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CONSTRUCTIVE
APPROXIMATION

Biorthogonal Laurent Polynomials, Töplitz Determinants, Minimal Toda Orbits and Isomonodromic Tau Functions

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Abstract. We consider the class of biorthogonal polynomials that are used to solve the inverse spectral problem associated to elementary co-adjoint orbits of the Borel group of upper triangular matrices; these orbits are the phase space of generalized integrable lattices of Toda type. Such polynomials naturally interpolate between the theory of orthogonal polynomials on the line and orthogonal polynomials on the unit circle and tie together the theory of Toda, relativistic Toda, Ablowitz–Ladik and Volterra lattices. We establish corresponding Christoffel–Darboux formulæ. For all these classes of polynomials a 2×2 system of Differential-Difference-Deformation equations is analyzed in the most general setting of pseudo-measures with arbitrary rational logarithmic derivative. They provide particular classes of isomonodromic deformations of rational connections on the Riemann sphere. The corresponding isomonodromic tau function is explicitly related to the shifted Töplitz determinants of the moments of the pseudo-measure. In particular, the results imply that any (shifted) Töplitz (Hänkel) determinant of a symbol (measure) with arbitrary rational logarithmic derivative is an isomonodromic tau function.

Contents

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1. Introduction

The connection between orthogonal polynomials on the line and Toda lattices is rather well known [3], as well as the relations to the KP hierarchy [1]. Dynamical variables of the Toda lattice are arranged into a tridiagonal Lax matrix, that can be viewed as a recurrence matrix for a system of orthogonal polynomials. In the (semi)finite case, the evolution of the corresponding measure provides a linearization of the Toda flows. More generally, one can set up (in)finite-dimensional Hamiltonian systems on $\mathbb{R}^{2n}$ $(n \leq \infty)$ with Hamiltonians

$$H_I(\underline{q}, \underline{p}) = \frac{1}{2}\sum_{i=1}^{n} {p_i}^2 + \sum_{i \notin I} p_i \, \mathrm{e}^{q_{i+1}-q_i} + \sum_{j=1}^{|I|} \mathrm{e}^{q_{i_{j+1}}-q_{i_j}}, \tag{1.1}$$

$$I := \{i_1 < i_2 < \cdots < i_k\}. \tag{1.2}$$

As is noted in [10] such a family of Hamiltonians (labelled by the multi-index I) contains integrable lattice hierarchies of Toda, relativistic Toda, Volterra and Ablowitz–Ladik type. These integrable Hamiltonian systems have a Lax representation with the Lax operator given as an $n \times n$ lower Hessenberg matrix which we denote by Q (in [10], [11] it was denoted by X), belonging to a certain "elementary" $(2n-2)$-dimensional co-adjoint orbit of the solvable group of upper triangular matrices. These systems are linearized by the Moser map

$$Q \mapsto \mathcal{W}(z; Q) := (z\mathbf{1} - Q)^{-1}_{11} \;=\; \sum_{j=0}^{\infty} \frac{\hat{\mu}_j(Q)}{z^{j+1}}, \tag{1.3}$$

$$\hat{\mu}_j(Q) \;=\; Q^j_{11}. \tag{1.4}$$

In the case of infinite lattices these expressions take on a formal meaning in terms of power series but the analysis is unchanged.

The *moments* $\hat{\mu}_j$ of Q define a *normalized* moment functional $\mathcal{L}$:

$$\mathcal{L}(x^j) = \hat{\mu}_j(Q) = Q^j_{11},$$

and the reconstruction of Q from its moments (the "inverse moment problem") can be accomplished [11] by constructing a suitable sequence of biorthogonal (Laurent) polynomials $\{r_i, p_i\}_{i\in\mathbb{N}}$, where the p_i's are polynomials in x of degree i while the r_i's are, in general, polynomials in x and x^{-1}, and r_i, p_i satisfy

$$p_i(Q)e_1 = e_{i+1}, \qquad e_1^T r_i(Q) = e_{i+1}^T, \qquad i = 0, 1, \ldots,$$

where e_j denote vectors of the standard basis in $\mathbb{C}^n$. As a result,

$$\mathcal{L}(r_i p_j) = \delta_{ij},$$

and the Lax operator Q corresponding to the chosen orbit is then reconstructed by

$$Q_{ij} = e_i^T Q e_j = \mathcal{L}(r_{i-1} x p_{j-1}).$$

Explicit formulæ for these biorthogonal polynomials in terms of shifted Töplitz determinants can be found in [10], [11] and will be recalled here in due time. Vice versa, one could assign an arbitrary (generic) moment functional $\mathcal{L} : \mathbb{C}[z, z^{-1}] \to \mathbb{C}$, a multi-index I and then reconstruct the Lax operator Q_I (i.e. view the Lax operator as a function of $\mathcal{L}$ rather than the other way around)

$$\mathcal{L} \mapsto Q_I(\mathcal{L}).$$

From this point of view, the linearization of the (infinite) Hamiltonian hierarchy is accomplished simply by

$$\mathcal{L}_{\mathbf{t}}(\bullet) = \mathcal{L}(\mathrm{e}^{\sum_J 1/J t_J z^J} \bullet), \tag{1.5}$$

where the series may have to be understood formally. This procedure displays the common nature of all the above-mentioned integrable lattices, inasmuch as the linearizing space is always the same (the space of moment functionals) and what changes from one lattice to another is only the orbit, namely the map Q_I.

Finite-dimensional systems (of dimension $2n - 2$) on an elementary orbit $\mathfrak{O}_I$ correspond to those moment-functionals for which certain shifted Töplitz determinants of size $\leq n$ do not vanish, whereas all the larger ones do. In such cases, the tau function of the hierarchy is defined by the (closed) differential

$$\mathrm{d} \ln \tau = \sum_{J=1}^{n} \frac{1}{J} \operatorname{Tr}((Q_I)^J)\, \mathrm{d}t_J \tag{1.6}$$

and coincides with the largest nonvanishing (shifted) Töplitz determinant.

One of the main purposes of this paper is to connect this determinant to a different notion of the "tau" function, namely, the one introduced by Jimbo, Miwa and Ueno in [14], [15]. It was shown in [6], [4] that the Hänkel determinants of an arbitrary (generic) "semiclassical" moment functional on the space of polynomials can be identified with the isomonodromic tau function introduced by our Japanese colleagues. Similarly, it was shown in [17] that Töplitz determinants of a particular class of symbols on the unit circle are also identifiable with the same kind of isomonodromic tau functions.

These two apparently distinct situations are in fact the two ends of a "continuous" spectrum of situations: in fact, the case of Hänkel determinants is dealt with in the setting of (generalized) ordinary orthogonal polynomials, whereas that of Töplitz determinants uses orthogonal polynomials on the unit circle; in this latter situation one considers polynomials $p_i(z)$ orthogonal in the usual $L^2(S^1, \mathrm{d}\mu)$ sense

$$\int_{S^1} p_j(z)\overline{p_k(z)}\,\mathrm{d}\mu(z) = \delta_{jk}. \tag{1.7}$$

Here one defines $r_j(z) = \overline{p_j}(z^{-1})$ and the orthogonality is recast into

$$\mathcal{L}(r_j p_k) = \delta_{jk}, \tag{1.8}$$

where, in this special case,

$$\mathcal{L} : \mathbb{C}[z, z^{-1}] \to \mathbb{C}; \qquad \mathcal{L}(z^j) = \int_{S^1} z^j\,\mathrm{d}\mu(z). \tag{1.9}$$

We see that we can regard the case of orthogonal polynomials on the circle as a special case of biorthogonal Laurent polynomials with respect to a moment functional satisfying the reality condition $\mu_k = \overline{\mu_{-k}}$.

According to the previous description of integrable lattices, the two situations correspond to two different elementary orbits and hence we should be able to treat them on a common ground, together with all the other lattices associated with the orbits $\mathfrak{O}_I$. Indeed, we will show that this is the case and that for the class of moment functionals of the semiclassical type introduced in [4] all the shifted Töplitz determinants which arise as tau functions of the corresponding integrable lattices are also isomonodromic tau functions for a rational 2×2 connection on $\mathbb{C}^1$ which will be explicitly constructed in the paper.

The approach to this problem follows the strategy used in [4] rather than the one in [17]; in the course of our analysis we will obtain generalized Christoffel–Darboux identities which naturally interpolate between the ordinary Christoffel–Darboux identity for orthogonal polynomials on the line and the one for orthogonal polynomials on the unit circle.

Moreover, we will show that the Töplitz and Hänkel determinants of the same size for one such moment-functional are connected by a sequence of elementary Schlesinger transformations, at each step of which we obtain tau functions associated to interpolating orbits; in figurative terms, we show that the papers [17] (see Example 10.1) and [4] are connected by a Schlesinger transformation (when specializing the semiclassical measure to the one relevant for [17]) and that “neighbouring” elementary co-adjoint orbits are also connected by an elementary Schlesinger transformation.

2. Setting

We start in the most general and abstract setting, without any reference to a (pseudo)-measure. We consider an arbitrary moment functional

$$\mathcal{L} : \mathbb{C}[z, z^{-1}] \to \mathbb{C} \tag{2.1}$$

on the space polynomials in z and z^{-1} and denote its moments with $\mu_j = \mathcal{L}(z^j)$, $j \in \mathbb{Z}$. We introduce the following *shifted Töplitz* determinants and polynomials

$$\Delta_n^\ell = \det \begin{pmatrix} \mu_\ell & \mu_{\ell+1} & \cdots & \mu_{\ell+n-1} \\ \mu_{\ell-1} & \mu_\ell & \cdots & \mu_{\ell+n-2} \\ & \ddots & \ddots & \\ \mu_{\ell-n+1} & \mu_{\ell-n+2} & \cdots & \mu_\ell \end{pmatrix}, \tag{2.2}$$

$$\Delta_0^\ell \equiv 1, \qquad \Delta_{-n}^\ell \equiv 0,$$

$$\wp_n^\ell(x) := \det \begin{pmatrix} \mu_\ell & \mu_{\ell+1} & \cdots & \mu_{\ell+n} \\ \mu_{\ell-1} & \mu_\ell & \cdots & \mu_{\ell+n-1} \\ & \ddots & \ddots & \\ \mu_{\ell-n+1} & \mu_{\ell-n+2} & \cdots & \mu_{\ell+1} \\ 1 & x & \cdots & x^n \end{pmatrix}. \tag{2.3}$$

Using some classical identities for determinants we can derive recurrence relations for the shifts $n \to n+1$ and $\ell \to \ell+1$ for the above polynomials. We first need the following:

Proposition 2.1. *For any* $(n+1)\times(n+1)$ *matrix* A *the following determinant identity holds true* (*Jacobi identity*)

$$A^{1..n}_{1..n} A^{2..n+1}_{2..n+1} - A^{2..n+1}_{1..n} A^{1..n}_{2..n+1} = A^{1..n+1}_{1..n+1} A^{2..n}_{2..n}, \tag{2.4}$$

where the sub/superscript ranges denote the rows/columns of the submatrix we are computing the determinant of. As a corollary, for any $(n+1)\times(n+2)$ *matrix* B, *we have*

$$B^{1..n+1}_{2..n+2} B^{1..n}_{1..n} + B^{1..n+1}_{1..n+1} B^{1..n}_{2..n,n+2} = B^{1..n+1}_{1..n,n+2} B^{1..n}_{2..n+1}, \tag{2.5}$$

which can be obtained from (2.4) *by adjoining an appropriate row.*

Using (2.4) on the determinant defining $\wp_n^\ell$ we find

$$x\Delta_n^\ell \wp_{n-1}^\ell - \Delta_n^{\ell+1} \wp_{n-1}^{\ell-1} = \Delta_{n-1}^\ell \wp_n^\ell. \tag{2.6}$$

Applying (2.5) to the determinant defining $\wp_n^\ell$ adjoined to the next row of moments on the top we find

$$\wp_n^{\ell-1}\Delta_n^\ell + \Delta_{n+1}^\ell \wp_{n-1}^{\ell-1} = \wp_n^\ell \Delta_n^{\ell-1}, \tag{2.7}$$

$$\wp_n^{\ell-1}\Delta_n^{\ell+1} + x\Delta_{n+1}^\ell \wp_{n-1}^\ell = \wp_n^\ell \Delta_n^\ell, \tag{2.8}$$

$$x\Delta_n^\ell \wp_{n-1}^\ell - \Delta_n^{\ell+1} \wp_{n-1}^{\ell-1} = \wp_n^\ell \Delta_{n-1}^\ell. \tag{2.9}$$

We now use these identities to express $\wp_n^\ell := [\wp_n^\ell, \wp_{n-1}^{\ell-1}]$ in terms of $\wp_{n-1}^\ell = [\wp_{n-1}^\ell, \wp_{n-2}^{\ell-1}]$,

$$\begin{bmatrix}\wp_n^\ell \\ \wp_{n-1}^{\ell-1}\end{bmatrix} = \begin{bmatrix} \frac{x\Delta_n^\ell}{\Delta_{n-1}^\ell} - \frac{\Delta_n^{\ell+1}\Delta_{n-1}^{\ell-1}}{(\Delta_{n-1}^\ell)^2} & \frac{\Delta_n^{\ell+1}\Delta_n^\ell}{(\Delta_{n-1}^\ell)^2}, \\ \frac{\Delta_{n-1}^{\ell-1}}{\Delta_{n-1}^\ell} & -\frac{\Delta_n^\ell}{\Delta_{n-1}^\ell} \end{bmatrix} \begin{bmatrix}\wp_{n-1}^\ell \\ \wp_{n-2}^{\ell-1}\end{bmatrix}, \tag{2.10}$$

$$\wp_n^\ell = \mathcal{C}_n^\ell \wp_{n-1}^\ell, \tag{2.11}$$

$$\det \mathcal{C}_n^\ell = -x\frac{(\Delta_n^\ell)^2}{(\Delta_{n-1}^\ell)^2}, \qquad \textbf{Circle Case} \tag{2.12}$$

$$\mathbf{j}\mathcal{C}_n^\ell(x)^{-1}\mathcal{C}_n^\ell(y) - \mathbf{j} = \left(1 - \frac{y}{x}\right)\begin{bmatrix} \frac{\Delta_{n-1}^{\ell-1}}{\Delta_n^\ell} & 0 \\ -1 & 0\end{bmatrix}, \tag{2.13}$$

where

$$\mathbf{j} := \begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}. \tag{2.14}$$

We have named this the "circle case" because this sort of recursion is relevant for orthogonal polynomials on the unit circle. Indeed, let us fix ℓ and define

$$\Phi_n = \begin{bmatrix}\varphi_n \\ \varphi_n^*\end{bmatrix} = \begin{bmatrix} \frac{1}{\Delta_n^\ell} & 0 \\ (-1)^{n+1}\frac{\Delta_n^{\ell-1}}{(\Delta_n^\ell)^2} & (-1)^n \frac{\Delta_{n+1}^\ell}{(\Delta_n^\ell)^2}\end{bmatrix}\wp_n^l.$$

Then a computation using the identity

$$(\Delta_n^\ell)^2 - \Delta_n^{\ell-1}\Delta_n^{\ell+1} = \Delta_{n+1}^\ell \Delta_{n-1}^\ell,$$

(another instance of the determinantal identities (2.4)) shows that vectors Φ_n satisfy a recurrence of Szegő type:

$$\Phi_n = \begin{bmatrix} x & -(-1)^n\frac{\Delta_n^{\ell+1}}{\Delta_n^\ell} \\ -(-1)^n\frac{\Delta_n^{\ell-1}}{\Delta_n^\ell} & 1\end{bmatrix}\Phi_{n-1}.$$

Moreover, if we set $l = 0$ and assume that $\mu_{-k} = \bar{\mu}_k$ for all k, that φ_n defined by the recurrence above are monic polynomials orthogonal with respect to the measure on the unit circle with the moments μ_k (see, e.g. [20]).

We next derive a recursion in ℓ,

$$\begin{bmatrix}\wp_n^\ell \\ \wp_{n-1}^{\ell-1}\end{bmatrix} = \begin{bmatrix} \frac{\Delta_n^\ell}{\Delta_n^{\ell-1}} + \frac{\Delta_{n+1}^\ell \Delta_{n-1}^{\ell-1}}{x(\Delta_n^{\ell-1})^2} & \frac{\Delta_{n+1}^\ell\Delta_n^\ell}{x(\Delta_n^{\ell-1})^2} \\ \frac{\Delta_{n-1}^{\ell-1}}{x\Delta_n^{\ell-1}} & \frac{\Delta_n^\ell}{x\Delta_n^{\ell-1}}\end{bmatrix}\begin{bmatrix}\wp_n^{\ell-1} \\ \wp_{n-1}^{\ell-2}\end{bmatrix}, \tag{2.15}$$

$$\wp_n^\ell = \mathcal{T}_n^\ell \wp_n^{\ell-1}, \tag{2.16}$$

$$\det \mathcal{T}_n^\ell = \frac{1}{x}\frac{(\Delta_n^\ell)^2}{(\Delta_n^{\ell-1})^2}, \tag{2.17}$$

Circle-to-Line Transform

$$\mathbf{j}\mathcal{T}_n^\ell(x)^{-1}\mathcal{T}_n^\ell(y) - \mathbf{j} = \left(1 - \frac{x}{y}\right)\begin{bmatrix} \frac{\Delta_{n-1}^{\ell-1}}{\Delta_n^\ell} & 1 \\ 0 & 0 \end{bmatrix}. \tag{2.18}$$

The name "circle-to-line" refers to the fact that this recursion relation interpolates between the previous "circle" case and the next one, which will be named the "line" case. Indeed, composing these two we can express $\wp_n^\ell = [\wp_n^\ell, \wp_{n-1}^{\ell-1}]$ in terms of $\wp_{n-1}^{\ell-1} = [\wp_{n-1}^{\ell-1}, \wp_{n-2}^{\ell-2}]$,

$$\begin{bmatrix} \wp_n^\ell \\ \wp_{n-1}^{\ell-1} \end{bmatrix} = \begin{bmatrix} \frac{\Delta_n^\ell}{\Delta_{n-1}^{\ell-1}}\left(x - \frac{\Delta_n^{\ell+1}\Delta_{n-1}^{\ell-1} - \Delta_{n-2}^{\ell-1}\Delta_n^\ell}{\Delta_{n-1}^\ell \Delta_{n-1}^{\ell-1}}\right) & \frac{(\Delta_n^\ell)^2}{(\Delta_{n-1}^{\ell-1})^2} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \wp_{n-1}^{\ell-1} \\ \wp_{n-2}^{\ell-2} \end{bmatrix}, \tag{2.19}$$

$$\wp_n^\ell = \mathcal{L}_n^\ell \wp_{n-1}^{\ell-1}, \tag{2.20}$$

Line case

$$\det \mathcal{L}_n^\ell = -\frac{(\Delta_n^\ell)^2}{(\Delta_{n-1}^{\ell-1})^2}, \tag{2.21}$$

$$\mathbf{j}\mathcal{L}_n^\ell(x)^{-1}\mathcal{L}_n^\ell(y) - \mathbf{j} = (x - y)\begin{bmatrix} \frac{\Delta_{n-1}^{\ell-1}}{\Delta_n^\ell} & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.22}$$

This recursion is called the "line" case because it is the relevant recursion relation for ordinary orthogonal polynomials on the line; indeed, the standard recursion relation in this case can be written as

$$\begin{bmatrix} p_{n+1} \\ p_n \end{bmatrix} = \begin{bmatrix} \frac{x - \beta_n}{\gamma_n} & -\frac{\gamma_{n-1}}{\gamma_n} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_n \\ p_{n-1} \end{bmatrix},$$

which has the same shape and x-dependence as the ladder matrix (2.19). Moreover, setting $\ell = n-1$ the Töplitz determinants are equivalent to Hänkel determinants involving only the moments $\mu_0, \dots, \mu_{2n}$ and one recovers the familiar formulæ for ordinary orthogonal polynomials (up to a normalization for the polynomials $\wp_n^{n-1}$).

2.1. *Second-Kind Polynomials*

Let us define the following *second-kind* polynomials

$$\mathcal{R}_n^\ell(x) = \mathcal{L}_z\left(\frac{\wp_n^\ell(x) - \wp_n^\ell(z)}{x - z}\right). \tag{2.23}$$

The three types of recursion (2.10), (2.15), (2.19) involve at most a multiplication or division by x and have otherwise constant coefficients (in x): moreover, we find

$$x\mathcal{R}_n^\ell(x) \ = \ \mathcal{L}_z\left(\frac{x\wp_n^\ell(x) - z\wp_n^\ell(z)}{x - z}\right) - \mathcal{L}_z(\wp_n^\ell(z)), \tag{2.24}$$

$$x^{-1}\mathcal{R}_n^\ell(x) \;=\; \mathcal{L}_z\left(\frac{x^{-1}\wp_n^\ell(x) - z^{-1}\wp_n^\ell(z)}{x-z}\right) - \frac{1}{x}\mathcal{L}_z(z^{-1}\wp_n^\ell(z)). \tag{2.25}$$

The last terms in these identities vanish because of the determinant structure of $\wp_n^\ell$, provided that $n \geq 1$ and $0 \leq \ell \leq n-1$ for the first case and $-1 \leq \ell \leq n-2$ for the second case. From this observation we find that these auxiliary sequences of polynomials satisfy the same recurrence relations in the following ranges:

$$\begin{bmatrix} \mathcal{R}_n^\ell \\ \mathcal{R}_{n-1}^{\ell-1}\end{bmatrix} = \begin{cases} \mathcal{L}_n^\ell \begin{bmatrix}\mathcal{R}_{n-1}^{\ell-1} \\ \mathcal{R}_{n-2}^{\ell-2}\end{bmatrix}, & 1 \leq \ell \leq n-1, \\ \mathcal{C}_n^\ell \begin{bmatrix}\mathcal{R}_{n-1}^{\ell} \\ \mathcal{R}_{n-2}^{\ell-1}\end{bmatrix}, & 0 \leq \ell \leq n-2, \\ \mathcal{T}_n^\ell \begin{bmatrix}\mathcal{R}_{n-1}^{\ell-1} \\ \mathcal{R}_{n-2}^{\ell-2}\end{bmatrix}, & 0 \leq \ell \leq n-1. \end{cases} \tag{2.26}$$

3. Christoffel–Darboux Formulæ

Consider $(n, l) \in \mathbb{N}\times\mathbb{N}$ and choose an arbitrary path starting at the origin of the following type:

$$\{(n_k, \ell_k),\ k = 0, 1, \ldots, (n_0, \ell_0) = (0, 0), (n_1, \ell_1) = (1, 0)\}, \tag{3.1}$$

and such that the possible subsequent moves are right, up or up-right. For the move $(n_{k-1}, \ell_{k-1}) \mapsto (n_k\ell_k)$ we introduce the **transfer matrices** following an idea of [13] used for orthogonal polynomials on the circle

$$T_k(x) := \begin{cases} \mathcal{C}_{n_k}^{\ell_k} & \text{if the move is right (\textbf{circle move}),} \\ \mathcal{T}_{n_k}^{\ell_k} & \text{if the move is up (\textbf{circle-to-line move}),} \\ \mathcal{L}_{n_k}^{\ell_k} & \text{if the move is up-right (\textbf{line move}).} \end{cases} \tag{3.2}$$

Using these transfer matrices we define the two dual auxiliary sequences of matrices as follows:

$$\Xi_k(x) \;=\; T_k(x)\Xi_{k-1}(x), \tag{3.3}$$

$$\Xi_k^\star(x) \;=\; \frac{1}{\det T_k(x)}\,\Xi_{k-1}^\star(x)T_k^t(x), \tag{3.4}$$

$$\Xi_0^\star \;=\; \Xi_0^t. \tag{3.5}$$

This definition in particular implies that

$$\Xi_k^\star = \frac{1}{\prod_{j=1}^k \det T_j}\,\Xi_k^t. \tag{3.6}$$

The choice of the initial conditions for the auxiliary sequences is arbitrary but it is convenient to choose Ξ_0 in such a way that the first column of Ξ_n will contain $\wp_n^\ell$

and $\wp_{n-1}^{\ell-1}$ and the second column the corresponding second-kind polynomials. Since the matrices constructed with the polynomials $\wp_{n_k}^{\ell_k}$ and the second-kind polynomials already satisfy the same recursion relation for $k \geq 1$, it is sufficient to impose the same initial conditions with the following choice (recall that the first move is always a circle-move)

$$\Xi_0 = (C_{n_1}^{\ell_1})^{-1} \begin{bmatrix} \wp_{n_1}^{\ell_1} & \mathcal{R}_{n_1}^{\ell_1} \\ \wp_{n_1-1}^{\ell_1-1} & \mathcal{R}_{n_1-1}^{\ell_1-1} \end{bmatrix} = \frac{1}{\mu_0^2 x} \begin{bmatrix} \mu_0 & \mu_1\mu_0 \\ 1 & \mu_1 - \mu_0 x \end{bmatrix} \begin{bmatrix} \mu_0 x - \mu_1 & \mu_0^2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \mu_0/x \\ 0 & 1/x \end{bmatrix}. \tag{3.7}$$

Recall that for any 2×2 matrix we have $A^t = \det(A)\mathbf{j}A^{-1}\mathbf{j}^{-1}$. We now compute

$$\begin{aligned} \Xi_k^\star(x)\mathbf{j}\Xi_k(y) &= \frac{1}{\det T_k(x)} \Xi_{k-1}^\star(x) T_k^t(x)\mathbf{j}T_k(y)\Xi_{k-1}(y) \\ &= \Xi_{k-1}^\star(x)\mathbf{j}T_k^{-1}(x)T_k(y)\Xi_{k-1}(y) \\ &= \Xi_{k-1}^\star(x)\mathbf{j}\Xi_{k-1}(y) + \Xi_{k-1}^\star(x)(\mathbf{j}T_k^{-1}(x)T_k(y) - \mathbf{j})\Xi_{k-1}(y). \end{aligned} \tag{3.8}$$

Let us define $\dot{\ell}_k := \ell_k - \ell_{k-1}$ and $\dot{n}_k := n_k - n_{k-1}$. Then the three formulæ (2.10), (2.15), (2.19) can be uniformly written

$$\mathbf{j}T_k^{-1}(x)T_k(y) - \mathbf{j} = (-1)^{1-\dot{n}_k} \left(\frac{1}{y} - \frac{1}{x}\right) x^{\dot{\ell}_k} y^{\dot{n}_k} \begin{bmatrix} \frac{\Delta_{n_k-1}^{\ell_k-1}}{\Delta_{n_k}^{\ell_k}} & 1 - \dot{n}_k \\ \dot{\ell}_k - 1 & 0 \end{bmatrix}, \tag{3.9}$$

$$\det T_k(x) = (-1)^{\dot{n}_k} x^{\dot{n}_k - \dot{\ell}_k} \left(\frac{\Delta_{n_k}^{\ell_k}}{\Delta_{n_{k-1}}^{\ell_{k-1}}}\right)^2, \tag{3.10}$$

$$\prod_{j=1}^{k} \det T_j(x) = (-1)^{n_k} x^{n_k - \ell_k} (\Delta_{n_k}^{\ell_k})^2. \tag{3.11}$$

Summing up both sides of (3.8) we obtain the following *master Christoffel–Darboux identity*

(3.12)

$$\begin{aligned} &\Xi_N^\star(x)\mathbf{j}\Xi_N(y) - \begin{bmatrix} 0 & -1/y \\ 1/x & 0 \end{bmatrix} \\ &= \left(\frac{1}{y} - \frac{1}{x}\right) \sum_{k=0}^{N-1} (-1)^{1-\dot{n}_{k+1}} x^{\dot{\ell}_{k+1}} y^{\dot{n}_{k+1}} \Xi_k^\star(x) \begin{bmatrix} \frac{\Delta_{n_{k+1}-1}^{\ell_{k+1}-1}}{\Delta_{n_{k+1}}^{\ell_{k+1}}} & 1 - \dot{n}_{k+1} \\ \dot{\ell}_{k+1} - 1 & 0 \end{bmatrix} \Xi_k(y) \\ &= \left(\frac{1}{x} - \frac{1}{y}\right) \sum_{k=0}^{N-1} (-1)^{-\dot{n}_{k+1}} x^{\dot{\ell}_{k+1}} y^{\dot{n}_{k+1}} \Xi_k^\star(x) \begin{bmatrix} \frac{\Delta_{n_{k+1}-1}^{\ell_{k+1}-1}}{\Delta_{n_{k+1}}^{\ell_{k+1}}} & 1 - \dot{n}_{k+1} \\ \dot{\ell}_{k+1} - 1 & 0 \end{bmatrix} \Xi_k(y). \end{aligned}$$

3.1. *Principal Christoffel–Darboux Identities*

We look at the (1, 1) entry of the above identity

$$
\begin{aligned}
&\frac{(-1)^{n_N}}{(\Delta^{\ell_N}_{n_N})^2}\left(\frac{\wp^{\ell_N-1}_{n_N-1}(x)}{x^{n_N-\ell_N}}\wp^{\ell_N}_{n_N}(y)-\frac{\wp^{\ell_N}_{n_N}(x)}{x^{n_N-\ell_N}}\wp^{\ell_N-1}_{n_N-1}(y)\right)\\
&\qquad=\left(\frac{1}{x}-\frac{1}{y}\right)\sum_{k=0}^{N-1}\frac{(-1)^{n_{k+1}}x^{\ell_{k+1}-n_k}y^{\dot n_{k+1}}}{(\Delta^{\ell_k}_{n_k})^2}\\
&\qquad\quad\times\left[\frac{\Delta^{\ell_{k+1}-1}_{n_{k+1}-1}}{\Delta^{\ell_{k+1}}_{n_{k+1}}}\wp^{\ell_k}_{n_k}(y)+(1-\dot n_{k+1})\wp^{\ell_k-1}_{n_k-1}(y)\right]\\
&\qquad\quad\times\left[\wp^{\ell_k}_{n_k}(x)-(1-\dot\ell_{k+1})\frac{\Delta^{\ell_{k+1}}_{n_{k+1}}}{\Delta^{\ell_{k+1}-1}_{n_{k+1}-1}}\wp^{\ell_k-1}_{n_k-1}(x)\right].
\end{aligned}
\tag{3.13}
$$

The two terms in the product inside the sum above can be simplified using (2.7) for the case $\dot\ell_{k+1}=0$ and (2.9) for the case $\dot n_{k+1}=0$, indeed,

$$
\begin{aligned}
\left[\frac{\Delta^{\ell_{k+1}-1}_{n_{k+1}-1}}{\Delta^{\ell_{k+1}}_{n_{k+1}}}\wp^{\ell_k}_{n_k}(y)+(1-\dot n_{k+1})\wp^{\ell_k-1}_{n_k-1}(y)\right]
&=\begin{cases} y\dfrac{\Delta^{\ell_k}_{n_k}}{\Delta^{\ell_k+1}_{n_k}}\wp^{\ell_k}_{n_{k+1}-1}(y) & \text{if } \dot n_{k+1}=0,\\[2ex] \dfrac{\Delta^{\ell_{k+1}-1}_{n_{k+1}-1}}{\Delta^{\ell_{k+1}}_{n_{k+1}}}\wp^{\ell_k}_{n_k}(y) & \text{if } \dot n_{k+1}=1,\end{cases}\\
&=y^{1-\dot n_{k+1}}\frac{\Delta^{\ell_{k+1}-1}_{n_k}}{\Delta^{\ell_{k+1}}_{n_{k+1}}}\wp^{\ell_k}_{n_{k+1}-1}(y),
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
\left[\wp^{\ell_k}_{n_k}(x)-(1-\dot\ell_{k+1})\frac{\Delta^{\ell_{k+1}}_{n_{k+1}}}{\Delta^{\ell_{k+1}-1}_{n_{k+1}-1}}\wp^{\ell_k-1}_{n_k-1}(x)\right]
&=\begin{cases} \dfrac{\Delta^{\ell_{k+1}}_{n_k}}{\Delta^{\ell_{k+1}-1}_{n_{k+1}-1}}\wp^{\ell_{k+1}-1}_{n_k}(x) & \text{if } \dot\ell_{k+1}=0,\\[2ex] \wp^{\ell_k}_{n_k}(x)=\wp^{\ell_{k+1}-1}_{n_k}(x) & \text{if } \dot\ell_{k+1}=1,\end{cases}\\
&=\frac{\Delta^{\ell_k}_{n_k}}{\Delta^{\ell_{k+1}-1}_{n_k}}\wp^{\ell_{k+1}-1}_{n_k}(x).
\end{aligned}
\tag{3.15}
$$

Using these expressions in the right-hand side of (3.13) the identity becomes

$$
\begin{aligned}
&\frac{(-1)^{n_N}}{(\Delta^{\ell_N}_{n_N})^2}\left(\frac{\wp^{\ell_N-1}_{n_N-1}(x)}{x^{n_N-\ell_N}}\wp^{\ell_N}_{n_N}(y)-\frac{\wp^{\ell_N}_{n_N}(x)}{x^{n_N-\ell_N}}\wp^{\ell_N-1}_{n_N-1}(y)\right)\\
&\qquad=\left(\frac{y}{x}-1\right)\sum_{k=0}^{N-1}(-1)^{n_{k+1}}\frac{\wp^{\ell_k}_{n_{k+1}-1}(y)\wp^{\ell_{k+1}-1}_{n_k}(x)x^{\ell_{k+1}-n_k}}{\Delta^{\ell_{k+1}}_{n_{k+1}}\Delta^{\ell_k}_{n_k}}.
\end{aligned}
\tag{3.16}
$$

We can repeat the same arguments for the second-kind polynomials appearing in the other matrix entries; care must be paid to the fact that $(\Xi_0)_{12}$ is not $\mathcal{R}_0^0 \equiv 0$.

We obtain the following supplementary Christoffel–Darboux Identities (CDIs) (provided that $0 \le \ell_k \le n_{k+1} - 2, k = 1, \ldots$)

$$\begin{aligned}(3.17)\quad &\frac{(-1)^{n_N}}{(\Delta_{n_N}^{\ell_N})^2}\left(\frac{\mathcal{R}_{n_N-1}^{\ell_N-1}(x)}{x^{n_N-\ell_N}}\wp_{n_N}^{\ell_N}(y) - \frac{\mathcal{R}_{n_N}^{\ell_N}(x)}{x^{n_N-\ell_N}}\wp_{n_N-1}^{\ell_N-1}(y)\right) - \frac{1}{x}\\
&\qquad = \left(\frac{y}{x}-1\right)\sum_{k=0}^{N-1}(-1)^{n_{k+1}}\frac{\wp_{n_{k+1}-1}^{\ell_k}(y)\mathcal{R}_{n_k}^{\ell_{k+1}-1}(x)x^{\ell_{k+1}-n_k}}{\Delta_{n_{k+1}}^{\ell_{k+1}}\Delta_{n_k}^{\ell_k}},\\
&\frac{(-1)^{n_N}}{(\Delta_{n_N}^{\ell_N})^2}\left(\frac{\wp_{n_N-1}^{\ell_N-1}(x)}{x^{n_N-\ell_N}}\mathcal{R}_{n_N}^{\ell_N}(y) - \frac{\wp_{n_N}^{\ell_N}(x)}{x^{n_N-\ell_N}}\mathcal{R}_{n_N-1}^{\ell_N-1}(y)\right) + \frac{1}{y}\\
&\qquad = \left(\frac{y}{x}-1\right)\left[\sum_{k=0}^{N-1}(-1)^{n_{k+1}}\frac{\mathcal{R}_{n_{k+1}-1}^{\ell_k}(y)\wp_{n_k}^{\ell_{k+1}-1}(x)x^{\ell_{k+1}-n_k}}{\Delta_{n_{k+1}}^{\ell_{k+1}}\Delta_{n_k}^{\ell_k}} - \frac{1}{y}\right],\\
&\frac{(-1)^{n_N}}{(\Delta_{n_N}^{\ell_N})^2}\left(\frac{\mathcal{R}_{n_N-1}^{\ell_N-1}(x)}{x^{n_N-\ell_N}}\mathcal{R}_{n_N}^{\ell_N}(y) - \frac{\mathcal{R}_{n_N}^{\ell_N}(x)}{x^{n_N-\ell_N}}\mathcal{R}_{n_N-1}^{\ell_N-1}(y)\right)\\
&\qquad = \left(\frac{y}{x}-1\right)\sum_{k=0}^{N-1}(-1)^{n_{k+1}}\frac{\mathcal{R}_{n_{k+1}-1}^{\ell_k}(y)\mathcal{R}_{n_k}^{\ell_{k+1}-1}(x)x^{\ell_{k+1}-n_k}}{\Delta_{n_{k+1}}^{\ell_{k+1}}\Delta_{n_k}^{\ell_k}}.\end{aligned}$$

The additional term in the second identity stems from the mentioned discrepancy in the definition of Ξ_0 with the definition of the auxiliary polynomials: indeed, the term with $k = 0$ in the sum (3.12) is not zero in the off-diagonal terms but $\begin{bmatrix}1 & -1/y\\ 0 & 0\end{bmatrix}$. Thus the second identity above is rewritten as

$$\begin{aligned}(3.18)\quad &\frac{(-1)^{n_N}}{(\Delta_{n_N}^{\ell_N})^2}\left(\frac{\wp_{n_N-1}^{\ell_N-1}(x)}{x^{n_N-\ell_N}}\mathcal{R}_{n_N}^{\ell_N}(y) - \frac{\wp_{n_N}^{\ell_N}(x)}{x^{n_N-\ell_N}}\mathcal{R}_{n_N-1}^{\ell_N-1}(y)\right) + \frac{1}{x}\\
&\qquad = \left(\frac{y}{x}-1\right)\sum_{k=0}^{N-1}(-1)^{n_{k+1}}\frac{\mathcal{R}_{n_{k+1}-1}^{\ell_k}(y)\wp_{n_k}^{\ell_{k+1}-1}(x)x^{\ell_{k+1}-n_k}}{\Delta_{n_{k+1}}^{\ell_{k+1}}\Delta_{n_k}^{\ell_k}}.\end{aligned}$$

3.2. *Christoffel–Darboux Identities for Biorthogonal Laurent Polynomials*

The formulæ derived in the previous sections for the CDIs are very general, however the (Laurent) polynomials that appear in the sum are not biorthogonal with respect to the moment functional $\mathcal{L}$ unless the sequence n_k is strictly increasing and the sequence ℓ_k is weakly increasing. This is the situation which interests us the most and hence from now on we will assume that $n_k = k$.[1] Moreover, all the elementary orbits of the integrable lattices we are considering are in correspondence with this situation.

[1] If n_k were not strictly increasing, then the polynomials would be biorthogonal only provided the moments satisfy some nongeneric condition of vanishing of certain determinants.

From the formulæ defining the polynomials $\wp_n^\ell$ it follows that

$$\mathcal{L}_z(\wp_n^{\ell_n}(z)\wp_m^{\ell_{m+1}-1}(z)z^{\ell_{m+1}-m}) = \delta_{mn}(-1)^n\Delta_n^{\ell_n}\Delta_{n+1}^{\ell_{n+1}}. \tag{3.19}$$

This suggests that we introduce the following monic polynomials:

$$\pi_n(x) \;=\; \frac{1}{\Delta_n^{\ell_n}}\wp_n^{\ell_n}(x), \tag{3.20}$$

$$\rho_n(x) \;=\; \frac{(-1)^n}{\Delta_n^{\ell_n}}x^{\ell_{n+1}-n}\wp_n^{\ell_{n+1}-1}(x). \tag{3.21}$$

It is understood that the determinants $\Delta_n^{\ell_n}$ must not vanish: this is our implicit assumption of genericity on the moment functional. While the π_n's are monic in the usual sense, the ρ_n's are normalized on either the highest or the lowest power depending on $\dot{\ell}_{n+1}$. Moreover, the π_n's are polynomials in x whereas the ρ_n's are polynomials in x and x^{-1}. They satisfy the orthogonality relations

$$\mathcal{L}_z(\rho_m(z)\pi_n(z)) = \delta_{mn}h_n, \qquad h_n := \frac{\Delta_{n+1}^{\ell_{n+1}}}{\Delta_n^{\ell_n}}. \tag{3.22}$$

Remark 3.1. Laurent polynomials (3.20) and (up to a sign) (3.21) appeared in [11, equations (3.23) and (3.32)], where they were constructed from a moment functional defined by $\mathcal{L}(x^i) = (Q^i)_{11}$, where Q belongs to a certain "elementary" co-adjoint orbit determined by a sequence ℓ_n. In particular, $\mathcal{L}$ in this case is normalized by $\mathcal{L}(1) = \mu_0 = 1$. However, since expressions defining $\pi_n(x)$, $\rho_n(x)$ are homogeneous of degree 0 in moments μ_i, all algebraic relations for $\pi_n(x)$, $\rho_n(x)$ that were derived in [11] remain valid in our current framework. Let us also point out for the interested reader that increments $\dot{\ell}_n$ correspond to $1-\varepsilon_n$ in the notations of [11].

We finally introduce the **(bi)-orthonormal polynomials** and the second-kind polynomials

$$\begin{aligned} p_n(x) &:= \frac{1}{\sqrt{h_n}}\pi_n(x) = \frac{\wp_n^{\ell_n}}{\sqrt{\Delta_n^{\ell_n}\Delta_{n+1}^{\ell_{n+1}}}},\\ \widetilde{p}_n(x) &:= \mathcal{L}_z\left(\frac{p_n(x)-p_n(z)}{x-z}\right),\\ r_n(x) &:= \frac{1}{\sqrt{h_n}}\rho_n(x) = x^{\ell_{n+1}-n}\frac{(-1)^n\wp_n^{\ell_{n+1}-1}}{\sqrt{\Delta_n^{\ell_n}\Delta_{n+1}^{\ell_{n+1}}}},\\ \widetilde{r}_n(x) &:= \mathcal{L}_z\left(\frac{r_n(x)-r_n(z)}{x-z}\right). \end{aligned} \tag{3.23}$$

and their "starred" polynomials

$$p_n^\star(x) \;:=\; x^{\ell_n-n+1}p_n(x), \qquad \widetilde{p}_n^\star(x) := x^{\ell_n-n+1}\widetilde{p}_n(x), \tag{3.24}$$

$$r_n^\star(x) \;:=\; x^{n-\ell_{n+1}}r_n(x), \qquad \widetilde{r}_n^\star(x) := x^{n-\ell_{n+1}}\widetilde{r}_n(x). \tag{3.25}$$

In terms of these (Laurent) polynomials the CDIs read

$$
\begin{aligned}
(y-x)\sum_{n=0}^{N-1} r_n(x)p_n(y) &= \gamma_N\big(p_N(y)r_{N-1}(x) - p_N^\star(x)r_{N-1}^\star(y)\big),\\
(y-x)\sum_{n=0}^{N-1} \widetilde{r}_n(x)p_n(y) &= \gamma_N\big(p_N(y)\widetilde{r}_{N-1}(x) - \widetilde{p}_N^\star(x)r_{N-1}^\star(y)\big)+1,\\
(y-x)\sum_{n=0}^{N-1} r_n(x)\widetilde{p}_n(y) &= \gamma_N\big(\widetilde{p}_N(y)r_{N-1}(x) - p_N^\star(x)\widetilde{r}_{N-1}^\star(y)\big)-1,\\
(y-x)\sum_{n=0}^{N-1} \widetilde{r}_n(x)\widetilde{p}_n(y) &= \gamma_N\big(\widetilde{p}_N(y)\widetilde{r}_{N-1}(x) - \widetilde{p}_N^\star(x)\widetilde{r}_{N-1}^\star(y)\big),
\end{aligned}
$$

$$\gamma_N := \sqrt{\frac{h_N}{h_{N-1}}}. \tag{3.26}$$

It is convenient to rewrite in matrix form the previous identities as follows:

$$
\begin{aligned}
&\mathbf{p}(x) := [p_0,\ldots]^t,\quad \widetilde{\mathbf{p}}(x) := [\widetilde{p}_0,\ldots]^t, &(3.27)\\
&\mathbf{r}(x) := [r_0,\ldots]^t,\quad \widetilde{\mathbf{r}}(x) := [\widetilde{r}_0,\ldots]^t,
\end{aligned}
$$

$$\mathbf{P}(x) := [\mathbf{p}(x), \widetilde{\mathbf{p}}(x)]\,,\qquad \mathbf{R}(x) := [\mathbf{r}(x), \widetilde{\mathbf{r}}(x)]\,, \tag{3.28}$$

$$(\Pi_{N-1})_{ij} := \sum_{k=0}^{N-1}\delta_{ik}\delta_{kj}, \tag{3.29}$$

$$
\begin{aligned}
&\mathbf{R}^t(x)\Pi_{N-1}\mathbf{P}(y) &(3.30)\\
&\quad = \frac{1}{y-x}\left\{\gamma_N\begin{bmatrix} p_N^\star(x) & r_{N-1}(x)\\ \widetilde{p}_N^\star(x) & \widetilde{r}_{N-1}(x)\end{bmatrix}\mathbf{j}\begin{bmatrix} p_N(y) & \widetilde{p}_N(y)\\ r_{N-1}^\star(y) & \widetilde{r}_{N-1}^\star(y)\end{bmatrix} + \mathbf{j}\right\}.
\end{aligned}
$$

Remark 3.2. Christoffel–Darboux identities proved crucial in establishing a 2×2 partial differential equation (PDE) for the orthogonal polynomials in [4] (the so-called "folding"); a completely parallel rôle will be played in Section 5.

Remark 3.3. A word about the relations with previously known (bi)-orthogonal polynomials is now in order. If all the moves (except the first one) are *line-moves*, namely, if $\ell_n = n-1$, then it is not hard to show that $\pi_n = \rho_n$ are just orthogonal polynomials with respect to the (restriction of the) moment functional $\mathcal{L}$ to positive moments. Moreover, the shifted Töplitz determinants Δ_n^{n-1} are (up to a sign) the same as the Hänkel determinants of the same size (by permuting appropriately the columns).

Vice versa, if all moves are *circle-moves* (i.e. $\ell_n \equiv 0$) (and we also impose certain reality conditions on the moments of the functional), then the π_n are orthogonal polynomials for a certain measure on the unit circle and the ρ_n's are their so-called "dual" Laurent polynomials. The determinants appearing then in our sequence are precisely the "standard" ones Δ_n^0.

A second remark is that all these polynomial *do* satisfy three-term recurrence relations, although of a different sort than the standard ones. Indeed, it is well known that orthogonal polynomials p_n satisfy relations of the form

$$x p_n = \gamma_n p_{n+1} + \beta_n p_n + \gamma_{n-1} p_{n-1}, \tag{3.31}$$

where the coefficients γ_n, β_n enter in the tridiagonal Jacobi matrix representing the multiplication by x in the basis of the p_n's. At the opposite "end of the spectrum", orthogonal polynomials on the circle satisfy a different sort of three-term recurrence relation of the form

$$x(p_n + \delta_n p_{n-1}) = \gamma_n p_{n+1} + \beta_n p_n. \tag{3.32}$$

It is not hard to show that the polynomials that we are considering precisely "interpolate" these two sorts of recurrence relations. Indeed, it was observed in [11] that polynomials π_n defined by (3.20) satisfy a three-term recurrence relation of the form

$$x(\pi_n + (1 - \dot{\ell}_n) d_n \pi_{n-1}) = \pi_{n+1} + b_n \pi_n + \dot{\ell}_n d_n \pi_{n-1},$$

(see Remark 4.1 and equation (3.4) in [11]). Then our orthonormal polynomials $p_n(x)$ satisfy a recursion

$$x(p_n + (1 - \dot{\ell}_n)\delta_n p_{n-1}) = \gamma_n p_{n+1} + \beta_n p_n + \dot{\ell}_n \delta_n p_{n-1}, \tag{3.33}$$

for certain coefficients $\gamma_n, \beta_n, \delta_n$ whose explicit expression in terms of Töplitz determinants can be obtained from the formulæ above but is irrelevant for this discussion. We see that "circle-moves" ($\dot{\ell}_n = 0$) correspond to a three-term recurrence relation of the type appearing for orthogonal polynomials on the circle, while "line-moves" ($\dot{\ell}_n = 1$) correspond to the "usual" recurrence relation.

We should also mention that recurrence (3.33) can be further generalized to a generalized eigenvalue problem for a pair of tridiagonal matrices. This situation leads to more general biorthogonal rational functions. This was studied extensively in [21], [22].

4. Infinitesimal Deformations of the Moment Functional

We study the infinitesimal deformations for the wave vectors $\mathbf{p}(x)$, $\widetilde{\mathbf{p}}(x)$, $\mathbf{r}(x)$ and $\widetilde{\mathbf{r}}(x)$ under an infinitesimal deformation of the moment functional. Let us introduce the matrix of recurrence for these sequences of polynomials

$$x\mathbf{p} = Q\mathbf{p}, \qquad x\mathbf{r}^t = \mathbf{r}^t Q, \qquad Q_{nm} := \mathcal{L}(z p_n r_m). \tag{4.1}$$

The matrix Q is of Hessenberg form, namely, has nonzero entries on the superdiagonal and possibly on the diagonal and all other nonzero entries in the lower triangular part. The biorthogonality relation can be rewritten as

$$\mathcal{L}[\mathbf{p}\mathbf{r}^t] = \mathbf{1}. \tag{4.2}$$

Suppose we infinitesimally deform the moment functional

$$\dot{\mathcal{L}}(\bullet) = -\mathcal{L}(F(z)\bullet). \tag{4.3}$$

Here $F(z)$ can be any function (even a generalized distribution as we will see) provided that the moments of the deformation are still well defined: if $\mathcal{L}$ is given by an analytical expression in terms of some integral representation (as we will assume later on), then this means some condition of analyticity on F: if the functional is only defined by its moments, then F should be interpreted as formal series. In any situation the typical case of F being a polynomial (corresponding to the usual formal Toda-type flows) will be well defined.

A little more generally we could even assume that F is a distribution, particularly delta functions or derivatives of it. For instance, we can consider deformation of the type

$$\delta\mathcal{L}(p(x)) \equiv \dot{\mathcal{L}}(p(x)) = -\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k p(x)\bigg|_{x=a} \tag{4.4}$$

for some constant a: this means that we (formally) have set F to be the kth derivative of the Dirac delta distribution for the given moment functional supported at $x = a$.

Corresponding to any of these deformations the BOPs deform as

$$\delta\mathbf{p} = \mathbb{U}^{(F)}\mathbf{p}, \qquad \delta\mathbf{r} = \widetilde{\mathbb{U}}^{(F)}\mathbf{r}, \tag{4.5}$$

where a priori $\mathbb{U}$ and $\widetilde{\mathbb{U}}$ are lower triangular matrices since the range of powers of x entering in the expressions p_n, r_n will not change. In order to find expressions for these matrices we note first that their diagonals are the same

$$(\mathbb{U}^{(F)})_{nn} = (\widetilde{\mathbb{U}}^{(F)})_{nn} = -\tfrac{1}{2}\delta\ln(h_n). \tag{4.6}$$

Indeed, we have

$$\delta p_n = \delta\frac{x^n}{\sqrt{h_n}} + \cdots = -\tfrac{1}{2}\delta\ln(h_n)p_n + \text{previous}, \tag{4.7}$$

$$\delta r_n = -\tfrac{1}{2}\delta\ln(h_n)r_n + \text{previous}. \tag{4.8}$$

Differentiating the orthogonality relation we obtain

(4.9)

$$\mathbb{U}^{(F)} + \widetilde{\mathbb{U}}^{(F)t} = \begin{cases} F(Q) & \text{for the case of an ordinary function } F, \\ \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \mathbf{p}(x)\mathbf{r}^t(x)\bigg|_{x=a} & \text{for a deformation supported at one point.} \end{cases}$$

and hence, according to the two types, the matrices describing the infinitesimal deformations are given by

$$\mathbb{U}^{(F)} = F(Q)_{-0}, \qquad \widetilde{\mathbb{U}}^{(F)} = F(Q)^t{}_{-0}, \tag{4.10}$$

$$\mathbb{U}^{(\delta_a^k)} = \partial_a^k(\mathbf{p}(a)\mathbf{r}^t(a))_{-0}, \qquad \widetilde{\mathbb{U}}^{(\delta_a^k)} = \partial_a^k(\mathbf{r}(a)\mathbf{p}^t(a))_{-0}, \tag{4.11}$$

where A_{-0} means the lower triangular part plus half of the diagonal. Note that from (4.6) and the definition of h_n it follows that

$$\Delta_n^{\ell_n} = \prod_{k=0}^{n-1} h_k, \tag{4.12}$$

$$\delta_F \ln \Delta_n^{\ell_n} = -\mathrm{Tr}_n\, F(Q),$$

$$\delta_{\delta_a^k} \ln \Delta_n^{\ell_n} = -\partial_a^k \sum_{j=0}^{n-1} p_j(a) r_j(a),$$

where we have used the notation for the **truncated trace** $\mathrm{Tr}_n\, A := \sum_{j=0}^{n-1} A_{jj}$.

4.1. *Deformations for the Second-Kind* (*Laurent*) *Polynomials*

Using Leibnitz's rule we obtain the following deformation equations for the second-kind wave vectors $\widetilde{\mathbf{p}}, \widetilde{\mathbf{r}}$. For a deformation by a function $F(x)$ we have

$$\delta_F \widetilde{\mathbf{p}} = (\mathbb{U}^{(F)} - F(x))\widetilde{\mathbf{p}} + \mathcal{L}_z \left(\frac{F(x) - F(z)}{x - z} \right) \mathbf{p} - \left(\frac{F(x) - F(Q)}{x - Q} \right) \mathbf{e}_1, \tag{4.13}$$

$$\delta_F \widetilde{\mathbf{r}} = (\widetilde{\mathbb{U}}^{(F)} - F(x))\widetilde{\mathbf{r}} + \mathcal{L}_z \left(\frac{F(x) - F(z)}{x - z} \right) \mathbf{r} - \left(\frac{F(x) - F(Q^t)}{x - Q^t} \right) \mathbf{e}_1,$$

while for $F = \delta_{\mathcal{L}}^{(k)}(z - a)$ we have

$$\delta_F \widetilde{\mathbf{p}} = \mathbb{U}^{(\delta_a^k)} \widetilde{p} - \frac{\partial^k}{\partial a^k} \frac{\mathbf{p}(x) - \mathbf{p}(a)}{x - a}, \tag{4.14}$$

$$\delta_F \widetilde{\mathbf{r}} = \widetilde{\mathbb{U}}^{(\delta_a^k)} \widetilde{r} - \frac{\partial^k}{\partial a^k} \frac{\mathbf{r}(x) - \mathbf{r}(a)}{x - a}.$$

5. Folded Version of the Deformation Equations

Let us define

$$\chi_n := \begin{bmatrix} p_n & \widetilde{p}_n \\ r_{n-1}^\star & \widetilde{r}_{n-1}^\star \end{bmatrix}. \tag{5.1}$$

We want to express the previous infinite-dimensional deformation equations in terms of χ_n alone; this process is conceptually identical to the one followed in [4] and which is named "folding". To this end, we formulate the following:

Theorem 5.1. *The infinite deformations* (4.10) *for the wave vectors* **p**, **r** *and for the second-kind wave vectors* $\widetilde{\mathbf{p}}, \widetilde{\mathbf{r}}$ (4.13), (4.14) *are equivalent to the following deformation equations for* χ_n, $n \geq 1$,

$$\delta_{(F)} \chi_n = \mathcal{U}_n^{(F)}(x) \chi_n + \chi_n \mathcal{U}^{(F),R}(x), \tag{5.2}$$

$$\delta_{(\delta_a^k)} \chi_n(a) = \mathcal{U}_n^{(\delta_a^k)}(x) \chi_n(x) + \chi_n(x) \mathcal{U}^{(\delta_a^k),R}(x),$$

where we have used the following definitions:

$$\mathcal{U}_n^{(F)} = \begin{bmatrix} \frac{1}{2}F(Q)_{nn} & 0 \\ 0 & F(x) - \frac{1}{2}F(Q)_{n-1,n-1} \end{bmatrix} + \gamma_n \begin{bmatrix} -(\nabla_Q F)_{n,n-1} & (\nabla_Q F)_{n,n^\star} \\ -(\nabla_Q F)_{(n-1)^\star,n-1} & (\nabla_Q F)_{n,n-1} \end{bmatrix}, \tag{5.3}$$

$$\mathcal{U}^{(F),R} = \begin{bmatrix} 0 & \mathcal{W}_F \\ 0 & -F(x) \end{bmatrix}, \qquad \mathcal{W}_F := \mathcal{L}_z\left(\frac{F(x)-F(z)}{x-z}\right),$$

$$\nabla_Q F := \frac{F(x)-F(Q)}{x-Q}$$

$$\mathcal{U}_n^{(\delta_a^k)}(x) = \frac{\partial^k}{\partial a^k}\frac{1}{2}\begin{bmatrix} p_n r_n & 0 \\ 0 & -p_{n-1}r_{n-1} \end{bmatrix}_{z=a} + \frac{\partial^k}{\partial a^k}\frac{\gamma_n}{x-a}\begin{bmatrix} -p_n r_{n-1} & p_n p_n^\star \\ -r_{n-1}r_{n-1}^\star & r_{n-1}p_n \end{bmatrix}_{z=a}, \tag{5.4}$$

$$\mathcal{U}^{(\delta_a^k),R}(x) = \partial_a^k \begin{bmatrix} 0 & 1/(a-x) \\ 0 & 0 \end{bmatrix}.$$

Here, for a function $f(z)$ we have set

$$f(Q)_{i,j^\star} := \mathcal{L}(r_i f(z) r_i^\star), \qquad f(Q)_{i^\star,j} := \mathcal{L}(p_i^\star f(z) p_j). \tag{5.5}$$

Proof. We compute the deformations of both rows of χ_n. We start with deformation involving a function $F(x)$: the first row deforms according to the equation

$$\delta_F[p_n(x), \widetilde{p}_n(x)] = \delta_F \mathbf{e}_n^t \cdot [\mathbf{p}, \widetilde{\mathbf{p}}] = \mathbf{e}_n^t \cdot \mathbb{U}^{(F)} \cdot [\mathbf{p}, \widetilde{\mathbf{p}}] + \mathbf{e}_n^t \cdot [\mathbf{p}, \widetilde{\mathbf{p}}]\begin{bmatrix} 0 & \mathcal{W}_F \\ 0 & -F(x) \end{bmatrix} - \mathbf{e}_n^t \cdot \frac{F(x)-F(Q)}{x-Q} \cdot [\mathbf{0}, \mathbf{e}_1], \tag{5.6}$$

where we have set

$$\mathcal{W}_F(x) := \mathcal{L}_z\left(\frac{F(x)-F(z)}{x-z}\right). \tag{5.7}$$

We can compute the folded version

$$\begin{aligned} \mathbf{e}_n^t \cdot \mathbb{U}^{(F)} \cdot [\mathbf{p}, \widetilde{\mathbf{p}}] &= \tfrac{1}{2}F(Q)_{nn}[p_n, \widetilde{p}_n] + \mathbf{e}_n^t \mathcal{L}_z(F(z)\mathbf{p}(z)\mathbf{r}^t(z)\Pi_{n-1}[\mathbf{p}(x), \widetilde{\mathbf{p}}(x)]) \\ &= \tfrac{1}{2}F(Q)_{nn}[p_n, \widetilde{p}_n] \\ &\quad + \mathbf{e}_n^t \mathcal{L}_z((F(z)-F(x))\mathbf{p}(z)\mathbf{r}^t(z)\Pi_{n-1}[\mathbf{p}(x), \widetilde{\mathbf{p}}(x)]) = (\star). \end{aligned}$$

In the equality above the term proportional to $F(x)$ is zero because of the projection along $\mathbf{e}_n$ and hence the equality is valid: using now the CDIs on the last term (in the matrix form provided by (3.27)), we continue the chain of equalities

$$(5.8)\quad (\star) = \tfrac{1}{2}F(Q)_{nn}[p_n, \widetilde{p}_n] + \mathbf{e}_n^t \mathcal{L}_z\left(\frac{F(z)-F(x)}{x-z}\mathbf{p}(z)(\gamma_n[p_n(z)^\star, r_{n-1}(z)]\mathbf{j}\chi_n(x) - [0,1])\right)$$
$$= \tfrac{1}{2}F(Q)_{nn}[p_n, \widetilde{p}_n] - \gamma_n\mathcal{L}_z\left(\frac{F(z)-F(x)}{z-x}p_n[p_n^\star, r_{n-1}]\right)\mathbf{j}\chi_n(x) + \mathbf{e}_n^t \cdot \frac{F(x)-F(Q)}{x-Q}\cdot[\mathbf{0}, \mathbf{e}_1].$$

This implies that

$$(5.9)\qquad \delta_F[p_n(x), \widetilde{p}_n(x)] = \tfrac{1}{2}F(Q)_{nn}[p_n, \widetilde{p}_n] - \gamma_n\mathcal{L}_z\left(\frac{F(z)-F(x)}{z-x}p_n[p_n^\star, r_{n-1}]\right)\mathbf{j}\chi_n(x) + \mathbf{e}_n^t\cdot[\mathbf{p}.\widetilde{\mathbf{p}}]\begin{bmatrix}0 & \mathcal{W}_F\\ 0 & -F(x)\end{bmatrix},$$

and thus completes the proof of the folded deformation equations associated to a function (or generalized function) F for the polynomials of the first and second kind. In a similar way we can compute the same deformations for the dual Laurent polynomials:

$$(5.10)\qquad \delta_F[r_{n-1}(x), \widetilde{r}_{n-1}(x)] = \delta_F\mathbf{e}_{n-1}^t\cdot[\mathbf{r}, \widetilde{\mathbf{r}}] = \mathbf{e}_{n-1}^t\cdot\mathbb{U}^{(F)^t}\cdot[\mathbf{r}, \widetilde{\mathbf{r}}] + \mathbf{e}_{n-1}^t\cdot[\mathbf{r}, \widetilde{\mathbf{r}}]\begin{bmatrix}0 & \mathcal{W}_F\\ 0 & -F(x)\end{bmatrix} - \mathbf{e}_{n-1}^t\cdot\frac{F(x)-F(Q^t)}{x-Q^t}\cdot[\mathbf{0}, \mathbf{e}_1].$$

In parallel with (5.8) above, the computation now involves

$$(5.11)\quad \mathbf{e}_{n-1}^t\cdot\mathbb{U}^{(F)^t}\cdot[\mathbf{r}, \widetilde{\mathbf{r}}] = -\tfrac{1}{2}F(Q)_{n-1,n-1}[r_{n-1}, \widetilde{r}_{n-1}] + \mathbf{e}_{n-1}^t\cdot F(Q^t)\Pi_{n-1}[\mathbf{r}, \widetilde{\mathbf{r}}]$$
$$= -\tfrac{1}{2}F(Q)_{n-1,n-1}[r_{n-1}, \widetilde{r}_{n-1}] + \mathbf{e}_{n-1}^t\mathcal{L}_z(F(z)\mathbf{r}(z)\mathbf{p}^t(z)\Pi_{n-1}[\mathbf{r}(x), \widetilde{\mathbf{r}}(x)])$$
$$= (F(x) - \tfrac{1}{2}F(Q)_{n-1,n-1})[r_{n-1}, \widetilde{r}_{n-1}] + \mathbf{e}_{n-1}^t\mathcal{L}_z\left((F(z)-F(x))\mathbf{r}(z)\mathbf{p}^t(z)\Pi_{n-1}[\mathbf{r}(x), \widetilde{\mathbf{r}}(x)]\right)$$
$$= (F(x) - \tfrac{1}{2}F(Q)_{n-1,n-1})[r_{n-1}, \widetilde{r}_{n-1}] + \mathbf{e}_{n-1}\mathcal{L}_z\left(\frac{F(z)-F(x)}{z-x}\mathbf{r}(z)(-\gamma_n[p_n(z), r_{n-1}^\star(z)] \times \mathbf{j}\chi_n^\star(x)^t + [0,1])\right)$$

$$
\begin{aligned}
&= (F(x) - \tfrac{1}{2}F(Q)_{n-1,n-1})[r_{n-1}, \widetilde{r}_{n-1}] \\
&\quad + -\gamma_n \mathcal{L}_z \left(\frac{F(z) - F(x)}{z - x} r_{n-1}[p_n, r^\star_{n-1}] \right) \mathbf{j} \chi_n^{\star t}(x) \\
&\quad + \mathbf{e}_{n-1} \frac{F(x) - F(Q^t)}{x - Q^t} \mathbf{e}_1,
\end{aligned}
$$

where we have used the following definition:

$$
\chi_n^\star = x^{\ell_n - n + 1} \chi_n^t. \tag{5.12}
$$

Summarizing, we have obtained the following deformation equation:

$$
\begin{aligned}
\delta_F[r_{n-1}(x), \widetilde{r}_{n-1}(x)] &= (F(x) - \tfrac{1}{2}F(Q)_{n-1,n-1})[r_{n-1}, \widetilde{r}_{n-1}] \\
&\quad + \mathbf{e}^t_{n-1} \cdot [\mathbf{r}, \widetilde{\mathbf{r}}] \begin{bmatrix} 0 & \mathcal{W}_F \\ 0 & -F(x) \end{bmatrix} \\
&\quad + \gamma_n \mathcal{L}_z \left(\frac{F(z) - F(x)}{z - x} r_{n-1}[p_n, r^\star_{n-1}] \right) \mathbf{j} \chi_n^{\star t}(x).
\end{aligned} \tag{5.13}
$$

By "starifying" both sides we obtain

$$
\begin{aligned}
\delta_F[r^\star_{n-1}(x), \widetilde{r}^\star_{n-1}(x)] &= (F(x) - \tfrac{1}{2}F(Q)_{n-1,n-1})[r^\star_{n-1}, \widetilde{r}^\star_{n-1}] \\
&\quad + [r^\star_{n-1}, \widetilde{r}^\star_{n-1}] \begin{bmatrix} 0 & \mathcal{W}_F \\ 0 & -F(x) \end{bmatrix} \\
&\quad + \gamma_n \mathcal{L}_z \left(\frac{F(z) - F(x)}{z - x} r_{n-1}[p_n, r^\star_{n-1}] \right) \mathbf{j} \chi_n(x).
\end{aligned} \tag{5.14}
$$

Putting together (5.9) and (5.14) we obtain finally the folded version of this kind of deformation equation

$$
\delta_F \chi_n = \mathcal{U}_n^{(F)}(x) \chi_n + \chi_n \mathcal{U}^{(F),R}(x), \tag{5.15}
$$

$$
\begin{aligned}
\mathcal{U}_n^{(F)} &= \begin{bmatrix} \tfrac{1}{2}F(Q)_{nn} & 0 \\ 0 & F(x) - \tfrac{1}{2}F(Q)_{n-1,n-1} \end{bmatrix} \\
&\quad - \gamma_n \mathcal{L}_z \left(\frac{F(x) - F(z)}{x - z} \begin{bmatrix} p_n p_n^\star & p_n r_{n-1} \\ p_n r_{n-1} & r^\star_{n-1} r_{n-1} \end{bmatrix} \right) \mathbf{j} \\
&= \begin{bmatrix} \tfrac{1}{2}F(Q)_{nn} & 0 \\ 0 & F(x) - \tfrac{1}{2}F(Q)_{n-1,n-1} \end{bmatrix} \\
&\quad + \gamma_n \begin{bmatrix} -(\nabla_Q F)_{n,n-1} & (\nabla_Q F)_{n,n^\star} \\ -(\nabla_Q F)_{(n-1)^\star,n-1} & (\nabla_Q F)_{n,n-1} \end{bmatrix}
\end{aligned} \tag{5.16}
$$

$$
\mathcal{U}^{(F),R} = \begin{bmatrix} 0 & \mathcal{W}_F \\ 0 & -F(x) \end{bmatrix}. \tag{5.17}
$$

We now consider a deformation supported at one point $z = a$ with $F = \delta_a^k$ (the kth derivative of the Dirac delta supported at $z = a$)

$$\delta_F[p_n(x), \widetilde{p}_n(x)] = \mathbf{e}_n^t \cdot \mathbb{U}^{(\delta_a^k)} \cdot [\mathbf{p}, \widetilde{\mathbf{p}}] - \mathbf{e}_n^t \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \bigg|_{z=a} \frac{\mathbf{p}(x) - \mathbf{p}(z)}{x - z}[0, 1]. \tag{5.18}$$

This time we have

$$\mathbf{e}_n^t \cdot \mathbb{U}^{(\delta_a^k)} \cdot [\mathbf{p}, \widetilde{\mathbf{p}}] = \tfrac{1}{2}\partial_a^k(p_n(a)r_n(a))[p_n, \widetilde{p}_n] + \partial_a^k \mathbf{e}_n^t \cdot \mathbf{p}(a)\mathbf{r}^t(a)\Pi_{n-1}[\mathbf{p}, \widetilde{\mathbf{p}}] \tag{5.19}$$

$$= \tfrac{1}{2}\partial_a^k(p_n(a)r_n(a))[p_n, \widetilde{p}_n] + \partial_a^k \mathbf{e}_n^t \cdot \mathbf{p}(a)\left(\frac{\gamma_n[p_n^\star(a), r_{n-1}(a)]}{x - a}\mathbf{j}\chi_n(x) - \frac{[0, 1]}{x - a}\right). \tag{5.20}$$

We thus have

$$\delta_F\,[p_n(x), \widetilde{p}_n(x)] = \tfrac{1}{2}\partial_a^k(p_n(a)r_n(a))[p_n, \widetilde{p}_n] + \partial_a^k \frac{\gamma_n[p_n(a)p_n^\star(a), p_n(a)r_{n-1}(a)]}{x - a}\mathbf{j}\chi_n(x) - \partial_a^k \frac{[0, p_n(x)]}{x - a}. \tag{5.21}$$

Similarly, for the Laurent polynomials,

$$\delta_F[r_{n-1}(x), \widetilde{r}_{n-1}(x)] = \mathbf{e}_{n-1}^t \cdot \mathbb{U}^{(\delta_a^k)^t} \cdot [\mathbf{r}, \widetilde{\mathbf{r}}] - \mathbf{e}_{n-1}^t \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \bigg|_{z=a} \frac{\mathbf{r}(x) - \mathbf{r}(z)}{x - z}[0, 1], \tag{5.22}$$

where now

$$\mathbf{e}_{n-1}^t \cdot \mathbb{U}^{(\delta_a^k)^t} \cdot [\mathbf{r}, \widetilde{\mathbf{r}}] = -\tfrac{1}{2}\partial_a^k(p_{n-1}(a)r_{n-1}(a))[r_{n-1}, \widetilde{r}_{n-1}] + \partial_a^k \mathbf{e}_{n-1}^t \cdot \mathbf{r}(a)\left(\frac{\gamma_n[p_n(a), r_{n-1}^\star(a)]}{x - a}\mathbf{j}\chi_n^{\star t}(x) - \frac{[0, 1]}{x - a}\right), \tag{5.23}$$

so that finally

$$\delta_F[r_{n-1}(x), \widetilde{r}_{n-1}(x)] = -\tfrac{1}{2}\partial_a^k(p_{n-1}(a)r_{n-1}(a))[r_{n-1}, \widetilde{r}_{n-1}] + \partial_a^k \frac{\gamma_n[r_{n-1}(a)p_n(a), r_{n-1}(a)r_{n-1}^\star(a)]}{x - a}\mathbf{j}\chi_n^{\star t}(x) - \partial_a^k \frac{[0, r_{n-1}(x)]}{x - a}. \tag{5.24}$$

Starifying this last identity and collecting it together with (5.21) we finally have

$$\delta\chi_n(a) = \mathcal{U}_n^{(\delta_a^k)}(x)\chi_n(x) + \chi_n(x)\mathcal{U}^{(\delta_a^k),R}(x), \tag{5.25}$$

$$(5.26)\qquad \mathcal{U}_n^{(\delta_a^k)}(x) = \frac{\partial^k}{\partial a^k}\left\{\frac{1}{2}\begin{bmatrix} p_n(a)r_n(a) & 0 \\ 0 & -p_{n-1}(a)r_{n-1}(a)\end{bmatrix} + \frac{\gamma_n}{x-a}\begin{bmatrix} p_n(a)p_n^\star(a) & p_n(a)r_{n-1}(a) \\ r_{n-1}(a)p_n(a) & r_{n-1}(a)r_{n-1}^\star(a)\end{bmatrix}\mathbf{j}\right\}$$
$$= \frac{\partial^k}{\partial a^k}\frac{1}{2}\begin{bmatrix} p_n r_n & 0 \\ 0 & -p_{n-1}r_{n-1}\end{bmatrix}_{z=a} + \frac{\partial^k}{\partial a^k}\frac{\gamma_n}{x-a}\begin{bmatrix} -p_n r_{n-1} & p_n p_n^\star \\ -r_{n-1}r_{n-1}^\star & r_{n-1}p_n\end{bmatrix}_{z=a},$$

$$(5.27)\qquad \mathcal{U}^{(\delta_a^k),R}(x) = \partial_a^k\begin{bmatrix} 0 & 1/(a-x) \\ 0 & 0\end{bmatrix}.$$

This concludes the proof. ■

6. Moment Functionals of Integral Type and Ordinary Differential Equations

We now assume that the moment functional that we are considering admits an actual integral representation

$$(6.1)\qquad \mathcal{L}(z^k) := \sum \varkappa_j \int_{\Gamma_j} \mathrm{e}^{-V(z)} z^k\, \mathrm{d}z.$$

As far as the previous discussion on deformations is concerned, the integral representation of the moment functional is largely irrelevant, the only issue being the convergence of the deformation function: therefore, the "potential" $V(z)$ as well as the sets of integration Γ_j could be completely arbitrary. However, in view of our intentions, we will assume that Γ_j are contours in the complex plane and that $V(z)$ is a locally defined smooth function on these contours with the only restriction coming from the fact that negative moments should be defined as well as the positive ones.

In fact—although many considerations would remain identical in more general situations—we will assume that V is a locally analytic function in the complex z-plane excepted at some punctures, identically to the case of **semiclassical moment functionals** studied in [5], [4] with the only extra restriction that all negative moments should be defined and finite.

Semiclassical Moment Functionals. For the reader's convenience we briefly recall how these semiclassical moment functionals are constructed [5], [4], [18], [19]. In this case the potential is such that the derivative is an arbitrary rational function

$$(6.2)\qquad V'(z) = \text{rational function},$$

and thus $V(z)$ is a rational function plus logarithmic singularities at those poles of V' where the residue does not vanish. For simplicity we assume that V' has either a pole or a nonzero limit at $z = \infty$. Once we have chosen the potential V we also choose an arbitrary collection of contours (avoiding $z = 0$) $\{\Gamma_j\}$ with the property that $\Re(V(x))$ is uniformly bounded from below on all the chosen contours and tends to ∞ polynomially (in the length parameter) on the contours that extend to $z = \infty$.

Remark 6.1. The interest in this class of moment functionals originates in [18], [19] and is motivated principally by the fact that they generalize the moment functionals of *classical* orthogonal polynomials.

In more detailed terms:

(a) Consider a pole $z = c$ of V' of order $k \geq 2$: we attach to it $k - 1$ "petals" approaching $z = c$ along asymptotic directions in the sectors where $\Re(V(x)) \to +\infty$. We also attach a "stem" extending to ∞ and asymptotic to a direction such that $\Re(V(x)) \to \infty$.

(b) For a simple pole $z = c$ of V', if the residue is a positive integer (i.e. e^{-V} has a pole at $z = c$) we choose a small loop around the point, if the residue is a negative integer we take a contour from $z = c$ to ∞, if the residue is noninteger we take a loop coming from ∞ and returning to ∞ (with the same restriction as above for the asymptotic direction).

(c) We also choose arbitrary segments joining a certain number of points $z = a$ to ∞ (along admissible directions). These latter contours are called "hard-edge" contours because the pseudo-measure $d\mu = e^{-V(z)}\, dz$ has a limit at $z = a$ and integration by parts yields a boundary term.

6.1. *Differential Equations*

We first analyze in this situation the infinite-dimensional differential equation that the BOPs satisfy. The natural differential operation in this setting is not ∂_x but rather $x\partial_x$.

Let us introduce the matrices of the recurrence relations involving multiplication by x and the orthogonality relations

$$x\mathbf{p}(x) = Q\mathbf{p}(x), \qquad x\mathbf{r}^t(x) = \mathbf{r}^t(x)Q, \tag{6.3}$$

$$x\mathbf{p}'(x) = D\mathbf{p}(x), \qquad x\mathbf{r}'(x) = \tilde{D}\mathbf{r}(x), \tag{6.4}$$

$$\int_{\varkappa} \mathbf{p}\mathbf{r}^t\, e^{-V}\, dz = \mathbf{1}. \tag{6.5}$$

The matrices Q, D, $\tilde{D}$ bear a certain relation which expresses the result of integration by parts: indeed if we integrate the total derivative

$$\partial_z(z\mathbf{p}\mathbf{r}^t\, e^{-V(z)}) = \mathbf{p}\mathbf{r}^t\, e^{-V(z)} + D\mathbf{p}\mathbf{r}^t\, e^{-V(z)} + \mathbf{p}\mathbf{r}^t\, e^{-V(z)}\tilde{D}^t - QV'(Q)\mathbf{p}\mathbf{r}^t\, e^{-V(z)} \tag{6.6}$$

over the contours defining our semiclassical moment functional and use the definitions above for the matrices D, $\tilde{D}$, Q we obtain the matricial identity

$$D + \tilde{D}^t - (z\mathbf{p}(z)\mathbf{r}^t(z)\, e^{-V(z)})|_{\partial\varkappa} = QV'(Q) - \mathbf{1}. \tag{6.7}$$

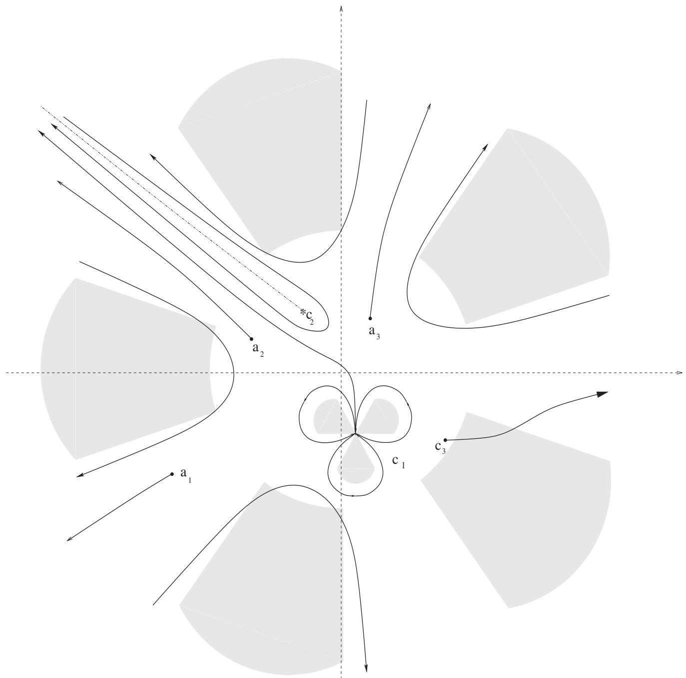

Fig. 1. The contours for a typical semiclassical moment functional. Here $V'(x)$ has a pole of order 4 at ∞, of order 4 at c_1 and simple poles at c_2, c_3 with noninteger and negative-integer residue, respectively. The contours originating from the a_i's are "hard-edge" contours. The shaded sectors represent the asymptotic "forbidden" directions for approaching a singularity. One of these sectors at ∞ in the figure does not have a contour surrounding it because such a contour would be "homologically" equivalent to minus the sum of all the others.

Here the notation that we have adopted is that $\int_\varkappa$ stands for the linear combination with coefficients $\varkappa_j$ of integrals on the oriented contours Γ_j and the evaluations $|_{\partial\varkappa}$ stand for the evaluations at all endpoints of the given contours, multiplied by the corresponding coefficient $\varkappa$ and the appropriate sign according to the orientation. The matrices D and $\tilde{D}$ are lower triangular and on the main diagonal they can be explicitly computed

$$xp_n' = np_n + \text{previous}, \tag{6.8}$$

$$xr_n' = ((-n)(1-\dot{\ell}_{n+1}) + \ell_{n+1})r_n + \text{previous}, \tag{6.9}$$

$$x\frac{\mathrm{d}}{\mathrm{d}x}r_n^\star = n\dot{\ell}_{n+1}r_n^\star + \text{previous}. \tag{6.10}$$

Formula (6.8) follows from the fact that the degree of p_n is n; formula (6.9) instead follows from the fact that the sequence of Laurent polynomials r_n (by construction in (3.23)) is such that

$$r_n(x) = Ax^{\ell_{n+1}} + \cdots + Bx^{\ell_{n+1}-n}$$

(here A, B are some constants irrelevant for the discussion) and, hence, if $\dot{\ell}_{n+1} = \ell_{n+1} - \ell_n$ is 1, then r_n contains the same monomials as r_{n-1} except for the top power, while if $\dot{\ell}_{n+1} = 0$, then r_n has only the monomial with the lowest power in addition to the monomials appearing in r_{n-1} (and its predecessors). Formula (6.10) follows simply from (6.9) and the definition of $r_n^\star = x^{n-\ell_{n+1}} r_n$ (3.24).

Collecting (6.6), (6.8), (6.9), (6.10) we obtain the following Virasoro scaling constraint:

$$(QV'(Q))_{nn} + (zp_n\, r_n\, \mathrm{e}^{-V})|_{\partial\varkappa} = 1 + \ell_{n+1} + n\dot{\ell}_{n+1}. \tag{6.11}$$

Note that we also have

$$\sum_{k=0}^{n-1}((QV'(Q))_{kk} + (zp_k\, r_k\, \mathrm{e}^{-V})|_{\partial\varkappa}) = \sum_{k=0}^{n-1}(1 + \ell_{k+1} + k\dot{\ell}_{k+1}) = n(\ell_n + 1). \tag{6.12}$$

The parts of D, $\widetilde{D}$ below the main diagonal are now expressed in terms of Q and the boundary terms only,

$$D_< = (QV'(Q))_< + (z(\mathbf{p}(z)\mathbf{r}^t(z))_< \mathrm{e}^{-V(z)})|_{\partial\varkappa}, \tag{6.13}$$

$$\widetilde{D}_< = (Q^tV'(Q^t))_< + (z(\mathbf{r}(z)\mathbf{p}^t(z))_< \mathrm{e}^{-V(z)})|_{\partial\varkappa}. \tag{6.14}$$

Note that, below the main diagonal, the matrices D and $\widetilde{D}$ are of the same form as the deformations we were considering previously; more precisely, they correspond to a variation by $F(z) = zV'(z)$ and a linear combination of variations supported at the endpoints of the contours Γ_j. The folded version of this ordinary differential equation (ODE) can be obtained from the formulæ (5.9), (5.14), (5.21), (5.23) with the only modification that comes from the diagonal part of D. Using (6.11) for the diagonal part the reader can check that the result is

$$\begin{aligned} \mathcal{D}_n &= \begin{bmatrix} n & 0 \\ 0 & xV'(x) - 1 - \ell_n \end{bmatrix} + \gamma_n \begin{bmatrix} -W_{n,n-1} & W_{n,n^\star} \\ -W_{(n-1)^\star,n-1} & W_{n,n-1} \end{bmatrix} \\ &\quad + \left(\frac{z\,\mathrm{e}^{-V(z)}\gamma_n}{x-z} \begin{bmatrix} -p_n r_{n-1} & p_n p_n^\star \\ -r_{n-1} r_{n-1}^\star & r_{n-1} p_n \end{bmatrix} \right)\Bigg|_{\partial\varkappa}, \\ W &:= \nabla_Q x V'(x) = \frac{QV'(Q) - xV'(x)}{Q - x}. \end{aligned} \tag{6.15}$$

We remark that the last "boundary" term (the term indicated as the evaluation $|_{\partial\varkappa}$) consists of simple poles with nilpotent residues located at the **hard-edges**.

For the full matrix χ_n the differential equation is

$$x\partial_x \chi_n(x) = \mathcal{D}_n(x)\chi_n(x) + \chi_n(x)\mathcal{D}^R(x), \tag{6.16}$$

$$\mathcal{D}^R(x) = \begin{bmatrix} 0 & \displaystyle\int_\varkappa \frac{xV'(x) - zV'(z)}{x-z}\,\mathrm{e}^{-V(z)}\,\mathrm{d}z + \frac{z\,\mathrm{e}^{-V(z)}}{z-x}\Bigg|_{\partial\varkappa} \\ 0 & -xV'(x) \end{bmatrix}. \tag{6.17}$$

Together with the differential equation and the deformation equations we recall that we also have difference equations

$$\chi_n = R_n(x)\chi_{n-1}, \qquad n \geq 1, \tag{6.18}$$

$$R_n(x) = \begin{cases} \begin{bmatrix} (x-\beta_n)/\gamma_n & \kappa_n \\ (-1)^{n+1} & 0 \end{bmatrix} & \text{if } \dot{\ell}_n = 1, \\ \begin{bmatrix} (x-\beta_n)/\gamma_n & \kappa_n \\ (-1)^{n+1} & \omega_n \end{bmatrix} & \text{if } \dot{\ell}_n = 0. \end{cases} \tag{6.19}$$

The ladder matrices R_n are simply obtained from the transfer matrices (3.2) by using the normalization of the polynomials as in (3.23). We have thus proved

Theorem 6.1. *The matrix χ_n satisfies the following system of difference-deformation-differential* (DDD *for short*) *equations*

$$\chi_n = R_n(x)\chi_{n-1}, \tag{6.20}$$

$$x\frac{\mathrm{d}}{\mathrm{d}x}\chi_n = \mathcal{D}_n\chi_n + \chi_n\mathcal{D}^R, \tag{6.21}$$

$$\delta_f\chi_n = \mathcal{U}_n^{(f)}\chi_n + \chi_n\mathcal{U}^{(f),R}, \tag{6.22}$$

where f denotes either any function or formal power series provided that $\mathcal{L}(f(z)z^k)$ is well defined for $k \in \mathbb{Z}$ or any derivative of the Dirac delta function supported at any point $a \neq 0$.

We observe that the right action of the differential-deformation equation is independent of n. This suggests that we can perform a "right gauge" change to dispose of this part. Indeed, we define the new object Γ_n which will be the focus in the rest of the paper,

$$\Gamma_n := \chi_n \begin{bmatrix} 1 & -\mathrm{e}^{V(x)}\displaystyle\int_{\varkappa}\frac{\mathrm{e}^{-V(z)}}{x-z}\,\mathrm{d}z \\ 0 & \mathrm{e}^{V(x)} \end{bmatrix}. \tag{6.23}$$

It is easy to verify that this change of gauge eliminates the right-actions for the differential equation and for any deformation of $V(x)$ and/or the endpoints of integration. The first column of Γ_n is the same as the first column of χ_n and hence contains the LOPs. The second column now contains the following **auxiliary functions**

$$\psi_n = \mathrm{e}^{V(x)}\int_{\varkappa}\frac{p_n(z)\,\mathrm{e}^{-V(z)}}{x-z}\,\mathrm{d}z, \tag{6.24}$$

$$\varphi^{\star}_{n-1} = x^{n-1-\ell_n}\varphi_{n-1} = x^{n-1-\ell_n}\,\mathrm{e}^{V(x)}\int_{\varkappa}\frac{r_{n-1}(z)\,\mathrm{e}^{-V(z)}}{x-z}\,\mathrm{d}z. \tag{6.25}$$

We note that the auxiliary functions are piecewise analytic functions off the contours Γ_j: it is a matter of routine inspection to read-off the relevant Riemann–Hilbert data. We defer this inspection to a later section.

In terms of the matrices Γ_n we have a DDD system of more standard form, without right multipliers.

Theorem 6.2. *The following system of difference-differential-deformation equations is Frobenius compatible*

$$\Gamma_n = R_n(x)\Gamma_{n-1}, \tag{6.26}$$

$$x\frac{\mathrm{d}}{\mathrm{d}x}\Gamma_n = \mathcal{D}_n\Gamma_n, \tag{6.27}$$

$$\delta_f\Gamma_n = \mathcal{U}_n^{(f)}\Gamma_n, \tag{6.28}$$

where f is as in Theorem 6.1.

A few remarks are in order here: by choosing f in Theorem 6.2 to be an ordinary function one can vary the potential V by $V \to V + \varepsilon f$ and hence all flows of the generalized Toda hierarchy are here included. However, we can also choose f as a distribution $\delta_a^{(k)}$ or linear combinations thereof. Clearly, if we choose the point a arbitrarily outside the singularities of $V(x)$ we still have a compatibility of the resulting system but we will change the structure of the singularities of $\mathcal{D}_n$, which falls outside the standard theory of isomonodromic deformations. For example, adding a δ_a corresponds to adding a term $\ln(x-a)$ in the potential **and** adjoining a small circle around a to the set of contours Γ_j's.

Vice versa the cases in which f is a distribution which does not alter the singularity structure of $\mathcal{D}_n$ are:

1. Movement of the endpoints which **contribute** to the boundary term:[2] then we have

$$f = \pm\varkappa\,\mathrm{e}^{-V(a)}\delta_a, \tag{6.29}$$

   where the coefficient $\varkappa$ is the coefficient of the contour Γ_j which has a as endpoint and the sign depends on the orientation of Γ_j.
2. Movements of **poles** of order k (if any) of the pseudo-measure $\mathrm{e}^{-V}\,\mathrm{d}z$: then we have

$$f(a) = \pm\varkappa k\delta_a^{(k+1)}(z)\,\mathrm{e}^{-V_r(z)}, \tag{6.30}$$

   where the coefficient $\varkappa$ is the coefficient of the loop encircling a, the sign is chosen according to the orientation of the contour and $V_r(z)$ is the part of V which is regular at $z = a$.

7. Spectral Curve

The goal of this section is to represent the spectral curve of the connection $\partial_x - (1/x)\mathcal{D}_n(x)$ in terms of the logarithmic derivatives of the shifted Töplitz determinants $\Delta_n^{\ell_n}$; this will be the essential bridge to connect with the isomonodromic tau function in the coming sections. This whole section is devoted to the proof of Theorem 7.1.

[2] They correspond to those endpoints of the contours Γ_j for which $\lim_{\Gamma_j \ni z \to \partial\Gamma_j} \mathrm{e}^{-V(z)} \neq 0$.

Theorem 7.1. *The following formula holds*:

$$\det\left(y\mathbf{1} - \frac{1}{x}\mathcal{D}_n(x)\right) = y^2 - \left(V'(x) + \frac{L_n}{x}\right)y + \frac{1}{x}\operatorname{Tr}_n\left(\frac{QV'(Q) - xV'(x)}{Q - x}\right) + \frac{1}{x}\left(\frac{z\,\mathrm{e}^{-V(z)}\mathbf{p}^t\Pi_{n-1}\mathbf{r}}{x - z}\right)\bigg|_{\partial\varkappa}, \tag{7.1}$$

$$L_n := n - 1 - \ell_n, \tag{7.2}$$

where $\Pi_{n-1} = \operatorname{diag}(1, 1, \ldots, 1, 0, \ldots)$ (*n nonzero entries*).

Before proceeding to the proof we make two remarks: this formula would be valid for an arbitrary *smooth* potential; quite clearly, however, in this case the spectral curve would not be an algebraic curve. The second remark is quite crucial to understand the relation of this formula with the logarithmic derivatives of the shifted Töplitz determinants and, later on, with the isomonodromic tau function, and thus deserves a small digression.

Remark 7.1. The coefficients of the spectral curve in Theorem 7.1 contain expressions of the form $\operatorname{Tr}_n(F(Q))$ (for some function $F(z)$): we recall that these expressions characterize the variations of the logarithm of the shifted Töplitz determinants $\Delta_n^{\ell_n}$ (the **tau functions** of the integrable lattice) as in (4.12). To simplify the matter let us consider the simplest case in which the potential $V(x) = \sum_{K=1}^{d}(t_K/K)x^K$ is a polynomial (or formally a series) and the coefficients t_K are the usual Toda times; in this case, the relevant expression in Theorem 7.1 is only (we take the simplest case without any hard-edge)

$$\operatorname{Tr}_n\left(\frac{QV'(Q) - xV'(x)}{Q - x}\right) = \sum_{K=1}^{d} t_K \operatorname{Tr}_n\left(\frac{Q^K - x^K}{Q - x}\right). \tag{7.3}$$

Taking the coefficients of the powers of x yields expressions (identifiable with Virasoro vector fields [4]) that contain truncated traces of the form

$$\operatorname{Tr}_n(Q^J) = J\partial_{t_J}\ln\Delta_n^{\ell_n}, \tag{7.4}$$

where the identity is the simplest case of (4.12) and gives the most common realization of the tau function recalled in the Introduction ((1.6)).[3] It should be clear that in our case of semiclassical moment functionals we have several different types of "times" corresponding to the coefficients of the partial fraction expansion of $V'(z)$ (which is a rational function) and to the position of the hard-edges: the derivatives of $\ln\Delta_n^{\ell_n}$ with respect to all these "generalized" Toda times are governed by (4.12) using for F the corresponding variation of the potential or a δ supported at the hard-edge.

[3] In (1.6) there are as many times as the size of the matrix n: in our setting here the matrices are infinite dimensional and the times would have to be reduced to a submanifold to give nonformal equations.

Proof of Theorem 7.1. We need to compute the two spectral invariants of the connection; the main tool is to use the compatibility between the ladder relations and the connections $\mathcal{D}_n(x)$. Indeed, from the compatibility between the difference-differential equations and from the explicit expression for $\mathcal{D}_n(x)$ (6.15) we can express a recurrence relation for the spectral invariants of $\mathcal{D}_n(x)$. The trace is computed by sight,

$$\mathrm{Tr}(\mathcal{D}_n(x)) \;=\; x V'(x) + n - 1 - \ell_n. \tag{7.5}$$

From the compatibility of difference-differential equations we have the gauge property

$$\mathcal{D}_{n-1} \;=\; {R_n}^{-1}\mathcal{D}_n R_n - x {R_n}^{-1} R_n'. \tag{7.6}$$

The gauge term is explicitly computed to be

$${R_n}^{-1}\mathcal{D}_n R_n \;=\; \mathcal{D}_{n-1} + x {R_n}^{-1} R_n', \tag{7.7}$$

$$x {R_n}^{-1} R_n' \;=\; \begin{cases} (-1)^n \dfrac{x}{\gamma_{n-1}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \text{if } \dot{\ell}_n = 1, \\[2ex] \begin{bmatrix} 1 & 0 \\ \dfrac{(-1)^n \Delta_{n-1}^{\ell_n - 1}}{\Delta_n^{\ell_n}\sqrt{\Delta_{n-2}^{\ell_{n-2}}}} & 0 \end{bmatrix} & \text{if } \dot{\ell}_n = 0. \end{cases} \tag{7.8}$$

These formulæ imply a recurrence relation for the quadratic invariant

$$\mathrm{Tr}(\mathcal{D}_n^2)) \;=\; \mathrm{Tr}(\mathcal{D}_{n-1}^2) + 2\,\mathrm{Tr}(\mathcal{D}_{n-1} x {R_n}^{-1} R_n') + \mathrm{Tr}((x {R_n}^{-1} R_n')^2). \tag{7.9}$$

For the *line case*, i.e. $\dot{\ell}_n = 1$ and using the form of the recursion matrices R_n together with the fact that in this case $r_{n-1}^\star = (-1)^{n-1} p_{n-1}$, we find

$$\begin{aligned} \mathrm{Tr}(\mathcal{D}_n^2) \;&=\; \mathrm{Tr}(\mathcal{D}_{n-1}^2) - 2x \left(\frac{Q V'(Q) - x V'(x)}{Q - x} \right)_{n-1,n-1} \\ &\quad - 2x \left(\frac{z\, \mathrm{e}^{-V(z) p_{n-1} r_{n-1}}}{x - z} \right)\Bigg|_{\partial\varkappa}. \end{aligned} \tag{7.10}$$

For the circle case $\dot{\ell}_n = 0$ instead we have

$$\begin{aligned} \mathrm{Tr}(\mathcal{D}_n^2) &= \mathrm{Tr}(\mathcal{D}_{n-1}^2) + 2(n-1) \\ &\quad + 2\gamma_{n-1} \left(-W_{n-1,n-2} + \frac{(-1)^n \Delta_{n-1}^{\ell_n - 1}}{\Delta_n^{\ell_n}\sqrt{\Delta_{n-2}^{\ell_{n-2}}}} W_{n-1,(n-1)^\star} \right) \\ &\quad + 2\gamma_{n-1} \left(\frac{z\, \mathrm{e}^{-V(z)} p_{n-1}}{x - z} \left(-r_{n-2} + \frac{(-1)^n \Delta_{n-1}^{\ell_n - 1}}{\Delta_n^{\ell_n}\sqrt{\Delta_{n-2}^{\ell_{n-2}}}} p_{n-1}^\star \right) \right)\Bigg|_{\partial\varkappa} + 1. \end{aligned} \tag{7.11}$$

Using the identity (2.7) together with the definitions of the biorthogonal polynomials and the various normalization factors (3.23) one can see that

$$-r_{n-2} + \frac{(-1)^n \Delta_{n-1}^{\ell_n - 1}}{\Delta_n^{\ell_n} \sqrt{\Delta_{n-2}^{\ell_{n-2}}}} p_{n-1}^\star = -\frac{z}{\gamma_{n-1}} r_{n-1}, \tag{7.12}$$

and hence,

$$-W_{n-1,n-2} + \frac{(-1)^n \Delta_{n-1}^{\ell_n - 1}}{\Delta_n^{\ell_n} \sqrt{\Delta_{n-2}^{\ell_{n-2}}}} W_{n-1,(n-1)^\star} = \frac{1}{\gamma_{n-1}} \mathcal{L}_z(zWp_{n-1}r_{n-1}). \tag{7.13}$$

Therefore the recursion for the circle case is

$$\begin{aligned} \operatorname{Tr}(\mathcal{D}_n^2) - \operatorname{Tr}(\mathcal{D}_{n-1}^2) \; &= \; 2(n-1) + 1 - 2\left(\frac{Q\left(QV'(Q) - xV'(x)\right)}{Q - x}\right)_{n-1,n-1} \\ &\quad - 2\left(\frac{z^2\,\mathrm{e}^{-V(z)} p_{n-1} r_{n-1}}{x - z}\right)\Bigg|_{\partial\varkappa} \\ &= \; 2(n-1) + 1 - 2(QV'(Q)_{n-1,n-1} \\ &\quad + (z\,\mathrm{e}^{-V(z)} p_{n-1} r_{n-1})|_{\partial\varkappa}) + 2xV'(x) \\ &\quad - 2x\left(\frac{QV'(Q) - xV'(x)}{Q - x}\right)_{n-1,n-1} \\ &\quad - 2x\left(\frac{z\,\mathrm{e}^{-V(z)} p_{n-1} r_{n-1}}{x - z}\right)\Bigg|_{\partial\varkappa} \\ &= \; 2(n-1) - 1 - 2\ell_n + 2xV'(x) \\ &\quad - 2x\left(\frac{QV'(Q) - xV'(x)}{Q - x}\right)_{n-1,n-1} \\ &\quad - 2x\left(\frac{z\,\mathrm{e}^{-V(z)} p_{n-1} r_{n-1}}{x - z}\right)\Bigg|_{\partial\varkappa}. \end{aligned} \tag{7.14}$$

Summarizing, in the two cases we have found

$$\begin{aligned} \operatorname{Tr}(\mathcal{D}_n^2) - \operatorname{Tr}(\mathcal{D}_{n-1}^2) \; &= \; 2(xV'(x) - \ell_n + (n-1) - \tfrac{1}{2})(1 - \dot{\ell}_n) \\ &\quad - 2x\left(\frac{QV'(Q) - xV'(x)}{Q - x}\right)_{n-1,n-1} \\ &\quad - 2x\left(\frac{z\,\mathrm{e}^{-V(z)} p_{n-1} r_{n-1}}{x - z}\right)\Bigg|_{\partial\varkappa}. \end{aligned} \tag{7.15}$$

To complete the computation we need to find $\mathrm{Tr}({\mathcal{D}_1}^2)$ or, equivalently, $\det(\mathcal{D}_1)$. We have

$$\det\left(\frac{1}{x}D_1\right) = \det\begin{bmatrix} p_1' & \psi_1' \\ r_0^{\star\prime} & \varphi_0^{\star\prime}\end{bmatrix}\begin{bmatrix} p_1 & \psi_1 \\ r_0^{\star} & \varphi_0^{\star}\end{bmatrix}^{-1} = \det\begin{bmatrix} p_1' & \psi_1' \\ r_0^{\star\prime} & \varphi_0^{\star\prime}\end{bmatrix}\mathrm{e}^{-V(x)} \tag{7.16}$$

$$= \sqrt{h_1}\,\mathrm{e}^{-V(x)}\det\begin{bmatrix} 1/\sqrt{h_1} & \psi_1' \\ 0 & \varphi_0^{\star\prime}\end{bmatrix}$$

$$= V'(x)\mathcal{L}_z\left(\frac{1}{x-z}\right) - \mathcal{L}_z\left(\frac{V'(z)}{x-z}\right) + \left(\frac{\mathrm{e}^{-V(z)}}{x-z}\right)\bigg|_{\partial\varkappa}$$

$$= \left(\frac{V'(Q)-V'(x)}{Q-x}\right)_{00} + \left(\frac{\mathrm{e}^{-V(z)}p_0r_0}{x-z}\right)\bigg|_{\partial\varkappa}.$$

This implies

$$\det\mathcal{D}_1(x) = x^2\left(\frac{V'(Q)-V'(x)}{Q-x}\right)_{00} + x^2\left(\frac{\mathrm{e}^{-V(z)}p_0r_0}{x-z}\right)\bigg|_{\partial\varkappa} \tag{7.17}$$

$$= x\overbrace{\left(V'(Q)_{00} + (p_0r_0\,\mathrm{e}^{-V(z)})|_{\partial\varkappa}\right)}^{=0} + x\left(\frac{QV'(Q)-xV'(x)}{Q-x}\right)_{00}$$

$$+\,x\left(\frac{z\,\mathrm{e}^{-V(z)}p_0r_0}{x-z}\right)\bigg|_{\partial\varkappa}$$

$$= x\left(\frac{QV'(Q)-xV'(x)}{Q-x}\right)_{00} + x\left(\frac{z\,\mathrm{e}^{-V(z)}p_0r_0}{x-z}\right)\bigg|_{\partial\varkappa}.$$

Hence ($\ell_1 = 0$),

$$\mathrm{Tr}({\mathcal{D}_1}^2) = (xV'(x))^2 - 2x\left(\frac{QV'(Q)-xV'(x)}{Q-x}\right)_{00} \tag{7.18}$$

$$-\,2x\left(\frac{z\,\mathrm{e}^{-V(z)}p_0r_0}{x-z}\right)\bigg|_{\partial\varkappa}$$

$$\mathrm{Tr}({\mathcal{D}_n}^2) = (xV'(x))^2 + 2xV'(x)\,(n-1-\ell_n) - (n-1-\ell_n) \tag{7.19}$$

$$+\,2\sum_{k=1}^{n}(k-1-\ell_k)(1-\dot{\ell}_k)$$

$$-\,2x\,\mathrm{Tr}_n\left(\frac{QV'(Q)-xV'(x)}{Q-x}\right)$$

$$-\,2x\left(\frac{z\,\mathrm{e}^{-V(z)}\mathbf{p}^t\Pi_{N-1}\mathbf{r}}{x-z}\right)\bigg|_{\partial\varkappa}.$$

Using this expression for the quadratic invariant we can obtain the following formula for the characteristic polynomial

$$
\begin{aligned}
(7.20)\qquad \det(\tilde{y}\mathbf{1} - \mathcal{D}_n(x)) \;&=\; \tilde{y}^2 - (xV'(x) + n - 1 - \ell_n)\tilde{y} + K_n \\
&\quad + x\,\mathrm{Tr}_n\left(\frac{QV'(Q) - xV'(x)}{Q - x}\right) \\
&\quad + x\left(\frac{z\,\mathrm{e}^{-V(z)}\mathbf{p}^t\Pi_{N-1}\mathbf{r}}{x - z}\right)\Bigg|_{\partial\varkappa}, \\
K_n \;&:=\; \frac{(n-1-\ell_n)(n-\ell_n)}{2} + \sum_{k=2}^{n}(\ell_k + 1 - k)(1 - \dot{\ell}_k).
\end{aligned}
$$

The last crucial observation is that $K_n \equiv 0$ for all n: this is nonobvious at first sight and it is true only because ℓ_n is a weakly increasing sequence of *integers*. Indeed, one can check that

$$
(7.21)\qquad K_{n+1} - K_n = \tfrac{1}{2}\dot{\ell}_{n+1}(1 - \dot{\ell}_{n+1}),
$$

so that $K_{n+1} = K_n = K_1 = 0$. To conclude the proof we note that the spectral curve of (7.20) is simply related to that of the connection by $\tilde{y} = xy$. This ends our proof. ■

8. Isomonodromic Deformations

By Theorem 6.2 we have compatible systems of DDD equations

$$
(8.1)\qquad \Gamma_n = R_n\Gamma_{n-1},
$$

$$
(8.2)\qquad \partial_x\Gamma_n = \frac{1}{x}\mathcal{D}_n\Gamma_n,
$$

$$
(8.3)\qquad \delta_f\Gamma_n = \mathcal{U}_n^{(f)}(x)\Gamma_n.
$$

The compatibility of this system entails **isomonodromic deformations** [14] for the connection $\partial_x - (1/x)\mathcal{D}_n$. Note that this connection has the same singularity structure of $V'(x)$. In order to have isomonodromic deformations in the sense of Jimbo, Miwa and Ueno (JMU) we need to impose that $V'(x)$ is a rational function. Then the deformations of $V(x)$ which give rise to the setting in JMU are those which do not alter the singularity structure of $V(x)$; this is why the most general setting compatible with this requirement is that of semiclassical moment functionals.

8.1. *Spectral Residue-Formulæ*

The logarithmic derivatives of the shifted Töplitz determinants $\Delta_n^{\ell_n}$ with respect to the generalized Toda times[4] can be obtained in terms of residue-formulæ involving the differential $y\,\mathrm{d}x$ on the spectral curve defined in Theorem 7.1. The same logic was used

[4] These are the coefficients of the partial fraction expansion of $V'(z)$, the location of the poles and the locations of the hard-edges.

in [4] in the context of orthogonal polynomials. Solving the equation of the eigenvalues of $\mathcal{D}_n(x)$ from Theorem 7.1 we find that $y = Y_\pm(x)$ where

$$Y_\pm(x) := \frac{1}{2}\left(V'(x) + \frac{L_n}{x}\right) \pm \frac{1}{2}\sqrt{\left(V'(x) + \frac{L_n}{x}\right)^2 - 4\mathcal{P}(x)}, \tag{8.4}$$

$$\mathcal{P}(x) := \frac{1}{x}\,\mathrm{Tr}_n\left(\frac{QV'(Q) - xV'(x)}{Q - x}\right) + \frac{1}{x}\left(\frac{z\,\mathrm{e}^{-V(z)}\mathbf{p}^t\Pi_{N-1}\mathbf{r}}{x - z}\right)\bigg|_{\partial\varkappa}. \tag{8.5}$$

Using these we can state the main theorem of this section.

Theorem 8.1. *Let $V'(x)$ be rational.*

(i) *Suppose that $x = c$ is a pole of order $d + 1$,*

$$V(x) = \sum_{J=1}^{d} \frac{t_J^{(c)}}{J(x-c)^J} - t_0^{(c)} \ln(x - c) + \mathcal{O}(1), \tag{8.6}$$

$$V'(x) = -\sum_{J=0}^{d} \frac{t_J^{(c)}}{(x-c)^{J+1}} + \mathcal{O}(1). \tag{8.7}$$

Then we have

$$t_J^{(c)} = -\mathop{\mathrm{res}}_{x=c} Y_+(x)(x-c)^J\,\mathrm{d}x, \qquad J = 0, \ldots, d, \tag{8.8}$$

$$\frac{\partial \ln \Delta_n^{\ell_n}}{\partial t_J^{(c)}} = \frac{1}{J}\mathop{\mathrm{res}}_{x=c} Y_-(x)(x-c)^{-J}\,\mathrm{d}x, \qquad J = 1, \ldots, d, \tag{8.9}$$

$$\frac{\partial \ln \Delta_n^{\ell_n}}{\partial c} = \mathop{\mathrm{res}}_{x=c} Y_-(x)\left(\sum_{J=0}^{d} \frac{t_J^{(c)}}{(x-c)^{J+1}}\right)\mathrm{d}x. \tag{8.10}$$

(ii) *Suppose that $x = \infty$ is a pole of V' with degree d, namely,*

$$V(x) = \sum_{J=1}^{d+1} \frac{t_J^{(\infty)}}{J} x^J + \mathcal{O}(\ln x), \tag{8.11}$$

$$V'(x) = \sum_{J=1}^{d+1} t_J^{(\infty)} x^{J-1} + \mathcal{O}(1/x). \tag{8.12}$$

Then we have

$$t_J^{(\infty)} = -\mathop{\mathrm{res}}_{x=\infty} Y_+(x)x^{-J}\,\mathrm{d}x, \qquad J = 1, \ldots d + 1, \tag{8.13}$$

$$\frac{\partial \ln \Delta_n^{\ell_n}}{\partial t_J^{(\infty)}} = \frac{1}{J}\mathop{\mathrm{res}}_{x=\infty} x^J Y_-(x)\,\mathrm{d}x, \qquad J = 1, \ldots, d + 1. \tag{8.14}$$

(iii) *Let* $x = a$ *be a* **hard-edge**,[5] *namely, a point of the boundary of one of the contours* $\{\Gamma_j\}$ *such that* $|V(a)| < \infty$. *Then*

$$\frac{\partial \ln \Delta_n^{\ell_n}}{\partial a} = \frac{1}{2} \operatorname*{res}_{x=a} \frac{1}{x^2} \operatorname{Tr}(\mathcal{D}_n)^2 \, \mathrm{d}x. \tag{8.15}$$

(iv) *Finally, we have*

$$\begin{aligned} \operatorname*{res}_{x=0} Y_+(x)\,\mathrm{d}x &= L_n = n - 1 - \ell_n - \sum_{c = \text{finite pole of } V'} t_0^{(c)}, \\ \operatorname*{res}_{x=\infty} Y_+(x)\,\mathrm{d}x &= \ell_n + 1 + t_0^{(\infty)}. \end{aligned} \tag{8.16}$$

Proof. We start by noticing that

$$\begin{aligned} Y_\pm &= \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \big(V'(x) + L_n/x\big) \mp \frac{\mathcal{P}(x)}{V'(x) + L_n/x} \\ &\quad + \begin{cases} \mathcal{O}((x-c)^{d+1}) & \text{for case (i),} \\ \mp \dfrac{n^2}{t_{d+1}^{(\infty)} x^{d+1}} + \mathcal{O}(x^{-d-2}) & \text{for case (ii).} \end{cases} \end{aligned} \tag{8.17}$$

At this point, formulæ (8.8), (8.13), (8.16) follow immediately by noticing that $\mathcal{P}/(V'(x)+L_n/x) = \mathcal{O}(1)$ in all cases and by straightforward computation of residues.[6] As for the remaining formulæ we have, for case (i),

$$\begin{aligned} \operatorname*{res}_{x=c}(x-c)^{-J} Y_-(x)\,\mathrm{d}x &= \operatorname*{res}_{x=c}(x-c)^{-J} \frac{\mathcal{P}(x)}{V'(x) + L_n/x} \\ &= \operatorname*{res}_{x=c} \frac{(x-c)^{-J}}{xV'(x)+L_n} \operatorname{Tr}_n \left(\frac{xV'(x) - QV'(Q)}{x - Q} \right) \\ &= \sum_{k=0}^{n-1} \mathcal{L}_z \left[\operatorname*{res}_{x=c} \frac{(x-c)^{-J}}{xV'(x)+L_n} \frac{xV'(x) - zV'(z)}{x - z} p_n(z) r_n(z) \right] \\ &= -\operatorname{Tr}_n(Q-c)^{-J} = J \partial_{t_J^{(c)}} \ln \Delta_n^{\ell_n}, \qquad J = 1, \ldots, d, \end{aligned} \tag{8.19}$$

and similar computation for the c-derivative. Here we have used the formulæ (4.12) expressing the variation of $\ln \Delta_n^{\ell_n}$ under an infinitesimal deformation of the type ensuing from an infinitesimal change of the parameters $t_J^{(c)}$.

---

[5] This means that this is one of the points contributing to the boundary terms.

[6] Note that at infinity

$$\frac{\mathcal{P}(x)}{V'(x) + L_n/x} = \frac{n}{x} + \mathcal{O}(x^{-2}). \tag{8.18}$$

For case (ii) the computation is completely parallel except for the last $J = d + 1$ residue. Indeed,

$$
\begin{aligned}
(8.20)\qquad \operatorname*{res}_{x=\infty} x^J Y_-(x)\,\mathrm{d}x &= \operatorname*{res}_{x=\infty} x^J \frac{\mathcal{P}(x)}{V'(x) + L_n/x} \\
&= \operatorname*{res}_{x=\infty} \frac{x^J}{xV'(x) + L_n} \operatorname{Tr}_n \left(\frac{xV'(x) - QV'(Q)}{x - Q} \right) \\
&= \sum_{k=0}^{n-1} \mathcal{L}_z \left[\operatorname*{res}_{x=\infty} \frac{x^J}{xV'(x) + L_n} \frac{xV'(x) - zV'(z)}{x - z} p_n(z) r_n(z) \right] \\
&= -\operatorname{Tr}_n Q^J = J \partial_{t_J^{(\infty)}} \ln \Delta_n^{\ell_n}, \qquad J = 1, \dots, d.
\end{aligned}
$$

For $J = d+1$ one has to use a similar manipulation but has to use the refined asymptotics (8.17): indeed, we have

$$
\begin{aligned}
(8.21)\qquad \operatorname*{res}_{x=\infty} xV'(x) Y_-(x)\,\mathrm{d}x &= -\operatorname{Tr}_n QV'(Q) + (n^2 - nL_n - \operatorname{Tr}_n QV'(Q) \\
&\qquad - z\,\mathrm{e}^{-V(z)} \mathbf{p}^t \Pi_{n-1} \mathbf{r}|_{\partial \varkappa}) \\
&= -\operatorname{Tr}_n QV'(Q),
\end{aligned}
$$

where we have used (6.12) together with the definition of $L_n = n - 1 - \ell_n$. This proves, together with the residues (8.20),

$$
(8.22)\qquad \operatorname*{res}_{x=\infty} x^{d+1} Y_-(x)\,\mathrm{d}x = -\operatorname{Tr}_n Q^{d+1} = (d+1) \partial_{t_{d+1}^{(\infty)}} \ln \Delta_n^{\ell_n}.
$$

Finally, for case (iii), the computation is immediate using the formula for $\operatorname{Tr} \mathcal{D}_n^2$ (7.19). ■

9. Riemann–Hilbert Problem

Not only does the matrix $\Gamma_n(x)$ solve a set of compatible PDEs and ODEs, but it is also the solution of a Riemann–Hilbert problem (RHP); in the context of orthogonal polynomials, such an RHP is the door to the asymptotic analysis for $n \to \infty$ (and rescaling of the potential $V \to nV$) of the orthogonal polynomials (see, e.g., [9]); for reference in a future work in this direction we want to specify in some detail the relevant RHP, although such details are not necessary for the main goals of the present paper.

Direct inspection of the asymptotic behaviour near the singularities of the biorthogonal polynomials and second-kind functions allows us to ascertain the Riemann–Hilbert data.

We start by noticing the following formal asymptotic behaviour of the auxiliary functions entering in Γ_n,

$$\psi_n = e^{V(x)}\int_{\varkappa}\frac{e^{-V(z)}p_n(z)}{x-z} \tag{9.1}$$

$$= \begin{cases} (-)^n x^{-\ell_n-2}\, e^{V(x)}\sqrt{\dfrac{\Delta_{n+1}^{\ell_{n+1}}}{\Delta_n^{\ell_n}}}(1+\mathcal{O}(x^{-1})) & \text{for } x\to\infty,\\ -x^{n-1-\ell_n}\, e^{V(x)}\dfrac{\Delta_{n+1}^{\ell_n}}{\sqrt{\Delta_n^{\ell_n}\Delta_{n+1}^{\ell_{n+1}}}}(1+\mathcal{O}(x)) & \text{for } x\to 0,\\ e^{V(x)}\sqrt{h_0}(Q-c)^{-1}_{n0} & \text{near poles of } V'(x), \end{cases}$$

$$\varphi^{\star}_{n-1} = x^{n-1-\ell_n}e^{V(x)}\int_{\varkappa}\frac{e^{-V(z)}r_{n-1}(z)}{x-z} \tag{9.2}$$

$$= \begin{cases} x^{-\ell_n-1}\, e^{V(x)}\sqrt{\dfrac{\Delta_n^{\ell_n}}{\Delta_{n-1}^{\ell_{n-1}}}}(1+\mathcal{O}(x^{-1})) & \text{for } x\to\infty,\\ x^{n-1-\ell_n}\, e^{V(x)}\dfrac{(-)^n\Delta_n^{\ell_n-1}}{\sqrt{\Delta_n^{\ell_n}\Delta_{n-1}^{\ell_{n-1}}}}(1+\mathcal{O}(x)) & \text{for } x\to 0,\\ e^{V(x)}\sqrt{h_0}c^{n-\ell_n+1}(Q-c)^{-1}_{0,n-1} & \text{near poles of } V'(x), \end{cases}$$

where we have used the definition of the LOPs (3.23) and the facts that

$$p_n \propto \wp_n^{\ell_n} \perp z^{\ell_n-n+1},\dots,z^{\ell_n}, \tag{9.3}$$

$$r_{n-1} \propto z^{\ell_n-n+1}\wp_{n-1}^{\ell_n-1} \perp z^0,\dots,z^{n-2}. \tag{9.4}$$

This implies the following formal asymptotic data for Γ_n near all the singularities.

At $x=0$ we have

$$\Gamma_n(x) \sim G_n^{(0)}\begin{bmatrix}1 & 0\\ 0 & x^{n-1-\ell_n}\, e^{V_{\mathrm{sing},0}(x)}\end{bmatrix}(\mathbf{1}+\mathcal{O}(x)), \tag{9.5}$$

$$G_n^{(0)} := \begin{bmatrix} \dfrac{(-)^n\Delta_n^{\ell_n+1}}{\sqrt{\Delta_n^{\ell_n}\Delta_{n+1}^{\ell_{n+1}}}} & -\dfrac{\Delta_{n+1}^{\ell_n}}{\sqrt{\Delta_n^{\ell_n}\Delta_{n+1}^{\ell_{n+1}}}}\\ \dfrac{\Delta_{n-1}^{\ell_n}}{\sqrt{\Delta_{n-1}^{\ell_{n-1}}\Delta_n^{\ell_n}}} & \dfrac{(-)^n\Delta_n^{\ell_n-1}}{\sqrt{\Delta_{n-1}^{\ell_{n-1}}\Delta_n^{\ell_n}}} \end{bmatrix}, \qquad \det G_n^{(0)} = \frac{1}{\Delta_{n+1}^{\ell_{n+1}}\Delta_{n-1}^{\ell_{n-1}}}. \tag{9.6}$$

At $x=\infty$ we have

$$\Gamma_n(x) = G_n^{(\infty)}\begin{bmatrix}x^n & 0\\ 0 & x^{-\ell_n-1}\, e^{V_{\mathrm{sing},\infty}(x)}\end{bmatrix}\left(\mathbf{1}+\mathcal{O}\left(\frac{1}{x}\right)\right), \tag{9.7}$$

$$G_n^{(\infty)} = \begin{bmatrix} 1/\sqrt{h_n} & 0 \\ 0 & \sqrt{h_{n-1}} \end{bmatrix}. \tag{9.8}$$

Near any other pole $x = c$ of $V'(x)$ we have

$$\Gamma_n(x) = G_n^{(c)} \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{e}^{V_{\mathrm{sing},c}(x)} \end{bmatrix} (\mathbf{1} + \mathcal{O}(x-c)), \tag{9.9}$$

$$G_n^{(c)} := \begin{bmatrix} p_n(c) & \sqrt{h_0}(Q-c)^{-1}_{n0}\,\mathrm{e}^{V_{\mathrm{reg},c}(c)} \\ r^\star_{n-1}(c) & \sqrt{h_0}c^{n-\ell_n+1}(Q-c)^{-1}_{0n-1}\,\mathrm{e}^{V_{\mathrm{reg},c}(c)} \end{bmatrix}, \tag{9.10}$$

where in all these formulæ the notation $V_{\mathrm{sing},p}$ ($V_{\mathrm{reg},p}$) denote the singular (regular) part of V at the point p.

Near a hard-edge point $x = a$ we have [4]

$$\Gamma_n \sim G_n^{(a)} \begin{bmatrix} 1 & \pm\varkappa \ln(x-a) \\ 0 & 1 \end{bmatrix} (\mathbf{1} + \mathcal{O}(x-a)), \tag{9.11}$$

$$G_n^{(a)} := \begin{bmatrix} p_n(a) & \mathrm{e}^{V(a)}\mathcal{L}\left(\dfrac{p_n(z)-p_n(a)}{a-z}\right) \\ r^\star_{n-1}(a) & a^{n-1-\ell_n}\mathrm{e}^{V(a)}\mathcal{L}\left(\dfrac{r_n(z)-r_n(a)}{a-z}\right) \end{bmatrix}. \tag{9.12}$$

Together with these data we also have the jumps across the contours Γ_j defining our moment functional: the situation in this respect is identical to [4]. In essence, the matrix $\Gamma_n(x)$ has the following jumps across the contour Γ_j:

$$\Gamma_n(x)_+ = \Gamma_n(x)_- \begin{bmatrix} 1 & 2i\pi\varkappa_i \\ 0 & 1 \end{bmatrix}. \tag{9.13}$$

Note that these jumps can be interpreted, depending on the point of view, as the Stokes multipliers of the problem near the singularities.

10. Isomonodromic Tau Function

In a seminal paper [14] the Japanese school defined a notion of the "isomonodromic tau function". We recall very briefly its definition and motivation.

Consider a rational covariant derivative operator of rank p over $\mathbb{C}P^1$,

$$\mathcal{D}_x = \partial_x - \mathcal{A}(x), \tag{10.1}$$

where the connection component $\mathcal{A}(x)$ is a $p \times p$ matrix, rational in x. Deformations of such an operator that preserve its (generalized) monodromy (i.e. including the Stokes data) are determined infinitesimally by requiring compatibility of the equations

$$\partial_x \Psi(x) = \mathcal{A}(x)\Psi(x), \tag{10.2}$$

$$\partial_{u_i} \Psi(x) = \mathcal{U}_i(x)\Psi(x), \qquad i = 1, \ldots, \tag{10.3}$$

where in the second set of equations $\mathcal{U}_i(x)$ are also $p \times p$ matrices, rational in x, viewed as components of a connection over the extended space consisting of the product of $\mathbb{C}P^1$ with the space of some deformation parameters $\{u_1, \ldots\}$. The invariance of the generalized monodromy of $\mathcal{D}_x$ follows [14] from the compatibility of this overdetermined system, which is equivalent to the zero-curvature equations

$$[\partial_x - \mathcal{A}(x), \partial_{u_i} - \mathcal{U}_i(x)] = 0, \qquad [\partial_{u_i} - \mathcal{U}_i(x), \partial_{u_j} - \mathcal{U}_j(x)] = 0. \tag{10.4}$$

Near a pole $x = c_\nu$ of $\mathcal{A}(x)$ a fundamental solution can be found that has the *formal* asymptotic behaviour, in a suitable sector:

$$\Psi(x) \sim C_\nu Y_\nu(x)\, \mathrm{e}^{T_\nu(x)}, \tag{10.5}$$

where C_ν is a constant invertible matrix,

$$Y_\nu(x) = \mathbf{1} + \mathcal{O}(x - c_\nu) \tag{10.6}$$

is a formal power series in the local parameter $(x - c_\nu)$ (or $1/x$ for the pole at infinity) and $T_\nu(x)$ is a Laurent-polynomial matrix in the local parameter, plus a possible logarithmic term $t_0 \ln(x - c)$. In the generic case $T_\nu(x)$ is a diagonal matrix; the locations of the poles c_ν and the coefficients of the nonlogarithmic part of $T_\nu(x)$ are the independent deformation parameters. The deformation of the connection matrix $A(x)$ is determined by the requirement that the (generalized) monodromy data be independent of all these isomonodromic deformation parameters.

The tau function has a very important property, in that its vanishing (in the space of "times") determines that the RHP (i.e. the reconstruction of the connection from its (generalized) monodromy) is impossible for those particular times.

Given a solution of such an isomonodromic deformation problem, one is led to consider the associated isomonodromic tau function [14], determined by integrating the following closed differential on the space of deformation parameters:

$$\omega := \sum_\nu \operatorname*{res}_{x=c_\nu} \mathrm{Tr}(Y_\nu^{-1} Y_\nu' \cdot dT_\nu(x)) = d \ln \tau_{\mathrm{JMU}}, \tag{10.7}$$

where the sum is over all poles of $\mathcal{A}(x)$ (including possibly one at $x = \infty$), and the differential is over all the independent isomonodromic deformation parameters.

In the case at hand in this paper, the connection is $(1/x)\mathcal{D}_n(x)$ and the content of Theorem 6.2 (with $f = \delta V(x)$ or a delta-function supported at the hard-edges) guarantees that the generalized monodromy is conserved: in other words, the deformations of our semiclassical moment functional within the same class induce isomonodromic deformations of the rational connection $\partial_x - x^{-1}\mathcal{D}_n(x)$.

In order to compute the associated isomonodromic tau function one would have to specialize formula (10.7) to our setting; this is, however, unnecessary. Indeed in [4] it was shown that (10.7) can be recast as a suitable computation of residues **over the spectral curve** of $\mathcal{D}_n(x)$: more precisely, the relevant result of [4] can be generalized to an arbitrary 2×2 rational connection (thus including our present case) and states that logarithmic derivatives of the JMU isomonodromic tau function [14] are given by the **same differential formulæ in Theorem 8.1** that yield the logarithmic derivatives of

$\Delta_n^{\ell_n}$ (i.e. the tau function of the lattice) **provided** that we substitute the spectral curve of the connection with the spectral curve of the connection in the *traceless gauge*.[7] In our situation the trace of $(1/x)\mathcal{D}_n(x)$ is $V'(x) + (n-1-\ell_n)/x$ so that we perform a scalar gauge transformation that recasts the connection in the form

$$\mathcal{A}_n^{(\mathrm{JMU})} = \frac{1}{x}\mathcal{D}_n(x) - \frac{1}{2}\left(V'(x) + \frac{n-1-\ell_n}{x}\right)\mathbf{1}_{2\times 2}. \tag{10.10}$$

This implies that the eigenvalue y_{JMU} has the following relation to the eigenvalue y of $(1/x)\mathcal{D}_n(x)$;

$$y_{\mathrm{JMU}} = y + \frac{1}{2}\left(V'(x) + \frac{n-1-\ell_n}{x}\right). \tag{10.11}$$

Using the same formulæ in Theorem 8.1 but replacing y by y_{JMU} one obtains an expression for the logarithmic derivatives of τ_{JMU}; the ratio $\mathcal{F}$ between $\Delta_n^{\ell_n}$ and τ_{JMU} is defined via the difference of the differential equations for the corresponding logarithms. For example, the derivative with respect to the usual Jth Toda time (the coefficient of the power x^J in the polynomial part of $V(x)$ at infinity) would give

$$\partial_{t_J} \ln \mathcal{F} = \partial_{t_J} \ln \Delta_n^{\ell_n} - \partial_{t_J} \ln \tau_{\mathrm{JMU}} = \operatorname*{res}_{x=\infty} \frac{x^J}{J}(y - y_{\mathrm{JMU}})\,\mathrm{d}x. \tag{10.12}$$

Since $y_{\mathrm{JMU}} - y = -\frac{1}{2}(V'(x) + (n-1-\ell_n)/x)$ is an explicit function of $V(x)$, it is a straightforward exercise (that already appears in [4]) to integrate the ensuing one-form for $\ln \mathcal{F}$: leaving the details to the interested reader we quote the result only. Up to multiplicative factors independent of the isomonodromic times we have

$$\Delta_n^{\ell_n} = \tau_{\mathrm{JMU}}\mathcal{F}(V), \tag{10.13}$$

$$\ln \mathcal{F}(V) = -\frac{1}{2}\sum_{c=\text{finite pole of } \widehat{V}'} \operatorname*{res}_{x=c} \widehat{V}'_{\mathrm{sing},c}(x)\widehat{V}_{\mathrm{reg},c}(x), \tag{10.14}$$

$$\widehat{V}'(x) := V'(x) + \frac{n-1-\ell_n}{x}, \tag{10.15}$$

where $\widehat{V}'_{\mathrm{sing},c}$ ($\widehat{V}_{\mathrm{reg},c}$) denotes the singular (regular) part of $\hat{V}'$ at the pole c.

Example 10.1. For an example let us consider the case relevant to the problem of the probability of the longest increasing sequence of random letters in a word of fixed length [17]

$$V(x) = -tx - \sum_{\alpha=1}^{M} k_\alpha \ln\left(\frac{x - r_\alpha}{x}\right). \tag{10.16}$$

---

[7] Given a (rational) connection $\partial_x - A(x)$ of dimension $p \times p$ the **rational gauge** is a connection in the same gauge class which is traceless and obtained from the first by a scalar gauge transformation. Specifically, this is accomplished simply by

$$\partial_x - A(x) \mapsto \mathrm{e}^{-W(x)}\,(\partial_x - A(x))\,\mathrm{e}W(x) = \partial_x - \widetilde{A}(x) \tag{10.8}$$

with the requirement $\mathrm{Tr}\,A(x) \equiv 0$. It is promptly seen that one can find such a gauge by choosing the scalar function $W(x)$ as a solution of

$$W'(x) = \mathrm{Tr}\,A(x). \tag{10.9}$$

In this case a direct computation (with $\ell_n \equiv 0$ since we are dealing with the usual orthogonal polynomials on the unit circle) gives for $\mathcal{F}$ the following expression:

$$\ln \mathcal{F} = -\frac{t}{2}\sum_{\alpha=1}^{M} k_\alpha r_\alpha + \frac{n-1}{2}\sum_{\alpha=1}^{M} k_\alpha \ln(-{r_\alpha}^2) + \frac{1}{2}\sum_{\alpha=1}^{M} {k_\alpha}^2 \ln(-{r_\alpha}^2) - \frac{1}{2}\sum_{\alpha=1}^{M}\sum_{\beta\neq\alpha} \ln\left(\frac{r_\beta - r_\alpha}{r_\alpha r_\beta}\right)^{k_\alpha k_\beta}, \tag{10.17}$$

which is the result also obtained in formula (3.76) in [17]: note that in that formula $r_0 = 0$ and $k_0 = n - \sum_{\alpha=1}^{M} r_\alpha$ and a short algebraic manipulation shows the equivalence. Moreover, the signs inside the logarithms in (10.17) are in fact irrelevant since omitting them would amount to multiplying $\mathcal{F}$ by a constant independent of the isomonodromic times, and hence could be reabsorbed in the definition of τ_{JMU}.

11. Schlesinger Transformations

From the asymptotics that the shift $n \mapsto n+1$ implemented by the matrices R_n are, in the language of isomonodromic deformations, what is known as *elementary Schlesinger* transformations. Specifically, the shift $n \mapsto n+1$ corresponds to the following two types of elementary Schlesinger transformations according of the type of move (*circle* or *line*) (refer to formulæ (9.6) and (9.7)).

> **Circle-move**. The Schlesinger transformation adds one to the first entry of the formal monodromy at ∞ and subtracts one from the second entry of the formal monodromy at zero.
>
> **Line-move**. The Schlesinger transformation adds one and subtracts one to the first and second entries (respectively) of the formal monodromy at infinity, leaving the formal monodromy at zero unchanged.

However, we can obtain a third type of elementary Schlesinger transformation by considering two distinct sequences of LOPs corresponding to two (weakly increasing) sequences of $\{\ell_n\}$'s. Suppose indeed that we consider another sequence of LOPs and the ensuing connection $x\partial_x - \widetilde{\mathcal{D}}_n(x)$ for some fixed n where the only difference between the two pairs of LOPs is that one (or more) circle-moves have been replaced by a line-move (or vice versa) along the chain for $n' \le n$: the only difference in the formulas will be that $\widetilde{\ell}_n = \ell_n \pm 1$. This is implemented by the "circle-to-line" transformation $\mathcal{T}_n$ (2.15) (suitably normalized),

$$\begin{bmatrix} p_n \\ r^\star_{n-1} \end{bmatrix} = \begin{bmatrix} a + \dfrac{b}{x} & \dfrac{c}{x} \\ \dfrac{d}{x} & \dfrac{e}{x} \end{bmatrix} \begin{bmatrix} \hat{p}_n \\ \hat{r}^\star_{n-1} \end{bmatrix}, \tag{11.1}$$

where the coefficients a, b, c, d, e above can be obtained explicitly in terms of shifted Töplitz determinants using the form of $\mathcal{T}_n$ (2.15) and the normalizations (3.23), and the polynomials $\hat{p}_n, \hat{r}^\star_{n-1}$ refer to the elements of the sequence of biorthogonal polynomials associated to the sequence $\{\hat{\ell}_k\}$: such a sequence differs from $\{\ell_k\}$ because $\hat{\ell}_n = \ell_n - 1$, namely, there is a $k_0 \le n$ such that $\hat{\ell}_k = \ell_k - 1, \forall k : k_0 \le k \le n$.

We therefore add the following third type of transformations.

> **Circle-to-line move**. The Schlesinger transformation subtracts one from the second entry of the formal monodromy at ∞ and adds one to the second entry of the formal monodromy at zero.

This last type of transformation shows that the orthogonal polynomials on the line and the orthogonal polynomials on the circle are related by a sequence $(n-1)$ Schlesinger transformations and at each step the Laurent biorthogonal polynomials that are obtained are those appearing in the solution of integrable lattice hierarchies associated to elementary orbits [10].

12. Conclusion

As a general "philosophy", it is acknowledged in the literature that KP tau functions and isomonodromic tau functions are often, if not always, related to one another, in the sense that a KP (or Toda) tau function is an isomonodromic tau function for a suitably chosen isomonodromic deformation. In the case of orthogonal polynomials, this relation was explored in [16] for some class and extended in [6], [4]. In this paper, this relation has been confirmed once more for the particular generalized Toda systems associated to "nonstandard" minimal orbits of the Borel subgroup: the natural bridge between the Hamiltonian and isomonodromic treatment is provided by the solution of the inverse spectral problem in terms of biorthogonal Laurent polynomials. It is to be expected that, whenever a description or formulation of an integrable dynamical problem in terms of (bi/multiple-orthogonal) polyomials is available, then a suitable definition of the tau function for the associated isomonodromic problem should tie the Hamiltonian tau function with the isomonodromic one. For instance, in the case of the biorthogonal polynomials arising in the study of two-matrix models [5], [7] a natural isomonodromic deformation of a polynomial connection can be derived; however, the connection is a highly resonant one and at present a definition of isomonodromic tau functions for resonant deformations of connections is not available. However, it is possible to formulate such a notion [8] and the connection can thus be positively established.

As is recalled in the appendix to follow, the Laurent orthogonal polynomials which we have investigated in the present paper are related to the solution of the inverse spectral problem for Toda-like systems associated to *certain* minimal (or elementary) irreducible orbits. There exist in fact other minimal orbits for which a treatment in terms of orthogonal polynomials of some sort is not readily and generally available, although inspection of specific examples leads us to expect that it is possible to overcome this difficulty. It is our intention to pursue the topic in future publications.

13. Appendix: Minimal Irreducible Co-Adjoint Orbits

As was mentioned earlier, every $n \times n$ principal submatrix of the Hessenberg matrix Q that defines recurrence relations (4.1) belongs to a $(2n-2)$-dimensional co-adjoint orbit of the Borel subgroup $\mathbf{B}_n$ of invertible upper triangular matrices in $sl(n)$. However, not every low-dimensional co-adjoint orbit can be obtained this way. In this appendix we give a description of all irreducible co-adjoint orbits of $\mathbf{B}_n$ in $sl(n)$ that have a minimal dimension $2n-2$.

First, we introduce some notations. Let $\mathfrak{b}_-$ be a subalgebra of lower triangular matrices in $sl(n)$. Denote by J an $n \times n$ shift matrix (1's on the first superdiagonal and 0's everywhere else) and let $Hess_n = J + \mathfrak{b}_-$ denote a set of lower Hessenberg matrices. An element $Q \in Hess_n$ is called **reducible** if it has a block upper triangular form $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{bmatrix}$, where Q_{11} is a $k \times k$ matrix $(0 < k < n)$. Q is called **irreducible** otherwise.

Orbits of the co-adjoint action of $\mathbf{B}_n$ on $Hess_n$ are given by

$$\mathcal{O}_{Q_0} = \{J + (\mathrm{Ad}_b\, Q_0)_{\leq 0} : b \in \mathbf{B}_n\}. \tag{13.1}$$

It is easy to see that if $\mathcal{O}_{Q_0}$ contains a reducible (resp., irreducible) element, then every element of $\mathcal{O}_{Q_0}$ is reducible (resp., irreducible). Therefore it makes sense to talk about *irreducible orbits of the co-adjoint action*. Our main goal in this appendix is to prove the following:

Theorem 13.1. *An irreducible co-adjoint orbit of* $\mathbf{B}_n$ *in* $Hess_n$ *has a minimal dimension* $(2n-2)$ *if and only if it contains an element* Q_0 *of the form*

$$Q_0 = J + H + \sum_{\alpha=1}^{k} E_{i_\alpha, i_{\alpha-1}-\varepsilon_{\alpha-1}}, \tag{13.2}$$

where

1. $\varepsilon_i \in \{0,1\}$ *and* $\varepsilon_0 = 0$;
2. $1 = i_0 < i_1 - \varepsilon_1 \leq i_1 < i_2 - \varepsilon_2 \leq i_2 < \cdots < i_{k-1} - \varepsilon_{k-1} \leq i_{k-1} < i_k = n$;
3. $H = \sum_{\alpha \in \{1,\ldots,n\} \setminus \{i_0,\ldots,i_{k-1}\}} h_\alpha E_{\alpha\alpha}$,

$$Q_0 = \begin{bmatrix}
0 & 1 & & & & & & & & & \\
\vdots & \star & 1 & & & & & & & & \\
\vdots & & 0 & 1 & & & & & & & \\
1 & \cdots & 0 & \star & 1 & & & & & & \\
 & & \vdots & 0 & \star & 1 & & & & & \\
 & & \vdots & \vdots & & \star & 1 & & & & \\
 & & \vdots & \vdots & & & \star & 1 & & & \\
 & & 1 & 0 & \cdots & \cdots & \cdots & 0 & 1 & & \\
 & & & & & & & \vdots & \star & 1 & \\
 & & & & & & & \vdots & & \star & 1 \\
 & & & & & & & 1 & 0 & \cdots & \star
\end{bmatrix}. \tag{13.3}$$

[*An example with* $n = 11$, $k = 3$, $i_1 = 4$, $i_2 = 8$, $i_3 = 11$, $\varepsilon_1 = \varepsilon_3 = 0$, $\varepsilon_2 = 1$.]

Remark 13.1. The case of $H = hE_{nn}$ and $\varepsilon_\alpha = 0$ ($\alpha = 1, \ldots, k-1$) was studied in [10], [11]. It is orbits of this type that can be studied via associated LOPs of the type appearing in this paper. Note that, in this case, parameters ℓ_j that were used in the main body of the paper are related to i_α via

$$\ell_j = \max\{i_\alpha : i_\alpha < j\}.$$

An investigation of the properties of moment functionals connected with a more general minimal orbits described in Theorem 13.1 will appear elsewhere.

Define a *staircase pattern* (I, ε) as a collection of pairs of indices

$$(I, \varepsilon) = \{(i_1, 1), (i_2, i_1 - \varepsilon_1), \ldots, (i_k = n, i_{k-1} - \varepsilon_{k-1})\}, \tag{13.4}$$

where

$$1 = i_0 < i_1 - \varepsilon_1 \le i_1 < i_2 - \varepsilon_2 \le i_2 < \cdots < i_{k-1} - \varepsilon_{k-1} \le i_{k-1} < i_k = n.$$

In what follows we will often use a notation

$$j_\alpha = i_{\alpha-1} - \varepsilon_{\alpha-1}.$$

We say that $Q \in Hess_n$ has a staircase pattern (I, ε) if

$$Q_{i_\alpha, j_\alpha} \neq 0 \quad \text{and} \quad Q_{ij} = 0 \quad \text{for} \quad i > i_\alpha,\ j < j_{\alpha+1} \quad (\alpha = 1, \ldots, k).$$

The set of all matrices in $Hess_n$ that have a staircase pattern (I, ε) will be denoted by $Hess(I, \varepsilon)$. For example, if $I = \{2, 3, \ldots, n\}$ and $\varepsilon = \{0, 0, \ldots, 0\}$, then $Hess(I, \varepsilon)$ coincides with the set of $n \times n$ Jacobi matrices. An immediate property of the set $Hess(I, \varepsilon)$ is that it is stable under the co-adjoint action of $\mathbf{B}_n$, since *corner entries* Q_{i_α, j_α} and the entries "under the staircase" $Q_{ij} = 0$, $i > i_\alpha$, $j < j_{\alpha+1}$ have only zeros to the left and below and, thus the former are being acted upon only by the diagonal part of $\mathbf{B}_n$ and the latter cannot be made nonzero by the co-adjoint action.

Let us fix a staircase pattern (I, ε). To begin the proof of Theorem 13.1, we first employ the strategy used in [12] to study *generic* staircase orbits.

Lemma 13.1. *If $Q \in Hess(I, \varepsilon)$, then there exist $\tilde{Q} \in \mathcal{O}_Q$ such that*

$$\begin{gathered} \tilde{Q}_{i_\alpha j_\alpha} = 1, \\ \tilde{Q}_{i j_\alpha} = 0 \quad (j_\alpha < i < i_\alpha), \\ \tilde{Q}_{i_\alpha j} = 0 \quad (j_\alpha < j < i_\alpha \text{ and } j \neq j_\beta : \beta < \alpha,\ j_\beta < i_\alpha), \\ (\alpha = 1, \ldots, k). \end{gathered} \tag{13.5}$$

Proof. First, we use a diagonal conjugation to reduce Q to an element with all corner entries equal to $1 : Q \to \mathrm{Ad}^*_D\, Q = D^{-1}(Q - J)D + J$, where $D = \mathrm{diag}(d_1, \ldots, d_n) = D_k \cdots D_1$ with diagonal matrices D_α defined by

$$(D_\alpha)_{ii} = \begin{cases} 1, & i \neq i_\alpha, \\ d_{i_\alpha} = \left(D_{\alpha-1}^{-1} \cdots D_1^{-1} Q D_1 \cdots D_{\alpha-1}\right)_{i_\alpha j_\alpha}, & i = i_\alpha. \end{cases}$$

Next, we use the co-adjoint action induced by a sequence of elementary upper-triangular matrices (each depending on one parameter only) to set as many as possible of the entries in rows and columns occupied by corner entries equal to zero. More precisely, to eliminate an (i, j_α)-entry ($j_\alpha \le i < i_\alpha$) using the corner entry (i_α, j_α), one employs $\mathrm{Ad}^*_{(\mathbf{1}+Q_{ij_\alpha}E_{ii_\alpha})}$. Similarly, to eliminate an (i_α, j)-entry ($j_\alpha < j < i_\alpha$), one uses $\mathrm{Ad}^*_{(\mathbf{1}-Q_{i_\alpha j}E_{j_\alpha j})}$. Note that when we write Q_{ij_α} (resp., $Q_{i_\alpha j}$), we refer to entries of the "current" value of Q, i.e. to the element that belongs to the orbit through the initial Q and that has been obtained through the sequence of transformations already applied.

The order in which we apply these elementary transformations is defined as follows: we first set to zero the entries in the first column (going down the column), then in the i_1st row (moving right), then in the j_2nd column (moving down), then in the i_2nd row (moving right), etc. Through the entire process, we want, for every $l < m$, to use an elementary matrix of the from $\mathbf{1} + xE_{lm}$ at most once. This means, in particular, that any (i_α, j_β)-entry, where $\beta < \alpha$ and $j_\beta < i_\alpha$ cannot be touched, since a matrix of the form $\mathbf{1}+xE_{j_\beta i_\alpha}$ has already been used to eliminate the (j_β, j_α)-entry. This explains why entries $\{(i_\alpha, j_\beta) : \beta < \alpha, j_\beta < i_\alpha\}$ are excluded from the list of entries in (13.5). On the other hand, all noncorner entries that are in the list can be set to 0, regardless of their initial values. ■

Corollary 13.1. *For each $Q \in Hess(I, \varepsilon)$ the matrix entries specified in* (13.5) *are independent functions on $\mathcal{O}_Q$.*

Proof. It suffices to notice that applying to $\tilde{Q}$ constructed in Lemma 13.1 elementary transformations of the same type that was used in its construction, but in the reverse order and with arbitrary parameters, one can obtain an element in $\mathcal{O}_Q$ with arbitrary nonzero values of the corner entries and arbitrary values of noncorner values specified in (13.5). ■

Lemma 13.2. *If, for some $1 \le \alpha < k$, $\varepsilon_\alpha > 1$, then, for any $Q \in Hess(I, \varepsilon)$,* $\dim \mathcal{O}_Q > 2n - 2$.

Proof. Denote by $M(I, \varepsilon)$ the set of pairs of indices that appear in the list given in (13.5). In view of the corollary above, we only need to show that, under conditions of the lemma, the number of elements in $M(I, \varepsilon)$ is greater than $2n - 2$. We will also show that, if $0 \le \varepsilon_\alpha \le 1$ for $\alpha = 1, \dots, k - 1$, then $\#M(I, \varepsilon) = 2n - 2$.

We will use an induction on k and n. Clearly, if $k = 1$, then $\varepsilon_0 = 0$ and $\#M(I, \varepsilon) = 2n-2$. Moreover, $\#M(I, \varepsilon) = 2n - 2$ for any k, provided $\varepsilon_\alpha = 0$ for all α. Now let $k = 2$ and $\varepsilon_1 > 0$. We are looking for a number of elements in the set $\{(1, 1), \dots, (i_1, 1), (i_1, 2), \dots, (i_1, i_1 - 1); (i_1 - \varepsilon_1, i_1 - \varepsilon_1), \dots, (i_1 - 1, i_1 - \varepsilon_1, i_1 - \varepsilon_1), (i_1 + 1, i_1 - \varepsilon_1, i_1 - \varepsilon_1), \dots, (n, i_1 - \varepsilon_1), \dots (n, i_1 - 1), (n, i_1 + 1), \dots, (n, n - 1)\}$, which is equal to

$$2(i_1 - 1) + 2(n - (i_1 - \varepsilon_1)) - 2 = 2(n + \varepsilon_1 - 2) \begin{cases} = 2n - 2 & \text{if } \varepsilon_1 = 1, \\ > 2n - 2 & \text{if } \varepsilon_1 > 1. \end{cases}$$

For $k > 2$, let s be such that $j_s < i_1 \le j_{s+1}$. We first consider the case when there is no r such that $j_r = i_1$. Then the set $M(I, \varepsilon)\backslash\{(1, 1), \dots, (i_1, 1), (i_1, 2), \dots, (i_1, i_1 - 1)\}$ has the same cardinality as a set $M(I', \varepsilon')$, where $(I', \varepsilon') = \{(i_2 - j_2, 1), (i_3 - j_2,$

$j_3-j_2+1), \ldots, (i_s-j_2, j_s-j_2+1), (i_{s+1}-j_2, j_{s+1}-j_2), \ldots, (i_k-j_2, j_k-j_2)\} = \{(i_2-i_1+\varepsilon_1, 1), (i_3-i_1+\varepsilon_1, i_2-i_1+\varepsilon_1-(\varepsilon_2-1)), \ldots, (i_s-i_1+\varepsilon_1, i_{s-1}-i_1+\varepsilon_1-(\varepsilon_{s-1}-1)), (i_{s+1}-i_1+\varepsilon_1, i_s-i_1+\varepsilon_1-\varepsilon_s), \ldots, (i_k-i_1+\varepsilon_1, i_{k-1}-i_1+\varepsilon_1-\varepsilon_{k-1})\}$, that is,

$$n' = i'_{k-1} = i_k - i_1 + \varepsilon_1,$$

$$i'_\alpha = i_{\alpha+1} - i_1 + \varepsilon_1, \qquad \alpha = 1, \ldots, k-1,$$

and

$$\varepsilon'_\alpha = \varepsilon_{\alpha+1} - 1 \quad (1 \le \alpha \le s-2), \qquad \varepsilon'_\alpha = \varepsilon_{\alpha+1} \quad (s-1 \le \alpha \le k-2).$$

If $s \ge 2$, then $\varepsilon_1 \ge 1$ and $\varepsilon_s = i_s - j_s > i_s - i_1 \ge s-1 \ge 1$, so that $\varepsilon'_{s-1} = \varepsilon_s \ge 2$ and, by the induction hypothesis, $\#M(I', \varepsilon') > 2(n - i_1 + \varepsilon_1 - 1) \ge 2(n - i_1)$ and $\#M(I, \varepsilon) > 2(i_1 - 1) + 2(n - i_1) = 2(n-1)$.

If $s = 1$, then $\varepsilon'_\alpha = \varepsilon_{\alpha+1}$ for $1 < \alpha \le k-2$ and

$$\#M(I', \varepsilon') \begin{cases} = 2(n - i_1 + \varepsilon_1 - 1) & \text{if all } \varepsilon'_\alpha \le 1, \\ > 2(n - i_1 + \varepsilon_1 - 1) & \text{if some } > \varepsilon'_\alpha > 1, \end{cases}$$

and thus, $\#M(I, \varepsilon) = 2(i_1 - 1) + \#M(I', \varepsilon')$ is greater than $2n-2$ if $\varepsilon_\alpha > 1$ for some $\alpha > 1$ and is equal to $2n-2$ otherwise.

Finally, consider the case when $j_r = i_1$ for some $r > 1$. If $r > 2$, then $\varepsilon_r = i_r - j_r = i_r - i_1 \ge r - 1 \ge 2$. Define $(\tilde{I}, \tilde{\varepsilon}) = (I, \varepsilon) \backslash \{(i_2, j_2), \ldots, (i_{r-1}, j_{r-1})\}$. Then $(\tilde{I}, \tilde{\varepsilon})$ still defines an irreducible staircase pattern, $\tilde{k} = \#(\tilde{I}, \tilde{\varepsilon}), k$ and $\#M(I, \varepsilon) > \#M(\tilde{I}, \tilde{\varepsilon}) > 2n-2$ by the induction hypothesis.

If $r = 2$, then $\varepsilon_1 = 0$, $j_2 = i_1$ and $\#M(I, \varepsilon) = 2(i_1 - 1) + \#M(I', \varepsilon')$, where $(I', \varepsilon') = \{(i_2-i_1+1, 1), (i_3-i_1+1, i_2-i_1+1-\varepsilon_2), \ldots, (n-i_1+1, (i_{k-1}-i_1+1-\varepsilon_{k-1})\}$ and, again by induction, the statement follows. ■

We are now ready to complete the proof of Theorem 13.1.

Proof of Theorem 13.1. Assume that $\dim \mathcal{O}_Q = 2n-2$. We have shown that if $Q \in Hess(I, \varepsilon)$, then $\varepsilon_\alpha \le 1$ for $\alpha = 1, \ldots, k-1$. Assume that the latter condition is satisfied and consider the element $\tilde{Q}$ constructed in Lemma 13.1. Suppose that some noncorner entry $\tilde{Q}_{ij}$ $(i > j)$ is nonzero. Then, by construction of $\tilde{Q}$, $j \ne j_\alpha$ $(\alpha = 1, \ldots, k)$. Define a diagonal matrix D by

$$D_{ll} = \begin{cases} 1 & \text{if } l \ne j \text{ or } l \ne j_\beta \text{ or } i \ne i_\beta, \\ d & \text{if } l = j \text{ or } (l = j_\beta \text{ and } i = i_\beta). \end{cases}$$

Then $\mathrm{Ad}^*_D \tilde{Q} = D^{-1} \tilde{Q} D$ has the same values as $\tilde{Q}$ in the entries specified by (13.5) but $(\mathrm{Ad}^*_D \tilde{Q})_{ij} = d^{-1} \tilde{Q}_{ij}$. This means that the matrix entry Q_{ij} viewed as a function on $\mathcal{O}_Q$ is independent of the matrix entries specified by (13.5), which is in contradiction with $\dim \mathcal{O}_Q = 2n-2$. Therefore, $\tilde{Q}_{ij} = 0$ for all $(i, j) \ne (i_\alpha, j\alpha)$. Since, by Lemma 13.1, $\tilde{Q}_{j_\alpha j_\alpha} = 0$ for $\alpha = 1, \ldots, k$, we proved that $\dim \mathcal{O}_Q = 2n-2$ implies that $\mathcal{O}_Q$ contains an element of the form (13.2).

To prove the converse consider an element Q_0 defined by (13.2). Clearly, for any $b \in \mathbf{B}_n$, $\mathrm{Ad}_b^* Q_0 = \mathrm{Ad}_b^*(Q_0 - H) + H$, therefore it is sufficient to consider the case where $H = 0$. In other words, we are interested in parametrizing the set

$$\{(b(Q_0 - H - J)b^{-1})_{\leq 0} : b \in \mathbf{B}_n\}.$$

Note that, for $i > j$, we have

$$(bE_{ij}b^{-1})_{\leq 0} = ((be_i)(e_j^T b^{-1}))_{\leq 0} = (uv^T)_{\leq 0},$$

where

$$u = (\Pi_i - \Pi_{j-1})(be_i), \qquad v^T = (e_j^T b^{-1})(\Pi_i - \Pi_{j-1}).$$

Thus,

$$(b(Q_0 - H - J)b^{-1})_{\leq 0} = \sum_{\alpha=1}^{k} (u_\alpha v_\alpha^T)_{\leq 0} \tag{13.6}$$

with

$$u_\alpha = (\Pi_{i_\alpha} - \Pi_{i_{\alpha-1}-\varepsilon_{\alpha-1}-1})(be_{i_\alpha}), \qquad v_\alpha^T = (e^T_{i_{\alpha-1}-\varepsilon_{\alpha-1}} b^{-1})(\Pi_{i_\alpha} - \Pi_{i_{\alpha-1}-\varepsilon_{\alpha-1}-1}).$$

Entries of vectors u_α, v_α cannot be arbitrary. First,

$$v_\alpha^T u_\alpha = e_{j_\alpha}^T b^{-1}(\Pi_{i_\alpha} - \Pi_{j_\alpha - 1}))be_{i_\alpha} = e_{j_\alpha}^T e_{i_\alpha} = 0. \tag{13.7}$$

Next, if $\varepsilon_\alpha = 0$, i.e. $j_\alpha = i_{\alpha-1}$, then

$$(v_\alpha)_{j_\alpha} = (b^{-1})_{j_\alpha j_\alpha} = (u_{\alpha-1})^{-1}_{j_\alpha}. \tag{13.8}$$

Finally, if $\varepsilon_\alpha = 1$, i.e. $j_\alpha = i_{\alpha-1} - 1$, then

$$\begin{aligned} v_\alpha^T u_{\alpha-1} &= (v_\alpha)_{j_\alpha}(u_{\alpha-1})_{i_{\alpha-1}-1} + (v_\alpha)_{j_\alpha+1}(u_{\alpha-1})_{i_{\alpha-1}} \\ &= (b^{-1})_{j_\alpha j_\alpha} b_{j_\alpha j_\alpha+1} + (b^{-1})_{j_\alpha j_\alpha+1} b_{j_\alpha+1 j_\alpha+1} = 0. \end{aligned} \tag{13.9}$$

We claim that (13.7), (13.8), (13.9) are the only restrictions on u_α, v_α. We will verify this claim for $k = 2$. The general case follows by an easy induction.

If $\varepsilon_1 = 0$, we set $u_1 = \mathrm{col}[u_{11}, u_{12}, u_{13}, 0, \ldots, 0]$ and $v_1^T = [v_{11}, v_{12}^T, v_{13}, 0, \ldots, 0]$, where $u_{11}, u_{13} \neq 0$, $v_{11} \neq 0$, $v_{13} \in \mathbb{C}$ and $u_{12}, v_{12} \in \mathbb{C}^{i_1-2}$. Similarly, $u_2 = \mathrm{col}[0, \ldots, 0, u_{21}, u_{22}, u_{23}]$ and $v_2^T = [0, \ldots, 0, v_{21} = u_{13}^{-1}, v_{22}^T, v_{23}]$, where $u_{21}, u_{23} \neq 0$, $v_{23} \in \mathbb{C}$ and $u_{22}, v_{22} \in \mathbb{C}^{n-i_1-1}$. We assume that conditions (13.7) are satisfied: $v_1^T u_1 = v_2^T u_2 = 0$ and define

$$b = \begin{pmatrix} v_{11}^{-1} & -v_{11}^{-1} v_{12}^T & u_{11} & 0 & 0 \\ 0 & \mathbf{1} & u_{12} & 0 & 0 \\ 0 & 0 & u_{13} & -u_{13} v_{22}^T & u_{21} \\ 0 & 0 & 0 & \mathbf{1} & u_{22} \\ 0 & 0 & 0 & 0 & u_{23} \end{pmatrix},$$

$$b^{-1} = \begin{pmatrix} v_{11} & v_{12}^T & v_{13} & * & * \\ 0 & \mathbf{1} & -u_{12}u_{13}^{-1} & * & * \\ 0 & 0 & v_{21} & v_{22}^T & v_{23} \\ 0 & 0 & 0 & \mathbf{1} & -u_{22}u_{23}^{-1} \\ 0 & 0 & 0 & 0 & u_{23}^{-1} \end{pmatrix}.$$

The specified entries are consistent with the relation $bb^{-1} = \mathbf{1}$ and entries marked by $*$'s are uniquely determined by this relation.

Similarly, if $\varepsilon_1 = 1$, we set $u_1 = \mathrm{col}[u_{11}, u_{12}, u_{13}, u_{14}, 0, \dots, 0]$ and $v_1^T = [v_{11}, v_{12}^T, v_{13}, v_{14}, 0, \dots, 0]$, where $u_{11}, u_{13}, u_{14} \neq 0$, $v_{11} \neq 0$, $v_{13}, v_{14} \in \mathbb{C}$ and $u_{12}, v_{12} \in \mathbb{C}^{i_1-3}$; and $u_2 = \mathrm{col}[0, \dots, 0, u_{21}, u_{22}, u_{23}, u_{24}]$ and $v_2^T = [0, \dots, 0, v_{21}, v_{22} = -v_{21}(u_{13}/u_{14}), v_{23}^T, v_{24}]$, where $u_{21}, u_{22}, u_{24} \neq 0$, $v_{21} \neq 0$, $v_{24} \in \mathbb{C}$ and $u_{23}, v_{23} \in \mathbb{C}^{n-i_1-2}$. Assuming again that $v_1^T u_1 = v_2^T u_2 = 0$, define

$$b = \begin{pmatrix} v_{11}^{-1} & -v_{11}^{-1}v_{12}^T & -v_{11}^{-1}v_{13} & u_{11} & 0 & 0 \\ 0 & \mathbf{1} & 0 & u_{12} & 0 & 0 \\ 0 & 0 & v_{21}^{-1} & u_{13} & -v_{21}^{-1}v_{23}^T & u_{21} \\ 0 & 0 & 0 & u_{14} & 0 & u_{22} \\ 0 & 0 & 0 & 0 & \mathbf{1} & u_{23} \\ 0 & 0 & 0 & 0 & 0 & u_{24} \end{pmatrix},$$

$$b^{-1} = \begin{pmatrix} v_{11} & v_{12}^T & v_{13} & v_{14} & * & * \\ 0 & \mathbf{1} & 0 & -u_{12}u_{14}^{-1} & * & * \\ 0 & 0 & v_{21} & v_{22} & v_{23}^T & v_{24} \\ 0 & 0 & 0 & u_{14}^{-1} & 0 & -u_{22}u_{24}^{-1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & -u_{23}u_{24}^{-1} \\ 0 & 0 & 0 & 0 & 0 & u_{24}^{-1} \end{pmatrix}$$

and observe that (13.7), (13.9) are consistent with $bb^{-1} = \mathbf{1}$ and entries marked by $*$'s can be uniquely determined.

To conclude the proof, observe that the right-hand side of (13.6) is invariant under a transformation $u_\alpha \to t_\alpha u_\alpha$, $v_\alpha \to t_\alpha^{-1} v_\alpha$, where t_α are arbitrary nonzero parameters. Therefore, we can assume that

$$(v_\alpha)_{j_\alpha} = 1 \quad \text{if} \quad \alpha = 1 \quad \text{or} \quad \varepsilon_{\alpha-1} = 1. \tag{13.10}$$

Recall, that, if $\varepsilon_{\alpha-1} = 0$, then $(v_\alpha)_{j_\alpha}$ is given by (13.8), while $(u_\alpha)_{j_\alpha}$ is determined by condition (13.7) for all α. Furthermore, if $\varepsilon_{\alpha-1} = 1$, then $j_\alpha = i_{\alpha-1} + 1$ and (j_α, j_α), $(j_\alpha + 1, j_\alpha)$ and $(j_\alpha + 1, j_\alpha + 1)$-entries of the right-hand side of (13.6) are given by

$$\begin{aligned} (j_\alpha, j_\alpha)&: \quad (u_{\alpha-1})_{j_\alpha}(v_{\alpha-1})_{j_\alpha} + (u_\alpha)_{j_\alpha+1}\frac{(u_{\alpha-1})_{j_\alpha}}{(u_{\alpha-1})_{j_\alpha+1}} - \sum_{s=j_\alpha+2}^{i_\alpha}(u_\alpha)_s(v_\alpha)_s, \\ (j_\alpha+1, j_\alpha)&: \quad (u_{\alpha-1})_{j_\alpha+1}(v_{\alpha-1})_{j_\alpha} + (u_\alpha)_{j_\alpha+1}, \\ (j_\alpha+1, j_\alpha+1)&: \quad (u_{\alpha-1})_{j_\alpha+1}(v_{\alpha-1})_{j_\alpha+1} - (u_\alpha)_{j_\alpha+1}\frac{(u_{\alpha-1})_{j_\alpha}}{(u_{\alpha-1})_{j_\alpha+1}}, \end{aligned} \tag{13.11}$$

where we have used (13.7), (13.9) and (13-9). Note that the entries in (13-10) are the only entries in (13.6) that depend on $(v_{\alpha-1})_{j_\alpha}$, $(v_{\alpha-1})_{j_\alpha}$ and $(u_\alpha)_{j_\alpha+1}$. Moreover, (13-10) does not change under a transformation

$$\begin{aligned}(u_\alpha)_{j_\alpha+1} &\to (u_\alpha)_{j_\alpha+1} - t(u_{\alpha-1})_{j_\alpha+1},\\ (v_{\alpha-1})_{j_\alpha} &\to (v_{\alpha-1})_{j_\alpha} + t,\\ (v_{\alpha-1})_{j_\alpha+1} &\to (v_{\alpha-1})_{j_\alpha} - t\frac{(u_{\alpha-1})_{j_\alpha}}{(u_{\alpha-1})_{j_\alpha+1}}.\end{aligned}$$

This means that we can set

$$(u_\alpha)_{j_\alpha+1} = (u_\alpha)_{i_{\alpha-1}} = 0 \qquad \text{or} \qquad \varepsilon_{\alpha-1} = 1. \tag{13.12}$$

Under the normalizations (13.10), (13.12) and restrictions (13.7), (13.8), (13.9), the rest of the parameters in (13.6),

$$(u_\alpha)_s, (v_\alpha)_s, \qquad s = i_{\alpha-1}+1, \dots, i_\alpha, \quad \alpha = 1, \dots, k,$$

can be chosen arbitrarily and, on the other hand, these parameters are uniquely determined by the right-hand side of (13.6). Thus, for Q_0 satisfying conditions of Theorem 13.1 we have found an explicit parametrization of $\mathcal{O}_{Q_0}$ by $2n-2$ independent parameters, which completes the proof. ■

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