{\mathrm{VI}} \to g M^{\mathrm{VI}} g^{-1},$$
$$g = \begin{pmatrix} 1 & 0 \\ 0 & q_2^{-1/8} \end{pmatrix}.$$

After applying the shift of the spectral parameter, equation of zero curvature (1.5) becomes

$$\partial_{\tau_1} L^{\mathrm{VI}} - \partial_{\widetilde{z}} \left(\frac{1}{2\pi\mathrm{i}} M^{\mathrm{VI}} + \frac{1}{4} L^{\mathrm{VI}} \right) = \left[L^{\mathrm{VI}}, M^{\mathrm{VI}} \right],$$

where $L^{\mathrm{VI}} = L^{\mathrm{VI}}(\widetilde{u} + \tau/4, v, \widetilde{z} + \tau/4, \tau)$, $M^{\mathrm{VI}} = M^{\mathrm{VI}}(\widetilde{u} + \tau/4, \widetilde{z} + \tau/4, \tau)$. This implies the following definition of the Lax matrices:

$$L^{\mathrm{III}} = 2\pi\mathrm{i} \lim_{\mathrm{Im}\tau_2 \to +\infty} g L^{\mathrm{VI}} g^{-1}, \qquad M^{\mathrm{III}} = \lim_{\mathrm{Im}\tau_2 \to +\infty} g \left(M^{\mathrm{VI}} + \frac{\pi\mathrm{i}}{2} L^{\mathrm{VI}} \right) g^{-1}.$$

Finally, the equation of zero curvature acquires the following form:

$$\partial_{\tau_1} L^{\mathrm{III}} - \partial_{\widetilde{z}} M^{\mathrm{III}} = \left[L^{\mathrm{III}}, M^{\mathrm{III}} \right], \tag{4.16}$$

where

$$\begin{aligned}
L^{\mathrm{III}} &= 2\pi \mathrm{i} \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} + \sum_{\alpha=0}^{3} \widetilde{\nu}_\alpha L_\alpha^{\mathrm{III}}, \\
L_0^{\mathrm{III}} &= 2\pi \begin{pmatrix} 0 & 1 \\ q_1^{1/4} \left(\mathrm{e}^{2\mathrm{i}\pi \widetilde{z}} - \mathrm{e}^{2\mathrm{i}\pi \widetilde{u}} \right) & 0 \end{pmatrix}, \qquad L_1^{\mathrm{III}} = 2\pi \mathrm{i} \begin{pmatrix} 0 & 1 \\ q_1^{1/4} \left(\mathrm{e}^{2\mathrm{i}\pi \widetilde{u}} + \mathrm{e}^{2\mathrm{i}\pi \widetilde{z}} \right) & 0 \end{pmatrix}, \\
L_2^{\mathrm{III}} &= -4\pi q_1^{1/8} \mathrm{e}^{-2\mathrm{i}\pi \widetilde{u}} \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\pi \widetilde{z}} \\ -q_1^{1/4} \mathrm{e}^{\mathrm{i}\pi \widetilde{z}} & 0 \end{pmatrix}, \qquad L_3^{\mathrm{III}} = 4\pi q_1^{1/8} \mathrm{e}^{\mathrm{i}\pi \widetilde{z}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\end{aligned} \tag{4.17a}$$

$$\begin{aligned}
M^{\mathrm{III}} &= \frac{\mathrm{i}\pi}{2} \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} + \sum_{\alpha=0}^{3} \widetilde{\nu}_\alpha M_\alpha^{\mathrm{III}}, \\
M_0^{\mathrm{III}} &= \frac{\pi}{2} \begin{pmatrix} 0 & 1 \\ q_1^{1/4} \left(3\mathrm{e}^{2\mathrm{i}\pi \widetilde{u}} + \mathrm{e}^{2\mathrm{i}\pi \widetilde{z}} \right) & 0 \end{pmatrix}, \qquad M_1^{\mathrm{III}} = \frac{\mathrm{i}\pi}{2} \begin{pmatrix} 0 & 1 \\ \mathrm{e}^{2\mathrm{i}\pi \widetilde{z}} - 3\mathrm{e}^{2\mathrm{i}\pi \widetilde{u}} & 0 \end{pmatrix}, \\
M_2^{\mathrm{III}} &= \pi q_1^{1/8} \begin{pmatrix} 0 & 3\mathrm{e}^{-\mathrm{i}\pi (2\widetilde{u} + \widetilde{z})} \\ 5 q_1^{1/4} \mathrm{e}^{-\mathrm{i}\pi (2\widetilde{u} - \widetilde{z})} & 0 \end{pmatrix}, \qquad M_3^{\mathrm{III}} = \pi q_1^{1/8} \mathrm{e}^{\mathrm{i}\pi \widetilde{z}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\end{aligned} \tag{4.17b}$$

As in Sects. 2.2 and 4.2, we can remove v from the second Lax matrix M^{III} (4.17b) by means of the gauge transformation

$$\begin{aligned}
L^{\mathrm{III}} &\to \widetilde{g} L^{\mathrm{III}} \widetilde{g}^{-1} - (\partial_{\widetilde{z}} \widetilde{g})\, \widetilde{g}^{-1}, \qquad M^{\mathrm{III}} \to \widetilde{g} M^{\mathrm{III}} \widetilde{g}^{-1} - \left(\partial_{\tau_1} \widetilde{g} \right) \widetilde{g}^{-1}, \\
\widetilde{g} &= \begin{pmatrix} q_1^{1/32} \mathrm{e}^{\mathrm{i}\pi (\widetilde{u} + \widetilde{z})/4} & 0 \\ 0 & q_1^{-1/32} \mathrm{e}^{-\mathrm{i}\pi (\widetilde{u} + \widetilde{z})/4} \end{pmatrix}.
\end{aligned}$$

Applying this transformation, we obtain the first Lax matrix L^{III} (4.17a) with the same shifted velocity $v - \frac{1}{8}$ as in the Hamiltonian (4.15).

Equation of zero curvature (4.16) is equivalent to the Hamilton equations of motion

$$\begin{cases} \dfrac{\mathrm{d}\widetilde{u}}{\mathrm{d}\tau_1} = \left\{ H^{\mathrm{III}}, \widetilde{u} \right\} = 2v - \dfrac{1}{4}, \\ \dfrac{\mathrm{d}v}{\mathrm{d}\tau_1} = -2\mathrm{i}\pi q_1^{1/4} \left(\widetilde{\nu}_0^2 + \widetilde{\nu}_1^2 \right) \mathrm{e}^{2\mathrm{i}\pi \widetilde{u}} + 16\mathrm{i}\pi q_1^{1/2} \widetilde{\nu}_2^2 \mathrm{e}^{-4\mathrm{i}\pi \widetilde{u}} + 8\mathrm{i}\pi q_1^{1/4} \widetilde{\nu}_2 \widetilde{\nu}_3 \mathrm{e}^{-2\mathrm{i}\pi \widetilde{u}}. \end{cases}$$

Eliminating v from this system we get the second-order differential equation

$$\frac{\mathrm{d}^2\widetilde{u}}{\mathrm{d}\tau_1^2} = -4\mathrm{i}\pi q_1^{1/4}\left(\widetilde{\nu}_0^2 + \widetilde{\nu}_1^2\right)\mathrm{e}^{2\mathrm{i}\pi\widetilde{u}} + 32\mathrm{i}\pi q_1^{1/2}\widetilde{\nu}_2^2\mathrm{e}^{-4\mathrm{i}\pi\widetilde{u}} + 16\mathrm{i}\pi q_1^{1/4}\widetilde{\nu}_2\widetilde{\nu}_3\mathrm{e}^{-2\mathrm{i}\pi\widetilde{u}},$$

which coincides with (4.2) up to a change of arbitrary constants.

5 Linear Problem for the Painlevé II Equation

We construct a linear problem for the Painlevé II equation by means of the degeneration process from the linear problem constructed in Sect. 4.1. This process involves substitutions

$$\tau_1 = \frac{tw^2}{2\pi\mathrm{i}}, \qquad \widetilde{u} = U\frac{w}{2\pi\mathrm{i}}, \qquad z = Z\frac{w}{2\pi\mathrm{i}}, \qquad v = \frac{V}{w}, \tag{5.1}$$

scalings of coupling constants, and the limit $w \to 0$. Note that the limit $w \to 0$ describes the transformation of an infinite complex cylinder from Sect. 4 to a complex plane. After applying (5.1), the canonical Poisson bracket transforms into

$$\{V, U\} = 2\pi\mathrm{i}.$$

From the decomposition of Lax matrices (4.9a), (4.9b) as series in q, one can determine the scalings of coupling constants

$$\widetilde{\nu}_0 = \mathrm{i}\frac{1+w^2}{4w^3}, \quad \widetilde{\nu}_1 = -\mathrm{i}\frac{2-w^3}{2w^3}, \quad \widetilde{\nu}_2 = -\mathrm{i}\frac{1-w^2}{4w^3}, \quad \widetilde{\nu}_3 = \frac{\mathrm{i}}{2}\left(\alpha - 1 + \frac{2}{w^3}\right).$$

Since Hamiltonian (4.6) for the Painlevé III equation diverges as $w \to 0$, we use the equivalent form (4.7) to obtain the Hamiltonian of the limiting system

$$H^{\mathrm{II}} = \lim_{w\to 0}\frac{w^2}{2\pi\mathrm{i}}\widetilde{H}^{\mathrm{III}}.$$

The Lax matrices of the limiting system are defined as

$$L^{\mathrm{II}} = \lim_{w\to 0} wL^{\mathrm{III}}, \qquad M^{\mathrm{II}} = \lim_{w\to 0}\frac{w^2}{2\pi\mathrm{i}}M^{\mathrm{III}}.$$

Thus, we get the limiting Hamiltonian and the equation of zero curvature in the following form:

$$\begin{aligned} H^{\mathrm{II}} &= -\frac{\mathrm{i}V^2}{2\pi} + \frac{\mathrm{i}\alpha\,(U-Z)}{4\pi} + \frac{\mathrm{i}t\left(U^2 - Z^2\right)}{8\pi} + \frac{\mathrm{i}\left(U^4 - Z^4\right)}{8\pi}, \\ \partial_t L^{\mathrm{II}} - \partial_Z M^{\mathrm{II}} &= \left[L^{\mathrm{II}}, M^{\mathrm{II}}\right] = \left\{H^{\mathrm{II}}, L^{\mathrm{II}}\right\}, \end{aligned} \tag{5.2}$$

where

$$L^{\mathrm{II}} = \begin{pmatrix} V & \frac{t}{4} - \frac{\alpha}{Z} + \frac{Z^2}{16} + \frac{UZ}{4} + \frac{U^2}{2} \\ -\frac{t}{4} - \frac{\alpha}{Z} - \frac{Z^2}{16} + \frac{UZ}{4} - \frac{U^2}{2} & -V \end{pmatrix}, \tag{5.3a}$$

$$M^{\mathrm{II}} = \begin{pmatrix} 0 & U + \frac{Z}{4} \\ U - \frac{Z}{4} & 0 \end{pmatrix}. \tag{5.3b}$$

One can rewrite (5.2) as the following system of the first-order differential equations:

$$\begin{cases} \frac{\mathrm{d}U}{\mathrm{d}t} = 2V, \\ \frac{\mathrm{d}V}{\mathrm{d}t} = U^3 + \frac{tU}{2} + \frac{\alpha}{2}, \end{cases}$$

which coincides with the Hamilton equations of motion. Eliminating v from this system, we get the Painlevé II equation

$$\frac{\mathrm{d}^2U}{\mathrm{d}t^2} = 2U^3 + tU + \alpha.$$

6 Linear Problem for the Painlevé I Equation

In this section, we construct a linear problem for the Painlevé I equation via the degeneration process from the linear problem constructed in Sect. 4.1. This process consists of the substitutions

$$\tau_1 = \frac{w^2 t}{2\pi \mathrm{i}}, \qquad \widetilde{u} = U\frac{w}{2\pi \mathrm{i}}, \qquad v = \frac{V}{w}, \qquad z = -\frac{1}{2} + Z\frac{w}{2\pi \mathrm{i}}, \tag{6.1}$$

scalings of coupling constants, and the limit $w \to 0$. After applying (6.1), the canonical Poisson bracket transforms into

$$\{V, U\} = 2\pi \mathrm{i}.$$

The simplest way to determine scalings of coupling constants is to analyze the decomposition of the Hamiltonian (4.6) as a series in q. This gives

$$\widetilde{\nu}_0 = \frac{\mathrm{i}}{2\sqrt{2}w^{5/2}}, \qquad \widetilde{\nu}_1 = -\frac{\mathrm{i}}{\sqrt{2}w^{5/2}}, \qquad \widetilde{\nu}_2 = -\frac{1}{2\sqrt{2}w^{5/2}}, \qquad \widetilde{\nu}_3 = \frac{1}{\sqrt{2}w^{5/2}}.$$

To obtain the convergent Lax matrices, we have to make the following gauge transformation:

$$L^{\mathrm{III}} \to g L^{\mathrm{III}} g^{-1}, \qquad M^{\mathrm{III}} \to g M^{\mathrm{III}} g^{-1},$$
$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{w} \end{pmatrix},$$

and consider the limit

$$L^{\mathrm{I}} = \lim_{w\to 0} w g L^{\mathrm{III}} g^{-1}, \qquad M^{\mathrm{I}} = \lim_{w\to 0} \frac{w^2}{2\pi \mathrm{i}} g M^{\mathrm{III}} g^{-1}.$$

Using the Hamiltonian (4.6) for the Painlevé III equation, we derive the Hamiltonian of the limiting system

$$H^{\mathrm{I}} = \lim_{w\to 0} \frac{w^2}{2\pi \mathrm{i}} H^{\mathrm{III}} = -\frac{\mathrm{i}V^2}{2\pi} + \frac{\mathrm{i}tU}{4\pi} + \frac{\mathrm{i}U^3}{2\pi}.$$

After taking the limit, the equation of zero curvature becomes

$$\partial_t L^{\mathrm{I}} - \partial_Z M^{\mathrm{I}} = \left[L^{\mathrm{I}}, M^{\mathrm{I}}\right] = \left\{H^{\mathrm{I}}, L^{\mathrm{I}}\right\}, \tag{6.2}$$

where

$$L^{\mathrm{I}} = \begin{pmatrix} V & \frac{1}{4\sqrt{2}}\left(2t + Z^2 + 2UZ + 4U^2\right) \\ \frac{1}{\sqrt{2}}(Z - 2U) & -V \end{pmatrix},$$
$$M^{\mathrm{I}} = \begin{pmatrix} 0 & \frac{1}{2\sqrt{2}}(4U + Z) \\ \sqrt{2} & 0 \end{pmatrix}. \tag{6.3}$$

One can rewrite (6.2) as a system of the first-order differential equations, which is equivalent to the Painlevé I equation

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = 2V, \\ \dfrac{\mathrm{d}V}{\mathrm{d}t} = 3U^2 + \dfrac{t}{2} \end{cases} \quad \Longleftrightarrow \quad \frac{d^2U}{dt^2} = 6U^2 + t.$$

7 Conclusion

We have constructed linear problems for the Painlevé equations I–V via the degeneration processes from the linear problem for the Painlevé VI equation. These degeneration processes can be described by the following diagram:

$$\begin{array}{ccccc} & & \mathrm{PV} & \longrightarrow & \mathrm{PIV} \\ & \nearrow & & & \\ \mathrm{PVI} & \longrightarrow & \mathrm{PIII} & \longrightarrow & \mathrm{PII} \\ & & & \searrow & \\ & & & & \mathrm{PI} \end{array} \tag{7.1}$$

Thus, we have supplemented the known relations (1.1) between the Painlevé equations with the degeneration scheme (7.1) for the 2×2-matrix linear problems discussed in this paper. Moreover, the derived linear problems for the Painlevé equations III and V appear to be new.

Since one can obtain the Calogero–Inozemtsev system via the reduction from the 2×2 elliptic Schlesinger system with four marked points, it is possible to apply the proposed degeneration process to the general case of the elliptic Schlesinger system. This process can probably give new nonautonomous systems that describe the interaction between the nonautonomous Toda and Calogero–Moser systems. We study such a degeneration process in the subsequent work [3].

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8 Appendix 1: Painlevé Equations

In this section, we present a connection between the different forms of the Painlevé equations V, IV, and III considered in this paper.

8.1 Painlevé V

The Painlevé V equation has the following rational form:

$$\frac{\mathrm{d}^2\lambda}{\mathrm{d}t^2} = \left(\frac{1}{2\lambda} + \frac{1}{\lambda - 1}\right)\left(\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}\lambda}{\mathrm{d}t} + \frac{(\lambda-1)^2}{t^2}\left(\alpha\lambda + \frac{\beta}{\lambda}\right) + \gamma\frac{\lambda}{t} + \delta\frac{\lambda\,(\lambda+1)}{(\lambda-1)}. \tag{8.1}$$

In order to obtain the equivalent form of (8.1) which we use in Sect. 2, one can perform the following change of variables:

$$\lambda\,(t) = \lambda\,(u\,(t)) = -\tan^2\left(\pi u\,(t)\right).$$

As a result, $(\mathrm{d}\lambda/\mathrm{d}t)^2$ becomes zero. After the substitution $t(\tau) = \mathrm{e}^{\tau}$, the derivative $\mathrm{d}\lambda/\mathrm{d}t$ becomes zero as well, which leads to

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = \frac{\alpha}{2\pi}\frac{\sin(\pi u)}{\cos^3(\pi u)} + \frac{\beta}{2\pi}\frac{\cos(\pi u)}{\sin^3(\pi u)} + 2\gamma\mathrm{e}^{\tau}\sin(2\pi u) + \delta\mathrm{e}^{2\tau}\sin(4\pi u)\,. \tag{8.2}$$

8.2 Painlevé IV

The Painlevé IV equation has the following rational form:

$$\frac{\mathrm{d}^2\lambda}{\mathrm{d}t^2} = \frac{1}{2\lambda}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)^2 + \frac{3}{2}\lambda^3 + 4t\lambda^2 + 2\left(t^2-\alpha\right)\lambda + \frac{\beta}{\lambda}. \tag{8.3}$$

In order to obtain the equivalent form of (8.3) considered in Sect. 3, one can make the change of variables

$$\lambda(t) = u^2(t)\,,$$

which leads to

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = \frac{3u^5}{4} + 2tu^3 + \left(t^2-\alpha\right)u + \frac{\beta}{2u^3}. \tag{8.4}$$

8.3 Painlevé III

The Painlevé III equation has the following rational form:

$$\frac{\mathrm{d}^2\lambda}{\mathrm{d}t^2} = \frac{1}{\lambda}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}\lambda}{\mathrm{d}t} + \frac{1}{t}\left(\alpha\lambda^2+\beta\right) + \gamma\lambda^3 + \frac{\delta}{\lambda}. \tag{8.5}$$

In order to obtain the equivalent form of (8.5) which we use in Sect. 4, one can make the change of variables

$$\lambda(t) = \mathrm{e}^{u(t)}.$$

Substituting $t = \mathrm{e}^{\tau}$ into (8.5), we get

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = \alpha\mathrm{e}^{\tau+u} + \beta\mathrm{e}^{\tau-u} + \gamma\mathrm{e}^{2(\tau+u)} + \delta\mathrm{e}^{2(\tau-u)}. \tag{8.6}$$

9 Appendix 2: Elliptic Functions

The definitions and properties of elliptic functions used in the paper can be found in [25,36]. The main object is the theta function defined by

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau)=\sum_{j\in\mathbb{Z}} q^{\frac{1}{2}(j+a)^2}\mathbf{e}\left((j+a)(z+b)\right),$$

where $q=\mathbf{e}(\tau)\equiv\exp(2\pi\mathrm{i}\tau)$.

We also use the Eisenstein functions

$$\begin{aligned}\varepsilon_k(z)&=\lim_{M\to+\infty}\sum_{n=-M}^{M}(z+n)^{-k}, \qquad k\in\mathbb{N},\\ E_k(z)&=\lim_{M\to+\infty}\sum_{n=-M}^{M}\varepsilon_k(z+n\tau).\end{aligned} \tag{9.1}$$

To determine limits of Lax matrices, we use the series expansions of the following functions:

$$\vartheta(z)=\theta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(z,\tau)=\sum_{j\in\mathbb{Z}} q^{\frac{1}{2}\left(j+\frac{1}{2}\right)^2}\mathbf{e}\left(\left(j+\frac{1}{2}\right)\left(z+\frac{1}{2}\right)\right), \tag{9.2}$$

$$\begin{aligned}\phi(u,z)&=\frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)},\\ \varphi_\alpha\left(u+\omega_\beta,z\right)&=\mathbf{e}\left(z\partial_\tau\omega_\alpha\right)\phi\left(u+\omega_\beta,z\right),\\ f_\alpha\left(u+\omega_\beta,z\right)&=\mathbf{e}\left(z\partial_\tau\omega_\alpha\right)\partial_w\phi\left(w,z\right)|_{w=u+\omega_\beta},\end{aligned} \tag{9.3}$$

where $\omega_\alpha=\left\{0,\frac{1}{2},\frac{\tau}{2},\frac{1+\tau}{2}\right\}$. The functions satisfy the following well-known identities:

$$\begin{aligned}\phi(u,z)\phi(-u,z)&=E_2(z)-E_2(u),\\ \partial_u\phi(u,z)&=\phi(u,z)(E_1(u+z)-E_1(u)),\end{aligned} \tag{9.4}$$

parity

$$\begin{aligned}E_k(-z)&=(-1)^k E_k(z),\\ \vartheta(-z)&=-\vartheta(z),\\ \phi(u,z)&=\phi(z,u)=-\phi(-u,-z),\end{aligned}$$

and quasi-periodicity

$$\begin{aligned}E_1(z+1)&=E_1(z), \quad E_1(z+\tau)=E_1(z)-2\pi\mathrm{i},\\ E_2(z+1)&=E_2(z), \quad E_2(z+\tau)=E_2(z),\\ \vartheta(z+1)&=-\vartheta(z), \quad \vartheta(z+\tau)=-q^{-\frac{1}{2}}\mathbf{e}(-z)\vartheta(z),\\ \phi(u+1,z)&=\phi(u,z), \quad \phi(u+\tau,z)=\mathbf{e}(-z)\phi(u,z).\end{aligned} \tag{9.5}$$

Using definition (9.3), we reduce the expansion of $\varphi_\alpha\left(u+\omega_\beta, z\right)$ to the expansion of theta functions:

$$\phi(u-\sigma\tau, z-\varsigma\tau)=\frac{\vartheta(u+z-(\sigma+\varsigma)\tau)\vartheta'(0)}{\vartheta(u-\sigma\tau)\vartheta(z-\varsigma\tau)},$$

and for the expansion of theta functions we have:

$$\vartheta\,(z+\sigma\tau)=[1+\boldsymbol{o}\,(1)]\,q^{\left(-\lfloor\sigma\rfloor^2/2+\frac{1}{8}-\lfloor\sigma\rfloor\{\sigma\}-\{\sigma\}/2\right)}\,\mathrm{e}\left(-\lfloor\sigma\rfloor z-\frac{\lfloor\sigma\rfloor}{2}\right)\times\begin{cases}-2\sin(\pi z)\,, & \{\sigma\}=0,\\ -\mathrm{i}\;\mathrm{e}\left(-\frac{z}{2}\right), & \{\sigma\}>0,\end{cases}$$

where $\lfloor\sigma\rfloor$ is the integer part of σ and $\{\sigma\}$ is the fractional part of σ. This gives the following answer:

$$\phi(u+\sigma_u\tau, z+\sigma_z\tau)=(1+\boldsymbol{o}\,(1))$$
$$\times\begin{cases}-2\pi\mathrm{i}q^{-\sigma_u\sigma_z+\{\sigma_u\}\{\sigma_z\}}\mathrm{e}\,(-\lfloor\sigma_z\rfloor u-\lfloor\sigma_u\rfloor z)\,, & \{\sigma_u\}>0,\ \{\sigma_z\}>0,\ \{\sigma_u\}+\{\sigma_z\}<1,\\ 4\pi q^{-\sigma_u\sigma_z+\{\sigma_u\}\{\sigma_z\}}\sin(\pi\,(u+z)) \\ \times\mathrm{e}\left(-\left(\frac{1}{2}+\lfloor\sigma_z\rfloor\right)u-\left(\frac{1}{2}+\lfloor\sigma_u\rfloor\right)z\right), & \{\sigma_u\}>0,\ \{\sigma_z\}>0,\ \{\sigma_u\}+\{\sigma_z\}=1,\\ 2\pi\mathrm{i}q^{-\sigma_u\sigma_z+\{\sigma_u\}\{\sigma_z\}-\{\sigma_u+\sigma_z\}} \\ \times\mathrm{e}\,(-(1+\lfloor\sigma_z\rfloor)u-(1+\lfloor\sigma_u\rfloor)z), & \{\sigma_u\}>0,\ \{\sigma_z\}>0,\ \{\sigma_u\}+\{\sigma_z\}>1,\\ \pi q^{-\sigma_u\sigma_z}\mathrm{e}\,(-\lfloor\sigma_z\rfloor u-\sigma_u z)\,\dfrac{\mathrm{e}\left(-\frac{u}{2}\right)}{\sin(\pi u)}, & \{\sigma_u\}=0,\ \{\sigma_z\}>0,\\ \pi q^{-\sigma_u\sigma_z}\mathrm{e}\,(-\sigma_z u-\lfloor\sigma_u\rfloor z)\,\dfrac{\mathrm{e}\left(-\frac{z}{2}\right)}{\sin(\pi z)}, & \{\sigma_u\}>0,\ \{\sigma_z\}=0,\\ \pi q^{-\sigma_u\sigma_z}\mathrm{e}\,(-\sigma_z u-\sigma_u z)\,\dfrac{\sin(\pi(u+z))}{\sin(\pi z)\sin(\pi u)}, & \{\sigma_u\}=0,\ \{\sigma_z\}=0.\end{cases}$$

To evaluate the limits of $f_\alpha\left(u+\omega_\beta, z\right)$, we use the identity (9.4) and the expansion of $E_1(u-\sigma\tau)$ from [2]:

$$E_1(u-\sigma\tau)=2\pi\mathrm{i}\lfloor\sigma\rfloor+\begin{cases}\pi\cot(\pi u)+\mathbf{o}\,(1)\,, & \{\sigma\}=0,\\ \pi\mathrm{i}+2\pi\mathrm{i}q^{\{\sigma\}}\mathbf{e}(-u)+\mathbf{o}\left(q^{\{\sigma\}}\right), & 0<\{\sigma\}<\dfrac{1}{2},\\ \pi\mathrm{i}+2\pi\mathrm{i}q^{\frac{1}{2}}\left(\mathbf{e}(-u)-\mathbf{e}(u)\right)+\mathbf{o}\left(q^{\frac{1}{2}}\right), & \{\sigma\}=\dfrac{1}{2},\\ \pi\mathrm{i}-2\pi\mathrm{i}q^{1-\{\sigma\}}\mathbf{e}\,(u)+\mathbf{o}\left(q^{1-\{\sigma\}}\right), & \dfrac{1}{2}<\{\sigma\}<1.\end{cases}$$

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