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CONSTRUCTIVE APPROXIMATION

Linearization Coefficients of Bessel Polynomials and Properties of Student t-Distributions

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Abstract. We prove positivity results about linearization and connection coefficients for Bessel polynomials. The proof is based on a recursion formula and explicit formulas for the coefficients in special cases. The result implies that the distribution of a finite convex combination of independent Student t-variables with arbitrary odd degrees of freedom has a density which is a finite convex combination of certain Student t-densities with odd degrees of freedom.

1. Introduction

In this paper we consider the Bessel polynomials q_n of degree n,

$$q_n(u) = \sum_{k=0}^{n} \alpha_k^{(n)} u^k, \tag{1}$$

where

$$\alpha_k^{(n)} = \frac{\binom{n}{k}}{\binom{2n}{k}} \frac{2^k}{k!} = \frac{(-n)_k\, 2^k}{(-2n)_k\, k!}, \tag{2}$$

where we have used the Pochhammer symbol $(z)_n := z(z+1)\cdots(z+n-1)$ for $z \in \mathbf{C}$, $n = 0, 1, \ldots$.

Using hypergeometric functions, see [1], we therefore have

$$q_n(u) = {}_1F_1(-n; -2n; 2u). \tag{3}$$

The first examples of these polynomials are

$$q_0(u) = 1, \qquad q_1(u) = 1 + u, \qquad q_2(u) = 1 + u + \frac{u^2}{3}.$$

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They are normalized according to

$$q_n(0) = 1,$$

and thus differ from the polynomials $\theta_n(u)$ in Grosswald's monograph [12] by the constant factor $(2n)!/n!\,2^n$, i.e.,

$$\theta_n(u) = \frac{(2n)!}{n!\,2^n} q_n(u).$$

The polynomials θ_n are sometimes called the reverse Bessel polynomials and $y_n(u) = u^n \theta_n(1/u)$ the ordinary Bessel polynomials. Two-parameter extensions of these polynomials are studied in [12], and we refer to this work concerning references to the vast literature and the history about Bessel polynomials. For a study of the zeros of the Bessel polynomials we refer to [5].

For $\nu > 0$ the probability density on **R**,

$$f_\nu(x) = \frac{A_\nu}{(1+x^2)^{\nu+1/2}}, \qquad A_\nu = \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\nu)}, \tag{4}$$

is called a Student t-density with $f = 2\nu$ degrees of freedom. The characteristic function is given by

$$\int_{-\infty}^{\infty} e^{ixy} f_\nu(x)\,dx = k_\nu(|y|), \qquad y \in \mathbf{R}, \tag{5}$$

where

$$k_\nu(u) = \frac{2^{1-\nu}}{\Gamma(\nu)} u^\nu K_\nu(u), \qquad u \geq 0, \tag{6}$$

and K_ν is the modified Bessel function of the second kind, or the Macdonald function, see [10, 17.34(9)].

If $\nu = n + \frac{1}{2}$ with $n = 0, 1, 2, \ldots$, then

$$k_\nu(u) = e^{-u} q_n(u), \qquad u \geq 0, \tag{7}$$

and f_ν is a Student t-density with $2\nu = 2n+1$ degrees of freedom. For $\nu = \frac{1}{2}$ then f_ν is density of a Cauchy distribution. Note that for simplicity we have avoided the usual scaling of the Student t-distribution.

In this paper, we solve the following three problems:

1. We find explicit expressions for the connection coefficients $c_k^{(n)}(a)$ and prove their nonnegativity for $a \in [0, 1]$ in the expansion

$$q_n(au) = \sum_{k=0}^{n} c_k^{(n)}(a) q_k(u). \tag{8}$$

2. We find explicit expressions for the linearization coefficients $\beta_i^{(n)}(a)$ and prove their nonnegativity for $a \in [0, 1]$ in the expansion

$$q_n(au)q_n((1-a)u) = \sum_{i=0}^{n} \beta_i^{(n)}(a) q_{n+i}(u). \tag{9}$$

3. We prove nonnegativity of the linearization coefficients $\beta_k^{(n,m)}(a)$ for $a \in [0,1]$ in the expansion

$$q_n(au)q_m((1-a)u) = \sum_{k=n\wedge m}^{n+m} \beta_k^{(n,m)}(a)q_k(u). \tag{10}$$

Note that $\beta_i^{(n)}(a) = \beta_{n+i}^{(n,n)}(a)$ and that (8) is a special case of (10) corresponding to $m = 0$ with $c_k^{(n)}(a) = \beta_k^{(n,0)}(a)$. Note also that $u = 0$ in (10) yields

$$\sum_{k=n\wedge m}^{n+m} \beta_k^{(n,m)}(a) = 1,$$

so (10) is a convex combination. Being polynomial identities, (8)–(10) of course hold for all complex a, u, but as we will see later, the nonnegativity of the coefficients can only be inferred for $0 \le a \le 1$.

Although (10) is more general than (8), (9), we stress that we give explicit formulas below for $c_k^{(n)}(a)$ and $\beta_i^{(n)}(a)$ from which the nonnegativity is clear. The nonnegativity of $\beta_k^{(n,m)}(a)$ for the general case can be deduced from the special case $m = 0$ via a recursion formula, see Lemma 3.6 below.

Because of (5) and (7) we note that formula (10) is equivalent with the following identity between Student t-densities

$$\frac{1}{a} f_{n+1/2}\left(\frac{x}{a}\right) * \frac{1}{1-a} f_{m+1/2}\left(\frac{x}{1-a}\right) = \sum_{k=n\wedge m}^{n+m} \beta_k^{(n,m)}(a) f_{k+1/2}(x) \tag{11}$$

for $0 < a < 1$ and $*$ is the ordinary convolution of densities.

We shall use (11) to explain directly, why there are no terms $q_k(u)$ with $k < n \wedge m$ appearing in (10) or, equivalently, why there are no terms $f_{k+1/2}$ with $k < n \wedge m$ in (11).

The density $f_{n+1/2}$ has a moment of order p if and only if $p \le 2n$. The odd moments are zero by symmetry and the even moments are given by

$$s_{2p}(n) = A_{n+1/2} \int_{-\infty}^{\infty} \frac{x^{2p}\,dx}{(1+x^2)^{n+1}} = \frac{(\frac{1}{2})_p(\frac{1}{2})_{n-p}}{(\frac{1}{2})_n}, \qquad 0 \le p \le n. \tag{12}$$

For $p \le n \wedge m$ the $(2p)$th moment of the left-hand side of (11) exists and is given as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)^{2p} \frac{1}{a} f_{n+1/2}\left(\frac{x}{a}\right) \frac{1}{1-a} f_{m+1/2}\left(\frac{y}{1-a}\right) dx\,dy$$
$$= \sum_{j=0}^{2p} \binom{2p}{2j} a^{2j}(1-a)^{2p-2j} s_{2j}(n)s_{2p-2j}(m).$$

Assume now that the expression (10) contains a nonzero term $\beta_{k_0}^{(n,m)}(a)q_{k_0}(u)$, where k_0 is the smallest index with this property and that $k_0 < n \wedge m$. The corresponding term $\beta_{k_0}^{(n,m)}(a) f_{k_0+1/2}$ on the right-hand side of (11) does not have a moment of order $2(k_0+1)$, but all the other terms $f_{k+1/2}$ with $k > k_0$ as well as the left-hand side of (11) have moment of order $2(k_0+1)$. This however, contradicts equation (11).

Our use of the words "linearization coefficients" does not agree completely with the terminology of [2], which defines the linearization coefficients for a polynomial system $\{q_n\}$ as the coefficients $a(k, n, m)$, such that

$$q_n(u)q_m(u) = \sum_{k=0}^{n+m} a(k, n, m)q_k(u).$$

Since, in the Bessel case,

$$q_1(u)^2 = -q_0(u) - q_1(u) + 3q_2(u),$$

the linearization coefficients in the proper sense are not nonnegative.

It is interesting to note that in [14] Koornwinder proved that the Laguerre polynomials

$$L_n^{(\alpha)}(u) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-u)^k}{k!}$$

satisfy a positivity property like (10), i.e.,

$$L_n^{(\alpha)}(au)L_m^{(\alpha)}((1-a)u) = \sum_{k=0}^{n+m} \kappa_k^{(n,m)}(a)L_k^{(\alpha)}(u),$$

with $\kappa_k^{(n,m)}(a) \geq 0$ for $a \in [0, 1]$ provided $\alpha \geq 0$.

In this connection it is worth pointing out that there is an easily established relationship between q_n and the Laguerre polynomials with α outside the range of orthogonality for the Laguerre polynomials, namely,

$$q_n(u) = \frac{(-1)^n}{\binom{2n}{n}} L_n^{(-2n-1)}(2u).$$

The problems discussed have an interesting application in statistics: the Behrens–Fisher problem consists in testing the equality of the means of two normal populations. Fisher [7][1] has shown that this test can be performed using the d-statistics defined as

$$d_{f_1,f_2,\theta} = t_1 \sin\theta - t_2 \cos\theta,$$

where t_1 and t_2 are two independent Student t-variables with respective degrees of freedom f_1 and f_2 and $\theta \in [0, \pi/2]$. Many different results have been obtained on the behaviour of the d-statistics. Tables of the distribution of $d_{f_1,f_2,\theta}$ have been provided in 1938 by Sukhatme [18] at Fisher's suggestion. In 1956, Fisher and Healy [8] explicited the distribution of $d_{f_1,f_2,\theta}$ as a mixture of Student t-distributions (Student t-distribution with a random, discrete number of degrees of freedom) for small, odd values of f_1 and f_2. This work was extended by Walker and Saw [19] who provided, still in the case of odd numbers of degrees of freedom, an explicit way of computing the coefficients of the mixture as solutions of a linear system; however, they did not prove the nonnegativity

[1] The collected papers of R. A. Fisher are available at the following address: http://www.library.adelaide.edu.au/digitised/fisher/

of these coefficients, claiming only

> Extensive numerical investigation indicates also that $\eta_i \geq 0$ for all i; however, an analytic proof has not been found.

This conjecture is proved in Theorem 2.4 below. Section 2 of this paper gives the explicit solutions to Problems 1, 2, and 3, whereas Section 3 is dedicated to their proofs.

To relate the Behrens–Fisher problem to our discussion we note that due to symmetry $d_{f_1, f_2, \theta}$ has the same distribution as $t_1 \sin\theta + t_2 \cos\theta$, which for $\theta \in \,]0, \pi/2[$ is a scaling of a convex combination of independent Student t-variables.

In this connection we recall a theorem of Ruben, see [16], which can be stated as follows:

> *Let X and Y be independent Student t-variables with 2μ (resp., 2ν) degrees of freedom. For $a, b > 0$ the variable $aX + bY$ has a density of the form*
>
> $$D(x) = \int_{a+b}^{\infty} \frac{1}{t} f_{\mu+\nu}\left(\frac{x}{t}\right) h(t)\, dt$$
>
> *for a certain probability density h on $]a + b, \infty[$, i.e., D is a mixture of scaled Student t-distributions with $2\mu + 2\nu$ degrees of freedom.*

For a short proof see [4, p. 66].

In Section 4 we use the fact that the Student t-distribution is a scale mixture of normal distributions by an inverse Gamma distribution to prove that Theorem 2.4 is equivalent to a result about inverse Gamma distributions. This is Theorem 4.1. Such a result has been observed for small values of the degrees of freedom in [21]. In [9] the coefficients are claimed to be nonnegative but the paper does not contain any arguments to prove it.

We finally indicate that our results can be extended to rotation invariant N-variate Student t-distributions.

2. Results

2.1. *Solution of Problem 1 and a Stochastic Interpretation*

Theorem 2.1. *The coefficients $c_k^{(n)}(a)$ in (8) are expressed as follows:*

$$\begin{aligned} c_k^{(n)}(a) &= a^k (1-a) \frac{\binom{n}{k}}{\binom{2n}{2k}} \sum_{r=0}^{(n-k-1)\wedge k} \binom{n+1}{k-r} \binom{n-k-1}{r} (1-a)^r \\ &= a^k (1-a) \frac{\binom{n}{k}\binom{n+1}{k}}{\binom{2n}{2k}} \, {}_2F_1\left(\begin{matrix} -k, -n+k+1 \\ n-k+2 \end{matrix}; 1-a\right) \end{aligned}$$

for $0 \leq k \leq n-1$ while $c_n^{(n)}(a) = a^n$. Hence, they are nonnegative for $0 \leq a \leq 1$.

A stochastic interpretation of Theorem 2.1 is obtained as follows: replacing u by $|u|$ and multiplying equation (8) by $\exp(-|u|)$, we get

$$e^{-(1-a)|u|}e^{-a|u|}q_n(a|u|) = \sum_{k=0}^{n} c_k^{(n)}(a)q_k(|u|)e^{-|u|}. \tag{13}$$

Equation (13) expresses that the convex combination of a Cauchy variable C and an independent Student t-variable X_n with $2n+1$ degrees of freedom follows a Student t-distribution with a random number $2f+1$ of degrees of freedom:

$$(1-a)C + aX_n \stackrel{d}{=} X_f,$$

where f is a discrete random variable with integer values in $[0, n]$ such that

$$\Pr\{f = k\} = c_k^{(n)}(a), \qquad 0 \le k \le n.$$

2.2. *Solution of Problem* 2 *and a Probabilistic Interpretation*

Theorem 2.2. *The coefficients $\beta_i^{(n)}(a)$ in* (9) *are expressed as follows*:

$$\begin{aligned}\beta_i^{(n)}(a) &= (4a(1-a))^i \left(\frac{n!}{(2n)!}\right)^2 2^{-2n} \frac{(2n-2i)!\,(2n+2i)!}{(n-i)!\,(n+i)!} \\ &\quad \times \sum_{j=0}^{n-i} \binom{2n+1}{2j}\binom{n-j}{i}(2a-1)^{2j} \\ &= \frac{(4a(1-a))^i}{2^{2n}} \frac{(-n)_i(n+\frac{1}{2})_i}{i!\,(-n+\frac{1}{2})_i} {}_2F_1\left(\begin{matrix}-n+i, -n-\frac{1}{2} \\ \frac{1}{2}\end{matrix}; (2a-1)^2\right).\end{aligned}$$

Hence they are nonnegative for $0 \le a \le 1$.

A probabilistic interpretation of this result can be formulated as follows.

Corollary 2.3. *Let X, Y be independent Student t-variables with $2n+1$ degrees of freedom. For $0 \le a \le 1$ the distribution of $aX + (1-a)Y$ follows a Student t-distribution with a random number $2f+1$ of degrees of freedom distributed according to*

$$\Pr\{f = n+i\} = \beta_i^{(n)}(a), \qquad 0 \le i \le n.$$

2.3. *Problem* 3

Theorem 2.4. *The coefficients $\beta_k^{(n,m)}(a)$ in* (10) *are nonnegative for* $0 \le a \le 1$.

We are unable to derive the explicit values of the coefficients $\beta_k^{(n,m)}(a)$. The proof of the nonnegativity is based on a recurrence formula given in Lemma 3.6 below. The nonnegativity allows the following probabilistic interpretation.

Corollary 2.5. *Let X, Y be independent Student t-variables with, respectively, $2n+1$, $2m+1$ degrees of freedom. For $0 \le a \le 1$ the distribution of $aX + (1-a)Y$ follows a Student t-distribution with a random number $2f+1$ of degrees of freedom distributed according to*

$$\Pr\{f = k\} = \beta_k^{(n,m)}(a), \qquad n \wedge m \le k \le n+m.$$

The result of Theorem 2.4 can be extended to yield

Theorem 2.6. *For $k \ge 2$, let $n_1, \ldots, n_k$ be nonnegative integers and let $a_1, \ldots, a_k$ be positive real numbers with sum 1. Then*

$$q_{n_1}(a_1 u) q_{n_2}(a_2 u) \cdots q_{n_k}(a_k u) = \sum_{j=l}^{L} \beta_j q_j(u), \qquad u \in \mathbf{R}, \tag{14}$$

where the coefficients β_j are nonnegative with sum 1, $l = \min(n_1, \ldots, n_k)$ and $L = n_1 + \cdots + n_k$.

3. Proofs

3.1. *Generalities about Bessel Polynomials*

As a preparation to the proofs we give some recursion formulas for q_n. They follow from corresponding formulas for θ_n from [12], but they can also be proved directly from the definitions (1) and (2). The formulas are

$$q_{n+1}(u) \ = \ q_n(u) + \frac{u^2}{4n^2 - 1} q_{n-1}(u), \qquad n \ge 1, \tag{15}$$

$$q_n'(u) \ = \ q_n(u) - \frac{u}{2n-1} q_{n-1}(u), \qquad n \ge 1. \tag{16}$$

We can write

$$u^n = \sum_{i=0}^{n} \delta_i^{(n)} q_i(u), \qquad n = 0, 1, \ldots, \tag{17}$$

and $\delta_i^{(n)}$ is given by a formula due to Carlitz [6], see [12, p. 73] or [19]:

$$\delta_i^{(n)} = \begin{cases} \dfrac{(n+1)!}{2^n} \dfrac{(-1)^{n-i}(2i)!}{(n-i)!\, i!\, (2i+1-n)!} & \text{for } \dfrac{n-1}{2} \le i \le n, \\ 0 & \text{for } 0 \le i < \dfrac{n-1}{2}. \end{cases} \tag{18}$$

Later we need the following extension of (15).

Lemma 3.1. *For $0 \le k \le n$ we have*

$$u^{2k} q_{n-k}(u) = \sum_{i=0}^{k} \gamma_i^{(n,k)} q_{n+i}(u), \tag{19}$$

where

$$\gamma_i^{(n,k)} = 2^{2k}\binom{k}{i}(n-k+\tfrac{1}{2})_{k+i}(-n-\tfrac{1}{2})_{k-i}. \tag{20}$$

Proof. The lemma is trivial for $k=0$ and reduces to the recursion (15) for $k=1$ written as

$$u^2 q_{n-1}(u) = 2^2(n-\tfrac{1}{2})_2(q_{n+1}(u)-q_n(u)). \tag{21}$$

We will prove formula (20) by induction in n, so assume it holds for some n and all $0 \le k \le n$. Multiplying the formula of the lemma by u^2 we get

$$u^{2k+2}q_{n-k}(u) = \sum_{i=0}^{k}\gamma_i^{(n,k)}u^2 q_{n+i}(u),$$

hence, by (21),

$$\begin{aligned} u^{2(k+1)}q_{n+1-(k+1)}(u) &= \sum_{i=0}^{k}\gamma_i^{(n,k)}2^2(n+i+\tfrac{1}{2})_2[q_{n+i+2}(u)-q_{n+i+1}(u)] \\ &= \gamma_k^{(n,k)}2^2(n+k+\tfrac{1}{2})_2 q_{n+k+2}(u) \\ &\quad + \sum_{i=1}^{k}2^2(n+i+\tfrac{1}{2})[\gamma_{i-1}^{(n,k)}(n+i-\tfrac{1}{2})-\gamma_i^{(n,k)}(n+i+\tfrac{3}{2})]q_{n+1+i}(u) \\ &\quad - \gamma_0^{(n,k)}2^2(n+\tfrac{1}{2})_2 q_{n+1}(u). \end{aligned}$$

Using the induction hypothesis we easily get

$$\gamma_k^{(n,k)}2^2(n+k+\tfrac{1}{2})_2 = 2^{2k+2}(n-k+\tfrac{1}{2})_{2k+2} = \gamma_{k+1}^{(n+1,k+1)},$$

and

$$-\gamma_0^{(n,k)}2^2(n+\tfrac{3}{2})(n+\tfrac{1}{2}) = 2^{2k+2}(n-k+\tfrac{1}{2})_{k+1}(-n-\tfrac{3}{2})_{k+1} = \gamma_0^{(n+1,k+1)}.$$

Concerning the coefficient C to $q_{n+1+i}(u)$ above we have

$$\begin{aligned} C &= 2^{2k+2}(n+i+\tfrac{1}{2})\left[\binom{k}{i-1}(n-k+\tfrac{1}{2})_{k+i-1}(-n-\tfrac{1}{2})_{k-i+1}(n+i-\tfrac{1}{2})\right. \\ &\qquad \left. - \binom{k}{i}(n-k+\tfrac{1}{2})_{k+i}(-n-\tfrac{1}{2})_{k-i}(n+i+\tfrac{3}{2})\right] \\ &= 2^{2k+2}(n-k+\tfrac{1}{2})_{k+1+i}(-n-\tfrac{1}{2})_{k-i} \\ &\quad \times\left[\binom{k}{i-1}(-n-\tfrac{1}{2}+k-i)-\binom{k}{i}(n+i+\tfrac{3}{2})\right] \\ &= 2^{2k+2}(n-k+\tfrac{1}{2})_{k+1+i}(-n-\tfrac{1}{2})_{k-i}\left[\binom{k+1}{i}(-n-\tfrac{3}{2})\right] \\ &= 2^{2k+2}\binom{k+1}{i}(n-k+\tfrac{1}{2})_{k+1+i}(-n-\tfrac{3}{2})_{k+1-i} = \gamma_i^{(n+1,k+1)}. \end{aligned}$$

■

Remark 3.2. An alternative proof of Lemma 3.1 has kindly been suggested by a referee: Inserting (1) and (20) in the right-hand side of (19) and changing the order of summation, one gets

$$\sum_{i=0}^{k} \gamma_i^{(n,k)} q_{n+i}(u) \;=\; \sum_{j=0}^{n+k} \frac{2^{2k+j}(n-k+\frac{1}{2})_k(-n-\frac{1}{2})_k n!\,(2n-j)!}{(n-j)!\,(2n)!\,j!} \times {}_3F_2\left(\begin{matrix}-k, n-j/2+\frac{1}{2}, n-j/2+1\\ n-k+\frac{3}{2}, n-j+1\end{matrix}; 1\right) u^j.$$

Applying the Pfaff–Saalschütz identity to the ${}_3F_2$, see [1, p. 69], the expression reduces to

$$\sum_{j=2k}^{n+k} \frac{(-n+k)_{j-2k}\, 2^{j-2k}}{(-2n+2k)_{j-2k}\,(j-2k)!} u^j,$$

which is the left-hand side of (19).

We stress that Lemma 3.1 is the special case $\nu = n + \frac{1}{2}$ of the following recursion for modified Bessel functions of the second kind.

Lemma 3.3. *For all* $\nu > 0$ *and all nonnegative integers* $j < \nu$ *we have, for* $u > 0$,

$$u^{\nu+j} K_{\nu-j}(u) = \sum_{i=0}^{j} (-2)^{j-i} \binom{j}{i} \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-(j-i))} u^{\nu+i} K_{\nu+i}(u)$$

and

$$u^{2j} k_{\nu-j}(u) = \sum_{i=0}^{j} (-1)^{j-i} 2^{2j} \binom{j}{i} \frac{\Gamma(\nu+1)\Gamma(\nu+i)}{\Gamma(\nu+1-(j-i))\,\Gamma(\nu-j)} k_{\nu+i}(u).$$

Proof. The second formula follows from the first using formula (6), and the first can be proved by induction using the following recursion formula for modified Bessel functions of the second kind, see [20, p. 79],

$$K_{\nu-1}(u) = K_{\nu+1}(u) - \frac{2\nu}{u} K_\nu(u).$$

We skip the details. ∎

3.2. *Proof of Theorem* 2.1

From (1) and (17) we get

$$q_n(au) = \sum_{j=0}^{n} \alpha_j^{(n)} a^j \sum_{i=0}^{j} \delta_i^{(j)} q_i(u) = \sum_{k=0}^{n} c_k^{(n)}(a) q_k(u) \tag{22}$$

with

$$\begin{aligned} c_k^{(n)}(a) &= \sum_{j=k}^{n} a^j \alpha_j^{(n)} \delta_k^{(j)} \\ &= a^k \frac{n!}{(2n)!} \frac{(2k)!}{k!} \sum_{j=k}^{n\wedge(2k+1)} (-a)^{j-k} \frac{(2n-j)!\,(j+1)}{(n-j)!\,(j-k)!\,(2k+1-j)!}. \end{aligned}$$

In particular, $c_n^{(n)}(a) = a^n$ and, for $0 \le k \le n-1$,

$$c_k^{(n)}(a) = a^k \frac{n!}{(2n)!} \frac{(2k)!}{k!} p(a), \tag{23}$$

where

$$p(a) = \sum_{i=0}^{(n-k)\wedge(k+1)} (-a)^i \frac{(2n-k-i)!\,(k+i+1)}{(n-k-i)!\,i!\,(k+1-i)!}.$$

Putting $a = 1$ in (22) we see that $c_k^{(n)}(1) = 0$ for $0 \le k \le n-1$, so $p(1) = 0$. By Taylor's formula we therefore have

$$p(a) = \sum_{r=1}^{(n-k)\wedge(k+1)} (-1)^r \frac{p^{(r)}(1)}{r!} (1-a)^r$$

with

$$p^{(r)}(1) = \sum_{i=r}^{(n-k)\wedge(k+1)} (-1)^i \frac{(2n-k-i)!\,(k+i+1)}{(n-k-i)!\,(i-r)!\,(k+1-i)!},$$

and we only consider $1 \le r \le (n-k) \wedge (k+1)$. To sum this we shift the summation index by r. For simplicity, we define $T := (n-k-r) \wedge (k+1-r)$ and get

$$(-1)^r p^{(r)}(1) = \sum_{i=0}^{T} (-1)^i \frac{(2n-k-r-i)!\,(k+r+i+1)}{(n-k-r-i)!\,i!\,(k+1-r-i)!}.$$

We write $k+r+1+i = (2k+2) - (k+1-r-i)$ and split the above sum accordingly

$$\begin{aligned} (-1)^r p^{(r)}(1) &= (2k+2) \sum_{i=0}^{T} (-1)^i \frac{(2n-k-r-i)!}{(n-k-r-i)!\,i!\,(k+1-r-i)!} \\ &\quad - \sum_{i=0}^{T} (-1)^i \frac{(2n-k-r-i)!}{(n-k-r-i)!\,i!\,(k-r-i)!}. \end{aligned}$$

Note that for nonnegative integers a, b, c with $b, c \le a$ we have

$$\sum_{i=0}^{b\wedge c} (-1)^i \frac{(a-i)!}{(b-i)!\,(c-i)!\,i!} = \frac{a!}{b!\,c!} \sum_{i=0}^{b\wedge c} \frac{(-b)_i(-c)_i}{(-a)_i i!} = \frac{a!}{b!\,c!}\, {}_2F_1(-b, -c; -a; 1),$$

where we use the fact that the sum is an ${}_2F_1$ evaluated at 1. Its value is given by the Chu–Vandermonde formula, see [1], hence

$$\sum_{i=0}^{b\wedge c}(-1)^i \frac{(a-i)!}{(b-i)!\,(c-i)!\,i!} = \frac{a!\,(c-a)_b}{(-a)_b b!\,c!}.$$

The two sums above are of this form and we get

$$(-1)^r p^{(r)}(1) = \frac{(2n-k-r)!}{(n-k-r)!\,(k+1-r)!}Q,$$

where

$$Q = (2k+2)\frac{(2k-2n+1)_{n-k-r}}{(k+r-2n)_{n-k-r}} - (k+1-r)\frac{(2k-2n)_{n-k-r}}{(k+r-2n)_{n-k-r}}.$$

For $r = n-k$ we have $Q = n+1$ and for $1 \le r \le n-k-1$ we find

$$\begin{aligned} Q &= \frac{(2k-2n+1)_{n-k-r-1}}{(k+r-2n)_{n-k-r}}[(2k+2)(k-r-n)-(k+1-r)(2k-2n)] \\ &= 2r(n+1)\frac{(n+r+1-k)_{n-k-r-1}}{(n+1)_{n-k-r}}, \end{aligned}$$

where we used $(a)_n = (-1)^n(1-a-n)_n$ twice. In both cases we get

$$(-1)^r p^{(r)}(1) = 2r\binom{n+1}{k+1-r}\frac{(2n-2k-1)!}{(n-k-r)!}$$

and, finally,

$$p(a) = \sum_{r=1}^{(n-k)\wedge(k+1)} (1-a)^r \frac{2r}{r!}\binom{n+1}{k+1-r}\frac{(2n-2k-1)!}{(n-k-r)!}.$$

If we insert this expression for $p(a)$ in (23), we get the formulas of Theorem 2.1. ■

Remark 3.4. The evaluation above of $(-1)^r p^{(r)}(1)$ can be done using generating functions as in [19]. The authors want to thank Mogens Esrom Larsen for the idea to use the Chu–Vandermonde identity twice.

3.3. *Proof of Theorem* 2.2

The starting point is the following formula of Macdonald, see [20, 13.71(1)]

$$K_\nu(z)K_\nu(X) = \frac{1}{2}\int_0^\infty \exp\left[-\frac{s}{2}-\frac{z^2+X^2}{2s}\right]K_\nu\left(\frac{zX}{s}\right)\frac{ds}{s}, \tag{24}$$

which we will use for $\nu = n+\frac{1}{2}$, $z = au$, $X = (1-a)u$. Multiplying (24) by

$$\left(\frac{2^{1-\nu}}{\Gamma(\nu)}\right)^2 (a(1-a)u^2)^\nu$$

and using (6) we find

$$k_\nu(au)k_\nu((1-a)u) = \frac{1}{2^\nu\Gamma(\nu)}\int_0^\infty \exp\left[-\frac{s}{2}-u^2\frac{a^2+(1-a)^2}{2s}\right]s^{\nu-1}k_\nu\left(\frac{a(1-a)u^2}{s}\right)\,ds.$$

We now put $\nu = n+\frac{1}{2}$ and use (7) to get

$$e^{-|u|}q_n(a|u|)q_n((1-a)|u|) = \frac{1}{2^{n+1/2}\Gamma(n+\frac{1}{2})}\int_0^\infty \exp\left[-\frac{s}{2}-\frac{u^2}{2s}\right]s^{n-1/2}q_n\left(\frac{a(1-a)u^2}{s}\right)\,ds.$$

We next insert the expression (1) for q_n under the integral sign. This gives

$$\begin{aligned} e^{-|u|}q_n(a|u|)q_n((1-a)|u|) &= \sum_{k=0}^n \alpha_k^{(n)}(a(1-a))^k u^{2k}\frac{1}{2^{n+1/2}\Gamma(n+\frac{1}{2})} \\ &\quad\times\int_0^\infty \exp\left[-\frac{s}{2}-\frac{u^2}{2s}\right]s^{n-k-1/2}\,ds. \end{aligned}$$

Using the following formula, see [10, 3.471(9)] or [20, 6.22(15)],

$$\int_0^\infty x^{\nu-1}\exp\left(-\frac{\beta}{x}-\gamma x\right)\,dx = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2}K_\nu(2\sqrt{\beta\gamma}) \tag{25}$$

and again (6), the above is equal to

$$= \sum_{k=0}^n \alpha_k^{(n)}(a(1-a))^k\frac{2^{n-k+1/2}\Gamma(n-k+\frac{1}{2})}{2^{n+1/2}\Gamma(n+\frac{1}{2})}e^{-|u|}u^{2k}q_{n-k}(|u|).$$

Finally, using Pochhammer symbols and skipping absolute values since we are now dealing with a polynomial identity, we get

$$q_n(au)q_n((1-a)u) = \sum_{k=0}^n \alpha_k^{(n)}(a(1-a))^k\frac{(\frac{1}{2})_{n-k}}{2^k(\frac{1}{2})_n}u^{2k}q_{n-k}(u). \tag{26}$$

Using the expression for $u^{2k}q_{n-k}(u)$ from Lemma 3.1 and the expression (2) for $\alpha_k^{(n)}$ in (26) we get

$$\begin{aligned} q_n(au)q_n((1-a)u) &= \sum_{k=0}^n \frac{(-n)_k(\frac{1}{2})_{n-k}}{(-2n)_k(\frac{1}{2})_n k!}(a(1-a))^k\sum_{i=0}^k \gamma_i^{(n,k)}q_{n+i}(u) \\ &= \sum_{i=0}^n q_{n+i}(u)\sum_{k=i}^n (a(1-a))^k\frac{(-n)_k(\frac{1}{2})_{n-k}}{(-2n)_k(\frac{1}{2})_n k!}\gamma_i^{(n,k)}, \end{aligned}$$

hence,

$$q_n(au)q_n((1-a)u) = \sum_{i=0}^{n} \beta_i^{(n)}(a) q_{n+i}(u),$$

with

$$\begin{aligned}
\beta_i^{(n)}(a) &= \sum_{k=i}^{n} (a(1-a))^k \frac{(-n)_k(\frac{1}{2})_{n-k}}{(-2n)_k(\frac{1}{2})_n k!} 2^{2k} \binom{k}{i} (n-k+\tfrac{1}{2})_{k+i}(-n-\tfrac{1}{2})_{k-i} \\
&= \sum_{k=i}^{n} (4a(1-a))^k \frac{(-n)_k(\frac{1}{2})_{n+i}}{(-2n)_k(\frac{1}{2})_n i!\,(k-i)!} (-n-\tfrac{1}{2})_{k-i} \\
&= \frac{(\frac{1}{2})_{n+i}}{(\frac{1}{2})_n i!} (4a(1-a))^i \sum_{l=0}^{n-i} (4a(1-a))^l \frac{(-n)_{i+l}(-n-\frac{1}{2})_l}{(-2n)_{i+l} l!} \\
&= \frac{(\frac{1}{2})_{n+i}(-n)_i}{(\frac{1}{2})_n i!(-2n)_i} (4a(1-a))^i \sum_{l=0}^{n-i} (1-(2a-1)^2)^l \frac{(-n+i)_l(-n-\frac{1}{2})_l}{(-2n+i)_l l!} \\
&= \frac{(\frac{1}{2})_{n+i}(-n)_i}{(\frac{1}{2})_n i!(-2n)_i} (4a(1-a))^i {}_2F_1\left(\begin{matrix}-n+i, -n-\frac{1}{2} \\ -2n+i\end{matrix}; 1-(2a-1)^2\right).
\end{aligned}$$

To this formula we apply the Pfaff transformation formula

$${}_2F_1\left(\begin{matrix}-n, b \\ c\end{matrix}; 1-z\right) = \frac{(c-b)_n}{(c)_n} {}_2F_1\left(\begin{matrix}-n, b \\ b+1-n-c\end{matrix}; z\right),$$

see [1, (2.3.14), p. 79], and we get

$$\begin{aligned}
\beta_i^{(n)}(a) &= \frac{(4a(1-a))^i}{2^{2n}} \frac{(\frac{1}{2})_{n+i}(\frac{1}{2})_{n-i}}{(\frac{1}{2})_n^2} \binom{n}{i} {}_2F_1\left(\begin{matrix}-n+i, -n-\frac{1}{2} \\ \frac{1}{2}\end{matrix}; (2a-1)^2\right) \\
&= \frac{(4a(1-a))^i}{2^{2n}} \frac{(n+\frac{1}{2})_i(-n)_i}{(-n+\frac{1}{2})_i i!} {}_2F_1\left(\begin{matrix}-n+i, -n-\frac{1}{2} \\ \frac{1}{2}\end{matrix}; (2a-1)^2\right),
\end{aligned}$$

which is the second expression of Theorem 2.2. The first expression is a simple reformulation of the second. ■

Remark 3.5. Formula (26) is a consequence of a finite-sum rule for $K_{n+1/2}$ given in [15, Theorem 2.2]. Formula (26) is very similar to a formula of Bateman for Jacobi polynomials, see formula (2.19) in [13]. Both there and in the present paper, there is an intimate relation between a formula of this type on the one hand, and an integral representation and product formula on the other hand.

3.4. *Proof of Theorem* 2.4

For $n, m \geq 0$ and $a \in \mathbf{R}$, we can write

$$q_n(au)q_m((1-a)u) = \sum_{k=0}^{m+n} \beta_k^{(n,m)}(a) q_k(u) \tag{27}$$

for some uniquely determined coefficients since the left-hand side is a polynomial in u of degree $\le n+m$. Clearly $\beta_k^{(n,m)}(a)$ is a polynomial in a satisfying

$$\beta_k^{(n,m)}(a) = \beta_k^{(m,n)}(1-a). \tag{28}$$

We shall prove that $\beta_k^{(n,m)}(a) \ge 0$ for $0 \le a \le 1$ and that $\beta_k^{(n,m)}(a) = 0$ if $k < n \wedge m$, which will be a consequence of the following recursion formula.

Lemma 3.6. *For $n, m \ge 1$, we have*

$$\frac{1}{2k+1}\beta_{k+1}^{(n,m)}(a) = \frac{a^2}{2n-1}\beta_k^{(n-1,m)}(a) + \frac{(1-a)^2}{2m-1}\beta_k^{(n,m-1)}(a), \tag{29}$$

where $k = 0, 1, \dots, m+n-1$.

Furthermore, $\beta_0^{(n,m)}(a) = 0$.

Proof. Differentiating (27) with respect to u gives

$$aq_n'(au)q_m((1-a)u) + (1-a)q_n(au)q_m'((1-a)u) = \sum_{k=1}^{m+n} \beta_k^{(n,m)}(a)q_k'(u)$$

and, using formula (16), we find

$$\begin{aligned} a&\left(q_n(au) - \frac{au}{2n-1}q_{n-1}(au)\right) q_m((1-a)u) \\ &\qquad + (1-a)q_n(au)\left(q_m((1-a)u) - \frac{(1-a)u}{2m-1}q_{m-1}((1-a)u)\right) \\ &= \sum_{k=1}^{m+n} \beta_k^{(n,m)}(a)\left(q_k(u) - \frac{u}{2k-1}q_{k-1}(u)\right) \end{aligned}$$

and using (27) once more we get

$$\begin{aligned} -\frac{a^2u}{2n-1}&q_{n-1}(au)q_m((1-a)u) - \frac{(1-a)^2u}{2m-1}q_n(au)q_{m-1}((1-a)u) \\ &= -\beta_0^{(n,m)}(a) - u\sum_{k=0}^{n+m-1} \beta_{k+1}^{(n,m)}(a)(2k+1)^{-1}q_k(u). \end{aligned}$$

For $u = 0$ this gives $\beta_0^{(n,m)}(a) = 0$ and dividing by $-u$ and equating the coefficients of $q_k(u)$, we get the desired formula. ■

Now the proof of Theorem 2.4 is easy by induction in k. Let $0 \le a \le 1$. We prove for fixed k that $\beta_k^{(n,m)}(a) \ge 0$ if $k \le n+m$ and that $\beta_k^{(n,m)}(a) = 0$ if $k < n \wedge m$. This is true for $k = 0$ by Lemma 3.6 when $n, m \ge 1$, and for $n = 0$ or $m = 0$ it follows by Theorem 2.1 since

$$\beta_0^{(n,0)}(a) = \beta_0^{(0,n)}(1-a) = c_0^{(n)}(a) \ge 0.$$

Assume now that the result holds for $k = k_0$ and assume $k_0+1 \le n+m$. The nonnegativity for $k = k_0 + 1$ now follows by Lemma 3.6 when $n, m \ge 1$, and when $n = 0$ or $m = 0$ it follows again by Theorem 2.1 that

$$\beta_k^{(n,0)}(a) = \beta_k^{(0,n)}(1-a) = c_k^{(n)}(a) \ge 0.$$

Finally, if $k_0 + 1 < n \wedge m$, then $\beta_{k_0+1}^{(n,m)}(a) = 0$ by Lemma 3.6 because $k_0 < (n-1) \wedge (m-1)$. ∎

3.5. *Proof of Theorem* 2.6

By Theorem 2.4 the result holds for $k = 2$. Assuming it holds for $k - 1 \ge 2$ we have

$$q_{n_1}(a_1u)\cdots q_{n_{k-1}}(a_{k-1}u) = \sum_{j=l'}^{L'} \gamma_j q_j((1-a_k)u), \qquad u \in \mathbf{R}, \tag{30}$$

with $l' = \min(n_1, \ldots, n_{k-1})$, $L' = n_1 + \cdots + n_{k-1}$, and $\gamma_j \ge 0$ because we can write

$$a_j u = \frac{a_j}{1-a_k}(1-a_k)u, \qquad j = 1, \ldots, k-1.$$

If we multiply (21) with $q_{n_k}(a_k u)$ we get

$$\sum_{j=l'}^{L'} \gamma_j q_{n_k}(a_k u) q_j((1-a_k)u) = \sum_{j=l'}^{L'} \gamma_j \sum_{i=n_k\wedge j}^{n_k+j} \beta_i^{(n_k,j)}(a_k) q_i(u),$$

and the assertion follows. ∎

4. Inverse Gamma Distribution

Grosswald proved [11] that the Student t-distribution is infinitely divisible. This is a consequence of the infinite divisibility of the inverse Gamma distribution because of subordination. It was proved later that the inverse Gamma distribution is a generalized Gamma convolution in the sense of Thorin, which is stronger than self-decomposability and in particular stronger than infinite divisibility, see, e.g., Bondesson [4] and the recent book by Steutel and van Harn [17].

The following density on the half-line is an inverse Gamma density with scale parameter $\frac{1}{4}$ and shape parameter $\nu > 0$:

$$C_\nu \exp\left(-\frac{1}{4t}\right) t^{-\nu-1}, \qquad t > 0, \quad C_\nu = \frac{1}{2^{2\nu}\Gamma(\nu)}. \tag{31}$$

Let the corresponding probability measure be denoted $\tilde{\gamma}_\nu$ and let further

$$g_t(x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right), \qquad t > 0, \quad x \in \mathbf{R},$$

denote the Gaussian semigroup of normal densities (in the normalization of [3]). Then the mixture

$$f_\nu(x) = \int_0^\infty g_t(x)\, d\tilde{\gamma}_\nu(t) \tag{32}$$

is the Student t-density (4) with 2ν degrees of freedom. The corresponding probability measure is denoted by σ_ν. This formula says that σ_ν is subordinated to the Gaussian semigroup by an inverse Gamma distribution $\tilde{\gamma}_\nu$. Since the Laplace transformation is one-to-one, it is clear that if two probabilities γ_1, γ_2 on $]0,\infty[$ lead to the same subordinated density

$$\int_0^\infty g_t(x)\, d\gamma_1(t) = \int_0^\infty g_t(x)\, d\gamma_2(t), \qquad x \in \mathbf{R},$$

then $\gamma_1 = \gamma_2$.

If we denote $\tau_a(x) = ax$, the distribution $\tau_a(\sigma_{n+1/2}) * \tau_{1-a}(\sigma_{m+1/2})$ is given in (11). However, note that $\tau_a(g_t(x)\, dx) = g_{ta^2}(x)\, dx$, so

$$\tau_a(\sigma_\nu) = \int_0^\infty g_{ta^2}(x)\, d\tilde{\gamma}_\nu(t)\, dx, \tag{33}$$

hence

$$\begin{aligned}
\tau_a(\sigma_{\nu_1}) * \tau_{1-a}(\sigma_{\nu_2}) &= \int_0^\infty \int_0^\infty (g_{ta^2}\, dx) * (g_{s(1-a)^2}\, dx)\, d\tilde{\gamma}_{\nu_1}(t)\, d\tilde{\gamma}_{\nu_2}(s) \\
&= \int_0^\infty \int_0^\infty (g_{ta^2+s(1-a)^2}\, dx)\, d\tilde{\gamma}_{\nu_1}(t)\, d\tilde{\gamma}_{\nu_2}(s) \\
&= \int_0^\infty g_u(x)\, d\tau_{a^2}(\tilde{\gamma}_{\nu_1}) * \tau_{(1-a)^2}(\tilde{\gamma}_{\nu_2})(u)\, dx.
\end{aligned}$$

Therefore, using (33) we see that for $\nu_1 = n + \frac{1}{2}$, $\nu_2 = m + \frac{1}{2}$ with $n, m = 0, 1, \ldots$, formula (11), rewritten as

$$\tau_a(\sigma_{n+1/2}) * \tau_{1-a}(\sigma_{m+1/2}) = \sum_{k=n\wedge m}^{n+m} \beta_k^{(n,m)}(a)\sigma_{k+1/2},$$

is equivalent to

$$\tau_{a^2}(\tilde{\gamma}_{n+1/2}) * \tau_{(1-a)^2}(\tilde{\gamma}_{m+1/2}) = \sum_{k=n\wedge m}^{n+m} \beta_k^{(n,m)}(a)\tilde{\gamma}_{k+1/2}. \tag{34}$$

This shows that Theorem 2.4 is equivalent to the following result about inverse Gamma distributions.

Theorem 4.1. *Let Z_n, Z_m be independent inverse Gamma variables with scale parameters $\frac{1}{4}$ and shape parameters $n + \frac{1}{2}, m + \frac{1}{2}$, respectively. For $0 \le a \le 1$ the variable $a^2 Z_n + (1-a)^2 Z_m$ follows an inverse Gamma distribution with scale parameter $\frac{1}{4}$ and random shape parameter $k + \frac{1}{2}$, where $k \in [n \wedge m, n+m]$ is distributed according to*

$$\Pr\{k = j\} = \beta_j^{(n,m)}(a), \qquad j = n \wedge m, \ldots, n+m.$$

The theorem has been observed for small values of n and m in Witkovský [21]. It is also stated in [9], but there is no convincing proof of the positivity of the coefficients $\beta_j^{(n,m)}(a)$.

Theorem 4.1 can be used to extend our results to multivariate Student t-distributions as follows. A rotation invariant N-variate Student t-probability density is given for $\mathbf{x} = (x_1, \dots, x_N) \in \mathbf{R}^N$ by

$$f_{N,\nu}(\mathbf{x}) = A_{N,\nu}(1 + |\mathbf{x}|^2)^{-\nu-N/2}, \qquad A_{N,\nu} = \frac{\Gamma(\nu + N/2)}{\Gamma(\nu)(\Gamma(\frac{1}{2}))^N},$$

where

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{N} x_i y_i, \qquad |\mathbf{x}| = (\langle \mathbf{x}, \mathbf{x} \rangle)^{1/2}, \quad \mathbf{x}, \mathbf{y} \in \mathbf{R}^N.$$

It is easy to verify that $f_{N,\nu}(\mathbf{x})$ is subordinated to the N-variate Gaussian semigroup

$$g_{N,t}(\mathbf{x}) = (4\pi t)^{-N/2} \exp\left(-\frac{|\mathbf{x}|^2}{4t}\right), \qquad t > 0, \quad \mathbf{x} \in \mathbf{R}^N,$$

by the inverse Gamma density (31), i.e.,

$$f_{N,\nu}(\mathbf{x}) = \int_0^\infty g_{N,t}(\mathbf{x})\, d\tilde{\gamma}_\nu(t). \tag{35}$$

Therefore the characteristic function is given by

$$\int_{\mathbf{R}^N} e^{i\langle \mathbf{x}, \mathbf{y} \rangle} f_{N,\nu}(\mathbf{x})\, d\mathbf{x} = k_\nu(|\mathbf{y}|) \tag{36}$$

generalizing (5). In fact,

$$\begin{aligned} \int_{\mathbf{R}^N} e^{i\langle \mathbf{x}, \mathbf{y} \rangle} f_{N,\nu}(\mathbf{x})\, d\mathbf{x} &= \int_0^\infty \left(\int_{\mathbf{R}^N} e^{i\langle \mathbf{x}, \mathbf{y} \rangle} g_{N,t}(\mathbf{x})\, d\mathbf{x} \right) d\tilde{\gamma}_\nu(t) \\ &= \int_0^\infty e^{-t|\mathbf{y}|^2}\, d\tilde{\gamma}_\nu(t) \end{aligned}$$

and the result follows by (25) after substituting t by $1/t$.

As a conclusion, Theorems 2.1, 2.2, and 2.4 apply in the multivariate case. For example, in analogy with (11) we have, for $0 < a < 1$,

$$\frac{1}{a^N} f_{N,n+1/2}(a^{-1}\mathbf{x}) * \frac{1}{(1-a)^N} f_{N,m+1/2}((1-a)^{-1}\mathbf{x}) = \sum_{k=n\wedge m}^{n+m} \beta_k^{(n,m)}(a) f_{N,k+1/2}(\mathbf{x}).$$

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