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CONSTRUCTIVE APPROXIMATION

Function Spaces in Lipschitz Domains and Optimal Rates of Convergence for Sampling

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Abstract. Assume that we want to recover $f : \Omega \to \mathbf{C}$ in the L_r-quasi-norm ($0 < r \leq \infty$) by a linear sampling method

$$S_n f = \sum_{j=1}^{n} f(x^j) h_j,$$

where $h_j \in L_r(\Omega)$ and $x^j \in \Omega$ and $\Omega \subset \mathbf{R}^d$ is an arbitrary bounded Lipschitz domain. We assume that f is from the unit ball of a Besov space $B^s_{pq}(\Omega)$ or of a Triebel–Lizorkin space $F^s_{pq}(\Omega)$ with parameters such that the space is compactly embedded into $C(\overline{\Omega})$. We prove that the optimal rate of convergence of linear sampling methods is

$$n^{-s/d+(1/p-1/r)_+},$$

nonlinear methods do not yield a better rate. To prove this we use a result from Wendland (2001) as well as results concerning the spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$. Actually, it is another aim of this paper to complement the existing literature about the function spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ for bounded Lipschitz domains $\Omega \subset \mathbf{R}^d$. In this sense, the paper is also a continuation of a paper by Triebel (2002).

1. Introduction

Let us start with a question concerning the classical Sobolev spaces $W^k_p(\Omega)$ on an arbitrary bounded (nonempty) Lipschitz domain $\Omega \subset \mathbf{R}^d$. Assume that we want to recover $f \in W^k_p(\Omega)$ in the L_r-norm by a linear sampling method

$$S_n f = \sum_{j=1}^{n} f(x^j) h_j, \tag{1.1}$$

where $h_j \in L_r(\Omega)$ and $x^j \in \Omega$. This makes sense if $pk > d$, then we have $W^k_p(\Omega) \hookrightarrow C(\overline{\Omega})$ as a compact embedding. Just now we assume $1 \leq p \leq \infty$ and $1 \leq r \leq \infty$, later we will study much more general spaces. It is natural to consider the worst case error of

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S_n, on the unit ball of $W_p^k(\Omega)$, given by

$$\sup\{\|f - S_n f|L_r(\Omega)\| : \|f|W_p^k(\Omega)\| \le 1\}. \tag{1.2}$$

We also use the same worst case error for nonlinear sampling methods

$$S_n f = \varphi(f(x^1), f(x^2), \dots, f(x^n)), \tag{1.3}$$

where $\varphi : \mathbf{C}^n \to L_r(\Omega)$ is now an arbitrary mapping. What is the optimal rate of convergence for linear (1.1) or nonlinear (1.3) sampling methods?

There is a vast literature about this question for $\Omega = [0, 1]^d$ and also for the periodic case, i.e., for the torus. In these cases it is well known (but we do not know who proved this first) that

$$\inf_{S_n} \sup\{\|f - S_n f|L_r(\Omega)\| : \|f|W_p^k(\Omega)\| \le 1\} \asymp n^{-k/d+(1/p-1/r)_+}, \tag{1.4}$$

see, e.g., [4] or [9]. This is true if we allow only linear methods and it is also true if we allow arbitrary nonlinear methods. Hence, in this sense, linear methods are optimal. To prove the upper bound, the known proofs of (1.4) heavily use the fact that we can divide $\Omega = [0, 1]^d$ into ℓ^d equal smaller cubes. Then one can use piecewise polynomial interpolation to obtain an order optimal method. In this paper we use a result of Wendland [23] to prove that (1.4) is correct for arbitrary bounded Lipschitz domains. It is interesting to compare this order with the known order of the approximation numbers. Instead of (1.1) we now allow methods

$$S_n f = \sum_{j=1}^{n} L_j(f) h_j, \tag{1.5}$$

where the $L_j : W_p^k(\Omega) \to \mathbf{C}$ are arbitrary continuous linear functionals. It turns out that the rate of convergence for methods (1.5) "based on general information" is better than the rate (1.4) of methods "based on standard information (or function values)" if, and only if,

$$p < 2 < r. \tag{1.6}$$

Actually we consider much more general function spaces and, therefore, it is the first aim of this paper to complement the existing literature about function spaces of type B_{pq}^s or F_{pq}^s for bounded Lipschitz domains $\Omega \subset \mathbf{R}^d$, see Triebel [21]. These spaces are considered as subspaces of $D'(\Omega)$, where we restrict ourselves to

$$0 < p \le \infty, \quad 0 < q \le \infty \quad \text{and} \quad s > d\left(\frac{1}{p} - 1\right)_+ \tag{1.7}$$

(with $p < \infty$ for the F-spaces). Then one has the compact embeddings

$$B_{pq}^s(\Omega) \hookrightarrow L_1(\Omega) \quad \text{and} \quad F_{pq}^s(\Omega) \hookrightarrow L_1(\Omega). \tag{1.8}$$

These two scales cover many well-known distinguished spaces such as:

- the (fractional and classical) Sobolev spaces

$$F^s_{p,2}(\Omega) = H^s_p(\Omega) \quad \text{and} \quad F^k_{p,2} = W^k_p(\Omega) \quad \text{where} \quad s > 0, \quad k \in \mathbf{N}_0, \tag{1.9}$$

  and $1 < p < \infty$;
- the classical Besov spaces

$$B^s_{pq}(\Omega), \quad 1 \le p < \infty, \quad 1 \le q \le \infty, \quad s > 0; \tag{1.10}$$

- and the Hölder–Zygmund spaces

$$\mathcal{C}^s(\Omega) = B^s_{\infty\infty}(\Omega), \quad s > 0. \tag{1.11}$$

We define these spaces in Section 2 as restrictions of the corresponding spaces in $\mathbf{R}^d$ to Ω and discuss afterward the intrinsic characterizations in terms of differences and derivatives. This might be considered as a continuation of [21]. However, we now stress those specific assertions needed later on. In Section 3 we restrict (1.7), preferably by

$$0 < p \le \infty, \quad 0 < q \le \infty \quad \text{and} \quad s > d/p, \tag{1.12}$$

(again $p < \infty$ for the F-spaces). Then (1.8) can be strengthened by

$$B^s_{pq}(\Omega) \hookrightarrow C(\overline{\Omega}) \quad \text{and} \quad F^s_{pq}(\Omega) \hookrightarrow C(\overline{\Omega}), \tag{1.13}$$

where the embeddings are compact. The target space $C(\overline{\Omega})$ can be replaced by the larger space $L_r(\Omega)$ with $0 < r \le \infty$. For brevity, let either $A = B$ or $A = F$ (with $p < \infty$ in the F-case). Of interest is the degree of compactness of the embeddings

$$id : G_1(\Omega) = A^s_{pq}(\Omega) \hookrightarrow L_r(\Omega) = G_2(\Omega), \tag{1.14}$$

where p, q, s are restricted by (1.12) and $0 < r \le \infty$. For this purpose we introduce, for $n \in \mathbf{N}$, the *sampling numbers* g_n and g_n^{lin}. Here

$$g_n(id) = \inf[\sup\{\|f - S_n f|G_2(\Omega)\| \, : \, \|f|G_1(\Omega)\| \le 1\}] \tag{1.15}$$

with $S_n = \varphi_n \circ N_n$, where the *information map* N_n is of the form

$$\begin{aligned} &N_n : G_1(\Omega) \to \mathbf{C}^n, \\ &N_n f = (f(x^1), \ldots, f(x^n)), \end{aligned} \tag{1.16}$$

with $\{x^j\}_{j=1}^n \subset \Omega$. Since $\varphi_n : \mathbf{C}^n \to G_2(\Omega)$ we obtain

$$S_n f = \varphi_n(f(x^1), \ldots, f(x^n)) \in G_2(\Omega) \quad \text{with} \quad f \in G_1(\Omega). \tag{1.17}$$

The infimum in (1.15) is taken over all n-tuples $\{x^j\}_{j=1}^n \subset \Omega$ and all φ_n. If in (1.17) only linear mappings S_n,

$$S_n f = \sum_{j=1}^n f(x^j) h_j, \quad h_j \in G_2(\Omega), \quad f \in G_1(\Omega), \tag{1.18}$$

are admitted, then the resulting numbers are denoted by $g_n^{\mathrm{lin}}(id)$. We get

$$g_n(id) \asymp g_n^{\mathrm{lin}}(id) \asymp n^{-s/d+(1/p-1/r)_+}, \qquad n \in \mathbf{N}. \tag{1.19}$$

This might be considered as the main result of this paper, Theorem 23. See Remark 24 for further comments. Moreover, we compare these sampling numbers with the *approximation numbers* $a_n(id)$ and the *entropy numbers* $e_n(id)$ and get, for $r \geq 1$,

$$n^{-s/d} \asymp e_n(id) \preceq a_n(id) \preceq g_n(id) \asymp n^{-s/d+(1/p-1/r)_+}, \qquad n \in \mathbf{N}, \tag{1.20}$$

Theorem 26. We clarify for which parameters $\preceq$ in one or both occurrences can be replaced by $\asymp$. Longer proofs are to be found in Section 4.

2. Function Spaces in Lipschitz Domains

2.1. *Basic Notation*, *Spaces in* $\mathbf{R}^d$

We will use standard notation. Let $\mathbf{N}$ be the collection of all natural numbers. Let $\mathbf{R}^d$ be the Euclidean d-space, where $d \in \mathbf{N}$; put $\mathbf{R} = \mathbf{R}^1$; whereas $\mathbf{C}$ is the complex plane. Furthermore, $a_+ = \max(a, 0)$ if $a \in \mathbf{R}$.

Let $S(\mathbf{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing, infinitely differentiable functions on $\mathbf{R}^d$. By $S'(\mathbf{R}^d)$ we denote the topological dual, the space of all tempered distributions on $\mathbf{R}^d$. Furthermore, $L_p(\mathbf{R}^d)$, with $0 < p \leq \infty$, is the standard complex quasi-Banach space with respect to Lebesgue measure, quasi-normed by

$$\|f \,|L_p(\mathbf{R}^d)\| = \left(\int_{\mathbf{R}^d} |f(x)|^p \, dx \right)^{1/p} \tag{2.1}$$

with the obvious modification if $p = \infty$. If $\psi \in S(\mathbf{R}^d)$, then

$$\widehat{\psi}(\xi) = (F\psi)(\xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} e^{-ix\xi}\, \psi(x)\, dx, \qquad x \in \mathbf{R}^d, \tag{2.2}$$

denotes the Fourier transform of ψ. As usual, $F^{-1}\psi$ or $\psi^\vee$ stands for the inverse Fourier transform, given by the right-hand side of (2.2) with i in place of $-i$. Here $x\xi$ denotes the scalar product in $\mathbf{R}^d$. Both F and F^{-1} are extended to $S'(\mathbf{R}^d)$ in the standard way. Let $\psi \in S(\mathbf{R}^d)$ with

$$\psi(x) = 1 \quad \text{if } |x| \leq 1 \qquad \text{and} \qquad \psi(y) = 0 \quad \text{if } |y| \geq \tfrac{3}{2}. \tag{2.3}$$

We put $\psi_0 = \psi$ and

$$\psi_j(x) = \psi(2^{-j}x) - \psi(2^{-j+1}x), \qquad x \in \mathbf{R}^d, \quad j \in \mathbf{N}. \tag{2.4}$$

Then, since

$$\sum_{k=0}^{\infty} \psi_k(x) = 1 \qquad \text{for all} \quad x \in \mathbf{R}^d, \tag{2.5}$$

the ψ_k form a dyadic resolution of unity in $\mathbf{R}^d$. Recall that $(\psi_k \widehat{f})^\vee$ is an entire analytic function on $\mathbf{R}^d$ for any $f \in S'(\mathbf{R}^d)$. In particular, $(\psi_k \widehat{f})^\vee(x)$ makes sense pointwise.

Definition 1. Let $s \in \mathbf{R}$ and $0 < q \leq \infty$.

(i) Let $0 < p \leq \infty$. Then $B^s_{pq}(\mathbf{R}^d)$ is the collection of all $f \in S'(\mathbf{R}^d)$ such that

$$\|f|B^s_{pq}(\mathbf{R}^d)\|_\psi = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\psi_j \widehat{f})^\vee |L_p(\mathbf{R}^d)\|^q\right)^{1/q} \tag{2.6}$$

(with the usual modification if $q = \infty$) is finite.

(ii) Let $0 < p < \infty$. Then $F^s_{pq}(\mathbf{R}^d)$ is the collection of all $f \in S'(\mathbf{R}^d)$ such that

$$\|f|F^s_{pq}(\mathbf{R}^d)\|_\psi = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\psi_j \widehat{f})(\cdot)|^q\right)^{1/q} |L_p(\mathbf{R}^d)\right\| \tag{2.7}$$

(with the usual modification if $q = \infty$) is finite.

Remark 2. These spaces, including their forerunners and special cases, have a long history. Systematic treatments have been given in [18], [19], where Chapter 1 in the latter book is an historically orientated survey. Both $B^s_{pq}(\mathbf{R}^d)$ and $F^s_{pq}(\mathbf{R}^d)$ are quasi-Banach spaces which are independent of the function ψ according to (2.3), in the sense of equivalent quasi-norms. This justifies our omission of the subscript ψ in (2.6) and (2.7) in what follows. If $p \geq 1$ and $q \geq 1$, then both $B^s_{pq}(\mathbf{R}^d)$ and $F^s_{pq}(\mathbf{R}^d)$ are Banach spaces.

2.2. *Special Cases and Characterizations for Spaces in* $\mathbf{R}^d$

We are mainly interested in spaces of type B^s_{pq} and F^s_{pq} in bounded Lipschitz domains where p, q, s are restricted by (1.7) or even by (1.12). To prepare our respective considerations we now have a closer look at some special cases of the above spaces in $\mathbf{R}^d$ and those equivalent (quasi-)norms which will play a role later on.

(i) Let $1 < p < \infty$ and let $k \in \mathbf{N}$. Then

$$F^0_{p,2}(\mathbf{R}^d) = L_p(\mathbf{R}^d) \qquad \text{and} \qquad F^k_{p,2}(\mathbf{R}^d) = W^k_p(\mathbf{R}^d), \tag{2.8}$$

where the latter are the *classical Sobolev spaces* usually normed by

$$\|f|W^k_p(\mathbf{R}^d)\| = \sum_{|\alpha| \leq k} \|D^\alpha f|L_p(\mathbf{R}^d)\|. \tag{2.9}$$

This may be found in [18, 2.5.6], and the references given there.

(ii) Let $x \in \mathbf{R}^d$, $h \in \mathbf{R}^d$, and $M \in \mathbf{N}$. Then

$$(\Delta^{M+1}_h f)(x) = (\Delta^1_h \Delta^M_h f)(x) \quad \text{with} \quad (\Delta^1_h f)(x) = f(x+h) - f(x), \tag{2.10}$$

are the usual differences in $\mathbf{R}^d$. It is well known that the *classical Besov spaces*

$$B^s_{pq}(\mathbf{R}^d) \quad \text{with} \quad 1 \leq p < \infty, \quad 1 \leq q \leq \infty, \quad s > 0, \tag{2.11}$$

can be characterized in many ways in terms of these differences Δ_h^M or in combinations of some differences and some derivatives. This can be extended to the spaces

$$B^s_{pq}(\mathbf{R}^d) \quad \text{with} \quad 0 < p \le \infty, \quad 0 < q \le \infty, \quad s > \sigma_p = d\left(\frac{1}{p} - 1\right)_+. \tag{2.12}$$

We refer to [18, 2.5.12], and [19, 2.6.1]. We restrict ourselves to an example which will be of some service later on.

Let

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad \sigma_p < s < M \in \mathbf{N}, \quad \overline{p} = \max(p, 1). \tag{2.13}$$

Then $f \in L_{\overline{p}}(\mathbf{R}^d)$ *belongs to* $B^s_{pq}(\mathbf{R}^d)$ *if, and only if,*

$$\begin{aligned} &\|f|B^s_{pq}(\mathbf{R}^d)\|_M \\ &\qquad = \|f|L_{\overline{p}}(\mathbf{R}^d)\| + \left(\int_0^1 t^{-sq} \sup_{|h|\le t} \|\Delta_h^M f|L_p(\mathbf{R}^d)\|^q \frac{dt}{t}\right)^{1/q} < \infty \end{aligned} \tag{2.14}$$

(*equivalent quasi-norms*).

This is covered by the Theorem and Remark 3 in [18, pp. 110, 113], and by the embedding theorem, as far as the replacement of

$$\|f|L_p(\mathbf{R}^d)\| \quad \text{by} \quad \|f|L_1(\mathbf{R}^d)\|,$$

in the case of $p < 1$, is concerned.

(iii) As a special case of (2.12) we mention the Hölder–Zygmund spaces

$$\mathcal{C}^s(\mathbf{R}^d) = B^s_{\infty\infty}(\mathbf{R}^d), \qquad s > 0, \tag{2.15}$$

which can be characterized according to (2.14) with $0 < s < M \in \mathbf{N}$ as the collection of all $f \in L_\infty(\mathbf{R}^d)$ such that

$$\|f|\mathcal{C}^s(\mathbf{R}^d)\|_M = \|f|L_\infty(\mathbf{R}^d)\| + \sup |h|^{-s}|(\Delta_h^M f)(x)| < \infty, \tag{2.16}$$

where the supremum is taken over all $x \in \mathbf{R}^d$ and over all $h \in \mathbf{R}^d$ with $0 < |h| \le 1$.

(iv) In generalization of (2.8) one has

$$F^s_{p,2}(\mathbf{R}^d) = H^s_p(\mathbf{R}^d), \qquad 1 < p < \infty, \quad s \in \mathbf{R}, \tag{2.17}$$

where $H^s_p(\mathbf{R}^d)$ are the (fractional) *Sobolev spaces*, previously denoted as Bessel potential spaces. This may be found in [18, 2.5.6], and the references given there.

(v) The question arises whether one has similar characterizations for the spaces $F^s_{pq}(\mathbf{R}^d)$ with the special cases $H^s_p(\mathbf{R}^d)$ according to (2.17) as in (2.13), (2.14) for the spaces $B^s_{pq}(\mathbf{R}^d)$. There are assertions of this type. We refer to [18, 2.5.10], but we do not formulate them. However, later on we need characterizations of both the B^s_{pq} spaces and of the F^s_{pq} spaces in terms of the ball means of differences which we are now going to describe in some detail. Let $M \in \mathbf{N}$ and let Δ^M_h be the differences according to (2.10). Then, for $0 < u \le \infty$,

$$d^M_{t,u} f(x) = \left(t^{-d} \int_{|h|\le t} |\Delta^M_h f(x)|^u \, dh \right)^{1/u}, \qquad x \in \mathbf{R}^d, \quad t > 0, \tag{2.18}$$

(with the usual modification if $u = \infty$) are ball means. Then one has the following characterizations. Let $1 \le r \le \infty$ and let $\overline{p} = \max(1, p)$.

(B) *Let*

$$0 < p \le \infty, \quad 0 < q \le \infty, \qquad d\left(\frac{1}{p} - \frac{1}{r}\right)_+ < s < M \in \mathbf{N}, \tag{2.19}$$

and $0 < u \le r$. *Then* $B^s_{pq}(\mathbf{R}^d)$ *is the collection of all* $f \in L_{\max(p,r)}(\mathbf{R}^d)$ *such that*

$$\| f \,|\, L_{\overline{p}}(\mathbf{R}^d)\| + \left(\int_0^1 t^{-sq} \, \| d^M_{t,u} f \,|\, L_p(\mathbf{R}^d)\|^q \, \frac{dt}{t} \right)^{1/q} < \infty \tag{2.20}$$

(*modification if* $q = \infty$) *in the sense of equivalent quasi-norms.*

(F) *Let*

$$0 < p < \infty, \quad 0 < q \le \infty, \qquad d\left(\frac{1}{\min(p,q)} - \frac{1}{r}\right)_+ < s < M \in \mathbf{N}, \tag{2.21}$$

and $0 < u \le r$. *Then* $F^s_{pq}(\mathbf{R}^d)$ *is the collection of all* $f \in L_{\max(p,r)}(\mathbf{R}^d)$ *such that*

$$\| f \,|\, L_{\overline{p}}(\mathbf{R}^d)\| + \left\| \left(\int_0^1 t^{-sq} \, d^M_{t,u} f(\cdot)^q \, \frac{dt}{t} \right)^{1/q} \Big| L_p(\mathbf{R}^d) \right\| < \infty \tag{2.22}$$

(*modification if* $q = \infty$) *in the sense of equivalent quasi-norms.*

We refer to [19, 3.5.3], where one finds a proof of this assertion. The replacement of $\| f \,|\, L_p(\mathbf{R}^d)\|$ in [19] by $\| f \,|\, L_{\overline{p}}(\mathbf{R}^d)\|$ is immaterial and covered by embedding theorems.

2.3. *Spaces in Lipschitz Domains*

Let $d - 1 \in \mathbf{N}$. Recall that

$$x' \in \mathbf{R}^{d-1} \mapsto h(x') \in \mathbf{R} \tag{2.23}$$

is called a Lipschitz function (on $\mathbf{R}^{d-1}$) if there is a number $c > 0$ such that

$$|h(x') - h(y')| \le c\, |x' - y'| \qquad \text{for all} \quad x' \in \mathbf{R}^{d-1}, \quad y' \in \mathbf{R}^{d-1}. \tag{2.24}$$

Definition 3. Let $d - 1 \in \mathbf{N}$.

(i) A *special Lipschitz domain* in $\mathbf{R}^d$ is the collection of all points $x = (x', x_d)$ with $x' \in \mathbf{R}^{d-1}$ such that

$$h(x') < x_d < \infty, \tag{2.25}$$

where $h(x')$ is a Lipschitz function according to (2.23), (2.24).

(ii) A *bounded Lipschitz domain* in $\mathbf{R}^d$ is a bounded open connected set Ω in $\mathbf{R}^d$ where $\partial\Omega$ can be covered by finitely many open balls B_j in $\mathbf{R}^d$ where $j = 1, \ldots, J$, centered at $\partial\Omega$ such that

$$B_j \cap \Omega = B_j \cap \Omega_j \qquad \text{with} \quad j = 1, \ldots, J, \tag{2.26}$$

where Ω_j are rotations of suitable special Lipschitz domains in $\mathbf{R}^d$.

Again we use standard notation. Let $0 < p \leq \infty$. Then $L_p(\Omega)$ is the quasi-Banach space of all complex-valued Lebesgue-measurable functions in Ω such that

$$\|f|L_p(\Omega)\| = \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p} < \infty \tag{2.27}$$

(with the obvious modification if $p = \infty$). Let $D'(\Omega)$ be the usual space of complex-valued distributions on Ω. Let $g \in S'(\mathbf{R}^d)$. Then we denote by $g|\Omega$ its restriction to Ω, hence

$$g|\Omega \in D'(\Omega) : (g|\Omega)(\psi) = g(\psi) \qquad \text{for} \quad \psi \in D(\Omega), \tag{2.28}$$

where $D(\Omega) = C_0^\infty(\Omega)$ has the usual meaning as the collection of all complex-valued infinitely differentiable functions in $\mathbf{R}^d$ with compact support in Ω.

Definition 4. Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$. Let $s \in \mathbf{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$. Let A^s_{pq} stand either for B^s_{pq} or F^s_{pq} (with $p < \infty$ in the F-case). Then $A^s_{pq}(\Omega)$ is the collection of all $f \in D'(\Omega)$ such that there is a $g \in A^s_{pq}(\mathbf{R}^d)$ with $g|\Omega = f$. Furthermore,

$$\|f|A^s_{pq}(\Omega)\| = \inf \|g|A^s_{pq}(\mathbf{R}^d)\|, \tag{2.29}$$

where the infimum is taken over all $g \in A^s_{pq}(\mathbf{R}^d)$ such that its restriction $g|\Omega$ to Ω coincides in $D'(\Omega)$ with f.

Remark 5. By standard arguments $A^s_{pq}(\Omega)$ are quasi-Banach spaces (Banach spaces if $p \geq 1, q \geq 1$). Spaces of this type, and even more its special cases, have attracted a lot of attention for decades. As far as the above generality is concerned we refer to [19, Chap. 5] and to [21] where these spaces are studied in bounded smooth domains and in bounded Lipschitz domains, respectively. There one also finds many (historical) references. The above definition can be formalized by introducing the *restriction operator* re,

$$\operatorname{re}(g) = g|\Omega : S'(\mathbf{R}^d) \to D'(\Omega), \tag{2.30}$$

generating for all admitted $A = B$, $A = F$, and s, p, q, a linear and bounded operator,

$$\text{re} : A^s_{pq}(\mathbf{R}^d) \hookrightarrow A^s_{pq}(\Omega). \tag{2.31}$$

One of the key problems in this context is the question of whether there is a linear and bounded *extension operator* ext such that

$$\text{ext} : A^s_{pq}(\Omega) \hookrightarrow A^s_{pq}(\mathbf{R}^d) \tag{2.32}$$

with

$$\text{re} \circ \text{ext} = id \qquad (\text{identity in } A^s_{pq}(\Omega)). \tag{2.33}$$

A satisfactory solution of this problem in the case of $\mathbf{R}^d_+$ and bounded C^∞ domains may be found in [19, 4.5 and 5.1.3]. The final solution of this problem in the case of bounded Lipschitz domains is due to V. S. Rychkov. He proved in [13] that there is a universal extension operator of type (2.32), (2.33) for all admitted spaces $A^s_{pq}(\Omega)$. In [19, 4.5 and 5.1], [13], and [21, 2.4], one finds many references on this substantial problem.

2.4. *Intrinsic Characterizations I*

The question arises to which extent the above spaces $A^s_{pq}(\Omega)$ in bounded Lipschitz domains Ω can be characterized intrinsically. We shift some specific assertions, which will be needed later on, to Subsection 2.6 and discuss here some cases largely parallel to Subsection 2.2. As above, Ω is always a bounded Lipschitz domain in $\mathbf{R}^d$.

(i) Let $1 < p < \infty$ and let $k \in \mathbf{N}$. Then (2.8) has a counterpart in Ω. This is obvious for $L_p(\Omega)$. As for the classical Sobolev spaces, we define temporarily $W^k_p(\Omega)$ as the collection of all $f \in L_p(\Omega)$ such that

$$\|f|W^k_p(\Omega)\| = \sum_{|\alpha|\le k} \|D^\alpha f|L_p(\Omega)\| < \infty. \tag{2.34}$$

Then

$$W^k_p(\Omega) = W^k_p(\mathbf{R}^d)|\Omega \tag{2.35}$$

(restriction from $\mathbf{R}^d$ to Ω as above) in the sense of equivalent norms. This is a very classical famous result. A short proof, further equivalent norms, and, in particular, references, may be found in [17, 4.2.4, p. 316].

(ii) Several intrinsic descriptions of the spaces

$$B^s_{pq}(\Omega) \text{ and } F^s_{pq}(\Omega) \quad \text{with} \quad s > 0, \quad 1 < p < \infty, \quad 1 \le q \le \infty, \tag{2.36}$$

in bounded Lipschitz domains Ω, in terms of respective differences and ball means of differences, are known. We refer to [17, Theorem 4.4.2(2), p. 324], and to [19, 1.10, pp. 68–75], where one finds many references, especially to the Russian school, in particular to G. A. Kaljabin. Here we describe the counterparts of (ii) and (v) in Subsection 2.2. Let $\Delta^M_h f$ be the differences as introduced in (2.10) and let, for $x \in \Omega$,

$$(\Delta^M_{h,\Omega} f)(x) = \begin{cases} (\Delta^M_h f)(x) & \text{if } x + lh \in \Omega \text{ for } l = 0, \dots, M, \\ 0 & \text{otherwise.} \end{cases} \tag{2.37}$$

Let σ_p *be as in* (2.12),

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad \sigma_p < s < M \in \mathbf{N}, \tag{2.38}$$

and $\overline{p} = \max(1, p)$. *Then* $f \in B^s_{pq}(\Omega)$ *if, and only if,* $f \in L_{\overline{p}}(\Omega)$ *and*

$$\|f|L_{\overline{p}}(\Omega)\| + \left(\int_0^1 t^{-sq} \sup_{|h|\le t} \|\Delta^M_{h,\Omega} f|L_p(\Omega)\|^q \frac{dt}{t}\right)^{1/q} < \infty \tag{2.39}$$

(*equivalent quasi-norms*).

This has been proved recently by S. Dispa [5]. It extends (2.13), (2.14) from $\mathbf{R}^d$ to bounded Lipschitz domains Ω in $\mathbf{R}^d$. This also includes the well-known counterpart of (2.15), (2.16), characterizing the Hölder–Zygmund spaces

$$\mathcal{C}^s(\Omega) = B^s_{\infty\infty}(\Omega), \qquad s > 0, \tag{2.40}$$

as the collection of all $f \in L_\infty(\Omega)$ such that, for $0 < s < M \in \mathbf{N}$,

$$\|f|\mathcal{C}^s(\Omega)\|_M = \|f|L_\infty(\Omega)\| + \sup |h|^{-s} \, |(\Delta^M_{h,\Omega} f)(x)| < \infty \tag{2.41}$$

(equivalent norms) where the supremum is taken over all $x \in \Omega$ and all $h \in \mathbf{R}^d$ with $0 < |h| \le 1$.

(iii) Next we discuss the counterpart of the characterizations of the B-spaces and the F-spaces in terms of ball means according to (2.20) and (2.22), respectively. First we have to adapt the ball means (2.18) to the bounded Lipschitz domain Ω. Let $M \in \mathbf{N}$, $t > 0$, $x \in \Omega$. Then

$$V^M(x,t) = \{h \in \mathbf{R}^d : |h| < t \text{ and } x + \tau h \in \Omega \text{ for } 0 \le \tau \le M\} \tag{2.42}$$

is the maximal open subset of a ball of radius t, centered at the origin, star-shaped with respect to the origin, such that $x + MV^M(x,t) \subset \Omega$. Then for $0 < u \le \infty$,

$$d^{M,\Omega}_{t,u} f(x) = \left(t^{-d} \int_{h \in V^M(x,t)} |(\Delta^M_h f)(x)|^u \, dh\right)^{1/u}, \qquad x \in \Omega, \quad t > 0, \tag{2.43}$$

(with the usual modification if $u = \infty$) is the substitute of (2.18). It coincides with [19, Def. 3.5.2, p. 193] (now for bounded Lipschitz domains). Again let $\overline{p} = \max(p, 1)$. Then one has the following counterpart of the assertions (B) and (F) in Subsection 2.2.

Proposition 6. *Let* Ω *be a bounded Lipschitz domain in* $\mathbf{R}^d$ *and let* $d^{M,\Omega}_{t,u} f$ *be given by* (2.43).

(B) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $1 \le u \le r \le \infty$,

$$d\left(\frac{1}{p} - \frac{1}{r}\right)_+ < s < M \in \mathbf{N}. \tag{2.44}$$

Then $B^s_{pq}(\Omega)$ *is the collection of all* $f \in L_{\max(p,r)}(\Omega)$ *such that*

$$\|f|L_{\overline{p}}(\Omega)\| + \left(\int_0^1 t^{-sq}\, \|d^{M,\Omega}_{t,u} f|L_p(\Omega)\|^q \,\frac{dt}{t}\right)^{1/q} < \infty \tag{2.45}$$

in the sense of equivalent quasi-norms (*usual modification if* $q = \infty$).

(F) *Let* $0 < p < \infty$, $0 < q \le \infty$, $1 \le u \le r \le \infty$,

$$d\left(\frac{1}{\min(p,q)} - \frac{1}{r}\right)_+ < s < M \in \mathbf{N}. \tag{2.46}$$

Then $F^s_{pq}(\Omega)$ *is the collection of all* $f \in L_{\max(p,r)}(\Omega)$ *such that*

$$\|f|L_{\overline{p}}(\Omega)\| + \left\| \left(\int_0^1 t^{-sq}\,(d^{M,\Omega}_{t,u} f)(\cdot)^q \,\frac{dt}{t}\right)^{1/q} \Big| L_p(\Omega)\right\| < \infty \tag{2.47}$$

in the sense of equivalent quasi-norms (*usual modification if* $q = \infty$).

Remark 7. We shift the proof of this proposition to Subsection 4.1. If Ω is a bounded C^∞ domain in $\mathbf{R}^d$ then the above proposition is covered by [19, Theorem 5.2.2, p. 245], where the above assertion is proved under the slightly more general condition $1 \le r \le \infty$ and $0 < u \le r$ in analogy to (B) and (F) at the end of Subsection 2.2.

2.5. *Some Other Distinguished Spaces*

There are a few other interesting spaces which are not covered by the scales B^s_{pq} and F^s_{pq} but, nevertheless, fit into the context of this paper. The most distinguished are L_1, L_∞, C, and the corresponding smoothness spaces W^k_1, W^k_∞, C^k, with $k \in \mathbf{N}$ built on them. Here $C(\mathbf{R}^d)$ is the naturally normed space of all complex-valued uniformly continuous bounded functions in $\mathbf{R}^d$. Let $k \in \mathbf{N}$. Then

$$C^k(\mathbf{R}^d) = \{f \in C(\mathbf{R}^d) : D^\alpha f \in C(\mathbf{R}^d),\ |\alpha| \le k\}, \tag{2.48}$$

$$W^k_\infty(\mathbf{R}^d) = \{f \in L_\infty(\mathbf{R}^d) : D^\alpha f \in L_\infty(\mathbf{R}^d),\ |\alpha| \le k\}, \tag{2.49}$$

$$W^k_1(\mathbf{R}^d) = \{f \in L_1(\mathbf{R}^d) : D^\alpha f \in L_1(\mathbf{R}^d),\ |\alpha| \le k\}, \tag{2.50}$$

always naturally normed. Again let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$. Then $C(\overline{\Omega})$, $L_\infty(\Omega)$, $L_1(\Omega)$, and, for $k \in \mathbf{N}$,

$$C^k(\overline{\Omega}), \quad W^k_\infty(\Omega), \quad W^k_1(\Omega), \tag{2.51}$$

are the obvious, intrinsically normed, counterparts, hence (2.34) for $p = 1$ and $p = \infty$ and Ω in place of $\mathbf{R}^d$ in (2.48)–(2.50). On the other hand, there are spaces on Ω defined as restrictions of the corresponding spaces on $\mathbf{R}^d$ as in Definition 4, and one may ask whether (2.35) remains valid for $p = 1$ and $p = \infty$. But this assertion is covered by Stein's extension method [14, VI, §3, Theorem 5 on p. 181], hence (in obvious notation)

$$W^k_1(\Omega) = W^k_1(\mathbf{R}^d)|\Omega \qquad \text{and} \qquad W^k_\infty(\Omega) = W^k_\infty(\mathbf{R}^d)|\Omega. \tag{2.52}$$

Furthermore, according to [18, Prop. 2.5.7, p. 89], and [6, p. 44], we have

$$B^0_{1,1}(\mathbf{R}^d) \hookrightarrow L_1(\mathbf{R}^d) \hookrightarrow B^0_{1,\infty}(\mathbf{R}^d) \tag{2.53}$$

and

$$B^0_{\infty,1}(\mathbf{R}^d) \hookrightarrow C(\mathbf{R}^d) \hookrightarrow L_\infty(\mathbf{R}^d) \hookrightarrow B^0_{\infty,\infty}(\mathbf{R}^d). \tag{2.54}$$

This can be extended to derivatives, resulting in

$$B^k_{\infty,1}(\mathbf{R}^d) \hookrightarrow C^k(\mathbf{R}^d) \hookrightarrow W^k_\infty(\mathbf{R}^d) \hookrightarrow B^k_{\infty,\infty}(\mathbf{R}^d) \tag{2.55}$$

and a similar assertion for $p = 1$ based on (2.53). Here $k \in \mathbf{N}$. These inclusions remain valid when restricted to Ω. Together with (2.52) one gets

$$B^k_{1,1}(\Omega) \hookrightarrow W^k_1(\Omega) \hookrightarrow B^k_{1,\infty}(\Omega) \tag{2.56}$$

and

$$B^k_{\infty,1}(\Omega) \hookrightarrow C^k(\overline{\Omega}) \hookrightarrow W^k_\infty(\Omega) \hookrightarrow B^k_{\infty,\infty}(\Omega), \tag{2.57}$$

where $k \in \mathbf{N}$.

Remark 8. The asymptotics of the sampling numbers described so far in (1.19) with (1.14) is independent of q in (1.14). This applies to the corner spaces in (2.56) or (2.57) and can be extended immediately to the spaces in between. This observation is the main reason for the above considerations.

2.6. *Intrinsic Characterizations II*

We adapt the characterizing quasi-norms for the spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ in Proposition 6 to our later needs. Again let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$. Let $M \in \mathbf{N}$. Let $\mathcal{P}^M(\mathbf{R}^d)$ be the space of all complex-valued polynomials in $\mathbf{R}^d$ of degree smaller than M and let $\mathcal{P}^M(\Omega)$ be the restriction of $\mathcal{P}^M(\mathbf{R}^d)$ to Ω. Let

$$\{P_j^{\Omega,M}\}_{j=1}^{\dim^M} \quad \text{with} \quad \dim^M = \dim \mathcal{P}^M(\mathbf{R}^d) = \dim \mathcal{P}^M(\Omega), \tag{2.58}$$

be an $L_2(\Omega)$-orthonormal basis of real polynomials in $\mathcal{P}^M(\Omega)$.

Theorem 9. *Let Ω be a bounded connected Lipschitz domain in $\mathbf{R}^d$, let $d^{M,\Omega}_{t,u} f$ be the ball means according to* (2.43), *and let $\{P_j^{\Omega,M}\}$ be the above polynomial basis.*

(B) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $1 \le u \le r \le \infty$,

$$d\left(\frac{1}{p} - \frac{1}{r}\right)_+ < s < M \in \mathbf{N}. \tag{2.59}$$

Then $B^s_{pq}(\Omega)$ is the collection of all $f \in L_{\max(p,r)}(\Omega)$ such that

$$\begin{aligned} \|f|B^s_{pq}(\Omega)\|^*_{u,M} \;=\; &\sum_{j=1}^{\dim^M} \left| \int_\Omega f(x) P_j^{\Omega,M}(x)\,dx \right| \\ &+ \left(\int_0^1 t^{-sq} \|d^{M,\Omega}_{t,u} f|L_p(\Omega)\|^q \frac{dt}{t} \right)^{1/q} < \infty \end{aligned} \tag{2.60}$$

in the sense of equivalent quasi-norms (*usual modification if* $q = \infty$).

(F) *Let* $0 < p < \infty$, $0 < q \le \infty$, $1 \le u \le r \le \infty$,

$$d\left(\frac{1}{\min(p,q)} - \frac{1}{r}\right)_+ < s < M \in \mathbf{N}. \tag{2.61}$$

Then $F^s_{pq}(\Omega)$ *is the collection of all* $f \in L_{\max(p,r)}(\Omega)$ *such that*

$$\|f|F^s_{pq}(\Omega)\|^*_{u,M} = \sum_{j=1}^{\dim^M} \left|\int_\Omega f(x)P_j^{\Omega,M}(x)\,dx\right| + \left\|\left(\int_0^1 t^{-sq}(d^{M,\Omega}_{t,u}f)(\cdot)^q\,\frac{dt}{t}\right)^{1/q}\Big|L_p(\Omega)\right\| < \infty \tag{2.62}$$

in the sense of equivalent quasi-norms (*usual modification if* $q = \infty$).

Remark 10. We shift the proof to Subsection 4.2.

Corollary 11. *Let* Ω, $d^{M,\Omega}_{t,u}f$ *with* $M \in \mathbf{N}$, *and the polynomial basis* $\{P_j^{\Omega,M}\}$ *be as in Theorem* 9. *Let* $0 < p \le \infty$, $\overline{p} = \max(p,1)$ *and let*, *for* $f \in L_{\overline{p}}(\Omega)$,

$$g_f(x) = \sum_{j=1}^{\dim^M} a_j\,P_j^{\Omega,M}(x) \qquad \text{with} \qquad a_j = \int_\Omega f(x)P_j^{\Omega,M}(x)\,dx. \tag{2.63}$$

(B) *Then*, *under the hypotheses of part* (B) *of Theorem* 9,

$$\inf_{g\in\mathcal{P}^M(\Omega)} \|f-g|B^s_{pq}(\Omega)\|^*_{u,M} = \|f-g_f|B^s_{pq}(\Omega)\|^*_{u,M} = \left(\int_0^1 t^{-sq}\|d^{M,\Omega}_{t,u}f|L_p(\Omega)\|^q\,\frac{dt}{t}\right)^{1/q}. \tag{2.64}$$

(F) *Then*, *under the hypotheses of part* (F) *of Theorem* 9,

$$\inf_{g\in\mathcal{P}^M(\Omega)} \|f-g|F^s_{pq}(\Omega)\|^*_{u,M} = \|f-g_f|F^s_{pq}(\Omega)\|^*_{u,M} = \left\|\left(\int_0^1 t^{-sq}(d^{M,\Omega}_{t,u}f)(\cdot)^q\,\frac{dt}{t}\right)^{1/q}\Big|L_p(\Omega)\right\|. \tag{2.65}$$

Proof. This follows immediately from Theorem 9 and the assumption that $\{P_j^{\Omega,M}\}$ is a real orthonormal $L_2(\Omega)$-basis in $\mathcal{P}^M(\Omega)$. ■

Remark 12. We need a consequence of Corollary 11 if Ω is a ball,

$$\omega_\tau = \{x \in \mathbf{R}^d : |x| < \tau\}, \qquad 0 < \tau \le 1, \tag{2.66}$$

of radius τ and the dependence of the constants on τ.

Corollary 13. *Let* $d^{M,\omega_\tau}_{t,u}f$ *be the means*, *according to* (2.43), *with respect to the balls* ω_τ.

(B) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $1 \le u \le \infty$, *and*

$$d/p < s < M \in \mathbf{N}. \tag{2.67}$$

There is a positive constant c such that, for all τ *with* $0 < \tau \le 1$ *and all* $f \in B^s_{pq}(\omega_\tau)$,

$$\inf_{g \in \mathcal{P}^M(\omega_\tau)} \sup_{|x|<\tau} |f(x) - g(x)| \le c\tau^{s-d/p} \left(\int_0^\tau t^{-sq} \| d^{M,\omega_\tau}_{t,u} f | L_p(\omega_\tau) \|^q \frac{dt}{t} \right)^{1/q}. \tag{2.68}$$

(F) *Let* $0 < p < \infty$, $0 < q \le \infty$, $1 \le u \le \infty$, *and*

$$d/\min(p,q) < s < M \in \mathbf{N}. \tag{2.69}$$

There is a positive constant c such that, for all τ *with* $0 < \tau \le 1$ *and all* $f \in F^s_{pq}(\omega_\tau)$,

(2.70)

$$\inf_{g \in \mathcal{P}^M(\omega_\tau)} \sup_{|x|<\tau} |f(x) - g(x)| \le c\tau^{s-d/p} \left\| \left(\int_0^\tau t^{-sq} (d^{M,\omega_\tau}_{t,u} f)(\cdot)^q \frac{dt}{t} \right)^{1/q} \Big| L_p(\omega_\tau) \right\|.$$

Remark 14. We shift the proof of this homogeneity property to Subsection 4.3. It comes out that the optimal polynomials are the dilated optimal polynomials according to (2.63). In particular, they depend linearly on f.

3. Rates of Convergence

3.1. *Numbers Measuring Compactness*

Again let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$ and let $A^s_{pq}(\Omega)$ be the spaces introduced in Definition 4. We are mainly interested in studying sampling numbers of the compact embeddings

$$id : G_1(\Omega) = A^s_{pq}(\Omega) \hookrightarrow L_r(\Omega) = G_2(\Omega) \tag{3.1}$$

where

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s > d/p \quad \text{and} \quad 0 < r \le \infty, \tag{3.2}$$

with $p < \infty$ for the F-spaces. In addition, we wish to compare these numbers with the well-established approximation numbers a_n and the entropy numbers e_n of id given by (3.1) with (3.2). First we recall the definitions of a_n and e_n in their natural context. As usual, the family of all linear and bounded maps from a complex quasi-Banach space A into a complex quasi-Banach space B will be denoted by $L(A, B)$. Let U_A be the closed unit ball in A.

Definition 15. Let A and B be two complex quasi-Banach spaces and let $T \in L(A, B)$.

(i) Then for all $n \in \mathbf{N}$ the nth *entropy number* $e_n(T)$ of T is defined as the infimum over all $\varepsilon > 0$ such that $T(U_A)$ can be covered by 2^{n-1} balls in B of radius ε.

(ii) Then for all $n \in \mathbf{N}$ the nth *approximation number* $a_n(T)$ of T is defined by

$$a_n(T) = \inf\{\|T - R\| : R \in L(A, B),\ \operatorname{rank} R < n\}, \tag{3.3}$$

where rank R is the dimension of the range of R.

Remark 16. Both numbers have a long and substantial history and have been studied in great detail. One may consult [3] (Banach spaces) and [6] (quasi-Banach spaces) and the (historical) references given there. The latter book deals especially with these numbers for mappings between function spaces of the above type $A^s_{pq}(\Omega)$ in bounded C^∞ domains Ω. This has been extended in [21] to more general bounded domains Ω. In the present paper we are interested in these numbers only in comparison with the sampling numbers which we are going to define next. Let $\mathbf{C}^n$ be the collection of all n-tuples of complex numbers.

Definition 17. Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$ and let *id* be given by (3.1), (3.2) (with $p < \infty$ for the F-spaces). For $\{x^j\}_{j=1}^n \subset \Omega$ we define the *information map*

$$N_n : G_1(\Omega) \to \mathbf{C}^n \tag{3.4}$$

by

$$N_n f = (f(x^1), \ldots, f(x^n)), \qquad f \in G_1(\Omega). \tag{3.5}$$

For

$$\varphi_n : \mathbf{C}^n \to G_2(\Omega) \qquad \text{consider} \qquad S_n = \varphi_n \circ N_n. \tag{3.6}$$

(i) Then for all $n \in \mathbf{N}$ the nth *sampling number* $g_n(id)$ of *id* is defined by

$$g_n(id) = \inf[\sup\{\|f - S_n f | G_2(\Omega)\| : \|f | G_1(\Omega)\| \le 1\}], \tag{3.7}$$

where the infimum is taken over all n-tuples $\{x^j\}_{j=1}^n \subset \Omega$ and all $S_n = \varphi_n \circ N_n$ according to (3.6).

(ii) For all $n \in \mathbf{N}$ the nth *linear sampling number* $g_n^{\mathrm{lin}}(id)$ of *id* is defined by (3.7), where only linear mappings $S_n = \varphi_n \circ N_n$,

$$S_n f = \sum_{j=1}^n f(x^j) h_j, \qquad h_j \in G_2(\Omega), \quad f \in G_1(\Omega), \tag{3.8}$$

are admitted.

Remark 18. Obviously we have, by (3.6),

$$S_n f = \varphi_n(f(x^1), \ldots, f(x^n)) \in G_2(\Omega) \qquad \text{where} \quad f \in G_1(\Omega). \tag{3.9}$$

Hence one gets by the above definitions and by Definition 15(ii) that

$$g_n(id) \le g_n^{\mathrm{lin}}(id) \qquad \text{and} \qquad a_{n+1}(id) \le g_n^{\mathrm{lin}}(id), \qquad n \in \mathbf{N}. \tag{3.10}$$

We justify the above definition. Let A and B be two quasi-Banach spaces and let $T \in L(A, B)$. Then one gets (essentially as a reformulation of compactness) that

$$T \text{ is compact if, and only if, } e_n(T) \to 0 \text{ for } n \to \infty. \tag{3.11}$$

This applies in particular to *id* according to (3.1), (3.2), and also to

$$id : A^s_{pq}(\Omega) \hookrightarrow C(\overline{\Omega}) \tag{3.12}$$

with (3.2) ($p < \infty$ for the F-spaces). In these cases the asymptotics of the corresponding entropy numbers is known:

Let id be either (3.1), (3.2) *with* $r \geq 1$ *or* (3.12). *Then*

$$e_n(id) \asymp n^{-s/d}, \qquad n \in \mathbf{N}. \tag{3.13}$$

This is covered by [6, Section 3.3, especially Theorem 2 in Subsection 3.3.3, p. 118], in case of bounded C^∞ domains and has been extended in [20, Section 23], and [21] to arbitrary bounded domains. The incorporation of the target spaces $C(\overline{\Omega})$ and $L_1(\Omega)$ is justified by the respective remarks in the above Subsection 2.5. If $0 < r < 1$ in the target space $L_r(\Omega)$, then it follows, by Hölder's inequality,

$$e_n(id) \preceq n^{-s/d}, \qquad n \in \mathbf{N}. \tag{3.14}$$

But as we shall see later on in Corollary 28 the asymptotics (3.13) extends also to these cases. In particular, *id* given by (3.1), (3.2) is always compact. The following observation will be of crucial importance for us later on.

Proposition 19. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$ and let id be given by* (3.1), (3.2) (*with $p < \infty$ for the F-spaces*). *Then, for $n \in \mathbf{N}$,*

$$g_n(id) \asymp \inf[\sup\{\|f|G_2(\Omega)\| : \|f|G_1(\Omega)\| \leq 1,\ f(x^j) = 0\}], \tag{3.15}$$

where the infimum is taken over all sets $\{x^j\}_{j=1}^n \subset \Omega$.

Remark 20. We shift the proof of the proposition to Subsection 4.4. This assertion is known in the case of Banach spaces. Then (3.15) can be strengthened by

$$g^0_n(id) \leq g_n(id) \leq 2g^0_n(id) \tag{3.16}$$

denoting temporarily the right-hand side of (3.15) by $g^0_n(id)$, see [16, pp. 45 and 58]. This applies in our case to the spaces (3.1) with $p \geq 1$, $q \geq 1$, $r \geq 1$.

3.2. *Main Assertions*

Recall that $a_+ = \max(a, 0)$ if $a \in \mathbf{R}$. Let $F^s_{pq}(\Omega)$ and $\mathcal{C}^s(\Omega)$ be the spaces introduced in Definition 4 and (2.40), respectively.

Proposition 21. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$.*

(i) *Let*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s > d/p, \quad \text{and} \quad 0 < r \le \infty. \tag{3.17}$$

Then

$$g_n^{\text{lin}}(id : F^s_{pq}(\Omega) \hookrightarrow L_r(\Omega)) \preceq n^{-s/d+(1/p-1/r)_+}, \qquad n \in \mathbf{N}. \tag{3.18}$$

(ii) *Let $s > 0$ and $0 < r \le \infty$. Then*

$$g_n^{\text{lin}}(id : \mathcal{C}^s(\Omega) \hookrightarrow L_r(\Omega)) \preceq n^{-s/d}, \qquad n \in \mathbf{N}. \tag{3.19}$$

Remark 22. We shift the proof of this crucial proposition to Subsection 4.5. It paves the way to the proof of our main result which reads as follows.

Theorem 23. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$ and let id be given by* (3.1), (3.2). *Then*

$$g_n(id) \asymp g_n^{\text{lin}}(id) \asymp n^{-s/d+(1/p-1/r)_+}, \qquad n \in \mathbf{N}. \tag{3.20}$$

Remark 24. We shift the proof to Subsection 4.6. It is based on Proposition 21.

Special cases of Theorem 23 are known. Most references only study the case $\Omega = [0, 1]^d$ or the periodic case on the torus (from the point of view of the present paper, there is no major difference between these two cases), an exception is Wendland [23], who basically studies the cases $G_1(\Omega) = C^k(\overline{\Omega})$ and $G_2(\Omega) = L_\infty(\Omega)$. We should also say that so far only Banach spaces were studied, i.e., the case $p \ge 1$, $q \ge 1$, and $r \ge 1$. As we already said in the Introduction, the proof of the upper bound for $\Omega = [0, 1]^d$ cannot be generalized easily to general bounded Lipschitz domains. Special cases of Theorem 23 (for Banach spaces and $\Omega = [0, 1]^d$) are contained in [4], [9], [11], [12], and [15].

Other spaces are also studied in the literature, again for Banach spaces and only for the cube: for spaces of functions with dominating mixed derivatives, see [1] and [15], and for anisotropic Besov spaces, see [7]. Weighted Hilbert spaces and the problem of tractability were recently studied by [22]. Here the main interest is the question of how the constants also depend on the dimension d. The given list of papers is far from being complete, but hopefully useful.

This problem of optimal recovery was also studied for randomized (or Monte Carlo) methods, again only for special spaces and $\Omega = [0, 1]^d$. It is known that randomized algorithms are no better than deterministic ones, see [9] and [12]. This is true as long as we study "standard information," i.e., methods that are based on function values. To prove this, we only have to consider the lower bounds. These are based on the "bump function technique," as in the proof of Theorem 23. For this proof technique applied to Monte Carlo methods, see [12, p. 53]. Hence this equivalence of deterministic and randomized methods holds true in the general case of Theorem 23.

It is very remarkable that algorithms for the quantum computer have a better rate of convergence if $p < r$, see [10].

It is also very interesting that randomized algorithms that are based on arbitrary linear information (compare with the approximation numbers, or formula (1.5)) can be essentially smaller than the approximation numbers, see [8].

Furthermore, some other cases, which are not covered by (3.1), (3.2), are of interest. This applies, in particular, to the spaces $C^k(\overline{\Omega})$, $W^k_\infty(\Omega)$, and $W^k_1(\Omega)$ with $k \in \mathbf{N}$ according to Subsection 2.5.

Corollary 25. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$. Let $0 < r \le \infty$, and let*

$$id_{k,\infty} : C^k(\overline{\Omega}) \hookrightarrow L_r(\Omega) \quad \text{with} \quad k \in \mathbf{N}, \tag{3.21}$$

$$id^*_{k,\infty} : W^k_\infty(\Omega) \hookrightarrow L_r(\Omega) \quad \text{with} \quad k \in \mathbf{N}, \tag{3.22}$$

$$id_{k,1} : W^k_1(\Omega) \hookrightarrow L_r(\Omega) \quad \text{with} \quad d < k \in \mathbf{N}. \tag{3.23}$$

Then, for $n \in \mathbf{N}$,

$$g_n(id_{k,\infty}) \asymp g_n(id^*_{k,\infty}) \asymp g_n^{\text{lin}}(id_{k,\infty}) \asymp g_n^{\text{lin}}(id^*_{k,\infty}) \asymp n^{-k/d} \tag{3.24}$$

and

$$g_n(id_{k,1}) \asymp g_n^{\text{lin}}(id_{k,1}) \asymp n^{-k/d+(1-1/r)_+}. \tag{3.25}$$

Proof. On the one hand, we have the embeddings (2.56), (2.57). On the other hand, the sampling numbers in (3.20) do not depend on the index q in (3.1). Then the above assertions follow from Theorem 23. ■

3.3. *Relations to Approximation Numbers and Entropy Numbers*

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$. Let

$$-\infty < s_2 < s_1 < \infty \quad \text{and} \quad s_1 - d/p_1 > s_2 - d/p_2. \tag{3.26}$$

Then

$$id : A^{s_1}_{p_1 q_1}(\Omega) \hookrightarrow A^{s_2}_{p_2 q_2}(\Omega) \tag{3.27}$$

is compact where $p_1, p_2, q_1, q_2 \in (0, \infty]$ (with $p_1 < \infty$ and/or $p_2 < \infty$ in the case of the F-spaces). One has

$$e_n(id) \asymp n^{-(s_1-s_2)/d}, \qquad n \in \mathbf{N}, \tag{3.28}$$

for the respective entropy numbers. This is covered by [20, Sect. 23] (and by [6, Sect. 3.3] as far as C^∞ domains are concerned). Corresponding assertions for approximation numbers are more complicated. Nevertheless in the case of bounded C^∞ domains, one knows the respective asymptotics for $a_n(id)$ with the exception of a few limiting cases. We refer to [6, 3.3.4], and to [2]. According to [21] one can extend these results to bounded Lipschitz domains.

In our case we have, on the one hand,

$$A^{s_1}_{p_1 q_1}(\Omega) = A^s_{pq}(\Omega), \qquad 0 < p \le \infty, \quad 0 < q \le \infty, \quad s > d/p, \tag{3.29}$$

but, on the other hand, according to (2.8) and its restriction to Ω only

$$A^{s_2}_{p_2q_2}(\Omega) = F^0_{r,2}(\Omega) = L_r(\Omega) \qquad \text{if} \quad 1 < r < \infty. \tag{3.30}$$

Using the inclusions (2.53), (2.54), restricted to Ω, one can incorporate afterward $r = 1$, $r = \infty$. This explains our restriction to $1 \le r \le \infty$ in the following assertion.

Theorem 26. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$ and let id be given by* (3.1), (3.2), *now with the additional restriction $r \ge 1$. Let $a_n(id)$, $e_n(id)$, $g_n(id)$, $g_n^{\mathrm{lin}}(id)$ be as introduced in Definitions* 15 *and* 17. *Then*

$$n^{-s/d} \asymp e_n(id) \preceq a_n(id) \preceq g_n(id) \asymp g_n^{\mathrm{lin}}(id) \asymp n^{-s/d+(1/p-1/r)_+} \tag{3.31}$$

where $n \in \mathbf{N}$. Furthermore,

$$e_n(id) \asymp a_n(id) \quad \textit{if, and only if,} \quad r \le p, \tag{3.32}$$

and

$$a_n(id) \asymp g_n(id) \quad \textit{if, and only if,} \quad \begin{cases} \textit{either } 0 < p \le r \le 2, \\ \textit{or} \quad 2 \le p \le r \le \infty, \\ \textit{or} \quad 1 \le r \le p \le \infty. \end{cases} \tag{3.33}$$

Remark 27. We skip the proof of this result since the needed bounds on approximation numbers are known. In the case of the entropy numbers $e_n(id)$ we have the left-hand side of (3.31) for $L_r(\Omega)$ with $r \ge 1$ as the target space and the estimate (3.14) for the general case where $0 < r \le \infty$. However, the equivalence can be extended to all cases. Again we omit the details.

Corollary 28. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$ and let id be given by* (3.1), (3.2). *Then*

$$e_n(id) \asymp n^{-s/d}, \qquad n \in \mathbf{N}. \tag{3.34}$$

Furthermore,

$$e_n(id) \asymp g_n(id) \quad \textit{if, and only if,} \quad r \le p. \tag{3.35}$$

4. Proofs

4.1. *Proof of Proposition* 6

Step 1. Let

$$\|f|F^s_{pq}(\Omega)\|_{u,M} \quad \text{and} \quad \|f|F^s_{pq}(\mathbf{R}^d)\|_{u,M} \tag{4.1}$$

be the quasi-norms in (2.47) and (2.22), respectively. Let $f \in F^s_{pq}(\Omega)$. Then by Definition 4 and the equivalent quasi-norm (2.22) there is an element $g \in F^s_{pq}(\mathbf{R}^d)$ with $g|\Omega = f$

such that

$$\|f|F^s_{pq}(\Omega)\|_{u,M} \leq \|g|F^s_{pq}(\mathbf{R}^d)\|_{u,M} \leq c\|f|F^s_{pq}(\Omega)\| \tag{4.2}$$

where $c > 0$ is independent of f. Similarly for $B^s_{pq}(\Omega)$.

Step 2. As for the converse we rely on the characterization of $F^s_{pq}(\Omega)$ in Lipschitz domains in terms of local means according to [13, Theorem 3.2, p. 251]. As for the kernels of these local means one may choose the distinguished kernels constructed in [19, 3.3.2, especially formula (10) on p. 175], which can be estimated from above by

$$ct^{-d} \int_{h\in V^M(x,t)} |\Delta^M_h f(x)|\, dh. \tag{4.3}$$

Using Hölder's inequality one can estimate this expression from above by $d^{M,\Omega}_{t,u} f$ where one used (for the first and last time) that $u \geq 1$. This proves the converse. Similarly for $B^s_{pq}(\Omega)$.

4.2. *Proof of Theorem* 9

Step 1. It follows by (2.62), (2.47), the notation (4.1), and Hölder's inequality that

$$\|f|F^s_{pq}(\Omega)\|^*_{u,M} \preceq \|f|F^s_{pq}(\Omega)\|_{u,M}. \tag{4.4}$$

Similarly for $B^s_{pq}(\Omega)$.

Step 2. We prove the converse of (4.4) by contradiction assuming that there is no positive constant c such that

$$\|f|L_{\overline{p}}(\Omega)\| \leq c\|f|F^s_{pq}(\Omega)\|^*_{u,M}. \tag{4.5}$$

Then there is a sequence of functions $\{f_j\}^\infty_{j=1} \subset F^s_{pq}(\Omega)$ such that

$$1 = \|f_j|L_{\overline{p}}(\Omega)\| > j\|f_j|F^s_{pq}(\Omega)\|^*_{u,M}, \qquad j \in \mathbf{N}. \tag{4.6}$$

In particular, $\{f_j\}$ is bounded in $F^s_{pq}(\Omega)$ and hence precompact in $L_{\overline{p}}(\Omega)$. The latter follows from the discussions in Remark 18 extended to $s > d(1/p - 1)_+$ and at the beginning of Subsection 3.3. We may assume that

$$f_j \to f \quad \text{in } L_{\overline{p}}(\Omega), \qquad \text{hence} \quad \|f|L_{\overline{p}}(\Omega)\| = 1. \tag{4.7}$$

By (4.6) the sequence $\{f_j\}$ converges in $F^s_{pq}(\Omega)$ and

$$(d^{M,\Omega}_{t,u} f)(x) = 0 \quad \text{in } \Omega, \qquad \int_\Omega f(x) P^{\Omega,M}_l(x)\, dx = 0 \tag{4.8}$$

for $l = 1, \ldots, \dim^M$. Then we also have $(d^{N,\Omega}_{t,u} f)(x) = 0$ for any $\mathbf{N} \ni N > M$. Since (2.47) is a characterization, it follows that $f \in F^\sigma_{pq}(\Omega)$ for any $\sigma \in \mathbf{R}$. By well-known embedding theorems of type (3.1), (3.2) one has $D^\alpha f \in C(\overline{\Omega})$ for all α. Hence, $f \in C^\infty(\overline{\Omega})$. We have locally $(\Delta^M_h f)(x) = 0$. By Taylor expansion arguments it follows that f is locally, and hence globally, in the connected domain Ω, a polynomial of degree less than M, hence $f \in \mathcal{P}^M(\Omega)$. Now we obtain by the second part of (4.8) that $f = 0$. This contradicts (4.7). Similarly for the B-spaces.

4.3. *Proof of Corollary* 13

We prove part (F). The proof of part (B) is the same. Let $f \in F^s_{pq}(\omega_\tau)$. Then $f(\tau\cdot) \in F^s_{pq}(\omega)$ where $\omega = \omega_1$ is the unit ball. Let $g \in \mathcal{P}^M(\omega_\tau)$ be such that $g(\tau\cdot) \in \mathcal{P}^M(\omega)$ is the optimal polynomial according to (2.63) and (2.65) for $f(\tau\cdot)$ and $\Omega = \omega$. It follows by the embedding (3.12) and (2.65) that

$$\sup_{|x|<\tau} |f(x) - g(x)| = \sup_{|x|<1} |f(\tau x) - g(\tau x)| \preceq \left\| \left(\int_0^1 t^{-sq} (d^{M,\omega}_{t,u} f(\tau\cdot))(\cdot)^q \, \frac{dt}{t} \right)^{1/q} \middle| L_p(\omega) \right\|. \tag{4.9}$$

By (2.43) we have, for $|x| < 1$ and $0 < t < 1$, $0 < \tau \leq 1$,

$$\begin{aligned}(d^{M,\omega}_{t,u} f(\tau\cdot))(x) &= \left(t^{-d} \int_{h \in V^M(x,t)} |(\Delta^M_h f(\tau\cdot))(x)|^u \, dh \right)^{1/u} \\ &= \left((\tau t)^{-d} \int_{\tau h \in V^M(\tau x, \tau t)} |(\Delta^M_{\tau h} f)(\tau x)|^u \tau^d \, dh \right)^{1/u} \\ &= d^{M,\omega_\tau}_{\tau t,u} f(\tau x).\end{aligned} \tag{4.10}$$

Inserting (4.11) in (4.10) one gets (2.70).

4.4. *Proof of Proposition* 19

Step 1. We denote the right-hand side of (3.15) by $g^0_n(id)$ and prove in this step that

$$g^0_n(id) \preceq g_n(id), \qquad n \in \mathbf{N}. \tag{4.11}$$

Let f^0 be the identically vanishing function in Ω and let S^ε_n, for given $n \in \mathbf{N}$ and given $\varepsilon > 0$, be a map approximating $g_n(id)$ in (3.7) up to ε. In particular,

$$\|S^\varepsilon_n f^0 | G_2(\Omega)\| \leq g_n(id) + \varepsilon. \tag{4.12}$$

Furthermore,

$$g^0_n(id) \leq \sup \|f | G_2(\Omega)\| = \sup \|f - S^\varepsilon_n f + S^\varepsilon_n f^0 | G_2(\Omega)\|, \tag{4.13}$$

where the supremum is taken over all $f \in G_1(\Omega)$ with $\|f | G_1(\Omega)\| \leq 1$ and $f(x^j) = 0$. Enlarging the supremum on the right-hand side of (4.13) by taking the supremum over the whole unit ball in $G_1(\Omega)$ one gets, by the above assumption and (4.12),

$$g^0_n(id) \preceq g_n(id) + \varepsilon, \qquad n \in \mathbf{N}, \tag{4.14}$$

uniformly in n and ε. This proves (4.11).

Step 2. We prove the converse to (4.11). Let $\Gamma = \{x^j\}^n_{j=1} \subset \Omega$ be n pairwise different points. We interpret the information map according to (3.4), (3.5) as the trace operator tr_Γ,

$$\mathrm{tr}_\Gamma = N_n : G_1(\Omega) \to \mathbf{C}^n, \qquad n \in \mathbf{N}. \tag{4.15}$$

It generates a quasi-norm in $\mathbf{C}^n$,

$$\|\{c_j\}\|_\Gamma = \inf\{\|h\,|G_1(\Omega)\| : h(x^j) = c_j\}. \tag{4.16}$$

We choose as φ_n in (3.6) a respective (nonlinear) bounded extension operator ext_Γ,

$$\varphi_n = \mathrm{ext}_\Gamma : \mathbf{C}^n \to G_1(\Omega) \qquad (\text{and hence } \hookrightarrow G_2(\Omega)), \tag{4.17}$$

and put

$$S_n = \mathrm{ext}_\Gamma \circ \mathrm{tr}_\Gamma = \varphi_n \circ N_n. \tag{4.18}$$

In particular, S_n is a (nonlinear) bounded operator in $G_1(\Omega)$. For given $\varepsilon > 0$ we choose Γ such that

$$\|h|G_2(\Omega)\| \le g_n^0(id) + \varepsilon \quad \text{if} \quad \|h|G_1(\Omega)\| \le 1, \quad h(x^j) = 0, \tag{4.19}$$

for $j = 1, \dots, n$. Then one has, for

$$f \in G_1(\Omega) \quad \text{with} \quad \|f|G_1(\Omega)\| \le 1 \quad \text{and} \quad h = f - S_n f, \tag{4.20}$$

that $\|h|G_1(\Omega)\| \preceq 1$ with $h(x^j) = 0$ and hence

$$\|f - S_n f|G_2(\Omega)\| = \|h|G_2(\Omega)\| \preceq g_n^0(id) + \varepsilon. \tag{4.21}$$

One gets, finally, the converse to (4.11).

4.5. *Proof of Proposition* 21

Step 1. We begin with a preparation. Let $\tau > 0$ and let $\{x^j\}_{j=1}^n \subset \Omega$ be points having pairwise distance of at least τ such that for some $c > 0$ the balls B^j centered at x^j and of radius $c\tau$ cover Ω. We may assume that c is independent of τ and that $n \asymp \tau^{-d}$. Let $M \in \mathbf{N}$. We specify (3.8) by H. Wendland's polynomial reproducing map

$$S_n f = \sum_{j=1}^n f(x^j) h_j \tag{4.22}$$

such that, for all polynomials $P \in \mathcal{P}^M(\mathbf{R}^d)$,

$$(S_n P)(x) = P(x) \qquad \text{where} \quad x \in \Omega. \tag{4.23}$$

Here $h_j \in L_\infty(\Omega)$ are real functions with

$$\sum_{j=1}^n |h_j(x)| \le 2, \qquad x \in \Omega, \tag{4.24}$$

and

$$\mathrm{supp}\, h_j \subset bB^j \cap \Omega, \tag{4.25}$$

where bB^j is a ball centered at x^j and of radius $bc\tau$, and $b > 1$ is a suitably chosen number. For given $M \in \mathbf{N}$ there is a number $\tau_0 > 0$ such that there are mappings of this type for all τ with $0 < \tau \le \tau_0$. We refer to [23].

Step 2. We prove (3.18). The proof of (3.19) is the same. Since

$$F^s_{pq_1}(\Omega) \hookrightarrow F^s_{pq_2}(\Omega) \qquad \text{if} \quad q_1 \le q_2, \tag{4.26}$$

we may assume $q \ge p$, and by Hölder's inequality also $r \ge p$. Let $r < \infty$. Let $f \in F^s_{pq}(\Omega)$ and $\widetilde{f} \in F^s_{pq}(\mathbf{R}^d)$ with

$$\|\widetilde{f}|F^s_{pq}(\mathbf{R}^d)\| \le 2\|f|F^s_{pq}(\Omega)\|. \tag{4.27}$$

Let $\Omega_j = B^j \cap \Omega$ and $\widetilde{\Omega}_j = aB^j \cap \Omega$ for some $a > 1$ specified later on. Choosing S_n according to (4.22) we have, by (4.23) for $P_j \in \mathcal{P}^M(\mathbf{R}^d)$,

(4.28)
$$\begin{aligned}\|f - S_n f|L_r(\Omega)\|^r &\le \sum_{j=1}^n \|f - P_j + S_n P_j - S_n f|L_r(\Omega_j)\|^r \\ &\le c\tau^d \sum_{j=1}^n \left(\sup_{x\in\Omega_j} |f(x) - P_j(x)|^r + \sup_{x\in\widetilde{\Omega}_j} |f(x) - P_j(x)|^r\right),\end{aligned}$$

where the first term comes from $f - P_j$ and where we used (4.24), (4.25) in the second term assuming that a is chosen sufficiently large. Hence,

$$\|f - S_n f|L_r(\Omega)\|^r \le c\tau^d \sum_{j=1}^n \sup_{x\in aB^j} |\widetilde{f}(x) - P_j(x)|^r. \tag{4.29}$$

We wish to apply Corollary 13(F) to aB^j having radius $\lambda = ac\tau$ in place of ω_τ. Since $q \ge p$, (2.69) reduces to (2.67). We may choose $u = 1$ and simplify the notation by writing d^M_t instead of $d^{M,aB^j}_{t,1}$. Let P_j in (4.29) be optimal polynomials according to (2.70). Since $q \ge p$ and $r \ge p$ we obtain, by (4.29) and (2.70),

(4.30)
$$\begin{aligned}\|f - S_n f|L_r(\Omega)\|^r &\le c_1\tau^{(s-d/p+d/r)r} \sum_{j=1}^n \left(\int_{aB^j} \left(\int_0^\lambda t^{-sq}(d^M_t \widetilde{f})(x)^q \frac{dt}{t}\right)^{p/q} dx\right)^{r/p} \\ &\le c_1\tau^{(s-d/p+d/r)r} \left(\sum_{j=1}^n \int_{aB^j} \left(\int_0^\lambda t^{-sq}(d^M_t \widetilde{f})(x)^q \frac{dt}{t}\right)^{p/q} dx\right)^{r/p} \\ &\le c_2\tau^{(s-d/p+d/r)r} \left(\int_{\mathbf{R}^d} \left(\int_0^1 t^{-sq}(d^M_t \widetilde{f})(x)^q \frac{dt}{t}\right)^{p/q} dx\right)^{r/p} \\ &\le c_3\tau^{(s-d/p+d/r)r} \|\widetilde{f}|F^s_{pq}(\mathbf{R}^d)\|^r \\ &\le c_4\tau^{(s-d/p+d/r)r} \|f|F^s_{pq}(\Omega)\|^r,\end{aligned}$$

where we used (4.27). Now (3.18) follows from $n \asymp \tau^{-d}$. If $r = \infty$, then one has to modify in the usual way.

4.6. *Proof of Theorem* 23

Step 1. First we extend Proposition 21 by real interpolation from the F-spaces to the B-spaces. Let $p < \infty$,

$$0 < \theta < 1, \quad 0 < q_0 \leq \infty, \quad 0 < q_1 \leq \infty, \quad 0 < q \leq \infty, \tag{4.31}$$

and

$$s = (1-\theta)s_0 + \theta s_1 \qquad \text{with} \quad s_0 \neq s_1. \tag{4.32}$$

Let $(\cdot, \cdot)_{\theta,q}$ be the real interpolation method. Then

$$(F^{s_0}_{pq_0}(\mathbf{R}^d), F^{s_1}_{pq_1}(\mathbf{R}^d))_{\theta,q} = B^s_{pq}(\mathbf{R}^d). \tag{4.33}$$

Details, explanations, and references may be found in [18, 2.4.2, p. 64]. According to [21, Theorem 2.13], one can extend this assertion to bounded Lipschitz domains, hence

$$(F^{s_0}_{pq_0}(\Omega), F^{s_1}_{pq_1}(\Omega))_{\theta,q} = B^s_{pq}(\Omega). \tag{4.34}$$

We may assume that the linear operator S_n in (4.22) is the same for $F^{s_0}_{pq_0}(\Omega)$ and $F^{s_1}_{pq_1}(\Omega)$, where s_0 and s_1 are near to given s. Then it follows by (4.30) and the interpolation property that

$$\|f - S_n f | L_r(\Omega)\| \leq c\tau^{s-d/p+d/r} \|f | B^s_{pq}(\Omega)\|. \tag{4.35}$$

This can be extended to $p = \infty$ by the interpolation formula

$$(\mathcal{C}^{s_0}(\Omega), \mathcal{C}^{s_1}(\Omega))_{\theta,q} = B^s_{\infty,q}(\Omega), \qquad s_0 \neq s_1, \tag{4.36}$$

and (4.30) with $\mathcal{C}^s(\Omega)$ in place of $F^s_{pq}(\Omega)$ according to (3.19). Then one gets, in all cases (3.1), (3.2),

$$g_n(id) \preceq g^{\mathrm{lin}}_n(id) \preceq n^{-s/d+(1/p-1/r)_+}, \qquad n \in \mathbf{N}. \tag{4.37}$$

Step 2. We prove the respective estimate from below, hence

$$g_n(id) \succeq n^{-s/d+(1/p-1/r)_+}, \qquad n \in \mathbf{N}. \tag{4.38}$$

We begin with a preparation. There is a number $c > 0$ with the following property. For any set of points

$$\{x^j\}^{2^{ld}}_{j=1} \subset \Omega, \qquad l \in \mathbf{N}, \tag{4.39}$$

there are points $y^j \in \Omega$ with $j = 1, \ldots, 2^{ld}$, such that

$$|y^j - x^k| \geq c2^{-l+1} \qquad \text{for all} \quad 1 \leq j, k \leq 2^{ld}, \tag{4.40}$$

and

$$|y^j - y^m| \geq c2^{-l+1}, \qquad \mathrm{dist}(y^j, \partial\Omega) \geq c2^{-l+1}, \tag{4.41}$$

for $1 \le j, m \le 2^{ld}$ with $j \neq m$. Let φ be a nonnegative C^∞ function in $\mathbf{R}^d$ with support in the unit ball and, say, $\varphi(0) = 1$. Let $n = 2^{ld}$ and

$$f_n(x) = \sum_{k=1}^{n} \varphi(c^{-1} 2^l (x - y^k)), \qquad x \in \Omega. \tag{4.42}$$

Then

$$\| f_n | L_r(\Omega) \| = c_1, \qquad 0 < r \le \infty, \tag{4.43}$$

where $c_1 > 0$ depends on Ω, r, and c, but not on l and y^k. Furthermore, by atomic arguments, [20, Theorem 13.8, p. 75], or by the localization property for the above spaces $A^s_{pq}(\mathbf{R}^d)$ according to [6, 2.3.2, pp. 35–36], it follows that

$$\| f_n | A^s_{pq}(\Omega) \| \le c_2 2^{ls}, \qquad l \in \mathbf{N}, \tag{4.44}$$

where $c_2 > 0$ is independent of $l \in \mathbf{N}$ and of the points y^j.

Step 3. After these preparations we can prove (4.38). Again let $n = 2^{ld}$ and let y^k and f_n be as above. Then we have $f_n(x^j) = 0$ and according to (4.43), (4.44),

$$\| f_n | L_r(\Omega) \| = c_1 \ge c_3 2^{-ls} \| f_n | A^s_{pq}(\Omega) \| \tag{4.45}$$

for some $c_3 > 0$ which is independent of n. It follows by Proposition 19 that

$$g_n(id) \succeq n^{-s/d} \tag{4.46}$$

for $n = 2^{ld}$ and hence for all $n \in \mathbf{N}$. This proves (4.38) for $r \le p$. Let $p < r \le \infty$. For given points x^j according to (4.39) we now select one of the above points y^k. We assume, without restriction of generality, say, $y_1 = 0$. The respective substitutes of (4.42)–(4.44) are now

$$f_n(x) = \varphi(c^{-1} 2^l x), \qquad x \in \Omega, \quad l \in \mathbf{N}, \tag{4.47}$$

$$\| f_n | L_r(\Omega) \| = c_1 2^{-ld/r}, \qquad 0 < r \le \infty, \tag{4.48}$$

and

$$\| f_n | A^s_{pq}(\Omega) \| \le c_2 2^{l(s-d/p)}, \qquad l \in \mathbf{N}. \tag{4.49}$$

The counterpart of (4.45) is given by

$$\| f_n | L_r(\Omega) \| \ge c_3 2^{-l(s-d/p+d/r)} \| f_n | A^s_{pq}(\Omega) \|. \tag{4.50}$$

This proves (4.38) for $p < r \le \infty$ by the same arguments as above.

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