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CONSTRUCTIVE APPROXIMATION

# The Eigenfunctions of the Hilbert Matrix

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**Abstract** For each noninteger complex number $\lambda$, the Hilbert matrix

$$H_\lambda = \left(\frac{1}{n+m+\lambda}\right)_{n,m\geq 0}$$

defines a bounded linear operator on the Hardy spaces $\mathcal{H}^p$, $1 < p < \infty$, and on the Korenblum spaces $\mathcal{A}^{-\tau}$, $\tau > 0$. In this work, we determine the point spectrum with multiplicities of the Hilbert matrix acting on these spaces. This extends to complex $\lambda$ results by Hill and Rosenblum for real $\lambda$. We also provide a closed formula for the eigenfunctions. They are in fact closely related to the associated Legendre functions of the first kind. The results will be achieved through the analysis of certain differential operators in the commutator of the Hilbert matrix.

**Keywords** Hilbert matrix · Integral operator · Eingenvalues · Eigenfunctions · Differential operators · Hypergeometric function · Associated Legendre functions of the first kind

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## 1 Introduction

For each $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, the Hilbert matrix of parameter $\lambda$ is

$$H_\lambda = \left( \frac{1}{n+m+\lambda} \right)_{n,m \geq 0}.$$

Hill [4], see also the work by Rosenblum [9], showed that any nonnegative complex number is a latent root of $H_\lambda$. He also determined the multiplicities of all latent roots of $H_\lambda$ for nonnegative real $\lambda$ and in particular for all positive integers, thus solving the eigenvalue problem for $\lambda$ real. The aim of this work is to extend this result to $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Indeed, if we restrict ourselves to the Hardy space or the Korenblum classes, then the eigenvalue problem is completely solved for complex $\lambda$. We will also provide an explicit formula for the eigenfunctions of $H_\lambda$ and identify them with the associated Legendre functions of the first kind.

In Sect. 2, we will review some known integral representations of the Hilbert matrix based on the Hankel form. We will also show that the Hilbert matrix preserves the Hardy spaces and the Korenblum spaces. The fact that $H_\lambda$ acts boundedly on $\mathcal{H}^p$, $1 < p < \infty$, has already been proved in [2] for $\lambda = 1$.

In Sect. 3, we will show that $H_\lambda$ almost commutes with certain differential operators $D$. Indeed, we are interested in the operator defined formally by

$$f \to H_\lambda D f - D H_\lambda f, \tag{1.1}$$

where $D$ is a differential operator.

Finally, in Sect. 4, we will provide a description of the point spectrum of the Hilbert matrix $H_\lambda$ acting on the Hardy spaces $\mathcal{H}^p$, for $p > 1$, and on the Korenblum spaces $\mathcal{A}^{-\tau}$, $\mathcal{A}_0^{-\tau}$, for $0 < \tau < 1$.

## 2 Integral Operator Representations

Let $\mathbb{D}$ denote the unit disk. For each $1 \leq p < \infty$, the Hardy space $\mathcal{H}^p$ consists of those functions $f$ analytic on $\mathbb{D}$ for which the norm

$$\|f\|_p^p = \sup_{0<r<1} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^p \frac{d\theta}{2\pi}$$

is finite. We also denote by $\mathcal{H}^\infty$ the space of bounded analytic functions on $\mathbb{D}$ endowed with the supremum norm.

In addition, we will also consider the Korenblum spaces, which are special cases of weighted spaces of analytic functions. For each real number $\tau > 0$, the Banach space $\mathcal{A}^{-\tau}$ consists of those functions $f$ analytic on $\mathbb{D}$ for which the norm

$$\|f\|_\tau = \sup_{\mathbb{D}} \left(1 - |z|\right)^\tau \left| f(z) \right|$$

is finite. The Banach subspace $\mathcal{A}_0^{-\tau}$ consists of those $f$ in $\mathcal{A}^{-\tau}$ for which

$$\big(1-|z|\big)^\tau \big|f(z)\big| \to 0 \quad \text{as } |z| \to 1^-.$$

The Hilbert matrix of parameter $\lambda$ can be represented with the help of the Hankel form with symbol $t^\lambda/\kappa$, where $\kappa = e^{2\pi i\lambda} - 1$ and $\lambda \in \mathbb{C} \setminus \mathbb{Z}$. More precisely, from the identity

$$\frac{i}{\kappa}\int_0^{2\pi} e^{i\theta(m+n+\lambda)}\,\mathrm{d}\theta = \frac{1}{m+n+\lambda},$$

we see that the Hilbert matrix of parameter $\lambda$ extends to a bounded operator on the Hardy space $\mathcal{H}^2$, which will also be denoted by $H_\lambda$, defined by

$$(H_\lambda f)(z) = \frac{i}{\kappa}\int_0^{2\pi} \frac{f(e^{i\theta})e^{i\theta\lambda}}{1-e^{i\theta}z}\,\mathrm{d}\theta.$$

Obviously, it is also possible to define $H_\lambda f$ for $f$ in the Hardy space $\mathcal{H}^1$. However, this representation is not easy to work with. Therefore, we shall use alternative integral representations obtained by changing the path of integration in the above formula. The more classical representation for the case $f \in \mathcal{H}^1$ and $\Re\lambda > 0$ is obtained in the following way: the Hilbert matrix of parameter $\lambda$ is

$$H_\lambda = \left(\frac{1}{m+n+\lambda}\right)_{m,n\geq 0}.$$

Thus for $f(z) = \sum_{n=0}^\infty a_n z^n \in \mathcal{H}^1$, Hardy's inequality (see [3], for instance) implies that

$$\sum_{n=0}^\infty \frac{|a_n|}{n+1} \leq \pi \|f\|_{\mathcal{H}^1}.$$

Hence the power series

$$F(z) = \sum_{n=0}^\infty \left(\sum_{m=0}^\infty \frac{a_m}{m+n+\lambda}\right) z^n$$

has bounded coefficients, thus its radius of convergence is greater than or equal to 1 and we obtain a well-defined analytic function $H_\lambda f$ on the disc $\mathbb{D}$ for each $f \in \mathcal{H}^1$. A standard computation shows that, for $\Re\lambda > 0$, we have

$$(H_\lambda f)(z) = \lim_{r\to 1}\int_0^r \frac{f(t)t^{\lambda-1}}{1-tz}\,\mathrm{d}t = \int_0^1 \frac{f(t)t^{\lambda-1}}{1-tz}\,\mathrm{d}t.$$

Indeed, if we set

$$\big(H_\lambda^r f\big)(z) = \sum_{n=0}^\infty z^n \sum_{m=0}^\infty \frac{a_m r^{m+n+\lambda}}{m+n+\lambda},$$

then

$$\begin{aligned}
\big(H_\lambda^r f\big)(z) &= \sum_{n=0}^{\infty} z^n \sum_{m=0}^{\infty} a_m \int_0^r t^{m+n+\lambda-1}\,\mathrm{d}t \\
&= \sum_{n=0}^{\infty} z^n \int_0^r \left(\sum_{m=0}^{\infty} a_m t^m\right) t^{n+\lambda-1}\,\mathrm{d}t \\
&= \sum_{n=0}^{\infty} z^n \int_0^r f(t) t^{n+\lambda-1}\,\mathrm{d}t \\
&= \sum_{n=0}^{\infty} \int_0^r f(t) t^{\lambda-1}\big(t^n z^n\big)\,\mathrm{d}t \\
&= \int_0^r f(t) t^{\lambda-1}\left(\sum_{n=0}^{\infty} t^n z^n\right)\mathrm{d}t \\
&= \int_0^r \frac{f(t)t^{\lambda-1}}{1-tz}\,\mathrm{d}t.
\end{aligned}$$

Therefore,

$$(H_\lambda f)(z) = \int_0^1 \frac{f(t)t^{\lambda-1}}{1-tz}\,\mathrm{d}t.$$

A similar argument yields that the same representation holds true when $f$ is in the Korenblum space $\mathcal{A}^{-\tau}$, $0<\tau<1$. For the remaining case $\Re\lambda \le 0$, we shall use a similar approach. We will integrate along the boundary of a Stolz angle at 1. Throughout the rest of this work, $C$ will be the closed path defined by the boundary of the Stolz angle $\{z\in\mathbb{D} : |1-z| \le \sigma(1-|z|)\}$ where $\sigma > 1$ is fixed. We assume that $C$ is positively oriented, that is, in the counterclockwise sense. Now, for $0<\varepsilon<1$, we see that the straight line with equation $\Re z = \varepsilon$ and $C$ meet exactly at two conjugate points $a_\varepsilon$ and $\overline{a_\varepsilon}$, where $\Im a_\varepsilon > 0$. To fix notation, for any nonnegative intergers $n$ and $m$, we set $z^{n+m+\lambda-1} = e^{(n+m+\lambda-1)\log z}$, where $\log z = \ln|z| + i\arg z$ with $\arg z \in [0, 2\pi)$; of course, other definitions are possible in the argument below. Let $C_\varepsilon$ denote the subarc of $C$ that goes from $a_\varepsilon$ to $\overline{a_\varepsilon}$ in the counterclockwise sense. Next, consider the closed contour $C'_\varepsilon$ obtained by adding the line segments $\Im z = \pm\Im a_\varepsilon$ and a semicircle (on the left half-plane) of center 0 and radius $\Im a_\varepsilon$ to $C_\varepsilon$. With $f$ in $\mathcal{H}^1$ or $\mathcal{A}^{-\tau}$, using Cauchy's theorem, we have

$$\int_{C'_\varepsilon} \frac{f(t)t^{\lambda-1}}{1-tz}\,\mathrm{d}t = 0.$$

Thus, making $\varepsilon$ tend to 0, one easily sees that the following integral representation holds:

$$(H_\lambda f)(z) = \frac{1}{\kappa}\int_C \frac{f(t)t^{\lambda-1}}{1-tz}\,\mathrm{d}t. \tag{2.1}$$

Now, (2.1) makes sense also for $\Re\lambda \le 0$, which provides an integral representation for $H_\lambda$ whenever $\lambda \in \mathbb{C} \setminus \mathbb{Z}$. Moreover, as the above argument shows, this representation is independent of the aperture $\sigma$ of the Stolz angle.

**Theorem 2.1** *Let $f$ be analytic on $\mathbb{D}$ and integrable on $[0,1]$, and let $H_\lambda f$ be as in* (2.1). *Then*

(i) *if $f \in L^p([0,1])$, $1 < p < \infty$, then $H_\lambda f \in \mathcal{H}^p$;*
(ii) *if $0 < \tau < 1$ and $|f(x)| = O((1-x)^{-\tau})$ as $x \to 1$, then $H_\lambda f \in \mathcal{A}^{-\tau}$;*
(iii) *if $0 < \tau < 1$ and $|f(x)| = o((1-x)^{-\tau})$ as $x \to 1$, then $H_\lambda f \in \mathcal{A}_0^{-\tau}$.*

*In particular, if $X$ is one of the spaces $\mathcal{H}^p$, $1 < p < \infty$, $\mathcal{A}^{-\tau}$ or $\mathcal{A}_0^{-\tau}$, $0 < \tau < 1$, then $H_\lambda$ is a bounded linear operator from $X$ into itself. Moreover, $H_\lambda$ is the unique operator $T$ on $X$ such that*

$$Tz^n = \sum_{m=0}^{\infty} \frac{z^m}{m+n+\lambda},$$

*where by abuse of notation we write $z$ for the identity function on $\mathbb{D}$.*

*Proof* To prove (i), observe that if $h \in \mathcal{H}^q$, with $1/p + 1/q = 1$, a straightforward computation shows that

$$\int_0^{2\pi} (H_\lambda f)\big(re^{i\theta}\big)\overline{h\big(e^{i\theta}\big)}\,\frac{\mathrm{d}\theta}{2\pi} = \frac{1}{\kappa}\int_C f(t)\overline{h(r\bar{t})}t^{\lambda-1}\,\mathrm{d}t.$$

Since the arc length measure on $C$ is a Carleson measure, by Hölder's inequality, we find that

$$\left|\int_0^{2\pi} (H_\lambda f)\big(re^{i\theta}\big)\overline{h\big(e^{i\theta}\big)}\,\frac{\mathrm{d}\theta}{2\pi}\right| \le M\|f\|_p\|h\|_q,$$

where $M$ is a positive constant independent of $f$ and $h$. By duality, this implies

$$\int_0^{2\pi} \big|(H_\lambda f)\big(re^{i\theta}\big)\big|^p\,\frac{\mathrm{d}\theta}{2\pi} \le M^p\|f\|_p^p.$$

To prove (ii), observe that if $|f(x)| \le M(1-x)^{-\tau}$, where $M$ is a positive constant and $0 < \tau < 1$, then

$$\begin{aligned}\big(1-|z|\big)^\tau\big|(H_\lambda f)(z)\big| &\le M\big(1-|z|\big)^\tau\int_C \frac{|f(t)|}{1-|t||z|}|\mathrm{d}t| \\ &\le M\|f\|_\tau\big(1-|z|\big)^\tau\int_C \frac{(1-|t|)^{-\tau}}{1-|t||z|}|\mathrm{d}t|\end{aligned}$$

for all $z \in \mathbb{D}$. It is easy to see that the last term in the above display is uniformly bounded in $z \in \mathbb{D}$, which shows that $H_\lambda f \in \mathcal{A}^{-\tau}$.

To prove (iii), observe first that

$$\big(1-|z|\big)^\tau \int_{C\cap\{|t|\le 1-\delta\}} \frac{f(t)t^{\lambda-1}}{1-tz}\,\mathrm{d}t \to 0 \quad \text{as } |z|\to 1^-.$$

Hence, the previous estimate yields that

$$\overline{\lim_{|z|\to 1}}\big(1-|z|\big)^\tau \big|(H_\lambda f)(z)\big| \le M_1 \sup_{1-\delta\le|t|<1} \big(1-|t|\big)^\tau \big|f(t)\big|$$

for a positive constant $M_1>0$ and for any $0<\delta<1$. Now, the boundedness of $H_\lambda$ on the spaces considered in the statement of the theorem follows immediately from the closed graph theorem. Finally, the fact that

$$H_\lambda z^n = \sum_{m=0}^{\infty} \frac{z^m}{m+n+\lambda}$$

is obvious, since

$$\frac{1}{\kappa}\int_C t^{m+n+\lambda-1}\,\mathrm{d}t = \frac{1}{m+n+\lambda}.$$

The uniqueness assertion follows immediately from the fact that polynomials are dense in $\mathcal{H}^p$, $1<p<\infty$, and $\mathcal{A}_0^{-\tau}$, $0<\tau<1$ and weak-star dense in $\mathcal{A}^{-\tau}$. □

*Remark* The case in which $X=\mathcal{H}^p$, $1<p<\infty$, and $\lambda=1$ was proved earlier in [2].

## 3 Differential Operators in the Commutator

The purpose of this section is to prove that $H_\lambda$ almost commutes with certain differential operators $D$. Indeed, we are interested in the operator defined formally by

$$f \to H_\lambda D f - D H_\lambda f. \tag{3.1}$$

We shall only investigate linear differential operators of second order with polynomial coefficients. These are defined by

$$Df = q_3 f'' + q_2 f' + q_1 f, \tag{3.2}$$

where

$$q_3(z)=\sum_0^3 \alpha_j z^j, \qquad q_2(z)=\sum_0^2 \beta_j z^j, \qquad q_1(z)=\gamma_1 z.$$

The main result of this section provides a class of such operators where the commutator in question has rank one. We shall assume throughout that $\lambda\in\mathbb{C}\setminus\mathbb{Z}$.

**Theorem 3.1** *Let $D$ be a differential operator as in (3.2). Assume that the polynomials $q_1, q_2, q_3$ in (3.2) satisfy*

$$q_1(z) = \big[\alpha\lambda(\lambda+1) - \lambda\gamma\big]z,$$
$$q_2(z) = (z-1)\big[\big(2\alpha(\lambda+1) - \gamma\big)z - \gamma\big],$$
$$q_3(z) = \alpha z(z-1)^2$$

*for some constants $\alpha, \gamma \in \mathbb{C}$. Then for every $f \in \mathcal{A}^{-\tau}$, $0 < \tau < 1$, we have*

$$H_\lambda Df - DH_\lambda f = \frac{1}{\kappa}(\lambda-1)(\alpha\lambda - \gamma)\int_C f(t)t^{\lambda-2}\,\mathrm{d}t,$$

*where $C$ is the contour in (2.1).*

*Proof* By means of the integral representation formula (2.1), we see that

$$J(z) = \kappa(H_\lambda Df)(z) - \kappa(DH_\lambda f)(z) = \int_C \frac{(Df)(t)t^{\lambda-1}}{1-tz}\,\mathrm{d}t - D\int_C \frac{f(t)t^{\lambda-1}}{1-tz}\,\mathrm{d}t. \tag{3.3}$$

Observe that $Df \in L^1(C)$ whenever $f \in A^{-\tau}$ so that the integrals involved make sense. Now write $z = \zeta^{-1}$, for $|\zeta| > 1$, and for a polynomial $p$ of degree $n$, write

$$\tilde{p}(z) = z^n p\left(\frac{1}{z}\right).$$

With this notation, from (3.3), we obtain

$$J\left(\frac{1}{\zeta}\right) = \zeta\int_C \frac{(Df)(t)t^{\lambda-1}}{\zeta - t}\,\mathrm{d}t - 2\tilde{q}_3(\zeta)\int_C \frac{f(t)t^{\lambda+1}}{(\zeta-t)^3}\,\mathrm{d}t$$
$$- \tilde{q}_2(\zeta)\int_C \frac{f(t)t^{\lambda}}{(\zeta-t)^2}\,\mathrm{d}t - \tilde{q}_1(\zeta)\int_C \frac{f(t)t^{\lambda-1}}{\zeta-t}\,\mathrm{d}t.$$

Now assume that

$$f(z) = (1-z)^2 g(z), \tag{3.4}$$

where $g \in \mathcal{A}^{-\tau}$. On $C$, we have $1 - |t| \le |1-t| \le \sigma(1-|t|)$, where $\sigma$ is a fixed number greater than 1, and by (3.4), we see that $|f(t)| = O(|1-t|^{2-\tau})$ as $t \to 1$ on $C$. Thus we can integrate by parts in equality (3.3) to obtain

$$J\left(\frac{1}{\zeta}\right) = \zeta\int_C \frac{(Df)(t)t^{\lambda-1}}{\zeta - t}\,\mathrm{d}t - \tilde{q}_3(\zeta)\int_C \frac{(f(t)t^{\lambda+1})''}{\zeta-t}\,\mathrm{d}t$$
$$+ \tilde{q}_2(\zeta)\int_C \frac{(f(t)t^{\lambda})'}{\zeta-t}\,\mathrm{d}t - \tilde{q}_1(\zeta)\int_C \frac{f(t)t^{\lambda-1}}{\zeta-t}\,\mathrm{d}t.$$

Next we set

$$\tilde{q}_i(\zeta) = \tilde{q}_i(\zeta) - \tilde{q}_i(t) + \tilde{q}_i(t), \quad i = 1, 2, 3,$$
$$\zeta = \zeta - t + t,$$

and integrate by parts again using the same argument together with the fact that both $\tilde{q}_3$ and $\tilde{q}_2$ have degree 2. We have

$$\begin{aligned}
J\left(\frac{1}{\zeta}\right) &= \int_C \frac{(Df)(t)t^\lambda}{\zeta - t}\,\mathrm{d}t \\
&\quad - \int_C \frac{\tilde{q}_3(t)(f(t)t^{\lambda+1})'' - \tilde{q}_2(t)(f(t)t^\lambda)' + \tilde{q}_1(t)f(t)t^{\lambda-1}}{\zeta - t}\,\mathrm{d}t \\
&\quad + \int_C (Df)(t)t^{\lambda-1}\,\mathrm{d}t - \int_C \frac{\tilde{q}_3(\zeta) - \tilde{q}_3(t)}{\zeta - t}\big(f(t)t^{\lambda+1}\big)'' \\
&\quad + \int_C \big(\tilde{q}_2(\zeta) - \tilde{q}_2(t)\big)\big(f(t)t^\lambda\big)'\,\mathrm{d}t \\
&= \int_C \frac{(Df)(t)t^\lambda - \tilde{q}_3(t)(f(t)t^{\lambda+1})'' + \tilde{q}_2(t)(f(t)t^\lambda)' - \tilde{q}_1(t)f(t)t^{\lambda-1}}{\zeta - t}\,\mathrm{d}t \\
&\quad + \int_C (Df)(t)t^{\lambda-1}\,\mathrm{d}t - q_2(0)\int_C f(t)t^\lambda\,\mathrm{d}t.
\end{aligned}$$

Observe now that

$$\begin{aligned}
&(Df)(t)t^\lambda - \tilde{q}_3(t)\big(f(t)t^{\lambda+1}\big)'' + \tilde{q}_2(t)\big(f(t)t^\lambda\big)' - \tilde{q}_1(t)f(t)t^{\lambda-1} \\
&\quad = f''(t)\big[q_3(t)t^\lambda - \tilde{q}_3(t)t^{\lambda+1}\big] + f'(t)\big[q_2(t)t^\lambda - 2(\lambda+1)\tilde{q}_3(t)t^\lambda + \tilde{q}_2(t)t^\lambda\big] \\
&\qquad + f(t)\big[q_1(t)t^\lambda - \lambda(\lambda+1)\tilde{q}_3(t)t^{\lambda-1} + \tilde{q}_2(t)\lambda t^{\lambda-1} - \tilde{q}_1(t)t^{\lambda-1}\big],
\end{aligned}$$

which vanishes under the conditions in the hypotheses. Furthermore, integrating by parts once again, we obtain

$$\begin{aligned}
\int_C (Df)(t)t^{\lambda-1}\,\mathrm{d}t &= \int_C q_3(t)f''(t)t^{\lambda-1} + q_2(t)f'(t)t^{\lambda-1} + q_1(t)f(t)t^{\lambda-1}\,\mathrm{d}t \\
&= \int_C f(t)\big[\big(q_3(t)t^{\lambda-1}\big)'' - \big(q_2(t)t^{\lambda-1}\big)' + q_1(t)t^{\lambda-1}\big]\,\mathrm{d}t \\
&= \int_C f(t)\big[\gamma t^\lambda + (\lambda-1)(\alpha\lambda - \gamma)t^{\lambda-2}\big]\,\mathrm{d}t.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_C (Df)(t)t^{\lambda-1}\,\mathrm{d}t - q_2(0)\int_C f(t)t^\lambda\,\mathrm{d}t \\
&\quad = \int_C f(t)\big[\gamma t^\lambda + (\lambda-1)(\alpha\lambda - \gamma)t^{\lambda-2}\big]\,\mathrm{d}t - \int_C \gamma f(t)t^\lambda\,\mathrm{d}t \\
&\quad = \int_C f(t)(\lambda-1)(\alpha\lambda - \gamma)t^{\lambda-2}\,\mathrm{d}t,
\end{aligned}$$

and the equation in the statement is satisfied for every function $f$ as in (3.4). To remove this extra assumption, let $f \in \mathcal{A}^{-\tau}$ be arbitrary and for $0 < a < 1$ set

$$f_a(z) = \left(\frac{1-z}{1-az}\right)^2 f(z).$$

We then have

$$(H_\lambda D f_a)(z) - (D H_\lambda f_a)(z) = \frac{1}{\kappa} \int_C f_a(t)(\lambda-1)(\alpha\lambda-\gamma)t^{\lambda-2}\,\mathrm{d}t.$$

We claim that

(a) $(H_\lambda D f_a)(z) \to (H_\lambda D f)(z)$,

(b) $(D H_\lambda f_a)(z) \to (D H_\lambda f)(z)$,

(c) $\displaystyle\int_C f_a(t)(\lambda-1)(\alpha\lambda-\gamma)t^{\lambda-2}\,\mathrm{d}t \to \int_C f(t)(\lambda-1)(\alpha\lambda-\gamma)t^{\lambda-2}\,\mathrm{d}t$

as $a \to 1^-$, for all $z \in \mathbb{D}$.

To prove (a), we write

$$(H_\lambda D f_a)(z) = \frac{1}{\kappa} \int_C \frac{(D f_a)(t)}{1-tz}\,\mathrm{d}t \tag{3.5}$$

and note that

$$f_a(z) = \Phi_a(z) f(z),$$

where

$$\Phi_a(z) = \left(\frac{1-z}{1-az}\right)^2$$

satisfies $|\Phi_a(z)| \le 4$ for each $z \in \mathbb{D}$. It is also easy to see that

$$\left|\Phi_a'(z)\right| \le 2\left(1-|z|\right)^{-1} \quad \text{and} \quad \left|\Phi_a''(z)\right| \le 2\left(1-|z|\right)^{-2}$$

for each $z \in \mathbb{D}$. Using once again that on the contour $C$ we have that $1-|t| \le |1-t| \le \sigma(1-|t|)$, we may obtain the estimates

$$\left|f_a''(t)\right| \le M_1\left(\frac{|f(t)|}{|1-t|^2} + \frac{|f'(t)|}{|1-t|} + \left|f''(t)\right|\right),$$
$$\left|f_a'(t)\right| \le M_2\left(\frac{|f(t)|}{|1-t|} + \left|f'(t)\right|\right)$$

for all $t \in C$ and where $M_i$, $i = 1, 2$, are positive constants. Hence from the particular form of these differential operators, it follows that

$$\begin{aligned}\left|(D f_a)(t)\right| &\le M_3\left[\left(1-|t|\right)^2\left|f''(t)\right| + \left(1-|t|\right)\left|f'(t)\right| + \left|f(t)\right|\right] \\ &\le M_f\left(1-|t|\right)^{-\tau},\end{aligned}$$

where $M_3$ is another constant and $M_f$ is a constant that depends only on $f$. Thus (a) follows from (3.5) and the bounded Lebesgue convergence theorem.

The proof of (b) is much easier. Indeed, the bounded Lebesgue convergence theorem applied to the identity (2.1) immediately yields

$$(H_\lambda f_a)(z) \to (H_\lambda f)(z)$$

uniformly on compacts, as $a \to 1^-$. Hence

$$(DH_\lambda f_a)(z) \to (DH_\lambda f)(z)$$

as $a \to 1^-$, for all $z \in \mathbb{D}$.

Finally, (c) also follows by a further application of the bounded Lebesgue convergence theorem. □

We shall only make use of two operators from this family, namely, those obtained for $\alpha = 0;\ \gamma = -1$ and $\alpha = 1;\ \gamma = \lambda$; that is,

$$(D_{1,\lambda} f)(z) = \left(z^2 - 1\right) f'(z) + \lambda z f(z),$$
$$(D_{2,\lambda} f)(z) = z(z-1)^2 f''(z) + (z-1)\left[(\lambda+2)z - \lambda\right] f'(z) + \lambda z f(z).$$

Obviously, these two operators generate the linear space of operators considered in Theorem 3.1 The next proposition is valid for any complex region whenever we can define on it a branch of $(1-z)^{\frac{-\lambda+\nu}{2}}$ and $(1+z)^{\frac{-\lambda-\nu}{2}}$. In particular, since we will need them to be analytic on the unit disk $\mathbb{D}$, we can always define a branch of the former on $\mathbb{C} \setminus [1, +\infty)$ and a branch of the latter on $\mathbb{C} \setminus (-\infty - 1]$.

**Proposition 3.2**

(i) *For $\nu \in \mathbb{C}$, the space of solutions of the equation*

$$D_{1,\lambda} f = \nu f \tag{3.6}$$

*is one-dimensional and it is spanned by*

$$f_\lambda(z) = (1-z)^{\frac{-\lambda+\nu}{2}} (1+z)^{\frac{-\lambda-\nu}{2}}.$$

(ii) *The solutions of the equation*

$$D_{1,\lambda} g - \nu g = f_\lambda$$

*are*

$$g_\lambda(z) = \left(\frac{1}{2}\log\left(\frac{z-1}{z+1}\right) + k\right)(1-z)^{\frac{-\lambda+\nu}{2}} (1+z)^{\frac{-\lambda-\nu}{2}},$$

*where $k \in \mathbb{C}$ is arbitrary.*

*Proof* (i) Equation (3.6) is equivalent to

$$f'(z) = -\frac{\lambda z - \nu}{z^2 - 1} f(z).$$

Thus

$$f(z) = ce^{-\int \frac{\lambda z - \nu}{z^2 - 1} dz} = c(1 - z)^{\frac{-\lambda+\nu}{2}} (1 + z)^{\frac{-\lambda-\nu}{2}},$$

where $c$ is constant.

(ii) The equation now is

$$\left(z^2 - 1\right) g'(z) + (\lambda z - \nu) g(z) = f_\lambda(z). \tag{3.7}$$

The solution of the corresponding homogeneous equation is provided by (i), that is,

$$g(z) = k(1 - z)^{\frac{-\lambda+\nu}{2}} (1 + z)^{\frac{-\lambda-\nu}{2}}, \quad k \in \mathbb{C}.$$

Now we look for a particular solution of (3.7) of the form

$$g_p(z) = k_z (1 - z)^{\frac{-\lambda+\nu}{2}} (1 + z)^{\frac{-\lambda-\nu}{2}},$$

which we substitute in (3.7) to obtain

$$k_z' = \frac{1}{z^2 - 1} .$$

Thus,

$$k_z = \frac{1}{2} \log\left(\frac{z - 1}{z + 1}\right).$$

Therefore, the general solution of (3.7) is

$$g(z) = \left(\frac{1}{2} \log\left(\frac{z - 1}{z + 1}\right) + k\right) (1 - z)^{\frac{-\lambda+\nu}{2}} (1 + z)^{\frac{-\lambda-\nu}{2}}.$$

□

As might be expected, the spectral theory of $D_{2,\lambda}$ is more complicated. However, it turns out that the eigenvalue problem

$$D_{2,\lambda} f = \nu f$$

reduces to the classical hypergeometric equation

$$z(1 - z) f'' + \left[\gamma - (\alpha + \beta + 1) z\right] f' - \alpha\beta f = 0. \tag{3.8}$$

In fact, the reduction of the above equation to the hypergeometric one has been intensively studied (see [7], for instance).

As usual, we denote the hypergeometric function by

$$\begin{aligned}{}_2F_1(\alpha,\beta;\gamma;z) &= 1+\frac{\alpha\beta}{1!\gamma}z+\frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)}z^2+\cdots\\ &= \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!},\end{aligned}$$

where $(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1)$. In general, the radius of convergence is 1, except when $\alpha$ or $\beta$ are nonpositive integers, in which case the series reduces to a polynomial.

**Theorem 3.3** *For $\lambda\in\mathbb{C}\setminus\mathbb{Z}$ and $\nu\in\mathbb{C}$, the solutions of the eigenvalue problem*

$$D_{2,\lambda}f=\nu f$$

*analytic in $\mathbb{D}$ form a one-dimensional space spanned by*

$$f(z)=(1-z)^a\,{}_2F_1(a+1,a+\lambda;\lambda;z)=(1-z)^{a'}\,{}_2F_1\bigl(a'+1,a'+\lambda;\lambda;z\bigr),\quad(3.9)$$

*where $a$ and $a'$ are solutions of the quadratic equation*

$$w^2+w+\lambda=\nu.\quad(3.10)$$

*Moreover, if $a,\ a'$ are ordered by $\Re a'\ge-\frac{1}{2}\ge\Re a$, then*:

(i) *if $a\neq-\frac{1}{2}$ and $a+1-\lambda=-a'-\lambda\notin\mathbb{N}\cup\{0\}$, then the limit*

$$\lim_{x\to1^-}(1-x)^{-\Re a}\bigl|f(x)\bigr|$$

*exists, is finite and nonzero*;

(ii) *if $a\neq-\frac{1}{2}$ and $\lambda-a-1=a'+\lambda=-n$, $n\in\mathbb{N}\cup\{0\}$, then $\Re\lambda\le\frac{1}{2}$ and*

$$f(z)=(1-z)^{a'}Q(z),$$

*where $Q(z)$ is a polynomial of degree $n$*;

(iii) *if $a=-\frac{1}{2}$, then*

$$\bigl|f(z)\bigr|=O\biggl(\bigl(1-|z|\bigr)^{-1/2}\log\frac{1}{1-|z|}\biggr)\quad\text{as } |z|\to1^-.$$

*Proof* We begin with the substitution $f(z)=(1-z)^a g(z)$ to obtain the equation

$$\begin{aligned}&z(z-1)^2g''(z)+(z-1)\bigl(z(2a+\lambda+2)-\lambda\bigr)g'(z)\\ &\quad+\bigl(z\bigl(a^2+a+a\lambda+\lambda\bigr)-a\lambda-\nu\bigr)g(z)=0.\end{aligned}$$

If $a^2+a+\lambda=\nu$, the above equation becomes the well-known hypergeometric equation, see (3.8), with parameters $\alpha, \beta, \gamma$ subject to

$$\begin{aligned}\alpha+\beta&=2a+\lambda+1;\\ \alpha\beta&=a\lambda+\nu=a^2+a+a\lambda+\lambda=(a+1)(a+\lambda);\\ \gamma&=\lambda.\end{aligned}$$

Obviously, this leads to the choice $\alpha=a+1$, $\beta=a+\lambda$, $\gamma=\lambda$, and the solutions of the eigenvalue problem are those provided in the statement of the theorem. The fact that our solution is independent of the choice of the root $a$ of (3.10) is Euler's Formula, see [1] (in older literature it is also called Kummer's first formula, see [5] p. 248, formula (9.5.3) or [8]), a well-known identity for hypergeometric functions.

To prove (i), we use the results in [5], Sect. 9.3, to find the following known Gauss' formula

$$\lim_{x\to 1^-} {}_2F_1(a+1,a+\lambda;\lambda;x)=\frac{\Gamma(\lambda)\Gamma(-2a-1)}{\Gamma(\lambda-a-1)\Gamma(-a)}\neq 0,$$

and the right-hand side is finite and nonzero under the hypothesis of the statement.

The fact that (ii) holds follows directly from the power series expansion of the hypergeometric function ${}_2F_1(a+1,a+\lambda;\lambda;z)$.

Finally, to prove (iii), we observe that the coefficients of ${}_2F_1(a+1,a+\lambda;\lambda;z)$ satisfy

$$\left|\frac{(1/2)_n(-1/2+\lambda)_n}{n!(\lambda)_n}\right|=O\left(\frac{1}{n}\right)\quad n\to\infty,$$

hence

$$\left|{}_2F_1(a+1,a+\lambda;\lambda;z)\right|\leq M\log\frac{1}{1-|z|},$$

where $M$ is a positive constant, which gives the desired estimate. □

*Remark* Using the Pfaff transformation (see for example [8]), we have

$$\begin{aligned}f_a(z)&=(1-z)^a\,{}_2F_1(a+1,a+\lambda;\lambda;z)=\frac{1}{1-z}\,{}_2F_1\big(a+1,-a;\lambda;z/(z-1)\big)\\ &=\frac{\Gamma(\lambda)(-z)^{1/2-\lambda/2}}{1-z}P_a^{1-\lambda}\left(\frac{1+z}{1-z}\right),\end{aligned}$$

where $P_a^{1-\lambda}$ denotes the associated Legendre function of the first kind of parameters $a$ and $1-\lambda$ (see p. 192 in [5]). Observe that $P_a^{1-\lambda}((1+z)/(1-z))$ can be seen as a function analytic on the unit disk multiplied by $(-z)^{-1/2+\lambda/2}$, which justifies the last equality above.

**Corollary 3.4** *For each $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ and for each $a \in \mathbb{C}$, set*

$$f_a(z) = (1-z)^a \, {}_2F_1(a+1, a+\lambda; \lambda; z).$$

*Then, for $-\frac{1}{2} \geq \Re a > -1$, we have that $H_\lambda f_a$ is defined and satisfies*

$$H_\lambda f_a = -\frac{\pi}{\sin \pi a} f_a.$$

*Moreover,*

(i) *If $a \neq -\frac{1}{2}$ and $a+1-\lambda \notin \mathbb{N} \cup \{0\}$, then $f_a \in \mathcal{A}^{-\tau}$ if and only if $\tau > -\Re a$ and $f_a \in \mathcal{H}^p$ if and only if $\frac{1}{p} > \Re a$.*
(ii) *If $a \neq -\frac{1}{2}$ and $\lambda - a - 1 = -n$ with $n \in \mathbb{N} \cup \{0\}$, then $f_a \in \mathcal{A}^{-\tau}$ for all $\tau \geq \Re a + 1$, $f_a \in \mathcal{A}_0^{-\tau}$ for all $\tau > \Re a + 1$, and $f_a \in \mathcal{H}^p$ whenever $\frac{1}{p} > \Re a + 1$.*
(iii) *If $a = -\frac{1}{2}$, then $f_a \in \mathcal{A}_0^{-\tau}$ for $\tau > \frac{1}{2}$ and $f_a \in \mathcal{H}^p$ whenever $p < 2$.*

*Proof* For each $a \in \mathbb{C}$, we can find $\nu \in \mathbb{C}$ such that $a$ is a root of (3.10). If $a'$ denotes the other root of this equation, then $f_a = f_{a'}$. Hence, by Theorems 2.1 and 3.3, either $H_\lambda f_a$ or $H_\lambda f_{a'}$ is well defined. Moreover, from Theorem 3.3 we find that

$$(D_{2,\lambda} - \nu I) H_\lambda f_a = H_\lambda (D_{2,\lambda} - \nu I) f_a = 0;$$

i.e., $H_\lambda f_a \in \ker(D_{2,\lambda} - \nu I)$, which, by Theorem 3.3, has dimension one. Thus,

$$H_\lambda f_a = \mu f_a$$

for some $\mu \in \mathbb{C}$. Since $f_a(0) = 1$, we have

$$\mu = (H_\lambda f_a)(0) = \frac{1}{\kappa} \int_C t^{\lambda-1} f_a(t)\, \mathrm{d}t.$$

In order to compute the value of the right-hand side, we need the identity

$$\frac{1}{\kappa} \int_C t^{x-1}(1-t)^{y-1}\, \mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{3.11}$$

where $\kappa = e^{2\pi i x} - 1$. This equality holds whenever $\Re y > 0$ and $\Gamma(x)$, $\Gamma(y)$, and $\Gamma(x+y)$ are defined. Indeed, this is well known for $\Re x > 0$, hence for general $x$, it follows by the identity theorem for analytic functions. With (3.11) at hand, we compute

$$
\begin{aligned}
(H_\lambda f_a)(0) &= \frac{1}{\kappa}\int_C t^{\lambda-1} f_a(t)\,\mathrm{d}t \\
&= \frac{1}{\kappa}\int_C t^{\lambda-1}(1-t)^a\,{}_2F_1(a+1,a+\lambda;\lambda;t)\,\mathrm{d}t \\
&= \frac{1}{\kappa}\sum_{n=0}^{\infty}\frac{(a+1)_n(a+\lambda)_n}{n!(\lambda)_n}\int_C t^{\lambda+n-1}(1-t)^a\,\mathrm{d}t \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(a+1+n)}{\Gamma(a+1)}\frac{\Gamma(a+\lambda+n)}{\Gamma(a+\lambda)}\frac{\Gamma(\lambda)}{n!\Gamma(\lambda+n)}\frac{\Gamma(a+1)\Gamma(\lambda+n)}{\Gamma(a+1+\lambda+n)} \\
&= \frac{1}{\kappa}\frac{\Gamma(a+1)\Gamma(\lambda)}{\Gamma(a+\lambda)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n+1)}{n!\Gamma(a+1)}\int_C t^{\lambda+n+a-1}\,\mathrm{d}t \\
&= \frac{1}{\kappa}\frac{\Gamma(a+1)\Gamma(\lambda)}{\Gamma(a+\lambda)}\int_C t^{\lambda+a-1}(1-t)^{-a-1}\,\mathrm{d}t \\
&= \frac{\Gamma(a+1)\Gamma(\lambda)}{\Gamma(a+\lambda)}\frac{\Gamma(\lambda+a)\Gamma(-a)}{\Gamma(\lambda)} \\
&= -\frac{\pi}{\sin\pi a};
\end{aligned}
$$

the interchange of integration and summation in the above equalities is justified by standards estimates. Finally, (i)–(iii) are direct consequences of Theorem 2.1. □

## 4 Eigenfunctions of the Hilbert Matrix

The purpose of this section is to prove the following theorem, which describes the point spectrum of the operators $H_\lambda$ on the spaces we are considering. Recall that $X$ denotes one of the spaces $\mathcal{H}^p$, $p > 1$, or $\mathcal{A}^{-\tau}$, $\mathcal{A}_0^{-\tau}$, for $0 < \tau < 1$.

**Theorem 4.1** *Let $\lambda \in \mathbb{C}\setminus\mathbb{Z}$, and let*

$$\mathcal{S}(X) = \{a : (1-z)^a \in X\}.$$

*Assume that $\lambda \in \mathcal{S}(X)$; then*

(i) *The operator $H_\lambda$ has the eigenvalues $\pm\pi\csc\pi\lambda$ with multiplicities $[\frac{N}{2}]$ and $[\frac{N-1}{2}]$, respectively, where $N$ is the largest integer for which the function $z \to (1-z)^{-N-\lambda}$ belongs to $X$. Furthermore, if*

$$f_n(z) = (1-z)^{-n-\lambda}(1+z)^n, \quad 0 \le n \le N,$$

*then $\ker(H_\lambda - \pi\csc\pi\lambda\, I)$ is spanned by the functions $f_{2k}$, $0 \le k \le N/2$, and $\ker(H_\lambda + \pi\csc\pi\lambda\, I)$ is spanned by the functions $f_{2k+1}$, $0 \le k \le N/2 - 1/2$.*

(ii) *If $p \ge 2$ or $0 < \tau < 1/2$, then $H_\lambda$ has no other eigenvalues.*

(iii) *If $p<2$ or $1/2<\tau<1$, the point spectrum of $H_\lambda$ on $X$ is the image of the set*

$$\mathcal{S}(X)\cap\left(\left\{\Re a\le -\frac{1}{2}\right\}\cup\{-\lambda,-\lambda-1\}\right)$$

*by the map $a\to -\pi\csc\pi a$. Each eigenvalue $-\pi\csc\pi a$, $a\ne-\lambda,-\lambda-1$, has multiplicity one and the corresponding eigenspace is spanned by the hypergeometric function ${}_2F_1(a+1,a+\lambda;\lambda;\cdot)$. Finally, if $X=\mathcal{A}^{-1/2}$, the point spectrum of $H_\lambda$ contains the image of the set*

$$\mathcal{S}\big(A^{-1/2}\big)\cap\left(\left\{\Re a<-\frac{1}{2}\right\}\cup\{-\lambda,-\lambda-1\}\right)\Big\backslash\left\{-\frac{1}{2}\right\}$$

*by the map $a\to-\pi\csc\pi a$. Again, each eigenvalue $-\pi\csc\pi a$, $a\ne-\lambda$, $a\ne -\lambda-1$, has multiplicity one, and the corresponding eigenfunction is spanned by ${}_2F_1(a+1,a+\lambda;\lambda;\cdot)$.*

The proof of this result requires several steps which we shall treat separately. Throughout the remainder of this work, $X$ will be one of the spaces $\mathcal{H}^p$, $p>1$, or $\mathcal{A}^{-\tau}$, $\mathcal{A}_0^{-\tau}$, $0<\tau<1$ and $\lambda$ a fixed number in $\mathbb{C}\setminus\mathbb{Z}$. Finally, we continue to write $\kappa=e^{2\pi i\lambda}-1$.

**Lemma 4.2** *Let $g,h\in X$ satisfy*

$$(\mu I-H_\lambda)h(z)=g(z)$$

*for some $\mu\in\mathbb{C}\setminus\{0\}$. If $D_{1,\lambda}g\in X$, then $D_{1,\lambda}h\in X$.*

*Proof* We have

$$h(z)=\frac{1}{\mu}(H_\lambda h)(z)+\frac{1}{\mu}g(z),$$

which implies

$$(D_{1,\lambda}h)(z)=\frac{1}{\mu}(H_\lambda D_{1,\lambda}h)(z)-\frac{\lambda-1}{\mu\kappa}\int_C t^{\lambda-2}h(t)\,\mathrm{d}t+\frac{1}{\mu}D_{1,\lambda}g(z). \tag{4.1}$$

Now, if $h\in A^{-\tau}$, it follows that (see [3])

$$(1-x)\big|h'(x)\big|=O\big((1-x)^{-\tau}\big)\quad\text{as } x\to1^-,$$

and if $f\in\mathcal{H}^p$, then the following map (see [6]) belongs to $L^p([0,1])$:

$$x\to(1-x)h'(x).$$

Thus, by Theorem 2.1, $H_\lambda D_{1,\lambda}h\in X$, hence by (4.1), $D_{1,\lambda}h\in X$. □

**Lemma 4.3** *If $\mu\in\mathbb{C}\setminus\{0\}$ and $1\in(H_\lambda-\mu I)X$, then $(H_\lambda-\mu I)X$ contains all the polynomials.*

*Proof* We prove the statement by induction on the degree $n$ of the polynomial. Thus by assumption, the statement holds true when $n=0$. Assume that $(H_\lambda - \mu I)X$ contains all the polynomials of degree at most $n$. If $h \in X$ satisfies $(H_\lambda h - \mu h) = z^n$, then by Theorem 3.1, we have

$$(H_\lambda - \mu I)(D_{1,\lambda} h)(z) - D_{1,\lambda} z^n = \text{const},$$

which shows

$$(H_\lambda - \mu I)(D_{1,\lambda} h)(z) = (n+\lambda) z^{n+1} - n z^n + \text{const}.$$

By Lemma 4.2, we have that $D_{1,\lambda} h \in X$, and the result follows. □

**Lemma 4.4** *Let $\mu \in \mathbb{C} \setminus \{0\}$, and let $f \in \ker(H_\lambda - \mu I)$, $f \neq 0$. If $X$ is one of the spaces $\mathcal{H}^p$, $p \geq 2$, or $\mathcal{A}^{-\tau}$, $\mathcal{A}_0^{-\tau}$, $0 < \tau < \frac{1}{2}$, or $f \in \mathcal{H}^\infty$, then $(H_\lambda - \mu I)X$ cannot contain the constant functions. In particular,*

$$\int_C f(t) t^{\lambda-2}\, \mathrm{d}t = 0,$$

*and $D_{1,\lambda} f \in \ker(H_\lambda - \mu I)$.*

*Proof* By Theorem 3.1 and Lemma 4.2, it suffices to prove the first part of the statement, since

$$(H_\lambda - \mu I) D_{1,\lambda} f(z) - D_{1,\lambda}(H_\lambda - \mu I) f(z) = \frac{1}{\kappa}(\lambda - 1) \int_C f(t) t^{\lambda-2}\, \mathrm{d}t.$$

If we assume the contrary, that is,

$$k = \int_C f(t) t^{\lambda-2}\, \mathrm{d}t \neq 0,$$

then $f \neq 0$, and by Theorem 3.1, we have

$$(H_\lambda - \mu I) D_{1,\lambda} f = \frac{k}{\kappa}(\lambda - 1).$$

By Lemma 4.2, we conclude that $1 \in (H_\lambda - \mu I)X$. Thus by Lemma 4.3, we find that $(H_\lambda - \mu I)X$ contains all polynomials. Now let $p$ be a polynomial and $g \in X$ with $H_\lambda g - \mu g = p$. For $0 < r < 1$, we have

$$\begin{aligned}
&\int_0^{2\pi} f\big(re^{i\theta}\big) p\big(re^{-i\theta}\big) \frac{\mathrm{d}\theta}{2\pi} \\
&\quad = \int_0^{2\pi} f\big(re^{i\theta}\big)\big((H_\lambda g)\big(re^{-i\theta}\big) - \mu g\big(re^{-i\theta}\big)\big) \frac{\mathrm{d}\theta}{2\pi} \\
&\quad = \int_0^{2\pi} f\big(re^{i\theta}\big)(H_\lambda g)\big(re^{-i\theta}\big) \frac{\mathrm{d}\theta}{2\pi} - \int_0^{2\pi} (H_\lambda f)\big(re^{i\theta}\big) g\big(re^{-i\theta}\big) \frac{\mathrm{d}\theta}{2\pi}.
\end{aligned}$$

By the integral representation (2.1) and a direct computation based on Cauchy's formula, we obtain from the last two integrals

$$\int_0^{2\pi} f\big(re^{i\theta}\big)p\big(re^{-i\theta}\big)\,\frac{\mathrm{d}\theta}{2\pi} = \frac{1}{\kappa}\int_C f\big(r^2t\big)g(t)t^{\lambda-1}\,\mathrm{d}t - \frac{1}{\kappa}\int_C f(t)g\big(r^2t\big)t^{\lambda-1}\,\mathrm{d}t.$$

Taking into account the values of the parameters $p$ and $\tau$, or if $f \in \mathcal{H}^\infty$, then the right-hand side of this equality converges to zero when $r \to 1^-$, whenever $g \in X$. This implies

$$\lim_{r\to 1^-}\int_0^{2\pi} f\big(re^{i\theta}\big)p\big(re^{-i\theta}\big)\,\frac{\mathrm{d}\theta}{2\pi} = 0$$

for every polynomial $p$, which leads to a contradiction, since $f \neq 0$. □

The last step needed in the proof of Theorem 4.1 is a little bit more involved. We shall make extensive use of two identities. First, from the integral representation (2.1) of $H_\lambda$, we easily deduce that

$$z(H_\lambda f)(z) = \frac{z}{\kappa}\int_C \frac{f(t)t^{\lambda-1}}{1-tz}\,\mathrm{d}t = -\frac{1}{\kappa}\int_C f(t)t^{\lambda-2}\,\mathrm{d}t + \frac{1}{\kappa}\int_C \frac{f(t)t^{\lambda-1}}{1-tz}\,\mathrm{d}t;$$

that is,

$$z(H_\lambda f)(z) = -\frac{1}{\kappa}\int_C f(t)t^{\lambda-2}\,\mathrm{d}t + (H_{\lambda-1}f)(z). \tag{4.2}$$

We shall use this equality in the equivalent form

$$(1-z)(H_\lambda f)(z) = \frac{1}{\kappa}\int_C f(t)t^{\lambda-2}\,\mathrm{d}t + H_{\lambda-1}\big((z-1)f\big)(z), \tag{4.3}$$

where we have once more denoted by $z$ the identity function on $\mathbb{D}$.

As an immediate consequence, we note the following simple observation.

**Lemma 4.5** *Let $\mu \in \mathbb{C}\setminus\{0\}$ and $f \in \ker(H_\lambda - \mu I)$. Then $(z-1)f \in \mathcal{H}^\infty$.*

*Proof* By (4.3), we have that

$$(1-z)f(z) = \frac{1}{\kappa\mu}\int_C f(t)t^{\lambda-2}\,\mathrm{d}t + \frac{1}{\mu}H_{\lambda-1}(z-1)f(z),$$

and

$$\begin{aligned}\big|\big(H_{\lambda-1}(z-1)f\big)(\zeta)\big| &\le M\int_C \frac{|f(t)||t-1|}{1-t|\zeta|}|\mathrm{d}t| \\ &\le M_1\int_C \big|f(t)\big||\mathrm{d}t|\end{aligned}$$

for some constants $M$ and $M_1$, which completes the proof. □

Our second identity is a well-known characterization of Hankel operators. If $B$ denotes the backward shift

$$(Bf)(z) = \frac{f(z) - f(0)}{z},$$

then

$$BH_\lambda f = H_\lambda z f. \tag{4.4}$$

**Lemma 4.6** *If* $\mu \in \mathbb{C} \setminus \{0\}$, *then*

$$\dim \ker(H_\lambda - \mu I) < +\infty.$$

*Proof* Obviously, it will suffice to show that

$$\mathcal{M} = \left\{ f \in \ker(H_\lambda - \mu I) : \int_C f(t) t^{\lambda-2}\, \mathrm{d}t = 0 \right\}$$

has finite dimension. Note that by (4.3), we have

$$\big(H_{\lambda-1}(1-z)f\big)(z) = (H_{\lambda-1}f)(z) - (H_\lambda f)(z) = (z-1)(H_\lambda f)(z) = \mu(z-1)f(z) \tag{4.5}$$

for all $f \in \mathcal{M}$; that is,

$$(1-z)\mathcal{M} \subset \ker(H_{\lambda-1} + \mu I).$$

By Lemma 4.5, we also have $(1-z)\mathcal{M} \subset \mathcal{H}^\infty \subset \mathcal{H}^2$.

Next let $\mathcal{N}$ be the closure of $(1-z)\mathcal{M}$ in $\mathcal{H}^2$, which is a subset of $\ker(H_{\lambda-1} + \mu I)|_{\mathcal{H}^2}$. We construct the standard orthonormal basis in $\mathcal{N}$ formed with the functions $e_n$, $n \geq 1$, which solve the extremal problems

$$e_n^{(n)}(0) = \sup\big\{ \big|f^{(n)}(0)\big| : f \in \mathcal{N}, \|f\| \leq 1, f^{(k)}(0) = 0, k < n \big\}.$$

We claim that this orthonormal basis must be finite. Indeed, by Lemma 4.4, we have

$$\int_C e_n t^{\lambda-3}\, \mathrm{d}t = 0$$

for all $n$, so that (4.5) applies and shows that

$$H_{\lambda-2}(1-z)e_n = \mu(1-z)e_n.$$

By (4.4), we have also

$$\mu(1-z)\frac{e_n}{z^n} = H_{\lambda-2}(1-z)z^n e_n.$$

Now note that since $e_n$ is orthogonal to $ze_n$,

$$\left\| (1-z)\frac{e_n}{z^n} \right\|_2 = \big\|(1-z)e_n\big\|_2 = \sqrt{2}.$$

On the other hand, the standard duality argument yields

$$\begin{aligned}\left\| H_{\lambda-2}(1-z)z^n e_n\right\|_2 &= \sup_{\|h\|_2\le 1}\left|\frac{1}{c}\int_C (1-t)e_n(t)h(t)t^{n+\lambda-3}\,\mathrm{d}t\right| \\ &\le \frac{1}{|c|}\int_C |1-t|\bigl(1-|t|^2\bigr)^{-1}|t|^{n+\lambda-3}|\mathrm{d}t|,\end{aligned}$$

where we have used the standard estimate

$$\bigl|g(z)\bigr| \le \bigl(1-|z|^2\bigr)^{-\frac{1}{2}}\|g\|_2.$$

Thus the inequality

$$\sqrt{2}|\mu| \le \frac{1}{|c|}\int_C |1-t|\bigl(1-|t|^2\bigr)^{-1}|t|^{n+\lambda-3}|\mathrm{d}t|$$

implies that $\dim \mathcal{N} < +\infty$, and the proof is complete. □

We are now prepared to prove Theorem 4.1.

*Proof of Theorem 4.1* The proof is divided into three steps.

*First Step*. We show that the functions $f_n$ defined in the statement satisfy

$$(H_\lambda f_n)(z) = (-1)^n \pi \csc \pi\lambda\, f_n(z) \tag{4.6}$$

and

$$\int_C f_n(t)t^{\lambda-2}\,\mathrm{d}t = 0, \quad 0 \le n \le N.$$

*Second Step*. Conversely, we prove that every eigenspace $\mathcal{M}_\mu$ of $H_\lambda$ with

$$\int_C f(t)t^{\lambda-2}\,\mathrm{d}t = 0, \quad \forall f \in \mathcal{M}_\mu,$$

is spanned by a subset of $\{f_n : 0 \le n \le N\}$. Thus (i) and (ii) will follow by Lemma 4.4.

*Third Step*. We show that if $f$ is an eigenfunction of $H_\lambda$ with

$$\int_C f(t)t^{\lambda-2}\,\mathrm{d}t \ne 0,$$

then the corresponding eigenspace has dimension one and consequently $f$ is an eigenfunction of $D_{2,\lambda}$. Then the result follows by Corollary 3.4.

*Proof of Step 1*. Let $\lambda \in \mathcal{S}(X)$. Since $N$ is the largest integer for which the function $f_n(z) = (1-z)^{-N-\lambda}$ belongs to $X$, we must have $N + \lambda < 1$, so that $n + \lambda < 0$ for all integers $n < N$. Now, recall from Corollary 3.4 that $f_0$ satisfies

$$(H_\lambda f_0)(z) = \pi \csc \pi\lambda f_0(z).$$

To prove the claim for general $n \le N$, we proceed by induction. Assume that (4.6) holds for $n-1$, and note that

$$f_n(z) = \frac{z+1}{z-1} f_{n-1}(z).$$

In view of the fact that $n \le N$, we also have that

$$f_n(z) = \lim_{a\to 1^-} \left(-1 + \frac{2az}{az-1}\right) f_{n-1}(z)$$

in the norm of $X$. But then by (4.4), we can write

$$(H_\lambda f_n)(z) = -\lim_{a\to 1^-} \bigl(-I - 2aB(I-aB)^{-1}\bigr)(H_\lambda f_{n-1})(z)$$

in the norm of $X$, and a direct computation gives

$$\begin{aligned}(H_\lambda f_n)(z) &= \lim_{a\to 1^-} \left(-H_\lambda f_{n-1}(z) - 2a \frac{H_\lambda f_{n-1}(z) - H_\lambda f_{n-1}(a)}{z-a}\right)\\ &= \lim_{a\to 1^-} (-1)^n \pi \csc \pi\lambda \left( f_{n-1}(z) + 2a \frac{f_{n-1}(z) - f_{n-1}(a)}{z-a}\right)\\ &= (-1)^n \pi \csc \pi\lambda \left( f_{n-1}(z) + 2\frac{f_{n-1}(z)}{z-1}\right) = (-1)^n \pi \csc \pi\lambda\ f_n(z),\end{aligned}$$

where we have used the fact that $n-1+\lambda < 0$. Now observe that the functions $f_n$ satisfy

$$\int_C f_n(t) t^{\lambda-2} = 0, \quad 0 \le n \le N.$$

This follows, for instance, directly from Theorem 3.1, since $f_n$ are eigenfunctions for both $H_\lambda$ and $D_{1,\lambda}$.

*Proof of Step 2.* Let $\mathcal{M}_\mu = \ker(H_\lambda - \mu I)$ be a nonzero eigenspace with the additional property that

$$\int_C f(t) t^{\lambda-2}\,\mathrm{d}t = 0, \quad f \in \mathcal{M}_\mu. \tag{4.7}$$

By Theorem 3.1 and Lemma 4.2, we have that $D_{1,\lambda}\mathcal{M}_\mu \subset \mathcal{M}_\mu$, and by Lemma 4.5, $\mathcal{M}_\mu$ has finite dimension. The eigenfunctions of $D_{1,\lambda}|\mathcal{M}_\mu$ must be the form provided by Proposition 3.2, and as eigenfunctions of $H_\lambda$ they must extend analytically in $\mathbb{C} \setminus [1,\infty)$. This implies that the functions in question belong to the set $\{f_n : 0 \le n \le N\}$. Now another application of Proposition 3.2 shows that the equation

$$D_{1,\lambda} g - \nu g = f_n$$

cannot have solutions analytic near $-1$. Thus we conclude that $\mathcal{M}_\mu$ is spanned by a subset of $\{f_n : 0 \le n \le N\}$, which proves Step 2.

*Proof of Step 3.* We need to show that if $\mu \in \mathbb{C} \setminus \{0\}$ and $f \in \ker(H_\lambda - \mu I)$ with

$$\int_C f(t)t^{\lambda-2}\,\mathrm{d}t \neq 0,$$

then

$$\dim\ker(H_\lambda - \mu I) = 1$$

and $f$ is an eigenfunction of $D_{2,\lambda}$. To verify this statement, we use (4.3) again to conclude that

$$\mu(1-z)f(z) = \frac{1}{\kappa}\int_C f(t)t^{\lambda-2}\,\mathrm{d}t + \big(H_{\lambda-1}(z-1)f\big)(z),$$

which implies that $(H_{\lambda-1} + \mu I)X$ contains the constants.
If $\dim\ker(H_\lambda - \mu I) \geq 2$, then we can find a function $g \in \ker(H_\lambda - \mu I)$ with $g \neq 0$ and

$$\int_C g(t)t^{\lambda-2}\,\mathrm{d}t = 0.$$

Now an application of (4.3) to $g$ shows that

$$\mu(1-z)g(z) = H_{\lambda-1}(z-1)g(z).$$

By Lemma 4.5, we obtain that $(1-z)g \in \ker(H_{\lambda-1} + \mu I) \bigcap \mathcal{H}^\infty$, which leads to a contradiction by Lemma 4.4. Thus $\ker(H_\lambda - \mu I)$ is spanned by $f$ and by Theorem 3.3, $f$ must be an eigenvalue of $D_{2,\lambda}$ and Step 3 is proved. This completes the proof of the theorem. □

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