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CONSTRUCTIVE APPROXIMATION

Some Extremal Functions in Fourier Analysis, III

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Abstract We obtain the best approximation in $L^1(\mathbb{R})$, by entire functions of exponential type, for a class of even functions that includes $e^{-\lambda|x|}$, where $\lambda > 0$, $\log|x|$ and $|x|^{\alpha}$, where $-1 < \alpha < 1$. We also give periodic versions of these results where the approximating functions are trigonometric polynomials of bounded degree.

Keywords Approximation · Entire functions · Exponential type

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1 Introduction

An entire function $K : \mathbb{C} \to \mathbb{C}$ is of exponential type $\sigma \geq 0$ if, for any $\epsilon > 0$, there exists a constant C_ϵ such that for all $z \in \mathbb{C}$ we have

$$\left|K(z)\right| \leq C_\epsilon e^{(\sigma+\epsilon)|z|}.$$

Given a function $f : \mathbb{R} \to \mathbb{R}$, we address here the problem of finding an entire function $K(z)$ of exponential type at most π such that the integral

$$\int_{-\infty}^{\infty} \left|K(x) - f(x)\right| \mathrm{d}x \tag{1.1}$$

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is minimized. A typical variant of this problem occurs when we impose the additional condition that $K(z)$ is real on $\mathbb{R}$ and satisfies $K(x) \geq f(x)$ for all $x \in \mathbb{R}$. In this case a minimizer of the integral (1.1) is called an extremal majorant of $f(x)$. Extremal minorants are defined analogously.

The study of these extremal functions dates back to A. Beurling in the 1930's, who solved the problem (1.1) (and its majorizing version) for $f(x) = \operatorname{sgn}(x)$. A complete collection of his results and many applications to analytic number theory (including Selberg's proof of the large sieve inequality) can be found in the paper [15] by J.D. Vaaler. In [5], Graham and Vaaler constructed the extremal majorants and minorants for the function $f(x) = e^{-\lambda|x|}$, where $\lambda > 0$. Recently, Carneiro and Vaaler in [2] were able to extend the construction of extremal majorants for a wide class of even functions that includes $\log|x|$ and $|x|^\alpha$, where $-1 < \alpha < 1$. The case $f(x) = \log|x|$, which can be viewed as a Fourier conjugate of $f(x) = \operatorname{sgn}(x)$, is particularly important, providing a number of interesting applications. Other problems on approximation and majorization by entire functions have been discussed in [3, 4, 8–11] and [14]. Extensions of the problem to several variables are considered in [1, 6] and [7].

The purpose of this paper, the third in this series, is to settle the best approximation problem (1.1) for the function $f(x) = e^{-\lambda|x|}$, where $\lambda > 0$, and also for the same class of even functions considered in [2], that includes $\log|x|$ and $|x|^\alpha$, where $-1 < \alpha < 1$.

We start by defining the entire function $z \mapsto K(\lambda, z)$ by

$$K(\lambda, z) = \left(\frac{\cos \pi z}{\pi}\right)\left\{\sum_{n\in\mathbb{Z}} \frac{(-1)^n e^{-\lambda|n-\frac{1}{2}|}}{\left(z - n + \frac{1}{2}\right)}\right\}. \tag{1.2}$$

This function has exponential type π and interpolates $e^{-\lambda|x|}$ at the integers plus a half. The construction of such functions and how they appear as natural candidates to our problem are explained in [15, Sects. 2 and 3]. Our first result is the following:

Theorem 1.1 *The function $K(\lambda, z)$ defined in (1.2) satisfies the following extremal property*:

(i) *If $\widetilde{K}(z)$ is an entire function of exponential type at most $\pi\delta$, where $\delta > 0$, then*

$$\int_{-\infty}^{\infty} \left|e^{-\lambda|x|} - \widetilde{K}(x)\right| \mathrm{d}x \geq \frac{2}{\lambda} - \frac{2}{\lambda}\operatorname{sech}\left(\frac{\lambda}{2\delta}\right), \tag{1.3}$$

with equality if and only if $\widetilde{K}(z) = K(\delta^{-1}\lambda, \delta z)$.

(ii) *For $x \in \mathbb{R}$, we have*

$$\operatorname{sgn}(\cos \pi x) = \operatorname{sgn}\left\{e^{-\lambda|x|} - K(\lambda, x)\right\}. \tag{1.4}$$

From Theorem 1.1 we see that $x \mapsto K(\lambda, x)$ is integrable on $\mathbb{R}$. Its Fourier transform

$$\widehat{K}(\lambda, t) = \int_{-\infty}^{\infty} K(\lambda, x) e(-tx)\, \mathrm{d}x \tag{1.5}$$

is a continuous function of the real variable t supported on the interval $[-\frac{1}{2}, \frac{1}{2}]$. Here we write $e(z) = e^{2\pi i z}$. The Fourier transform in (1.5) is a non-negative function of t and is given explicitly in Lemma 4.2.

The description of the sign changes given by (1.4) is a key point in our argument. It will allow us to apply the techniques of [2] when we integrate with respect to the parameter λ. For this, let μ be a measure defined on the Borel subsets of $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{\lambda^2 + 1} \, \mathrm{d}\mu(\lambda) < \infty. \tag{1.6}$$

It follows from (1.6) that, for $x \neq 0$, the function

$$\lambda \mapsto e^{-\lambda|x|} - e^{-\lambda}$$

is integrable on $(0, \infty)$ with respect to μ. We then define $f_\mu : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ by

$$f_\mu(x) = \int_0^\infty \left\{e^{-\lambda|x|} - e^{-\lambda}\right\} \mathrm{d}\mu(\lambda), \tag{1.7}$$

where

$$f_\mu(0) = \int_0^\infty \left\{1 - e^{-\lambda}\right\} \mathrm{d}\mu(\lambda)$$

may take the value ∞. Using f_μ, we define $K_\mu : \mathbb{C} \to \mathbb{C}$ by

$$K_\mu(z) = \lim_{N \to \infty} \left(\frac{\cos \pi z}{\pi}\right) \left\{ \sum_{n=-N}^{N+1} \frac{(-1)^n f_\mu\left(n - \frac{1}{2}\right)}{\left(z - n + \frac{1}{2}\right)} \right\}. \tag{1.8}$$

We will show that the sequence on the right of (1.8) converges uniformly on compact subsets of $\mathbb{C}$ and therefore defines $K_\mu(z)$ as a real entire function. Then it is easy to check that K_μ interpolates the values of f_μ at real numbers x such that $x - \frac{1}{2}$ is an integer. That is, the identity

$$K_\mu\left(n - \frac{1}{2}\right) = f_\mu\left(n - \frac{1}{2}\right) \tag{1.9}$$

holds for each integer n. We will prove that the entire function $K_\mu(z)$ satisfies the following extremal property:

Theorem 1.2 *Assume that the measure μ satisfies* (1.6).

(i) *The real entire function $K_\mu(z)$ defined by* (1.8) *has exponential type at most π.*
(ii) *For real $x \neq 0$, the function*

$$\lambda \mapsto e^{-\lambda|x|} - K(\lambda, x)$$

is integrable on $(0, \infty)$ with respect to μ.

(iii) *For all real x, we have*

$$f_\mu(x) - K_\mu(x) = \int_0^\infty \left\{e^{-\lambda|x|} - K(\lambda, x)\right\} \mathrm{d}\mu(\lambda). \tag{1.10}$$

(iv) *The function $x \mapsto f_\mu(x) - K_\mu(x)$ is integrable on $\mathbb{R}$, and*

$$\int_{-\infty}^\infty \left|f_\mu(x) - K_\mu(x)\right| \mathrm{d}x = \int_0^\infty \left\{\frac{2}{\lambda} - \frac{2}{\lambda}\operatorname{sech}\left(\frac{\lambda}{2}\right)\right\} \mathrm{d}\mu(\lambda). \tag{1.11}$$

(v) *If $t \neq 0$, then*

$$\int_{-\infty}^\infty \left\{f_\mu(x) - K_\mu(x)\right\} e(-tx)\,\mathrm{d}x$$
$$= \int_0^\infty \frac{2\lambda}{\lambda^2 + 4\pi^2 t^2} \mathrm{d}\mu(\lambda) - \int_0^\infty \widehat{K}(\lambda, t)\,\mathrm{d}\mu(\lambda). \tag{1.12}$$

(vi) *If $\widetilde{K}(z)$ is an entire function of exponential type at most π, then*

$$\int_{-\infty}^\infty \left|f_\mu(x) - \widetilde{K}(x)\right| \mathrm{d}x \geq \int_0^\infty \left\{\frac{2}{\lambda} - \frac{2}{\lambda}\operatorname{sech}\left(\frac{\lambda}{2}\right)\right\} \mathrm{d}\mu(\lambda). \tag{1.13}$$

(vii) *There is equality in (1.13) if and only if $\widetilde{K}(z) = K_\mu(z)$.*

Theorem 1.2 was stated for the best approximation of exponential type at most π of $f_\mu(x)$. It is often useful to have results of the same sort in which the entire approximations have exponential type at most $\pi\delta$, where δ is a positive parameter. To accomplish this we introduce a second measure ν defined on Borel subsets $E \subseteq (0, \infty)$ by

$$\nu(E) = \mu(\delta E), \tag{1.14}$$

where

$$\delta E = \{\delta x : x \in E\}$$

is the dilation of E by δ. If μ satisfies (1.6), then ν also satisfies (1.6), and the two functions $f_\mu(x)$ and $f_\nu(x)$ are related by the identity

$$f_\nu(x) = \int_0^\infty \left\{e^{-\lambda|x|} - e^{-\lambda}\right\} \mathrm{d}\nu(\lambda) = \int_0^\infty \left\{e^{-\lambda\delta^{-1}|x|} - e^{-\lambda\delta^{-1}}\right\} \mathrm{d}\mu(\lambda)$$
$$= \int_0^\infty \left\{e^{-\lambda|\delta^{-1}x|} - e^{-\lambda}\right\} \mathrm{d}\mu(\lambda) - \int_0^\infty \left\{e^{-\lambda\delta^{-1}} - e^{-\lambda}\right\} \mathrm{d}\mu(\lambda)$$
$$= f_\mu\left(\delta^{-1}x\right) - f_\mu\left(\delta^{-1}\right). \tag{1.15}$$

We apply Theorem 1.2 to the functions $f_\nu(x)$ and $K_\nu(z)$. Then using (1.15) we obtain corresponding results for the functions

$$f_\mu(x) - f_\mu\left(\delta^{-1}\right) = f_\nu(\delta x) \quad \text{and} \quad K_\nu(\delta z),$$

where the entire function $z \mapsto K_\nu(\delta z)$ has exponential type at most $\pi\delta$. This leads easily to the following more general form of Theorem 1.2:

Theorem 1.3 *Assume that the measure μ satisfies* (1.6), *and let ν be the measure defined by* (1.14), *where δ is a positive parameter.*

(i) *The real entire function $z \mapsto K_\nu(\delta z) + f_\mu(\delta^{-1})$ has exponential type at most $\pi\delta$.*
(ii) *For real $x \neq 0$, the function*

$$\lambda \mapsto e^{-\lambda|x|} - K\left(\delta^{-1}\lambda, \delta x\right) \tag{1.16}$$

is integrable on $(0, \infty)$ with respect to μ.
(iii) *For all real x, we have*

$$f_\mu(x) - f_\mu\left(\delta^{-1}\right) - K_\nu(\delta x) = \int_0^\infty \left\{e^{-\lambda|x|} - K\left(\delta^{-1}\lambda, \delta x\right)\right\} \mathrm{d}\mu(\lambda). \tag{1.17}$$

(iv) *The function $x \mapsto f_\mu(x) - f_\mu(\delta^{-1}) - K_\nu(\delta x)$ is integrable on $\mathbb{R}$, and*

$$\int_{-\infty}^\infty \left|f_\mu(x) - f_\mu\left(\delta^{-1}\right) - K_\nu(\delta x)\right| \mathrm{d}x = \int_0^\infty \left\{\frac{2}{\lambda} - \frac{2}{\lambda}\operatorname{sech}\left(\frac{\lambda}{2\delta}\right)\right\} \mathrm{d}\mu(\lambda). \tag{1.18}$$

(v) *If $t \neq 0$, then*

$$\begin{aligned}
&\int_{-\infty}^\infty \left\{f_\mu(x) - f_\mu\left(\delta^{-1}\right) - K_\nu(\delta x)\right\} e(-tx)\,\mathrm{d}x \\
&\quad = \int_0^\infty \frac{2\lambda}{\lambda^2 + 4\pi^2 t^2}\,\mathrm{d}\mu(\lambda) - \delta^{-1}\int_0^\infty \widehat{K}\left(\delta^{-1}\lambda, \delta^{-1}t\right) \mathrm{d}\mu(\lambda).
\end{aligned} \tag{1.19}$$

(vi) *If $\widetilde{K}(z)$ is an entire function of exponential type at most $\pi\delta$, then*

$$\int_{-\infty}^\infty \left|f_\mu(x) - \widetilde{K}(x)\right| \mathrm{d}x \geq \int_0^\infty \left\{\frac{2}{\lambda} - \frac{2}{\lambda}\operatorname{sech}\left(\frac{\lambda}{2\delta}\right)\right\} \mathrm{d}\mu(\lambda). \tag{1.20}$$

(vii) *There is equality in* (1.20) *if and only if $\widetilde{K}(z) = K_\nu(\delta z) + f_\mu(\delta^{-1})$.*

To illustrate how these results can be applied, we consider the problem of approximating the function $x \mapsto \log|x|$ by an entire function $z \mapsto V(z)$ of exponential type at most π. We select μ to be a Haar measure on the multiplicative group $(0, \infty)$, so that

$$\mu(E) = \int_E \lambda^{-1}\,\mathrm{d}\lambda \tag{1.21}$$

for all Borel subsets $E \subseteq (0, \infty)$. For this measure μ we find that

$$f_\mu(x) = -\log|x|.$$

We apply Theorem 1.2 with $V(z) = -K_\mu(z)$, that is,

$$V(z) = \lim_{N\to\infty} \left(\frac{\cos \pi z}{\pi}\right) \left\{ \sum_{n=-N}^{N+1} \frac{(-1)^n \log\left|n - \frac{1}{2}\right|}{\left(z - n + \frac{1}{2}\right)} \right\}, \tag{1.22}$$

where the limit converges uniformly on compact subsets of $\mathbb{C}$. From Theorem 1.2 we conclude that $V(z)$ is the best approximation of exponential type at most π for $\log|x|$ with

$$\int_{-\infty}^{\infty} \left|\log|x| - V(x)\right| \mathrm{d}x = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \frac{4G}{\pi}, \tag{1.23}$$

where $G = 0.915965594\ldots$ is the Catalan's constant. This follows from (1.11) and standard contour integration.

In a similar manner, Theorem 1.3 can be applied to determine the entire function of exponential type at most $\pi\delta$ that best approximates $x \mapsto \log|x|$. Alternatively, the functional equation for the logarithm allows us to accomplish this directly. Clearly, the entire function

$$z \mapsto -\log\delta + V(\delta z)$$

has exponential type at most $\pi\delta$, and is the best approximation to $x \mapsto \log|x|$ on $\mathbb{R}$ satisfying

$$\int_{-\infty}^{\infty} \left|\log|x| + \log\delta - V(\delta x)\right| \mathrm{d}x = \frac{4G}{\pi\delta}. \tag{1.24}$$

Another interesting application of Theorem 1.2 arises when we choose measures μ_σ such that

$$\mu_\sigma(E) = \int_E \lambda^{-\sigma}\, \mathrm{d}\lambda \tag{1.25}$$

for all Borel subsets $E \subseteq (0, \infty)$. For $0 < \sigma < 2$ the measure μ_σ satisfies the condition (1.6). We find that

$$f_{\mu_\sigma}(x) = \int_0^\infty \left\{e^{-\lambda|x|} - e^{-\lambda}\right\} \lambda^{-\sigma}\, \mathrm{d}\lambda = \Gamma(1-\sigma)\left\{|x|^{\sigma-1} - 1\right\}, \quad \text{if } \sigma \neq 1. \tag{1.26}$$

Therefore, if we want to find the best approximation of exponential type at most π for the even function $x \mapsto |x|^{\sigma-1}$, where $0 < \sigma < 2$ and $\sigma \neq 1$, we should consider

$$V_\sigma(z) = \frac{K_{\mu_\sigma}(z)}{\Gamma(1-\sigma)} + 1 = \lim_{N\to\infty} \left(\frac{\cos \pi z}{\pi}\right) \left\{ \sum_{n=-N}^{N+1} \frac{(-1)^n \left|n - \frac{1}{2}\right|^{\sigma-1}}{\left(z - n + \frac{1}{2}\right)} \right\}.$$

From (1.11) and contour integration we conclude that

$$\begin{aligned} \int_{-\infty}^{\infty} \left||x|^{\sigma-1} - V_\sigma(x)\right| \mathrm{d}x &= \frac{1}{\Gamma(1-\sigma)} \int_0^\infty \left\{\frac{2}{\lambda} - \frac{2}{\lambda}\operatorname{sech}\left(\frac{\lambda}{2}\right)\right\} \lambda^{-\sigma}\, \mathrm{d}\lambda \\ &= \frac{1}{\Gamma(1-\sigma)} \frac{4}{\sin\left(\frac{\pi\sigma}{2}\right)\pi^\sigma} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{1+\sigma}}. \end{aligned} \tag{1.27}$$

Our results can also be used to approximate certain real-valued periodic functions by trigonometric polynomials. This is accomplished by applying the Poisson summation formula to the functions $x \mapsto e^{-\lambda|x|}$ and $x \mapsto K(\lambda, x)$, and then integrating the parameter λ with respect to a measure μ. We give a general account of this method in Sect. 6. An interesting special case of Theorem 6.2 occurs when we consider μ to be the Haar measure defined in (1.21). In this case, we obtain the trigonometric polynomial of degree N that best approximates in $L^1(\mathbb{R}/\mathbb{Z})$ the function $x \mapsto \log|1-e(x)|$. Here is the precise result:

Theorem 1.4 *Let N be a non-negative integer. There exists a real-valued trigonometric polynomial*

$$v_N(x) = \sum_{n=-N}^{N} \widehat{v}_N(n) e(nx) \tag{1.28}$$

that is the best approximation in $L^1(\mathbb{R}/\mathbb{Z})$ for the function $x \mapsto \log|1-e(x)|$. Precisely, if $\widetilde{v}(x)$ is a trigonometric polynomial of degree at most N, we have

$$\int_{\mathbb{R}/\mathbb{Z}} \big|\log|1-e(x)| - \widetilde{v}(x)\big| \, \mathrm{d}x \geq \frac{4G}{(2N+2)\pi}, \tag{1.29}$$

with equality if and only if $\widetilde{v}(x) = v_N(x)$. Here $G = 0.915965594\ldots$ is the Catalan's constant.

The trigonometric polynomial $v_N(x)$ is explicitly described in Sect 6, (6.26)–(6.28). With the notation of Sect. 6 we have $v_N(x) = -k_\mu(N; x)$, for this particular measure μ.

2 Proof of Theorem 1.1

By performing a change of variables, it suffices to prove (1.3) for $\delta = 1$ and all $\lambda > 0$. We start by defining the following entire function of exponential type π

$$A(\lambda, z) = \left(\frac{\sin \pi z}{\pi}\right) \sum_{n=0}^{\infty} (-1)^n \frac{e^{-\lambda n}}{(z-n)}.$$

We also define the function $B : \mathbb{R} \to \mathbb{R}$ by

$$B(w) = -\frac{e^w}{e^w + 1}.$$

Lemma 2.1 *If $\Re(z) < 0$, we have*

$$A(\lambda, z) = \left(\frac{\sin \pi z}{\pi}\right) \int_0^{\infty} B(\lambda + w) e^{zw} \, \mathrm{d}w, \tag{2.1}$$

and if $\Re(z) > 0$, *we have*

$$A(\lambda, z) = e^{-\lambda z} - \left(\frac{\sin \pi z}{\pi}\right) \int_{-\infty}^{0} B(\lambda + w) e^{zw}\, \mathrm{d}w. \tag{2.2}$$

Proof Let $\rho > 0$. If $\Re(z) \le -\rho$, then

$$\begin{aligned}\int_0^\infty B(\lambda + w) e^{zw}\, \mathrm{d}w &= e^{-\lambda z} \int_\lambda^\infty B(w) e^{zw}\, \mathrm{d}w \\ &= e^{-\lambda z} \int_\lambda^\infty \sum_{n=0}^\infty (-1)^{n+1} e^{(z-n)w}\, \mathrm{d}w.\end{aligned}$$

Now

$$\left|\sum_{n=0}^\infty (-1)^{n+1} e^{(z-n)w}\right| \le \sum_{n=0}^\infty e^{-\rho w - nw} = \left(\frac{e^w}{e^w - 1}\right) e^{-\rho w},$$

so by the dominated convergence theorem we have

$$\int_0^\infty B(\lambda + w) e^{zw}\, \mathrm{d}w = e^{-\lambda z} \sum_{n=0}^\infty (-1)^{n+1} \int_\lambda^\infty e^{(z-n)w}\, \mathrm{d}w = \sum_{n=0}^\infty (-1)^n \frac{e^{-\lambda n}}{(z-n)},$$

and this proves (2.1).

Suppose now that $\Re(z) \ge \rho > 0$. Then

$$\int_{-\infty}^0 B(\lambda + w) e^{zw}\, \mathrm{d}w = e^{-\lambda z} \left\{ \int_{-\infty}^0 B(w) e^{zw}\, \mathrm{d}w + \int_0^\lambda B(w) e^{zw}\, \mathrm{d}w \right\}. \tag{2.3}$$

The first of these integrals is equal to

$$\int_{-\infty}^0 B(w) e^{zw}\, \mathrm{d}w = \int_{-\infty}^0 \left(\sum_{n=1}^\infty -e^{(2n-1)w} + e^{2nw}\right) e^{zw}\, \mathrm{d}w. \tag{2.4}$$

For $w < 0$ we have

$$\left|\left(\sum_{n=1}^\infty -e^{(2n-1)w} + e^{2nw}\right) e^{zw}\right| \le -B(w) e^{\rho w},$$

and therefore we can use the dominated convergence theorem to conclude that (2.4) is equal to

$$\begin{aligned}\int_{-\infty}^0 B(w) e^{zw}\, \mathrm{d}w &= \sum_{n=1}^\infty \int_{-\infty}^0 \left(-e^{(z+2n-1)w} + e^{(z+2n)w}\, \mathrm{d}w\right) \\ &= \sum_{n=1}^\infty \left(-\frac{1}{(z+2n-1)} + \frac{1}{(z+2n)}\right) = \sum_{n=1}^\infty \frac{(-1)^n}{(z+n)}. \end{aligned} \tag{2.5}$$

Analogously, for the second integral in (2.3) we have

$$\begin{aligned}\int_0^\lambda B(w)e^{zw}\,\mathrm{d}w &= \int_0^\lambda \left(\sum_{n=0}^\infty -e^{-2nw}+e^{-(2n+1)w}\right)e^{zw}\,\mathrm{d}w\\ &= \sum_{n=0}^\infty \int_0^\lambda \left(-e^{(z-2n)w}+e^{(z-2n-1)w}\,\mathrm{d}w\right)\\ &= \sum_{n=0}^\infty \frac{(-1)^n}{(z-n)} + \sum_{n=0}^\infty (-1)^{n+1}\frac{e^{(z-n)\lambda}}{(z-n)}. \end{aligned} \tag{2.6}$$

Putting together (2.5) and (2.6) in expression (2.3) and using the identity

$$\frac{\pi}{\sin \pi z} = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{(z-n)},$$

we conclude the proof of (2.2). □

We now proceed to the proof of (1.4). As the function $x \mapsto K(\lambda, x)$ is even, it suffices to prove (1.4) assuming $x \geq 0$. We first observe that

$$K(\lambda, z) = e^{-\frac{\lambda}{2}}\left\{A\left(\lambda, z-\frac{1}{2}\right) + A\left(\lambda, -z-\frac{1}{2}\right)\right\}. \tag{2.7}$$

Note that the right-hand side of (2.2) defines an analytic function for $\Re(z) > -1$, and this implies that (2.2) is true for $\Re(z) > -1$ by analytic continuation. If $x \geq 0$, then $x - \frac{1}{2} > -1$, and (2.2) gives us

$$A\left(\lambda, x-\frac{1}{2}\right) = e^{-\lambda(x-\frac{1}{2})} + \left(\frac{\cos \pi x}{\pi}\right)\int_{-\infty}^0 B(\lambda+w)e^{xw-w/2}\,\mathrm{d}w, \tag{2.8}$$

and as we have $-x-\frac{1}{2} < 0$, (2.1) gives us

$$A\left(\lambda, -x-\frac{1}{2}\right) = -\left(\frac{\cos \pi x}{\pi}\right)\int_0^\infty B(\lambda+w)e^{-xw-w/2}\,\mathrm{d}w. \tag{2.9}$$

We define the function $C(w) = B(w)e^{-w/2}$, and use (2.8) and (2.9) in expression (2.7) to obtain

$$e^{-\lambda x} - K(\lambda, x) = \left(\frac{\cos \pi x}{\pi}\right)\int_0^\infty \left\{C(\lambda+w) - C(\lambda-w)\right\}e^{-xw}\,\mathrm{d}w. \tag{2.10}$$

Now it is just a matter of observing that

$$C(w) = -\frac{1}{e^{w/2}+e^{-w/2}}$$

is an even function, which is strictly increasing for $w > 0$. Therefore, for $\lambda > 0$ and $w > 0$, we have

$$C(\lambda - w) = C\big(|\lambda - w|\big) < C(\lambda + w),$$

and the integral in (2.10) above is strictly positive. This proves that the sign of $e^{-\lambda|x|} - K(\lambda, x)$ is the same as the sign of $\cos \pi x$, which is part (ii) of Theorem 1.1.

To prove part (i), we first verify that $x \mapsto e^{-\lambda|x|} - K(\lambda, x)$ is integrable. In fact,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \big|e^{-\lambda|x|} - K(\lambda, x)\big| \,\mathrm{d}x \\
&= 2\int_{0}^{\infty} \big|e^{-\lambda x} - K(\lambda, x)\big| \,\mathrm{d}x \\
&= 2\int_{0}^{\infty} \left|\frac{\cos \pi x}{\pi}\right| \int_{0}^{\infty} \big\{C(\lambda + w) - C(\lambda - w)\big\} e^{-xw} \,\mathrm{d}w \,\mathrm{d}x \\
&= \int_{0}^{\infty} \big\{C(\lambda + w) - C(\lambda - w)\big\} \int_{0}^{\infty} \left|\frac{\cos \pi x}{\pi}\right| e^{-xw} \,\mathrm{d}x \,\mathrm{d}w \\
&\le \int_{0}^{\infty} \frac{C(\lambda + w) - C(\lambda - w)}{\pi w} \,\mathrm{d}w < \infty. \qquad (2.11)
\end{aligned}$$

Let $\widetilde{K}(z)$ be a function of exponential type at most π such that $x \mapsto e^{-\lambda|x|} - \widetilde{K}(x)$ is integrable. This implies that $\widetilde{K}(x)$ is integrable. From a classical result of Polya and Plancherel (see (6.2) in Sect. 6), the function $\widetilde{K}(x)$ is bounded on $\mathbb{R}$. We write

$$\psi(x) = e^{-\lambda|x|} - \widetilde{K}(x).$$

From the Paley–Wiener theorem, the Fourier transform of $\widetilde{K}(x)$ is supported on the interval $[-\frac{1}{2}, \frac{1}{2}]$. Therefore,

$$\widehat{\psi}(t) = \frac{2\lambda}{\lambda^2 + 4\pi^2 t^2} \quad \text{for } |t| \ge \frac{1}{2}. \qquad (2.12)$$

The function $\operatorname{sgn}(\cos \pi x)$ has period 2 and Fourier series expansion

$$\operatorname{sgn}(\cos \pi x) = \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)} e\left(\left(k + \frac{1}{2}\right)x\right). \qquad (2.13)$$

As $\operatorname{sgn}(\cos \pi x)$ is a normalized function of bounded variation on $[0, 2]$, this Fourier expansion converges at every point x and the partial sums are uniformly bounded. Using (2.12) and (2.13) we obtain the lower bound

$$\begin{aligned}
\int_{-\infty}^{\infty} \big|e^{-\lambda|x|} - \widetilde{K}(x)\big| \,\mathrm{d}x &\ge \left|\int_{-\infty}^{\infty} \psi(x) \operatorname{sgn}(\cos \pi x) \,\mathrm{d}x\right| \\
&= \left|\frac{2}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)} \int_{-\infty}^{\infty} \psi(x) e\left(\left(k + \frac{1}{2}\right)x\right) \mathrm{d}x\right|
\end{aligned}$$

$$= \left| \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)} \widehat{\psi}\left(-\left(k+\frac{1}{2}\right)\right) \right|$$

$$= \left| \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)} \frac{2\lambda}{\left(\lambda^2 + 4\pi^2 \left(k+\frac{1}{2}\right)^2\right)} \right|$$

$$= \frac{2}{\lambda} - \frac{2}{\lambda} \operatorname{sech}\left(\frac{\lambda}{2}\right). \tag{2.14}$$

The last sum in (2.14) can be calculated by integrating the meromorphic function

$$H(z) = \frac{1}{z \cos \pi z} \left(\frac{2\lambda}{\lambda^2 + 4\pi^2 z^2} \right)$$

along the positively oriented square contour connecting the vertices $-N - Ni$, $N - Ni$, $N + Ni$, and $-N + Ni$, where N is a natural number with $N \to \infty$.

From part (ii) of Theorem 1.1 that we already proved, it is clear that equality occurs in (2.14) if $\widetilde{K}(z) = K(\lambda, z)$. On the other hand, if we assume that there is equality in (2.14), then $\psi(x)\,\mathrm{sgn}(\cos \pi x)$ does not change sign (either in its real or imaginary parts). Since $\widetilde{K}(x)$ is continuous, we deduce that

$$\widetilde{K}\left(n - \frac{1}{2}\right) = e^{-\lambda |n - \frac{1}{2}|}$$

for all $n \in \mathbb{Z}$. From classical interpolation formulas (see [17, vol. II, p. 275] or [15, p. 187]) we conclude that

$$\widetilde{K}(z) = K(\lambda, z) + \beta \cos(\pi z)$$

for some constant β. But we have seen that $\widetilde{K}(x)$ and $K(\lambda, x)$ are integrable, thus $\beta = 0$. This concludes the proof of Theorem 1.1.

3 Growth Estimates in the Complex Plane

Let $\mathcal{R} = \{z \in \mathbb{C} : 0 < \Re(z)\}$ denote the open right half-plane. Throughout this section we work with a function $\Phi(z)$ that is analytic on $\mathcal{R}$ and satisfies the following conditions: If $0 < a < b < \infty$, then

$$\lim_{y \to \pm\infty} e^{-\pi |y|} \int_a^b \left| \frac{\Phi(x+iy)}{x+iy} \right| \mathrm{d}x = 0; \tag{3.1}$$

if $0 < \eta < \infty$, then

$$\sup_{\eta \le x} \int_{-\infty}^{\infty} \left| \frac{\Phi(x+iy)}{x+iy} \right| e^{-\pi |y|} \,\mathrm{d}y < \infty; \tag{3.2}$$

and

$$\lim_{x \to \infty} \int_{-\infty}^{\infty} \left| \frac{\Phi(x+iy)}{x+iy} \right| e^{-\pi |y|} \,\mathrm{d}y = 0. \tag{3.3}$$

Lemma 3.1 *Assume that the analytic function $\Phi : \mathcal{R} \to \mathbb{C}$ satisfies the conditions* (3.1), (3.2), *and* (3.3), *and let* $0 < \delta$. *Then there exists a positive number* $c(\delta, \Phi)$, *depending only on* δ *and* Φ, *such that the inequality*

$$\left|\Phi(z)\right| \leq c(\delta, \Phi)|z|e^{\pi|y|} \tag{3.4}$$

holds for all $z = x + iy$ *in the closed half-plane* $\{z \in \mathbb{C} : \delta \leq \Re(z)\}$.

Proof Write $\eta = \min\{\frac{1}{4}, \frac{1}{2}\delta\}$, and set

$$c_1(\eta, \Phi) = \sup\left\{\int_{-\infty}^{\infty}\left|\frac{\Phi(u+iv)}{u+iv}\right|e^{-\pi|v|}\,dv : \eta \leq u\right\}.$$

Then $c_1(\eta, \Phi)$ is finite by (3.2). Let $z = x + iy$ satisfy $\delta \leq \Re(z)$ and let T be a positive real parameter such that $|y| + \eta < T$. Then write $\Gamma(z, \eta, T)$ for the simply connected, positively oriented, rectangular path connecting the points $x - \eta - iT$, $x + \eta - iT$, $x + \eta + iT$, $x - \eta + iT$, and $x - \eta - iT$. From Cauchy's integral formula we have

$$\frac{\Phi(z)}{z} = \frac{1}{2\pi i}\int_{\Gamma(z,\eta,T)} \frac{\Phi(w)}{w(w-z)\cos\pi(w-z)}\,dw. \tag{3.5}$$

At each point $w = u + iv$ on the path $\Gamma(z, \eta, T)$ we find that

$$\eta \leq |w - z| \tag{3.6}$$

and

$$\begin{aligned}\frac{1}{|\cos\pi(w-z)|^2} &= \frac{2}{(\cos 2\pi(u-x) + \cosh 2\pi(v-y))} \leq \frac{2}{(\cosh 2\pi(v-y))}\\ &\leq 4e^{-2\pi|v-y|} \leq 4e^{2\pi(|y|-|v|)},\end{aligned}$$

which implies

$$\frac{1}{|\cos\pi(w-z)|} \leq 2e^{\pi(|y|-|v|)}. \tag{3.7}$$

Using the estimates (3.6) and (3.7) together with (3.1), we get

$$\begin{aligned}&\limsup_{T\to\infty}\left|\int_{x-\eta\pm iT}^{x+\eta\pm iT}\frac{\Phi(w)}{w(w-z)\cos\pi(w-z)}\,dw\right|\\ &\quad\leq \limsup_{T\to\infty} 2\eta^{-1}e^{\pi(|y|-T)}\int_{x-\eta}^{x+\eta}\left|\frac{\Phi(u\pm iT)}{u\pm iT}\right|du = 0.\end{aligned} \tag{3.8}$$

It follows from (3.5) and (3.8) that

$$\frac{\Phi(z)}{z} = \frac{1}{2\pi i}\int_{x+\eta-i\infty}^{x+\eta+i\infty}\frac{\Phi(w)}{w(w-z)\cos\pi(w-z)}\,dw$$

$$-\frac{1}{2\pi i}\int_{x-\eta-i\infty}^{x-\eta+i\infty}\frac{\Phi(w)}{w(w-z)\cos\pi(w-z)}\,dw. \tag{3.9}$$

By applying (3.6) and (3.7) again, we find that

$$\begin{aligned}&\left|\int_{x\pm\eta-i\infty}^{x\pm\eta+i\infty}\frac{\Phi(w)}{w(w-z)\cos\pi(w-z)}\,dw\right|\\&\quad\le 2\eta^{-1}e^{\pi|y|}\int_{-\infty}^{\infty}\left|\frac{\Phi(x\pm\eta+iv)}{x\pm\eta+iv}\right|e^{-\pi|v|}\,dv\\&\quad\le 2c_1(\eta,\Phi)\eta^{-1}e^{\pi|y|}.\end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10) leads to the estimate

$$\left|\frac{\Phi(z)}{z}\right|\le 2(\pi\eta)^{-1}c_1(\eta,\Phi)e^{\pi|y|},$$

and this plainly verifies (3.4) with $c(\delta,\Phi)=2(\pi\eta)^{-1}c_1(\eta,\Phi)$. □

Let $w=u+iv$ be a complex variable. From (3.2) we find that for each positive real number β such that $\beta-\frac{1}{2}$ is not an integer, and each complex number z with $|\Re(z)|\ne\beta$, the function

$$w\mapsto\left(\frac{\cos\pi z}{\cos\pi w}\right)\left(\frac{2w}{z^2-w^2}\right)\Phi(w)$$

is integrable along the vertical line $\Re(w)=\beta$. We define a complex-valued function $z\mapsto I(\beta,\Phi;z)$ on the open set

$$\left\{z\in\mathbb{C}:\left|\Re(z)\right|\ne\beta\right\} \tag{3.11}$$

by

$$I(\beta,\Phi;z)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\left(\frac{\cos\pi z}{\cos\pi w}\right)\left(\frac{2w}{z^2-w^2}\right)\Phi(w)\,dw. \tag{3.12}$$

It follows using Morera's theorem that $z\mapsto I(\beta,\Phi;z)$ is analytic in each of the three connected components.

Next we prove a simple estimate for $I(\beta,\Phi;z)$.

Lemma 3.2 *Assume that the analytic function $\Phi:\mathcal{R}\to\mathbb{C}$ satisfies the conditions* (3.1), (3.2), *and* (3.3). *Let β be a positive real number, $z=x+iy$ a complex number such that $|\Re(z)|\ne\beta$, and write*

$$B(\beta,\Phi)=\frac{2}{\pi}\int_{-\infty}^{+\infty}\left|\frac{\Phi(\beta+iv)}{\beta+iv}\right|e^{-\pi|v|}\,dv. \tag{3.13}$$

If $\beta-\frac{1}{2}$ is not an integer, then

$$\left|I(\beta,\Phi;z)\right|\le B(\beta,\Phi)|\sec\pi\beta|\left(1+\frac{|z|}{||x|-\beta|}\right)e^{\pi|y|}. \tag{3.14}$$

Proof On the vertical line $\Re(w)=\beta$ we have

$$\big||x|-\beta\big| \le \min\big\{|z-w|, |z+w|\big\}$$

and

$$|z| \le \frac{1}{2}|z-w| + \frac{1}{2}|z+w| \le \max\big\{|z-w|, |z+w|\big\}.$$

Therefore,

$$\begin{aligned}\left|\frac{w^2}{z^2-w^2}\right| &\le 1 + \left|\frac{z^2}{z^2-w^2}\right| \\ &= 1 + |z|^2\big(\min\big\{|z-w|, |z+w|\big\}\max\big\{|z-w|, |z+w|\big\}\big)^{-1} \\ &\le 1 + \frac{|z|}{||x|-\beta|}. \end{aligned} \tag{3.15}$$

On the line $\Re(w)=\beta$ we also use the elementary inequality

$$\big|\cos\pi(\beta+iv)\big|^{-1} \le 2e^{-\pi|v|}|\sec\pi\beta|. \tag{3.16}$$

Then we use (3.15) and (3.16) to estimate the integral on the right of (3.12). The bound (3.14) follows easily. □

For each positive number ξ we define an even rational function $z \mapsto \mathcal{A}(\xi, \Phi; z)$ on $\mathbb{C}$ by

$$\mathcal{A}(\xi, \Phi; z) = \frac{\Phi(\xi)}{(z-\xi)} - \frac{\Phi(\xi)}{(z+\xi)}. \tag{3.17}$$

Lemma 3.3 *Assume that the analytic function $\Phi : \mathcal{R} \to \mathbb{C}$ satisfies the conditions (3.1), (3.2), and (3.3). Then the sequence of entire functions*

$$\left(\frac{\cos\pi z}{\pi}\right)\sum_{n=1}^{N}(-1)^n\mathcal{A}\left(n-\frac{1}{2}, \Phi; z\right), \quad \textit{where } N = 1, 2, 3, \dots, \tag{3.18}$$

converges uniformly on compact subsets of $\mathbb{C}$ as $N \to \infty$, and therefore

$$\mathcal{K}(\Phi, z) = \lim_{N\to\infty}\left(\frac{\cos\pi z}{\pi}\right)\sum_{n=1}^{N}(-1)^n\mathcal{A}\left(n-\frac{1}{2}, \Phi; z\right) \tag{3.19}$$

defines an entire function.

Proof We assume that z is a complex number in $\mathcal{R}$ such that $z-\frac{1}{2}$ is not an integer. Then

$$w \mapsto \left(\frac{\cos\pi z}{\cos\pi w}\right)\left(\frac{2w}{z^2-w^2}\right)\Phi(w) \tag{3.20}$$

defines a meromorphic function of w on the right half-plane $\mathcal{R}$. We find that (3.20) has a simple pole at $w = z$ with residue $-\Phi(z)$. And for each positive integer n, (3.20) has a pole of order at most one at $w = n - \frac{1}{2}$ with residue

$$\left(\frac{\cos \pi z}{\pi}\right)(-1)^n \mathcal{A}\left(n - \frac{1}{2}, \Phi; z\right).$$

Plainly, (3.20) has no other poles in $\mathcal{R}$. Let $0 < \beta < \frac{1}{2}$, let N be a positive integer, and T a positive real parameter. Write $\Gamma(\beta, N, T)$ for the simply connected, positively oriented rectangular path connecting the points $\beta - iT$, $N - iT$, $N + iT$, $\beta + iT$, and $\beta - iT$. If z satisfies $\beta < \Re(z) < N$ and $|\Im(z)| < T$, and $z - \frac{1}{2}$ is not an integer, then from the residue theorem we obtain the identity

$$\begin{aligned}\left(\frac{\cos \pi z}{\pi}\right)\sum_{n=1}^{N}(-1)^n \mathcal{A}\left(n - \frac{1}{2}, \Phi; z\right) - \Phi(z) \\ = \frac{1}{2\pi i}\int_{\Gamma(\beta,N,T)}\left(\frac{\cos \pi z}{\cos \pi w}\right)\left(\frac{2w}{z^2 - w^2}\right)\Phi(w)\,dw.\end{aligned} \tag{3.21}$$

We let $T \to \infty$ on the right-hand side of (3.21), and we use hypotheses (3.1) and (3.2). In this way we conclude that

$$\left(\frac{\cos \pi z}{\pi}\right)\sum_{n=1}^{N}(-1)^n \mathcal{A}\left(n - \frac{1}{2}, \Phi; z\right) - \Phi(z) = I(N, \Phi; z) - I(\beta, \Phi; z). \tag{3.22}$$

Initially (3.22) holds for $\beta < \Re(z) < N$ and $z - \frac{1}{2}$ not an integer. However, we have already observed that both sides of (3.22) are analytic in the strip $\{z \in \mathbb{C} : \beta < \Re(z) < N\}$. Therefore the condition that $z - \frac{1}{2}$ is not an integer can be dropped.

Now let $M < N$ be positive integers. From (3.22) we find that

$$\left(\frac{\cos \pi z}{\pi}\right)\sum_{n=M+1}^{N}(-1)^n \mathcal{A}\left(n - \frac{1}{2}, \Phi; z\right) = I(N, \Phi; z) - I(M, \Phi; z) \tag{3.23}$$

in the infinite strip $\{z \in \mathbb{C} : \beta < \Re(z) < M\}$. In fact we have seen that both sides of (3.23) are analytic in the infinite strip $\{z \in \mathbb{C} : |\Re(z)| < M\}$. Therefore, the identity (3.23) must hold in this larger domain by analytic continuation. Let $\mathcal{J} \subseteq \mathbb{C}$ be a compact set, and assume that L is an integer so large that $\mathcal{J} \subseteq \{z \in \mathbb{C} : 2|z| < L\}$. From (3.3), Lemma 3.2, and (3.23), it is obvious that the sequence of entire functions (3.18), where $L \le N$, is uniformly Cauchy on $\mathcal{J}$. This verifies the lemma showing that (3.19) defines an entire function. □

Lemma 3.4 *Assume that the analytic function $\Phi : \mathcal{R} \to \mathbb{C}$ satisfies the conditions* (3.1), (3.2) *and* (3.3). *Let the entire function $\mathcal{K}(\Phi, z)$ be defined by* (3.19). *If $0 < \beta < \frac{1}{2}$, then the identity*

$$\Phi(z) - \mathcal{K}(\Phi, z) = I(\beta, \Phi; z) \tag{3.24}$$

holds for all z in the half-plane $\{z \in \mathbb{C} : \beta < \Re(z)\}$, and the identity

$$-\mathcal{K}(\Phi, z) = I(\beta, \Phi; z) \tag{3.25}$$

holds for all z in the infinite strip $\{z \in \mathbb{C} : |\Re(z)| < \beta\}$.

Proof We argue as in the proof of Lemma 3.3, letting $N \to \infty$ on both sides of (3.22). Then we use (3.3) and Lemma 3.2, and obtain the identity

$$\Phi(z) - \mathcal{K}(\Phi, z) = I(\beta, \Phi; z)$$

at each point of the half-plane $\{z \in \mathbb{C} : \beta < \Re(z)\}$. This proves (3.24).

Next, we assume that $|\Re(z)| < \beta$. In this case the residue theorem provides the identity

$$\begin{aligned}\left(\frac{\cos \pi z}{\pi}\right) &\sum_{n=1}^{N} (-1)^n \mathcal{A}\left(n - \frac{1}{2}, \Phi; z\right) \\ &= \frac{1}{2\pi i} \int_{\Gamma(\beta, N, T)} \left(\frac{\cos \pi z}{\cos \pi w}\right)\left(\frac{2w}{z^2 - w^2}\right) \Phi(w)\, \mathrm{d}w.\end{aligned} \tag{3.26}$$

We let $T \to \infty$ and argue as before. In this way (3.26) leads to

$$\left(\frac{\cos \pi z}{\pi}\right) \sum_{n=1}^{N} (-1)^n \mathcal{A}\left(n - \frac{1}{2}, \Phi; z\right) = I(N, \Phi; z) - I(\beta, \Phi; z). \tag{3.27}$$

Then we let $N \to \infty$ on both sides of (3.27), and we use (3.3) and Lemma 3.2 again. We find that

$$-\mathcal{K}(\Phi, z) = I(\beta, \Phi; z),$$

and this verifies (3.25). □

Corollary 3.5 *Suppose that $\Phi(z) = 1$ is constant on $\mathcal{R}$. If $0 < \beta < \frac{1}{2}$, then*

$$I(\beta, 1; z) = 0 \tag{3.28}$$

in the open half-plane $\{z \in \mathbb{C} : \beta < \Re(z)\}$.

Proof We have

$$\begin{aligned}\mathcal{K}(1, z) &= \lim_{N\to\infty} \left(\frac{\cos \pi z}{\pi}\right) \sum_{n=1}^{N} (-1)^n \mathcal{A}\left(n - \frac{1}{2}, 1; z\right) \\ &= \lim_{N\to\infty} \left(\frac{\cos \pi z}{\pi}\right) \sum_{n=-N+1}^{N} (-1)^n \left(z - n + \frac{1}{2}\right)^{-1} = 1.\end{aligned}$$

Now the identity (3.28) follows from (3.24). □

Lemma 3.6 *Assume that the analytic function $\Phi : \mathcal{R} \to \mathbb{C}$ satisfies the conditions* (3.1), (3.2) *and* (3.3). *Let the entire function $\mathcal{K}(\Phi, z)$ be defined by* (3.19). *Then there exists a positive number $c(\Phi)$, depending only on Φ, such that the inequality*

$$\big|\mathcal{K}(\Phi, z)\big| \le c(\Phi)\big(1 + |z|\big)e^{\pi|y|} \tag{3.29}$$

holds for all complex numbers $z = x + iy$. In particular, $\mathcal{K}(\Phi, z)$ is an entire function of exponential type at most π.

Proof In the closed half-plane $\{z \in \mathbb{C} : \frac{1}{4} \le \Re(z)\}$ the identity (3.24) implies that

$$\big|\mathcal{K}(\Phi, z)\big| \le \big|\Phi(z)\big| + \left|I\left(\frac{1}{8}, \Phi; z\right)\right|.$$

Then an estimate of the form (3.29) in this half-plane follows from Lemma 3.1 and Lemma 3.2. In the closed infinite strip $\{z \in \mathbb{C} : |\Re(z)| \le \frac{1}{4}\}$ we have

$$\big|\mathcal{K}(\Phi, z)\big| = \left|I\left(\frac{3}{8}, \Phi; z\right)\right|$$

from the identity (3.25). Plainly, an estimate of the form (3.29) in this closed infinite strip follows from Lemma 3.2. This suffices to prove inequality (3.29) for all complex z, since $\mathcal{K}(\Phi, z)$ is an even function of z. □

4 Fourier Expansions

Lemma 4.1 *If $0 < \beta < \frac{1}{2}$, then at each point z in the half-plane $\{z \in \mathbb{C} : \beta < \Re(z)\}$ we have*

$$e^{-\lambda z} - K(\lambda, z) = \frac{1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty}\left(\frac{\cos \pi z}{\cos \pi w}\right)\left(\frac{2w}{z^2 - w^2}\right)e^{-\lambda w}\, \mathrm{d}w. \tag{4.1}$$

Proof We apply Lemma 3.3 with $\Phi(z) = e^{-\lambda z}$. It follows that

$$\mathcal{K}(\Phi, z) = K(\lambda, z).$$

The identity (4.1) follows now from Lemma 3.4. □

As $x \mapsto K(\lambda, x)$ is the restriction of a function of exponential type π, bounded and integrable on $\mathbb{R}$, its Fourier transform

$$\widehat{K}(\lambda, t) = \int_{-\infty}^{\infty} K(\lambda, x)e(-tx)\, \mathrm{d}x \tag{4.2}$$

is a continuous function of the real variable t supported on the interval $[-\frac{1}{2}, \frac{1}{2}]$. Then by Fourier inversion we have the representation

$$K(\lambda, z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \widehat{K}(\lambda, t)e(tz)\, \mathrm{d}t \tag{4.3}$$

for all complex z. It will be useful to have more explicit information about the Fourier transform of this function.

Lemma 4.2 *For* $|t| \le \frac{1}{2}$ *the Fourier transform* (4.2) *is given by*

$$\widehat{K}(\lambda, t) = \frac{\sinh\left(\frac{\lambda}{2}\right)\cos\pi t}{\sinh^2\left(\frac{\lambda}{2}\right) + \sin^2\pi t}. \tag{4.4}$$

From (4.4) *we conclude that*

$$\widehat{K}(\lambda, t) \ge 0 \tag{4.5}$$

for all $t \in \mathbb{R}$.

Proof The following entire function

$$H(z) = \frac{\cos\pi z}{\pi\left(z + \frac{1}{2}\right)}$$

has exponential type π and, when restricted to $\mathbb{R}$, belongs to $L^2(\mathbb{R})$. By the Paley–Wiener theorem we know that its Fourier transform is supported on $[-\frac{1}{2}, \frac{1}{2}]$, being explicitly given by

$$\widehat{H}(t) = e^{\pi i t} \tag{4.6}$$

for $t \in [-\frac{1}{2}, \frac{1}{2}]$. An adaptation of [15, Theorem 9], together with (4.6), shows that the entire function of exponential type at most π, integrable on $\mathbb{R}$,

$$K(\lambda, z) = \sum_{n \in \mathbb{Z}} e^{-\lambda|n-\frac{1}{2}|}\left(\frac{\cos\pi(z-n)}{\pi\left(z - n + \frac{1}{2}\right)}\right) \tag{4.7}$$

has a continuous Fourier transform supported on $[-\frac{1}{2}, \frac{1}{2}]$ given by

$$\widehat{K}(\lambda, t) = \sum_{n \in \mathbb{Z}} e^{-\lambda|n-\frac{1}{2}|} e^{-2\pi i t n} e^{\pi i t} \tag{4.8}$$

for $t \in [-\frac{1}{2}, \frac{1}{2}]$. This leads to (4.4). □

Lemma 4.3 *Let* ν *be a finite measure on the Borel subsets of* $(0, \infty)$. *For each complex number* z *the function* $\lambda \mapsto K(\lambda, z)$ *is* ν*-integrable on* $(0, \infty)$. *The complex-valued function*

$$K_\nu^\star(z) = \int_0^\infty K(\lambda, z)\,\mathrm{d}\nu(\lambda) \tag{4.9}$$

is an entire function which satisfies the inequality

$$\left|K_\nu^\star(z)\right| \le \nu\left\{(0, \infty)\right\} e^{\pi|y|} \tag{4.10}$$

for all $z = x + iy$. *In particular,* $K_\nu^\star(z)$ *is an entire function of exponential type at most* π.

Proof We apply (4.3) and the fact that $0 \le \widehat{K}(\lambda, t)$. We find that

$$\begin{aligned}
\int_0^\infty \big|K(\lambda, z)\big| \,\mathrm{d}\nu(\lambda) &= \int_0^\infty \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \widehat{K}(\lambda, t) e(tz) \,\mathrm{d}t \right| \mathrm{d}\nu(\lambda) \\
&\le \int_0^\infty \int_{-\frac{1}{2}}^{\frac{1}{2}} \widehat{K}(\lambda, t) e^{-2\pi t y} \,\mathrm{d}t \,\mathrm{d}\nu(\lambda) \\
&\le e^{\pi|y|} \int_0^\infty \int_{-\frac{1}{2}}^{\frac{1}{2}} \widehat{K}(\lambda, t) \,\mathrm{d}t \,\mathrm{d}\nu(\lambda) \\
&= e^{\pi|y|} \int_0^\infty K(\lambda, 0) \,\mathrm{d}\nu(\lambda). \qquad (4.11)
\end{aligned}$$

As $K(\lambda, 0) \le 1$ by (1.4), it follows from (4.11) that

$$\int_0^\infty \big|K(\lambda, z)\big| \,\mathrm{d}\nu(\lambda) \le \nu\big\{(0, \infty)\big\} e^{\pi|y|}.$$

This shows that $\lambda \mapsto K(\lambda, z)$ is ν-integrable on $(0, \infty)$ and verifies the bound (4.10). It follows easily using Morera's theorem that $z \mapsto K_\nu^\star(z)$ is an entire function. Then (4.10) implies that this entire function has exponential type at most π. □

Let ν be a finite measure on the Borel subsets of $(0, \infty)$. It follows that

$$\Psi_\nu(z) = \int_0^\infty e^{-\lambda z} \,\mathrm{d}\nu(\lambda) \qquad (4.12)$$

defines a function that is bounded and continuous in the closed half-plane $\{z \in \mathbb{C} : 0 \le \Re(z)\}$, and analytic in the interior of this half-plane.

Lemma 4.4 *If $0 < \beta < \frac{1}{2}$, then at each point z in the half-plane $\{z \in \mathbb{C} : \beta < \Re(z)\}$ we have*

$$\Psi_\nu(z) - K_\nu^\star(z) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \left(\frac{\cos \pi z}{\cos \pi w} \right) \left(\frac{2w}{z^2 - w^2} \right) \Psi_\nu(w) \,\mathrm{d}w. \qquad (4.13)$$

Proof We apply (4.1) and Fubini's theorem to get

$$\begin{aligned}
\Psi_\nu(z) - K_\nu^\star(z) &= \int_0^\infty \big\{ e^{-\lambda z} - K(\lambda, z) \big\} \,\mathrm{d}\nu(\lambda) \\
&= \int_0^\infty \left\{ \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \left(\frac{\cos \pi z}{\cos \pi w} \right) \left(\frac{2w}{z^2 - w^2} \right) e^{-\lambda w} \,\mathrm{d}w \right\} \mathrm{d}\nu(\lambda) \\
&= \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \left(\frac{\cos \pi z}{\cos \pi w} \right) \left(\frac{2w}{z^2 - w^2} \right) \Psi_\nu(w) \,\mathrm{d}w.
\end{aligned}$$

This proves (4.13). □

5 Proof of Theorem 1.2

Let μ be a measure defined on the Borel subsets of $(0,\infty)$ that satisfies (1.6). Let $z = x + iy$ be a point in the open right half-plane $\mathcal{R} = \{z \in \mathbb{C} : 0 < \Re(z)\}$. Using (1.6) we find that

$$\lambda \mapsto e^{-\lambda z} - e^{-\lambda}$$

is integrable on $(0,\infty)$ with respect to μ. We define $F_\mu : \mathcal{R} \to \mathbb{C}$ by

$$F_\mu(z) = \int_0^\infty \left\{e^{-\lambda z} - e^{-\lambda}\right\} \mathrm{d}\mu(\lambda). \tag{5.1}$$

It follows by applying Morera's theorem that $F_\mu(z)$ is analytic on $\mathcal{R}$. Also, at each point z in $\mathcal{R}$ the derivative of F_μ is given by

$$F'_\mu(z) = -\int_0^\infty \lambda e^{-\lambda z} \,\mathrm{d}\mu(\lambda). \tag{5.2}$$

Then (5.2) leads to the bound

$$\left|F'_\mu(x+iy)\right| \le \int_0^\infty \lambda e^{-\lambda x} \,\mathrm{d}\mu(\lambda) = \left|F'_\mu(x)\right|. \tag{5.3}$$

From (5.3) and the dominated convergence theorem we conclude that

$$\lim_{x\to\infty} \left|F'_\mu(x+iy)\right| = 0 \tag{5.4}$$

uniformly in y. Clearly, the functions $f_\mu(x)$, defined by (1.7), and $F_\mu(z)$, defined by (5.1), satisfy the identities

$$f_\mu(x) = F_\mu\big(|x|\big) \quad \text{and} \quad f'_\mu(x) = \operatorname{sgn}(x) F'_\mu\big(|x|\big) \tag{5.5}$$

for all real $x \neq 0$.

Lemma 5.1 *The analytic function $F_\mu(z)$ defined by (5.1) satisfies each of the three conditions (3.1), (3.2), and (3.3).*

Proof Let $0 < \xi \le 1$. If $\xi \le \Re(z)$, then from (5.3) we obtain the inequality

$$\begin{aligned} \left|F_\mu(z)\right| &= \left|\int_1^z F'_\mu(w)\,\mathrm{d}w\right| \le |z-1| \max\left\{\left|F'_\mu(\theta z + 1 - \theta)\right| : 0 \le \theta \le 1\right\} \\ &\le \big(|z|+1\big)\left|F'_\mu(\xi)\right|, \end{aligned}$$

and therefore

$$\left|\frac{F_\mu(z)}{z}\right| \le \big(1+\xi^{-1}\big)\left|F'_\mu(\xi)\right|. \tag{5.6}$$

The conditions (3.1) and (3.2) follow from the bound (5.6).

Now assume that $1 \le x = \Re(z)$. We have

$$\begin{aligned}\left|F_\mu(x+iy)\right| &= \left|\int_1^x F'_\mu(u)\,\mathrm{d}u + i\int_0^y F'_\mu(x+iv)\,\mathrm{d}v\right| \\ &\le \int_1^x \left|F'_\mu(u)\right|\mathrm{d}u + |y|\left|F'_\mu(x)\right|,\end{aligned}$$

and therefore

$$\left|\frac{F_\mu(x+iy)}{x+iy}\right| \le \frac{1}{x}\int_1^x \left|F'_\mu(u)\right|\mathrm{d}u + \left|F'_\mu(x)\right|. \tag{5.7}$$

Then (5.4) and (5.7) imply that

$$\lim_{x\to\infty}\left|\frac{F_\mu(x+iy)}{x+iy}\right| = 0$$

uniformly in y. The remaining condition (3.3) follows from this. □

We are now in a position to apply the results of Sects. 3 and 4 to the function $F_\mu(z)$. In view of the identities (5.5), the entire function $K_\mu(z)$, defined by (1.8), and the entire function $\mathcal{K}(F_\mu, z)$, defined by (3.19), are equal. If $0 < \beta < \frac{1}{2}$, and $\beta < \Re(z)$, then from (3.24) of Lemma 3.4 we have

$$F_\mu(z) - K_\mu(z) = I(\beta, F_\mu; z). \tag{5.8}$$

Applying Lemma 3.6 we conclude that $K_\mu(z)$ is an entire function of exponential type at most π. This verifies (i) in the statement of Theorem 1.2.

Next we define a sequence of measures $\nu_1, \nu_2, \nu_3, \dots$ on Borel subsets $E \subseteq (0,\infty)$ by

$$\nu_n(E) = \int_E \left(e^{-\lambda/n} - e^{-\lambda n}\right)\mathrm{d}\mu(\lambda), \quad \text{for } n = 1, 2, \dots. \tag{5.9}$$

Then

$$\begin{aligned}\nu_n\{(0,\infty)\} &= \int_0^\infty \int_{1/n}^n \lambda e^{-\lambda u}\,\mathrm{d}u\,\mathrm{d}\mu(\lambda) = -\int_{1/n}^n F'_\mu(u)\,\mathrm{d}u \\ &= F_\mu(1/n) - F_\mu(n) < \infty,\end{aligned}$$

and therefore ν_n is a finite measure for each n. It will be convenient to simplify the notation used in (4.9) and (4.12). For z in $\mathbb{C}$ and n a positive integer we write

$$K_n(z) = \int_0^\infty K(\lambda, z)\,\mathrm{d}\nu_n(\lambda), \tag{5.10}$$

and for z in $\mathcal{R}$ we write

$$\Psi_n(z) = \int_0^\infty e^{-\lambda z}\,\mathrm{d}\nu_n(\lambda). \tag{5.11}$$

From Lemma 4.3 we learn that $K_n(z)$ is an entire function of exponential type at most π. If $0<\beta<\frac{1}{2}$, then (3.28) and (4.13) imply that

$$\Psi_n(z)-K_n(z)=I(\beta,\Psi_n;z)=I\big(\beta,\Psi_n-\Psi_n(1);z\big) \tag{5.12}$$

for all complex z such that $\beta<\Re(z)$. From the definitions (5.9), (5.10), and (5.11), we find that

$$\Psi_n(x)-K_n(x)=\int_0^\infty \big(e^{-\lambda x}-K(\lambda,x)\big)\big(e^{-\lambda/n}-e^{-\lambda n}\big)\,\mathrm{d}\mu(\lambda) \tag{5.13}$$

for all positive real x.

Let $w=u+iv$ be a point in $\mathcal{R}$. Then

$$\Psi_n(w)-\Psi_n(1)=\int_0^\infty \big(e^{-\lambda w}-e^{-\lambda}\big)\big(e^{-\lambda/n}-e^{-\lambda n}\big)\,\mathrm{d}\mu(\lambda), \tag{5.14}$$

and

$$\big|e^{-\lambda/n}-e^{-\lambda n}\big|\le 1$$

for all positive real λ and positive integers n. We let $n\to\infty$ on both sides of (5.14) and apply the dominated convergence theorem. In this way we conclude that

$$\lim_{n\to\infty}\Psi_n(w)-\Psi_n(1)=F_\mu(w) \tag{5.15}$$

at each point w in $\mathcal{R}$. If $0<\beta<\frac{1}{2}$ then, as in the proof of Lemma 5.1, on the line $\beta=\Re(w)$ we have

$$\big|\Psi_n(w)-\Psi_n(1)\big|\le\int_0^\infty\left|\int_1^w \lambda e^{-\lambda t}\,\mathrm{d}t\right|\mathrm{d}\mu(\lambda)\le\big(|w|+1\big)\big|F_\mu'(\beta)\big|.$$

It follows that

$$\left|\frac{\Psi_n(w)-\Psi_n(1)}{w}\right|$$

is bounded on the line $\beta=\Re(w)$. From this observation, together with (5.12) and (5.15), we conclude that

$$\lim_{n\to\infty}\Psi_n(z)-K_n(z)=\lim_{n\to\infty}I\big(\beta,\Psi_n-\Psi_n(1);z\big)=I(\beta,F_\mu;z)=F_\mu(z)-K_\mu(z) \tag{5.16}$$

at each complex number z with $\beta<\Re(z)$. In particular, we have

$$\lim_{n\to\infty}\Psi_n(x)-K_n(x)=F_\mu(x)-K_\mu(x) \tag{5.17}$$

for all positive x. We combine (5.13), (5.17) and (1.4) to use the monotone convergence theorem. This leads to the identity

$$F_\mu(x)-K_\mu(x)=\int_0^\infty\big(e^{-\lambda x}-K(\lambda,x)\big)\,\mathrm{d}\mu(\lambda) \tag{5.18}$$

for all positive x. Then we use the identity on the left of (5.5), and the fact that $x \mapsto K_\mu(x)$ is an even function, to write (5.18) as

$$f_\mu(x) - K_\mu(x) = \int_0^\infty \left(e^{-\lambda|x|} - K(\lambda, x)\right) \mathrm{d}\mu(\lambda) \tag{5.19}$$

for all $x \neq 0$. If $f_\mu(0)$ is finite, then (5.19) holds at $x = 0$ by continuity. If $f_\mu(0) = \infty$, then both sides of (5.19) are ∞. This establishes both (ii) and (iii) in the statement of Theorem 1.2.

Because of (1.4), we get

$$\begin{aligned}\int_{-\infty}^{\infty} \left|f_\mu(x) - K_\mu(x)\right| \mathrm{d}x &= \int_{-\infty}^{\infty}\int_0^\infty \left|e^{-\lambda|x|} - K(\lambda, x)\right| \mathrm{d}\mu(\lambda)\,\mathrm{d}x \\ &= \int_0^\infty \int_{-\infty}^{\infty} \left|e^{-\lambda|x|} - K(\lambda, x)\right| \mathrm{d}x\,\mathrm{d}\mu(\lambda) \\ &= \int_0^\infty \left\{\frac{2}{\lambda} - \frac{2}{\lambda}\operatorname{sech}\left(\frac{\lambda}{2}\right)\right\} \mathrm{d}\mu(\lambda) \end{aligned} \tag{5.20}$$

by Fubini's theorem. This proves (iv) of Theorem 1.2. Similarly, if $t \neq 0$, we find that

$$\begin{aligned}&\int_{-\infty}^{\infty} \left\{f_\mu(x) - K_\mu(x)\right\} e(-tx)\,\mathrm{d}x \\ &\quad= \int_{-\infty}^{\infty} \left\{\int_0^\infty \left(e^{-\lambda|x|} - K(\lambda, x)\right) \mathrm{d}\mu(\lambda)\right\} e(-tx)\,\mathrm{d}x \\ &\quad= \int_0^\infty \left\{\int_{-\infty}^{\infty} \left(e^{-\lambda|x|} - K(\lambda, x)\right) e(-tx)\,\mathrm{d}x\right\} \mathrm{d}\mu(\lambda) \\ &\quad= \int_0^\infty \frac{2\lambda}{\lambda^2 + 4\pi^2 t^2}\,\mathrm{d}\mu(\lambda) - \int_0^\infty \widehat{K}(\lambda, t)\,\mathrm{d}\mu(\lambda). \end{aligned} \tag{5.21}$$

This proves (v) in Theorem 1.2.

Finally, we assume that $\widetilde{K}(z)$ is an entire function of exponential type at most π and that

$$\int_{-\infty}^{\infty} \left|f_\mu(x) - \widetilde{K}(x)\right| \mathrm{d}x < \infty. \tag{5.22}$$

By the triangle inequality $K_\mu(x) - \widetilde{K}(x)$ is integrable, and since it has exponential type at most π, we know that its Fourier transform is supported on $[-\frac{1}{2}, \frac{1}{2}]$. Moreover, by a result of Polya and Plancherel (see (6.2) in Sect. 6) the function $K_\mu(x) - \widetilde{K}(x)$ is bounded. We write

$$\psi(x) = f_\mu(x) - \widetilde{K}(x) = \left\{f_\mu(x) - K(x)\right\} + \left\{K(x) - \widetilde{K}(x)\right\}. \tag{5.23}$$

From (5.23) and (5.21) we conclude that the Fourier transform of $\psi(x)$ is given by

$$\widehat{\psi}(t) = \int_0^\infty \frac{2\lambda}{\lambda^2 + 4\pi^2 t^2}\,\mathrm{d}\mu(\lambda) \quad \text{for } |t| \geq \frac{1}{2}. \tag{5.24}$$

We simply proceed as in (2.14) to conclude part (vi) of Theorem 1.2. From this we also note that $\widetilde{K}(z)$ minimizes the integral (5.22) if and only if

$$\widetilde{K}\left(n-\frac{1}{2}\right)=f_\mu\left(n-\frac{1}{2}\right) \tag{5.25}$$

for all $n\in\mathbb{Z}$. Therefore,

$$\left(\widetilde{K}-K_\mu\right)\left(n-\frac{1}{2}\right)=0$$

for all $n\in\mathbb{Z}$. From the interpolation formulas (see [17, vol. II, p. 275] or [15, p. 187]) we observe that

$$(\widetilde{K}-K_\mu)(z)=\beta\cos(\pi z)$$

for some constant β. But we have seen that $(\widetilde{K}-K_\mu)(x)$ is integrable, thus $\beta=0$. This concludes the proof of (vii) in Theorem 1.2.

6 Extremal Trigonometric Polynomials

We consider in this section the problem of approximating certain real-valued periodic functions by trigonometric polynomials of bounded degree. We identify functions defined on $\mathbb{R}$ and having period 1 with functions defined on the compact quotient group $\mathbb{R}/\mathbb{Z}$. For real numbers x we write

$$\|x\|=\min\left\{|x-m|:m\in\mathbb{Z}\right\}$$

for the distance from x to the nearest integer. Then $\|\ \|:\mathbb{R}/\mathbb{Z}\to[0,\frac{1}{2}]$ is well defined, and $(x,y)\to\|x-y\|$ defines a metric on $\mathbb{R}/\mathbb{Z}$ which induces its quotient topology. Integrals over $\mathbb{R}/\mathbb{Z}$ are with respect to the Haar measure normalized so that $\mathbb{R}/\mathbb{Z}$ has measure 1.

Let $F:\mathbb{C}\to\mathbb{C}$ be an entire function of exponential type at most $\pi\delta$, where δ is a positive parameter, and assume that $x\mapsto F(x)$ is integrable on $\mathbb{R}$. Then the Fourier transform

$$\widehat{F}(t)=\int_{-\infty}^{\infty}F(x)e(-tx)\,\mathrm{d}x \tag{6.1}$$

is a continuous function on $\mathbb{R}$. By classical results of Plancherel and Polya [12] (see also [16, Chap. 2, Part 2, Sect. 3]), we have

$$\sum_{m=-\infty}^{\infty}\left|F(\alpha_m)\right|\le C_1(\epsilon,\delta)\int_{-\infty}^{\infty}\left|F(x)\right|\mathrm{d}x, \tag{6.2}$$

where $m\mapsto\alpha_m$ is a sequence of real numbers such that $\alpha_{m+1}-\alpha_m\ge\epsilon>0$, and

$$\int_{-\infty}^{\infty}\left|F'(x)\right|\mathrm{d}x\le C_2(\delta)\int_{-\infty}^{\infty}\left|F(x)\right|\mathrm{d}x. \tag{6.3}$$

Plainly, (6.2) implies that F is uniformly bounded on $\mathbb{R}$, and therefore $x \mapsto |F(x)|^2$ is integrable. Then it follows from the Paley–Wiener theorem (see [13, Theorem 19.3]) that $\widehat{F}(t)$ is supported on the interval $[-\frac{\delta}{2}, \frac{\delta}{2}]$.

The bound (6.3) implies that $x \mapsto F(x)$ has bounded variation on $\mathbb{R}$. Therefore, the Poisson summation formula (see [17, vol. I, Chap. 2, Sect. 13]) holds as a pointwise identity

$$\sum_{m=-\infty}^{\infty} F(x+m) = \sum_{n=-\infty}^{\infty} \widehat{F}(n)e(nx), \tag{6.4}$$

for all real x. It follows from (6.2) that the sum on the left of (6.4) is absolutely convergent. As the continuous function $\widehat{F}(t)$ is supported on $[-\frac{\delta}{2}, \frac{\delta}{2}]$, the sum on the right of (6.4) has only finitely many nonzero terms, and so defines a trigonometric polynomial in x.

Next we consider the entire function $z \mapsto K(\delta^{-1}\lambda, \delta z)$. This function has exponential type at most $\pi\delta$. We apply (6.4) to obtain the identity

$$\sum_{m=-\infty}^{\infty} K\big(\delta^{-1}\lambda, \delta(x+m)\big) = \delta^{-1} \sum_{|n| \le \frac{\delta}{2}} \widehat{K}\big(\delta^{-1}\lambda, \delta^{-1}n\big)e(nx) \tag{6.5}$$

for all real x, and for all positive values of the parameters δ and λ. For our purposes it will be convenient to use (6.5) with $\delta = 2N+2$, where N is a non-negative integer, and to modify the constant term. For each non-negative integer N we define a trigonometric polynomial $k(\lambda, N; x)$, of degree at most N, by

$$k(\lambda, N; x) = -\frac{2}{\lambda} + \frac{1}{2N+2} \sum_{n=-N}^{N} \widehat{K}\left(\frac{\lambda}{2N+2}, \frac{n}{2N+2}\right)e(nx). \tag{6.6}$$

For $\lambda > 0$, the function $x \mapsto e^{-\lambda|x|}$ is continuous, integrable on $\mathbb{R}$, and has bounded variation. Therefore, the Poisson summation formula also provides the pointwise identity

$$\sum_{m=-\infty}^{\infty} e^{-\lambda|x+m|} = \sum_{n=-\infty}^{\infty} \frac{2\lambda}{\lambda^2 + 4\pi^2 n^2} e(nx). \tag{6.7}$$

And we find that

$$\sum_{m=-\infty}^{\infty} e^{-\lambda|x+m|} = \frac{\cosh\big(\lambda\big(x - [x] - \frac{1}{2}\big)\big)}{\sinh\big(\frac{\lambda}{2}\big)}, \tag{6.8}$$

where $[x]$ is the integer part of the real number x. For our purposes it will be convenient to define

$$p : (0, \infty) \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$$

by

$$p(\lambda, x) = -\frac{2}{\lambda} + \sum_{m=-\infty}^{\infty} e^{-\lambda|x+m|}. \tag{6.9}$$

Then $p(\lambda, x)$ is continuous on $(0, \infty) \times \mathbb{R}/\mathbb{Z}$, and differentiable with respect to x at each non-integer point x. It follows from (6.7) that the Fourier coefficients of $x \mapsto p(\lambda, x)$ are given by

$$\int_{\mathbb{R}/\mathbb{Z}} p(\lambda, x)\,\mathrm{d}x = 0, \tag{6.10}$$

and

$$\int_{\mathbb{R}/\mathbb{Z}} p(\lambda, x) e(-nx)\,\mathrm{d}x = \frac{2\lambda}{\lambda^2 + 4\pi^2 n^2} \tag{6.11}$$

for integers $n \neq 0$.

Theorem 6.1 *Let λ be a positive real number and N a non-negative integer.*

(i) *If $\widetilde{k}(x)$ is a trigonometric polynomial of degree at most N, then*

$$\int_{\mathbb{R}/\mathbb{Z}} \left|p(\lambda, x) - \widetilde{k}(x)\right| \mathrm{d}x \geq \frac{2}{\lambda} - \frac{2}{\lambda}\operatorname{sech}\left(\frac{\lambda}{4N+4}\right) \tag{6.12}$$

with equality if and only if $\widetilde{k}(x) = k(\lambda, N; x)$.

(ii) *For $x \in \mathbb{R}/\mathbb{Z}$ we have*

$$\operatorname{sgn}\bigl(\cos \pi(2N+2)x\bigr) = \operatorname{sgn}\bigl\{p(\lambda, x) - k(\lambda, N; x)\bigr\}. \tag{6.13}$$

Proof Throughout this proof we consider $\delta = 2N + 2$. From (6.5), (6.6), (6.7), and (6.9) we obtain

$$\begin{aligned} p(\lambda, x) - k(\lambda, N; x) &= \sum_{n=-\infty}^{\infty} \left\{\frac{2\lambda}{\lambda^2 + 4\pi^2 n^2} - \delta^{-1}\widehat{K}(\delta^{-1}\lambda, \delta^{-1}n)\right\} e(nx) \\ &= \sum_{m=-\infty}^{\infty} \left\{e^{-\lambda|x+m|} - K\bigl(\delta^{-1}\lambda, \delta(x+m)\bigr)\right\} \end{aligned} \tag{6.14}$$

for all $x \in \mathbb{R}/\mathbb{Z}$. Identity (6.13) now follows from (6.14) and (1.4). Now using (1.4) and (1.3), we arrive at

$$\begin{aligned} &\int_{\mathbb{R}/\mathbb{Z}} \left|p(\lambda, x) - k(\lambda, N; x)\right| \mathrm{d}x \\ &\quad = \int_{\mathbb{R}/\mathbb{Z}} \left|\sum_{m=-\infty}^{\infty} \left\{e^{-\lambda|x+m|} - K\bigl(\delta^{-1}\lambda, \delta(x+m)\bigr)\right\}\right| \mathrm{d}x \end{aligned}$$

$$= \int_{\mathbb{R}/\mathbb{Z}} \sum_{m=-\infty}^{\infty} \left| e^{-\lambda|x+m|} - K\left(\delta^{-1}\lambda, \delta(x+m)\right) \right| \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \left| e^{-\lambda|x|} - K(\delta^{-1}\lambda, \delta x) \right| \mathrm{d}x = \frac{2}{\lambda} - \frac{2}{\lambda}\operatorname{sech}\left(\frac{\lambda}{2\delta}\right), \tag{6.15}$$

and this proves that equality occurs in (6.12) when $\widetilde{k}(x) = k(\lambda, N; x)$.

Now let $\widetilde{k}(x)$ be a general trigonometric polynomial of degree at most N. Using identity (2.13) we obtain

$$\int_{\mathbb{R}/\mathbb{Z}} \left| p(\lambda, x) - \widetilde{k}(x) \right| \mathrm{d}x$$

$$\geq \left| \int_{\mathbb{R}/\mathbb{Z}} \left(p(\lambda, x) - \widetilde{k}(x) \right) \operatorname{sgn}\{\cos \pi \delta x\} \, \mathrm{d}x \right|$$

$$= \left| \int_{\mathbb{R}/\mathbb{Z}} p(\lambda, x) \operatorname{sgn}\{\cos \pi \delta x\} \, \mathrm{d}x \right|$$

$$= \left| \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)} \int_{\mathbb{R}/\mathbb{Z}} p(\lambda, x) e\left(\left(k + \frac{1}{2} \right) \delta x \right) \mathrm{d}x \right|$$

$$= \left| \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)} \frac{2\lambda}{\left(\lambda^2 + 4\pi^2\left(\left(k + \frac{1}{2}\right)\delta\right)^2\right)} \right|$$

$$= \frac{2}{\lambda} - \frac{2}{\lambda}\operatorname{sech}\left(\frac{\lambda}{2\delta}\right), \tag{6.16}$$

which proves (6.12). If equality occurs in (6.16), we must have (recall that $\delta = 2N + 2$)

$$\widetilde{k}\left(\frac{1}{2N+2} \left(k + \frac{1}{2} \right) \right) = p\left(\lambda, \frac{1}{2N+2} \left(k + \frac{1}{2} \right) \right) \quad \text{for } k = 0, 1, 2, \ldots, 2N+1. \tag{6.17}$$

Since the degree of $\widetilde{k}(x)$ is at most N, such polynomial exists and is unique [17, vol. II, p. 1]. Observe that $k(\lambda, N; x)$ already satisfies (6.17), this being a consequence of (6.13). Therefore, we must have $\widetilde{k}(x) = k(\lambda, N; x)$, which finishes the proof. □

It follows from (6.8) and (6.9) that

$$-\left\{ \frac{2}{\lambda} - \operatorname{csch}\left(\frac{\lambda}{2}\right) \right\} = p\left(\lambda, \frac{1}{2}\right) \leq p(\lambda, x) \leq p(\lambda, 0) = \coth\left(\frac{\lambda}{2}\right) - \frac{2}{\lambda}. \tag{6.18}$$

Then (6.18) provides the useful inequality

$$\begin{aligned} \left| p(\lambda, x) \right| &\leq \left| p(\lambda, x) - p\left(\lambda, \frac{1}{2}\right) \right| + \left| p\left(\lambda, \frac{1}{2}\right) \right| \\ &= p(\lambda, x) - p\left(\lambda, \frac{1}{2}\right) - p\left(\lambda, \frac{1}{2}\right) \end{aligned}$$

$$= p(\lambda, x) + 2\left\{\frac{2}{\lambda} - \operatorname{csch}\left(\frac{\lambda}{2}\right)\right\} \tag{6.19}$$

at each point (λ, x) in $(0, \infty) \times \mathbb{R}/\mathbb{Z}$. From (6.10) and (6.19) we conclude that

$$\int_{\mathbb{R}/\mathbb{Z}} \left|p(\lambda, x)\right| \mathrm{d}x \le 2\left\{\frac{2}{\lambda} - \operatorname{csch}\left(\frac{\lambda}{2}\right)\right\}. \tag{6.20}$$

Let μ be a measure on the Borel subsets of $(0, \infty)$ that satisfies (1.6). For $0 < x < 1$ it follows from (6.8) and (6.9) that $\lambda \mapsto p(\lambda, x)$ is integrable on $(0, \infty)$ with respect to μ. We define $q_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R} \cup \{\infty\}$ by

$$q_\mu(x) = \int_0^\infty p(\lambda, x)\,\mathrm{d}\mu(\lambda), \tag{6.21}$$

where

$$q_\mu(0) = \int_0^\infty \left\{\coth\left(\frac{\lambda}{2}\right) - \frac{2}{\lambda}\right\} \mathrm{d}\mu(\lambda) \tag{6.22}$$

may take the value ∞. Using (6.20) and Fubini's theorem we have

$$\begin{aligned}\int_{\mathbb{R}/\mathbb{Z}} \left|q_\mu(x)\right| \mathrm{d}x &\le \int_0^\infty \int_{\mathbb{R}/\mathbb{Z}} \left|p(\lambda, x)\right| \mathrm{d}x\,\mathrm{d}\mu(\lambda)\\ &\le 2\int_0^\infty \left\{\frac{2}{\lambda} - \operatorname{csch}\left(\frac{\lambda}{2}\right)\right\} \mathrm{d}\mu(\lambda) < \infty,\end{aligned}$$

so that q_μ is integrable on $\mathbb{R}/\mathbb{Z}$. Using (6.10) and (6.11), we find that the Fourier coefficients of q_μ are given by

$$\widehat{q}_\mu(0) = \int_{\mathbb{R}/\mathbb{Z}} q_\mu(x)\,\mathrm{d}x = \int_0^\infty \int_{\mathbb{R}/\mathbb{Z}} p(\lambda, x)\,\mathrm{d}x\,\mathrm{d}\mu(\lambda) = 0, \tag{6.23}$$

and

$$\begin{aligned}\widehat{q}_\mu(n) &= \int_{R/\mathbb{Z}} q_\mu(x)e(-nx)\,\mathrm{d}x = \int_0^\infty \int_{\mathbb{R}/\mathbb{Z}} p(\lambda, x)e(-nx)\,\mathrm{d}x\,\mathrm{d}\mu(\lambda)\\ &= \int_0^\infty \frac{2\lambda}{\lambda^2 + 4\pi^2 n^2}\,\mathrm{d}\mu(\lambda)\end{aligned} \tag{6.24}$$

for integers $n \neq 0$. As $n \mapsto \widehat{q}_\mu(n)$ is an even function of n, and $\widehat{q}_\mu(n) \ge \widehat{q}_\mu(n+1)$ for $n \ge 1$, the partial sums

$$q_\mu(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \widehat{q}_\mu(n)e(nx) \tag{6.25}$$

converge uniformly on compact subsets of $\mathbb{R}/\mathbb{Z} \setminus \{0\}$, (see [17, Chap. I, Theorem 2.6]). In particular, $q_\mu(x)$ is continuous on $\mathbb{R}/\mathbb{Z} \setminus \{0\}$.

For each non-negative integer N, we define a trigonometric polynomial $k_\mu(N;x)$, of degree at most N, by

$$k_\mu(N;x) = \sum_{n=-N}^{N} \widehat{k}_\mu(N;n)e(nx), \tag{6.26}$$

where the Fourier coefficients are given by (recall here Lemma 4.2)

$$\begin{aligned}\widehat{k}_\mu(N;0) &= \int_0^\infty \left\{-\frac{2}{\lambda} + \frac{1}{2N+2}\widehat{K}\left(\frac{\lambda}{2N+2}, 0\right)\right\} \mathrm{d}\mu(\lambda) \\ &= -\int_0^\infty \left\{\frac{2}{\lambda} - \frac{1}{2N+2}\operatorname{csch}\left(\frac{\lambda}{4N+4}\right)\right\} \mathrm{d}\mu(\lambda),\end{aligned} \tag{6.27}$$

and

$$\widehat{k}_\mu(N;n) = \frac{1}{2N+2}\int_0^\infty \widehat{K}\left(\frac{\lambda}{2N+2}, \frac{n}{2N+2}\right) \mathrm{d}\mu(\lambda) \tag{6.28}$$

for $n \neq 0$.

Theorem 6.2 *Let N be a non-negative integer, and assume that μ satisfies* (1.6).

(i) *If $\widetilde{k}(x)$ is a trigonometric polynomial of degree at most N, then*

$$\int_{\mathbb{R}/\mathbb{Z}} \left|q_\mu(x) - \widetilde{k}(x)\right| \mathrm{d}x \geq \int_0^\infty \left\{\frac{2}{\lambda} - \frac{2}{\lambda}\operatorname{sech}\left(\frac{\lambda}{4N+4}\right)\right\} \mathrm{d}\mu(\lambda), \tag{6.29}$$

with equality if and only if $\widetilde{k}(x) = k_\mu(N;x)$.

(ii) *For $x \in \mathbb{R}/\mathbb{Z}$ we have*

$$\operatorname{sgn}\big(\cos\pi(2N+2)x\big) = \operatorname{sgn}\big\{q_\mu(x) - k_\mu(N;x)\big\}. \tag{6.30}$$

Proof We use the elementary identity

$$k_\mu(N;x) = \int_0^\infty k(\lambda, N; x)\, \mathrm{d}\mu(\lambda). \tag{6.31}$$

Expression (6.13), together with (6.21) and (6.31), imply (6.30). Using (6.13) and (6.12) we observe that

$$\begin{aligned}\int_{\mathbb{R}/\mathbb{Z}} \left|q_\mu(x) - k_\mu(N;x)\right| \mathrm{d}x &= \int_{\mathbb{R}/\mathbb{Z}} \left|\int_0^\infty \{p(\lambda,x) - k(\lambda,N;x)\}\, \mathrm{d}\mu(\lambda)\right| \mathrm{d}x \\ &= \int_{\mathbb{R}/\mathbb{Z}} \int_0^\infty \left|p(\lambda,x) - k(\lambda,N;x)\right| \mathrm{d}\mu(\lambda)\, \mathrm{d}x \\ &= \int_0^\infty \int_{\mathbb{R}/\mathbb{Z}} \left|p(\lambda,x) - k(\lambda,N;x)\right| \mathrm{d}x\, \mathrm{d}\mu(\lambda)\end{aligned}$$

$$= \int_0^\infty \left\{ \frac{2}{\lambda} - \frac{2}{\lambda} \operatorname{sech}\left(\frac{\lambda}{4N+4} \right) \right\} d\mu(\lambda). \qquad (6.32)$$

This proves that equality occurs in (6.29) when $\widetilde{k}(x) = k_\mu(N; x)$. The proof of the lower bound (6.29) and the uniqueness part are similar to the ones given in Theorem 6.1. □

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