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CONSTRUCTIVE APPROXIMATION

On the Equivalence Between Existence of B-Spline Bases and Existence of Blossoms

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Abstract. In spline spaces with sections in arbitrary extended Chebyshev spaces and with connections defined by arbitrary lower triangular matrices with positive diagonal elements, we prove that existence of B-spline bases is equivalent to existence of blossoms. As is now classical, we construct blossoms with the help of osculating flats. As for B-spline bases, this expression denotes normalized basis consisting of minimally supported functions which are positive on the interior of their supports and which satisfy an additional "end point condition."

1. Introduction

Proving the existence of B-spline bases is a topic of concern for anyone interested in geometric design. Indeed, the minimum to expect from a "good" basis in a spline space is that it guarantees the location of any spline curve in the convex hull of its control points, with most local possible influence of each pole on the curve. Whence the classical requirements on B-spline bases: minimal support, positivity, normalization.

The existence of such bases in polynomial spline spaces of any degree with arbitrary regular lower triangular totally positive connection matrices was proved by N. Dyn and C. Micchelli [3]. This was the generalization of a similar result obtained by T. Goodman [4] for one-banded lower triangular connection matrices. In [1], P. J. Barry considered the case of splines with sections in arbitrary extended Chebyshev spaces. Extending the result of N. Dyn and C. Micchelli he proved that again an assumption of total positivity ensured the existence of B-spline bases. However, in that new context, this assumption does not concern the usual connection matrices, but matrices obtained when expressing the connections in terms of differential operators associated with the extended Chebyshev spaces. Total positivity is thus a useful sufficient condition, classically used to ensure existence of B-spline bases. Nevertheless, it may be a far too restrictive assumption: for instance, it does not include the classical case of C^2 trigonometric splines, for which existence of B-spline bases is known (see [9]).

Apart from the classical case of polynomial splines with parametric continuity, the first to establish relations between blossoms and B-spline bases was H.-P. Seidel. In-

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deed, in [16],17], he addressed the problem of deducing blossoms for geometrically continuous polynomial splines from the existence of B-spline bases obtained by N. Dyn and C. Micchelli. The values of the blossoms were defined by means of intersections of osculating flats, which has since become a classical way to introduce blossoms (see, for instance, [13]). On the other hand, in [8], we considered $(n+1)$-dimensional piecewise extended Chebyshev spaces, that is, spaces of functions with sections in arbitrary $(n+1)$-dimensional extended Chebyshev spaces, the connections between any two consecutive sections being given by arbitrary lower triangular matrices of order n with positive diagonals. In this particular context, assuming that blossoms, defined by means of osculating flats, exist **everywhere** (i.e., **on the set of all n-tuples**), we managed to prove that they satisfy a pseudoaffinity property with respect to each variable. Added to the obvious symmetry and diagonal property of the blossoms, the pseudoaffinity was the key-point to prove, in particular, the existence of B-spline bases in any associated spline spaces.

Hence, blossoms and B-spline bases are mathematical entities which obviously overlap. The question of the equivalence between their existence thus naturally comes up, and this is the main subject of the present paper. However, in the previous context, existence of blossoms **everywhere** is only a sufficient condition to ensure existence of B-spline bases and, like total positivity, it is a far too restrictive one. Indeed, B-spline bases naturally emerge from de Boor like algorithms, and such algorithms do not involve all values of the blossoms, but only those on a very limited set of n-tuples, known as the admissible n-tuples with respect to the knot vector. Therefore, when proving the equivalence between existence of blossoms and existence of B-spline bases, we inevitably mean existence of blossoms **on the set of admissible n-tuples only**.

On the other hand, if the desirable definition of blossoms is now clear from a geometrical point of view, the expression "B-spline basis" is not precise, apart from the fact that it always implies the three classical axioms we already mentioned: minimal support, positivity, normalization. In a previous paper [12], we pointed out that they are not sufficient to clearly identify a B-spline basis. We showed that it was natural to introduce an additional requirement in order to ensure both the unicity of a B-spline basis and the fact that the poles of a spline can be defined in terms of osculating flats at consecutive knots.

The paper is organized as follows. In the second section we introduce blossoms in the general context of spline spaces containing the constants, with sections in arbitrary $(n+1)$-dimensional spaces with nonvanishing Wronskians, the connections between consecutive sections being defined by arbitrary lower triangular matrices with positive diagonal elements. We discuss some preliminary facts, and we recall some essential results emerging from previous papers. In the third section we state the main theorem of the paper, i.e., in which sense exactly, in such W-spline spaces, existence of B-spline bases is equivalent to existence of blossoms. In the same section we also show that the results of [12] lead directly to the fact that existence of B-spline bases implies existence of blossoms. The difficult part of the theorem (i.e., existence of blossoms implies existence of B-spline bases) is proved in Section 4. Finally, Section 5 considers some possible extensions of this equivalence and it gives indications on how to effectively ensure existence of B-spline bases.

Note that the results of the present paper enabled us to show that existence of blossoms automatically ensures that the (unique) B-spline basis on a closed bounded interval is the optimal totally positive basis [11], in the sense of J.-M. Carnicer and J.-M. Peña [2].

2. The General Setting

2.1. *Piecewise Regular Functions and Blossoms*

Throughout the paper, a sequence $\mathcal{T} = (t_\ell)_{\ell\in\mathbf{Z}}$ is given, with $t_\ell < t_{\ell+1}$, and $|t_\ell| \to +\infty$ when $|\ell| \to +\infty$. We shall say that a function defined on $\mathbf{R}$ is *piecewise smooth* if it is C^∞ on each $I_\ell := [t_\ell, t_{\ell+1}]$ implying, in particular, that it is continuous on $\mathbf{R}$.

For each $\ell \in \mathbf{Z}$, we denote by $\mathcal{E}_\ell$ an $(n+1)$-dimensional linear subspace of $C^\infty(I_\ell)$ containing the constants, and we assume that it is a *W-space* on I_ℓ, in the sense that the Wronskian of one (or of any) basis of $\mathcal{E}_\ell$ never vanishes on I_ℓ. Given a sequence $\mathcal{M}_\ell$, $\ell \in \mathbf{Z}$, of lower triangular matrices of order n with positive diagonals, we consider the space $\mathcal{E}$ of all piecewise smooth functions $U : \mathbf{R} \to \mathbf{R}$ the restrictions of which to I_ℓ belong to $\mathcal{E}_\ell$, and which satisfy the connection conditions

$$(2.1)\quad (U'(t_\ell{}^+), \ldots, U^{(n)}(t_\ell{}^+))^T = \mathcal{M}_\ell \cdot (U'(t_\ell{}^-), \ldots, U^{(n)}(t_\ell{}^-))^T, \qquad \ell \in \mathbf{Z}.$$

The space $\mathcal{E}$ is clearly $(n+1)$-dimensional and it contains the constants. Picking up a basis $(\mathbf{I}, \Phi^1, \ldots, \Phi^n)$ of $\mathcal{E}$, consider the function $\Phi : \mathbf{R} \to \mathbf{R}^n$ defined by $\Phi := (\Phi^1, \ldots, \Phi^n)$. Our assumptions first imply that Φ satisfy the following property:

(I) *for all* $x \in \mathbf{R}$, *the* n *vectors* $\Phi'(x), \ldots, \Phi^{(n)}(x)$ *are linearly independent*;

these vectors being meant as either $\Phi'(t_\ell{}^-), \ldots, \Phi^{(n)}(t_\ell{}^-)$ or $\Phi'(t_\ell{}^+), \ldots, \Phi^{(n)}(t_\ell{}^+)$ when $x = t_\ell$. Applying the Gram–Schmidt orthonormalization process to these n vectors provides the Frénet frame of Φ at x. In particular, for $x = t_\ell$, we obtain the Frénet frame at t_ℓ^- and at t_ℓ^+. The kind of connection matrices we require is necessary and sufficient to ensure that:

(II) Φ *is Frénet-continuous of order* n *at each* t_ℓ;

in the sense that its Frénet frames of order n at t_ℓ^+ and at t_ℓ^- are equal.

An equivalent approach consists in starting with a piecewise smooth function Φ defined on $\mathbf{R}$ and with values in an n-dimensional affine space, supposed to meet the two requirements (I) and (II) above, and in introducing then the space $\mathcal{E} := \operatorname{span}(\Phi^0, \ldots, \Phi^n)$ where, given an affine frame $(A_0, \ldots, A_n)$ in the affine space, the function Φ is defined by

$$(2.2)\qquad \Phi(x) := \sum_{i=0}^{n} \Phi^i(x) A_i, \qquad \sum_{i=0}^{n} \Phi^i(x) = 1, \qquad x \in \mathbf{R}.$$

Apart from the fact that here the sequence of intervals I_ℓ is bi-infinite instead of finite, the situation is the same as in [8], which will allow us to use the results obtained there, possibly after slighty adapting them. In particular, using the same terminology as in [8], we shall say that such a function Φ is a *piecewise smooth geometrically regular function of order* n *on* $\mathbf{R}$.

At any point $x \in \mathbf{R}$, we can consider the ith-order osculating flat ($i \le n$) of Φ, that is, the affine flat which passes through $\Phi(x)$ and the direction of which is spanned by

$\Phi'(x), \ldots, \Phi^{(i)}(x)$, i.e., $\mathrm{Osc}_i\, \Phi(x) := \{\Phi(x) + \lambda_1 \Phi'(x) + \cdots + \lambda_i \Phi^{(i)}(x) \mid \lambda_1, \ldots, \lambda_i \in \mathbf{R}\}$. It is well defined even for $x = t_\ell$ due to the equality (2.1) and to $\mathcal{M}_\ell$ being regular and lower triangular.

Throughout the paper, the notation $\tau^{[\mu]}$ will mean that the point τ is repeated μ times. Given an n-tuple $(x_1, \ldots, x_n) \in \mathbf{R}^n$, up to a permutation, we can always write it as $(x_1, \ldots, x_n) = (\tau_1{}^{[\mu_1]}, \ldots, \tau_r{}^{[\mu_r]})$, where $\tau_1, \ldots, \tau_r$ are pairwise distinct real numbers and where $\mu_1, \ldots, \mu_r$ are positive integers such that $\sum_{i=1}^{r} \mu_i = n$. We are interested in how to ensure the following property:

$$\text{(2.3)} \qquad \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i}\, \Phi(\tau_i) \text{ consists of a single point.}$$

If (2.3) is satisfied, we shall set

$$\text{(2.4)} \qquad \{\varphi(x_1, \ldots, x_n)\} := \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i}\, \Phi(\tau_i),$$

We obtain a function φ, called *the blossom* of Φ, defined on the set $\mathcal{B}$ of all n-tuples $(x_1, \ldots, x_n)$ satisfying (2.3). Note that $\mathcal{B}$ is a symmetric set which contains at least the diagonal of $\mathbf{R}^n$, i.e., $\{(x^{[n]}) \mid x \in \mathbf{R}\}$. By construction, the blossom φ is symmetric on $\mathcal{B}$ and it gives Φ on the diagonal, i.e., $\varphi(x^{[n]}) = \Phi(x)$ for any $x \in \mathbf{R}$.

A function F defined on $\mathbf{R}$, with values in a finite-dimensional affine space, is called an *$\mathcal{E}$-function* when all its coordinates in an affine frame belong to $\mathcal{E}$. Equivalently, $\mathcal{E}$-functions are images of Φ under affine maps. Given an $\mathcal{E}$-function F, denote by h the unique affine map such that $F = h \circ \Phi$. Then we define the blossom f of F by $f := h \circ \varphi$. We shall say that F is *nondegenerate* if one of the following three equivalent properties is satisfied:

- the affine flat spanned by the image of F is of maximal dimension, that is, of dimension n;
- for all $i \leq n$, and all $x \in \mathbf{R}$ (or for a given $x \in \mathbf{R}$), $\mathrm{Osc}_i\, F(x)$ is i-dimensional; and
- h is injective.

Note that F is also nondegenerate iff its restriction to some I_ℓ is nondegenerate.

Accordingly, condition (2.3) is satisfied by Φ iff it is satisfied by any given nondegenerate $\mathcal{E}$-function. Hence, the set $\mathcal{B}$ does not depend on the chosen nondegenerate function. It is actually the domain of definition of all blossoms for $\mathcal{E}$-functions (in short, $\mathcal{E}$-blossoms). Taking images of φ under affine maps, we can state that the blossom f of any $\mathcal{E}$-function F satisfies the following two properties:

$(\mathrm{B})_1$ *symmetry property*: for all $(x_1, \ldots, x_n) \in \mathcal{B}$ and all permutation ϱ of $\{1, \ldots, n\}$, $f(x_{\varrho(1)}, \ldots, x_{\varrho(n)}) = f(x_1, \ldots, x_n)$; and

$(\mathrm{B})_2$ *diagonal property*: for all $x \in \mathbf{R}$, $f(x^{[n]}) = F(x)$.

2.2. *Blossoms for W-Splines*

We shall now consider a space of splines based on the space $\mathcal{E}$. With such splines we shall associate blossoms, which can be interpreted in terms of osculating flats thanks to some results proved in [8].

2.2.1. *W-Splines and Admissible Tuples*. To each t_ℓ, let us allocate a multiplicity $m_\ell \in \{0, \dots, n\}$. Associated with the corresponding knot vector $\mathcal{K} := ({t_\ell}^{[m_\ell]})_{\ell \in \mathbf{Z}}$, we define the spline space based on $\mathcal{E}$ as the space $\mathcal{S}$ of all piecewise smooth functions $S : \mathbf{R} \to \mathbf{R}$ the restrictions of which to I_ℓ belong to $\mathcal{E}_\ell$ for any $\ell \in \mathbf{Z}$, and which satisfy the connection conditions

$$(2.5)\quad (S'({t_\ell}^+), \dots, S^{(n-m_\ell)}({t_\ell}^+))^T = \widehat{\mathcal{M}}_\ell \cdot (S'({t_\ell}^-), \dots, S^{(n-m_\ell)}({t_\ell}^-))^T, \qquad \ell \in \mathbf{Z},$$

where the $(n - m_\ell)$-order square matrix $\widehat{\mathcal{M}}_\ell$ is obtained by deleting the last m_ℓ rows and columns of $\mathcal{M}_\ell$. We suppose that the knot vector is bi-infinite, i.e., that infinitely many multiplicities of positive or negative indices are not equal to zero. When necessary we shall rewrite the knot vector as $\mathcal{K} = (\xi_\ell)_{\ell \in \mathbf{Z}}$, with $\xi_\ell \le \xi_{\ell+1}$.

A function Σ defined on $\mathbf{R}$, with values in a given finite-dimensional affine space, is called an $\mathcal{S}$-*spline* when all its coordinates in an affine frame belong to $\mathcal{S}$. For each $\mathcal{S}$-spline Σ and each $i \in \mathbf{Z}$, there exists a (unique) $\mathcal{E}$-function Φ_i with values in the given affine space such that

$$(2.6)\qquad \Sigma(x) = \Phi_i(x), \qquad x \in I_i.$$

We shall say that S is *nondegenerate* if each function Φ_i is nondegenerate.

Associated with the knot vector $\mathcal{K}$ we also introduce the set $\mathcal{A}$ of admissible n-tuples, which plays a fundamental rôle in the study of $\mathcal{S}$-splines. An n-tuple $(\zeta_1, \dots, \zeta_n)$ is said to be admissible if, whenever the knot t_ℓ satisfies $\mathrm{Min}(\zeta_1, \dots, \zeta_n) < t_\ell < \mathrm{Max}(\zeta_1, \dots, \zeta_n)$, then at least m_ℓ among the points $x_1, \dots, x_n$ are equal to t_ℓ. We also define admissible p-tuples for any $p \le n + 1$. If $p < n$ and if $\zeta_1 = \cdots = \zeta_p = t_\ell$, we say that the p-tuple $(\zeta_1, \dots, \zeta_p)$ is admissible when $p \ge m_\ell$. In any other cases, the definition is similar to that for n-tuples. Furthermore, if $p \le n$, the domain of a given admissible p-tuple $(\zeta_1, \dots, \zeta_p)$ is defined as the set

$$\mathcal{D}(\zeta_1, \dots, \zeta_p) := \{x \in \mathbf{R} \mid (\zeta_1, \dots, \zeta_p, x) \text{ is admissible}\}.$$

It is easy to check that $\mathcal{D}(\zeta_1, \dots, \zeta_p) = \bigcup_{\ell \in \mathcal{J}(\zeta_1, \dots, \zeta_p)} I_\ell$, where $\mathcal{J}(\zeta_1, \dots, \zeta_p)$ is a nonempty set of consecutive integers. Due to the knot vector being bi-infinite, this set is finite. For instance, if $p \ge m_\ell$, $\mathcal{D}(t^{[p]}) = [t_{\ell-1}, t_{\ell+1}]$. More generally, given $p \le n + 1$, the p-tuple $(\zeta_1, \dots, \zeta_p)$ is admissible iff, up to a permutation, it is of the following form:

$$(2.7)\qquad (\zeta_1, \dots, \zeta_p) = ({t_\ell}^{[\alpha]}, {t_{\ell+1}}^{[m_{\ell+1}]}, \dots, {t_{\ell+s}}^{[m_{\ell+s}]}, {t_{\ell+s+1}}^{[\beta]}, y_1, \dots, y_r),$$

where the integers $\ell \in \mathbf{Z}$, $s, \alpha, \beta \in \mathbf{N}$ satisfy

$$m_\ell \ne 0, \quad m_{\ell+s+1} \ne 0, \quad \alpha < m_\ell, \quad \beta < m_{\ell+s+1}, \qquad r := p - \alpha - \beta - \sum_{i=\ell+1}^{\ell+s} m_i \ge 0,$$

and where $y_1, \dots, y_r \in]t_\ell, t_{\ell+s+1}[$. Moreover, if (2.7) holds (up to a permutation), then we have $\mathcal{D}(\zeta_1, \dots, \zeta_p) = [t_\ell, t_{\ell+s+1}]$.

2.2.2. *Spline Blossoms*. The definition of blossoms for $\mathcal{S}$-splines (in short, $\mathcal{S}$-blossoms) will naturally come out of the following result:

Lemma 2.1. *Let Σ be an $\mathcal{S}$-spline, given by* (2.6) *and, for each $i \in \mathbf{Z}$, denote by h_i the affine map such that $\Phi_i = h_i \circ \Phi$. Consider any r points $\tau_1 < \cdots < \tau_r$ and any r positive integers $\mu_1, \dots, \mu_r$, with $\sum_{i=1}^r \mu_i = p \le n$, such that the p-tuple $(\zeta_1, \dots, \zeta_p) := ({\tau_1}^{[\mu_1]}, \dots, {\tau_r}^{[\mu_r]})$ is admissible. Then, all affine maps h_j, $j \in \mathcal{J}(\zeta_1, \dots, \zeta_p)$, coincide on the set $\bigcap_{i=1}^r \mathrm{Osc}_{n-\mu_i} \Phi(\tau_i)$.*

Proof. Let $\ell - 1$ and ℓ be two consecutive integers in $\mathcal{D}(\zeta_1, \dots, \zeta_p)$. Due to the admissibility of $(\zeta_1, \dots, \zeta_p)$, there exists an integer $i_0 \in \{1, \dots, r\}$ such that $\tau_{i_0} = t_\ell$, and we know that $\mu_{i_0} \ge m_\ell$. From Σ being an $\mathcal{S}$-spline and all Φ_i being $\mathcal{E}$-functions, we can deduce that the two functions $\Phi_{\ell-1}$ and Φ_ℓ take the same value at the point t_ℓ and, at this point, they have the same left (or right) derivatives up to order $(n - \mu_{i_0})$. Since $\Phi_{\ell-1} = h_{\ell-1} \circ \Phi$ and $\Phi_\ell = h_\ell \circ \Phi$, it follows that $h_{\ell-1}$ and h_ℓ coincide on $\mathrm{Osc}_{n-\mu_{i_0}} \Phi(\tau_{i_0})$. Hence, in particular, $h_{\ell-1}(P) = h_\ell(P)$ for all $P \in \bigcap_{i=1}^r \mathrm{Osc}_{n-\mu_i} \Phi(\tau_i)$. ■

Suppose now that $p = n$ and that, in addition to being admissible, the n-tuple $(\zeta_1, \dots, \zeta_n) := ({\tau_1}^{[\mu_1]}, \dots, {\tau_r}^{[\mu_r]})$ belongs to the set $\mathcal{B}$. The intersection $\bigcap_{i=1}^r \mathrm{Osc}_{n-\mu_i} \Phi(\tau_i)$ then consists of the single point $\varphi(\zeta_1, \dots, \zeta_n)$. On the other hand, for all $i \in \mathbf{Z}$, $h_i \circ \varphi$ is actually the blossom φ_i of the $\mathcal{E}$-function Φ_i. The previous lemma thus proves the consistency of the definition below:

Definition 2.2. Let Σ be the $\mathcal{S}$-spline defined by (2.6) and, for each $i \in \mathbf{Z}$, denote by φ_i the blossom of the $\mathcal{E}$-function Φ_i. The *blossom* of Σ is the function σ defined on the set $\mathcal{B} \cap \mathcal{A}$ by

$$\sigma(\zeta_1, \dots, \zeta_n) := \varphi_j(\zeta_1, \dots, \zeta_n), \tag{2.8}$$

for any $j \in \mathcal{D}(\zeta_1, \dots, \zeta_n)$.

The set $\mathcal{B} \cap \mathcal{A}$ is symmetric and it contains at least the diagonal of $\mathbf{R}^n$. The latter definition makes it clear that the blossom σ of an $\mathcal{S}$-spline Σ satisfies the following two properties:

- $(\mathrm{SB})_1$ *symmetry property*: for all $(\zeta_1, \dots, \zeta_n) \in \mathcal{B} \cap \mathcal{A}$ and all permutation ϱ of $\{1, \dots, n\}$, $\sigma(\zeta_{\varrho(1)}, \dots, \zeta_{\varrho(n)}) = \sigma(\zeta_1, \dots, \zeta_n)$; and
- $(\mathrm{SB})_2$ *diagonal property*: for all $x \in \mathbf{R}$, $\sigma(x^{[n]}) = \Sigma(x)$.

If the $\mathcal{S}$-spline Σ given by (2.6) is nondegenerate, then we know from [8] that, as soon as the n-tuple $(\zeta_1, \dots, \zeta_n) := ({\tau_1}^{[\mu_1]}, \dots, {\tau_r}^{[\mu_r]})$ is admissible (with $\tau_1 < \cdots < \tau_r$), we have

$$\bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i} \Sigma(\tau_i) = \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i} \Phi_j(\tau_i) \tag{2.9}$$

for all $j \in \mathcal{J}(\zeta_1, \dots, \zeta_n)$. In the latter equality note that, when $r > 1$, in case $\tau_1 = t_\ell$ and $\mu_1 < m_\ell$, $\mathrm{Osc}_{n-\mu_1} \Sigma(\tau_1)$ stands for $\mathrm{Osc}_{n-\mu_1} \Sigma(t_\ell^+)$, and in case $\tau_r = t_{\ell'}$ and $\mu_r < m_{\ell'}$, then $\mathrm{Osc}_{n-\mu_r}\Sigma(\tau_r)$ stands for $\mathrm{Osc}_{n-\mu_r}\Sigma(t_{\ell'}^-)$. In case $(\zeta_1, \ldots, \zeta_n)$ belongs to $\mathcal{B}$, we know that

$$\{\varphi_j(\zeta_1, \ldots, \zeta_n)\} = \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i}\Phi_j(\tau_i) \ .$$

Equality (2.9) thus provides us with the following geometrical interpretation of blossoms of nondegenerate $\mathcal{S}$-splines:

Proposition 2.3. *If the $\mathcal{S}$-spline Σ is nondegenerate then, for any $(\zeta_1, \ldots, \zeta_n)$ in $\mathcal{B}\cap\mathcal{A}$, the value of the blossom σ of Σ at $(\zeta_1, \ldots, \zeta_n)$ satisfies*

$$\{\sigma(\zeta_1, \ldots, \zeta_n)\} = \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i}\Sigma(\tau_i), \tag{2.10}$$

where $(\zeta_1, \ldots, \zeta_n) = ({\tau_1}^{[\mu_1]}, \ldots, {\tau_r}^{[\mu_r]})$ with positive integers $\mu_1, \ldots, \mu_r$ and pairwise distinct $\tau_1, \ldots, \tau_r$, and with the convention adopted in (2.9).

Remark 2.4. We are actually interested in the space $\mathcal{S}$ rather than in the space $\mathcal{E}$ itself, and several spaces $\mathcal{E}$ can generate the same spline space $\mathcal{S}$ associated with the given knot vector $\mathcal{K}$. The space $\mathcal{E}$ (or, equivalently, the function Φ) is in fact just an interesting tool to achieve our results. Although it is not the approach we followed up to now, we could as well start with defining the space $\mathcal{S}$ by means of the sequence $\mathcal{E}_\ell$ and of lower triangular connection matrices $\widehat{\mathcal{M}}_\ell$ of order $n - m_\ell$ with positive diagonal. Then, we could introduce the space $\mathcal{E}$ as we did in Subsection 2.1 after completing each matrix $\widehat{\mathcal{M}}_\ell$ into an nth-order lower triangular matrix $\mathcal{M}_\ell$ with positive diagonal elements.

Our definition of $\mathcal{S}$-blossoms does involve the space $\mathcal{E}$. Nevertheless, even though their domain of definition $\mathcal{B} \cap \mathcal{A}$ seems to depend on $\mathcal{E}$, $\mathcal{S}$-blossoms do not depend on it. Indeed, this is guaranteed by equality (2.10) which gives an intrinsic geometrical interpretation of the blossom σ of any nondegenerate $\mathcal{S}$-spline Σ, and also by the fact that any value of any degenerate $\mathcal{S}$-spline can be obtained as the image under an affine map of the corresponding value of a nondegenerate one. We can thus consider that the $(n + 1)$-dimensional space $\mathcal{E}$ which the spline space $\mathcal{S}$ is based on is fixed once and for all, and this justifies our approach.

2.3. *Piecewise Smooth Chebyshev Functions*

In the present subsection, we assume that $\mathcal{E}$-blossoms exist on the whole of $\mathbf{R}^n$. In other words, in addition to (I) and (II), the piecewise smooth geometrically regular function Φ satisfies the following property:

(III) *the blossom φ of Φ is defined on the whole of $\mathbf{R}^n$, i.e., $\mathcal{B} = \mathbf{R}^n$*;

which means that (2.3) holds whatever r, whatever the distinct real numbers $\tau_1, \ldots, \tau_r$, and whatever the positive integers $\mu_1, \ldots, \mu_r$. We shall then say that Φ is a *piecewise smooth Chebyshev function of order n on* $\mathbf{R}$. This situation allows us to use all results on piecewise smooth Chebyshev functions obtained in [8]. In particular, it ensures the validity of *the subblossoming principle* which says that, for any point $a \in \mathbf{R}$, the function $\widetilde{\Phi}(x) := \varphi(a, x^{[n-1]})$, $x \in \mathbf{R}$, is a piecewise smooth Chebyshev function of order $n-1$ on $\mathbf{R}$, with values in the osculating hyperplane Osc $\Phi_{n-1}(a)$, and its blossom is defined by

$$\widetilde{\varphi}(x_1, \dots, x_{n-1}) := \varphi(a, x_1, \dots, x_{n-1}), \qquad x_1, \dots, x_{n-1} \in \mathbf{R}.$$

By iteration of the subblossoming principle we obtain that, for any $x_1, \dots, x_{n-1} \in \mathbf{R}$, the function $\varphi(x_1, \dots, x_{n-1}, \cdot)$ is a piecewise smooth Chebyshev function of order 1 on $\mathbf{R}$, with values in an affine line. This exactly means that $\varphi(x_1, \dots, x_{n-1}, \cdot)$ is piecewise smooth and strictly monotone on $\mathbf{R}$. By taking the images under affine maps we can deduce that, in addition to $(B)_1$ and $(B)_2$, $\mathcal{E}$-blossoms then satisfy the third property:

$(B)_3$ *pseudoaffinity property*: for any $x_1, \dots, x_{n-1}, a, b \in \mathbf{R}$, with $a \neq b$, there exists a function $\beta(x_1, \dots, x_{n-1}; a, b; \cdot) : \mathbf{R} \to \mathbf{R}$ (independent of F), piecewise smooth and strictly monotone, such that the blossom f of any $\mathcal{E}$-function F satisfies

$$f(x_1, \dots, x_{n-1}, x) = [1 - \beta(x_1, \dots, x_{n-1}; a, b; x)] f(x_1, \dots, x_{n-1}, a) + \beta(x_1, \dots, x_{n-1}; a, b; x) f(x_1, \dots, x_{n-1}, b), \qquad x \in \mathbf{R}. \tag{2.11}$$

It is known that from the three properties $(B)_i$, $i = 1, 2, 3$, it is possible to develop all classical tools of geometric design in the space $\mathcal{E}$: de Casteljau-like algorithms, Bernstein-type bases, subdivision, ...

Remark 2.5. The blossom σ of a given $\mathcal{S}$-spline Σ, satisfying (2.6), is defined by (2.8). Given an admissible $(n-1)$-tuple $(\zeta_1, \dots, \zeta_{n-1})$, let us choose an integer $j_0 \in \mathcal{J}(\zeta_1, \dots, \zeta_{n-1})$. For all $x \in \mathcal{D}(\zeta_1, \dots, \zeta_{n-1})$, the integer j_0 also belongs to $\mathcal{J}(\zeta_1, \dots, \zeta_{n-1}, x)$. Consequently,

$$\sigma(\zeta_1, \dots, \zeta_{n-1}, x) = \varphi_{j_0}(\zeta_1, \dots, \zeta_{n-1}, x).$$

Hence, applying (2.11) to φ_{j_0} we can see that, in addition to $(SB)_1$ (on $\mathcal{A}$) and $(SB)_2$, $\mathcal{S}$-blossoms also satisfy:

$(SB)_3$ *pseudoaffinity property*: for any $(\zeta_1, \dots, \zeta_{n-1}) \in \mathcal{A}$ and any $a, b \in \mathcal{D}(\zeta_1, \dots, \zeta_{n-1})$, $a \neq b$, we have

$$\begin{aligned} \sigma(\zeta_1, \dots, \zeta_{n-1}, x) = {} & [1 - \beta(\zeta_1, \dots, \zeta_{n-1}; a, b; x)] \sigma(\zeta_1, \dots, \zeta_{n-1}, a) \\ & + \beta(\zeta_1, \dots, \zeta_{n-1}; a, b; x) \sigma(\zeta_1, \dots, \zeta_{n-1}, b), \\ & \qquad x \in \mathcal{D}(\zeta_1, \dots, \zeta_{n-1}). \end{aligned} \tag{2.12}$$

Under the assumption $\mathcal{B} = \mathbf{R}^n$, because $\mathcal{S}$-blossoms are symmetric on $\mathcal{A}$ and satisfy $(SB)_2$, $(SB)_3$, it is possible to develop a de Boor-like algorithm in the space $\mathcal{S}$, as we shall see later on. But the assumption $\mathcal{B} = \mathbf{R}^n$ is only a sufficient condition. Indeed, in order to satisfy $(SB)_3$, it is sufficient to require that the set $\mathcal{B}$ contain the set $\mathcal{A}$ and that (2.11) be valid only for any admissible $(x_1, \dots, x_{n-1})$ and only for any $a, b, x \in \mathcal{D}(x_1, \dots, x_{n-1})$, with $a \neq b$. We shall show in the fourth section that this can actually be derived from the inclusion $\mathcal{B} \supset \mathcal{A}$.

Remark 2.6. More generally, if $I := \bigcup_{\ell\in\mathcal{L}} I_\ell$, where $\mathcal{L} \subset \mathbf{Z}$ is composed of consecutive integers, we shall say that a function defined on I is piecewise smooth if it is C^∞ on each I_ℓ, $\ell \in \mathcal{L}$. Given a piecewise smooth function Φ defined as in (2.2), but only for $x \in I$, it will be said to be a piecewise smooth geometrically regular function of order n on I when it satisfies (I) and (II) on I. The blossoms φ of Φ is then defined on a subset $\mathcal{B}$ of I^n, and Φ is said to be a piecewise smooth Chebyshev function of order n on I (and simply a Chebyshev function of order n on I if $I = I_\ell$) when $\mathcal{B} = I^n$. If so, the subblossoming principle still holds (and therefore so does the pseudoaffinity property), but of course only on I.

3. B-Splines and Blossoms

In order to guarantee the nice behaviour of $\mathcal{S}$-splines, it is essential to ensure the existence of a B-spline basis which, in the present context, is classically meant as a sequence $\mathcal{N}_j$, $j \in \mathbf{Z}$, meeting the following four requirements:

(BSB)$_1$ *support property*: for each $\ell \in \mathbf{Z}$ the support of $\mathcal{N}_\ell$ is equal to $[\xi_\ell, \xi_{\ell+n+1}]$;
(BSB)$_2$ *decomposition property*: for any $\mathcal{S}$-spline Σ with values in $\mathbf{R}^d$, there exists a sequence $(T_\ell)_{\ell\in\mathbf{Z}}$ of points of $\mathbf{R}^d$ such that

(3.1) $$\Sigma(x) = \sum_{\ell\in\mathbf{Z}} \mathcal{N}_\ell(x) T_\ell, \qquad x \in \mathbf{R};$$

(BSB)$_3$ *positivity property*: for each $\ell \in \mathbf{Z}$, $\mathcal{N}_\ell$ is positive on the interior of its support; and
(BSB)$_4$ *normalization property*: $\sum_{\ell\in\mathbf{Z}} \mathcal{N}_\ell(x) = 1$ for all $x \in \mathbf{R}$.

The latter requirements automatically imply that each function $\mathcal{N}_j$ is an element of $\mathcal{S}$ and that any decomposition (3.1) is unique (see [12]). Hence, defining an $\mathcal{S}$-spline Σ is equivalent to choosing in a finite-dimensional affine space the sequence T_j, $j \in \mathbf{Z}$, of its *poles*, associated with Σ through equality (3.1). The spline curve defined by Σ is then located in the convex hull of the poles, and the influence of a pole T_j is as local as possible.

Nevertheless, as soon as at least one knot T_j is multiple (i.e., $m_j > 1$), the poles of a spline Σ are not clearly identified. To be correct we should speak of the poles of Σ related to the sequence $\mathcal{N}_j$, $j \in \mathbf{Z}$. Indeed, as soon as there exists one such sequence, there exists infinitely many others. In [12] we justified the introduction of an additional requirement, according the following definition:

Definition 3.1. We say that a sequence $\mathcal{N}_\ell$, $\ell \in \mathbf{Z}$, is a B-spline basis in the space $\mathcal{S}$ if it satisfies the four properties (BSB)$_i$, $1 \le i \le 4$, along with the following one:
(BSB)$_5$ *end point property*: for all $\ell \in \mathbf{Z}$,

(3.2) $$\lim_{x\to\xi_\ell^+} \frac{\mathcal{N}_{\ell+1}(x)}{\mathcal{N}_\ell(x)} = 0, \qquad \lim_{x\to\xi_{\ell+n+1}^-} \frac{\mathcal{N}_{\ell-1}(x)}{\mathcal{N}_\ell(x)} = 0.$$

In our present context of W-splines (i.e., splines with sections in W-spaces), the end point property can equivalently be stated as follows [12]:

(BSB)$'_5$ given $\ell \in \mathbf{Z}$, if $\xi_\ell = t_k$ and $\xi_{\ell+n+1} = t_{k'}$, the function $\mathcal{N}_\ell$ vanishes exactly $n - m_k + p + 1$ times at t_k^+ and exactly $n - m_{k'} + p' + 1$ times at $t_{k'}^-$, where

$$p := \sharp\{j < \ell \mid \xi_j = t_k\}, \qquad p' := \sharp\{j > \ell + n + 1 \mid \xi_j = t_{k'}\}. \tag{3.3}$$

This additional end point property ensures the unicity of a possible B-spline basis. Equivalently, it ensures the possible poles of a spline to be clearly identified. Let us recall their geometrical meaning, proved in [12].

Proposition 3.2. *A sequence $\mathcal{N}_\ell$, $\ell \in \mathbf{Z}$, supposed to satisfy the conditions* (BSB)$_i$, $1 \le i \le 4$, *is a B-spline basis* (*i.e., it satisfies the end point property* (BSB)$_5$ *too*) *iff all poles T_q, $q \in \mathbf{Z}$, of a nondegenerate $\mathcal{S}$-spline Σ are obtained as intersections of osculating flats of Σ, as follows: for any $q \in \mathbf{Z}$,*

$$\{T_q\} = \mathrm{Osc}_{n-\alpha}\Sigma(t_\ell^+) \cap \bigcap_{i=\ell+1}^{\ell+s} \mathrm{Osc}_{n-m_i}\Sigma(t_i) \cap \mathrm{Osc}_{n-\beta}\Sigma(t_{\ell+s+1}^-), \tag{3.4}$$

where the n-tuple $(\xi_{q+1}, \dots, \xi_{q+n})$ is given by

$$(\xi_{q+1}, \dots, \xi_{q+n}) = ({t_\ell}^{[\alpha]}, {t_{\ell+1}}^{[m_{\ell+1}]}, \dots, {t_{\ell+s}}^{[m_{\ell+s}]}, {t_{\ell+s+1}}^{[\beta]}), \tag{3.5}$$

with $m_\ell \neq 0$, $m_{\ell+s+1} \neq 0$, $0 \le \alpha < m_\ell$, $0 \le \beta < m_{\ell+s+1}$.

To each knot t_ℓ, let us allocate a "new" multiplicity $\widetilde{m}_\ell \ge m_\ell$. Associated with the "new" knot vector $\widetilde{\mathcal{K}} := ({t_\ell}^{[\widetilde{m}_\ell]})_{\ell \in \mathbf{Z}}$, we consider the "new" spline space $\widetilde{\mathcal{S}}$, said to be obtained from $\mathcal{S}$ by *insertion of knots*. It satifies $\mathcal{S} \subset \widetilde{\mathcal{S}}$. The "new" set $\widetilde{\mathcal{A}}$ of admissible n-tuples is clearly contained in the "old" one $\mathcal{A}$. On the other hand, the space $\mathcal{E}$ is not modified when adding a given point $u \in]t_\ell, t_{\ell+1}[$ to the sequence $\mathcal{T}$, provided that the space $\mathcal{E}_\ell$ is replaced by its restrictions to the two intervals $[t_\ell, u]$ and $[u, t_{\ell+1}]$, the connections at the point u being defined by the idendity matrix of order n. Provided that the point u is allocated the multiplicity 0, the knot vector $\mathcal{K}$ and the spline space $\mathcal{S}$ are not modified either. The insertion of $(u^{[r]})$, $r \le n$, to the knot vector $\mathcal{K}$ then follows the previous description.

The aim of the present paper is to establish the exact links between existence of blossoms and existence of B-spline bases in W-spline spaces. The expression "existence of blossoms" in the spline space $\mathcal{S}$ is meant as the fact that $\mathcal{S}$-blossoms exist on the largest possible subset of $\mathbf{R}^n$, that is, on the set $\mathcal{A}$ of all admissible n-tuples or, equivalently, as the fact that $\mathcal{B} \supset \mathcal{A}$. We shall actually prove that existence of blossoms is equivalent to existence of B-spline bases in the sense of the following theorem:

Theorem 3.3. *The following two statements are equivalent:*

(i) *$\mathcal{S}$-blossoms exist on the set $\mathcal{A}$ of all admissible n-tuples, i.e., $\mathcal{B} \supset \mathcal{A}$;*
(ii) *there exists a* (*unique*) *B-spline basis in the space $\mathcal{S}$, and also in any spline space derived from $\mathcal{S}$ by insertion of knots.*

Let us first observe that, as a simple consequence of Proposition 3.2, we obtain the following result:

Proposition 3.4. *If there exists a (unique) B-spline basis in the space $\mathcal{S}$, then the set $\mathcal{B}$ contains all points $(\xi_{q+1}, \dots, \xi_{q+n})$, $q \in \mathbf{Z}$.*

Proof. Let Σ be a nondegenerate spline with poles T_q, $q \in \mathbf{Z}$. Suppose that the space $\mathcal{S}$ possesses a B-spline basis $\mathcal{N}_q$, $q \in \mathbf{Z}$. Then, according to Proposition 3.2, for any $q \in \mathbf{Z}$, the pole T_q of Σ is given by (3.4), where the integers ℓ, s, α, β come from (3.5). Select an integer k, $\ell \le k \le \ell + s$. It belongs to $\mathcal{J}(\xi_{q+1}, \dots, \xi_{q+n})$. Therefore, according to (2.10) and (2.8), the $\mathcal{E}$-function Φ_k, which coincides with Σ on $[t_k, t_{k+1}]$, satisfies

$$\text{(3.6)} \qquad \mathrm{Osc}_{n-\alpha}\Phi_k(t_\ell) \cap \bigcap_{i=\ell+1}^{\ell+s} \mathrm{Osc}_{n-m_i}\Phi_k(t_i) \cap \mathrm{Osc}_{n-\beta}\Phi_k(t_{\ell+s+1}) = \{T_q\}.$$

As already observed, the intersection of osculating flats obtained by replacing Φ_i by Φ in the left-hand side of (3.5) consists of a single point too. This means that the n-tuple $(\xi_{q+1}, \dots, \xi_{q+n})$ belongs to the set $\mathcal{B}$. ∎

Any admissible n-tuple being composed of n consecutive points of an appropriate knot vector obtained from the knot vector $\mathcal{K}$ by insertion of knots, Proposition 3.3 readily proves part (ii) $\Rightarrow$ (i) of Theorem 3.3. It therefore only remains to show that (i) $\Rightarrow$ (ii), which is what the next section is devoted to. The proof contains tricky points which may escape the reader's attention at first glance. We shall thus emphasize them whenever necessary.

4. Proof of Theorem 3.3: (i) Implies (ii)

Given two integers $\ell \in \mathbf{Z}$, $s > 0$, such that $p := n - \sum_{i=\ell+1}^{\ell+s} m_i \ge 1$, the $(n-p)$-tuple $({t_{\ell+1}}^{[m_{\ell+1}]}, \dots, {t_{\ell+s}}^{[m_{\ell+s}]})$ is admissible. Its domain satisfies

$$\mathcal{D}({t_{\ell+1}}^{[m_{\ell+1}]}, \dots, {t_{\ell+s}}^{[m_{\ell+s}]}) \supset [t_\ell, t_{\ell+s+1}],$$

and the latter inclusion is an equality when both m_ℓ and $m_{\ell+s+1}$ are positive. Let us introduce the following function:

$$\text{(4.1)} \qquad \Psi_{\ell,s}(x) := \varphi({t_{\ell+1}}^{[m_{\ell+1}]}, \dots, {t_{\ell+s}}^{[m_{\ell+s}]}, x^{[p]}).$$

Without any additional assumption on Φ, the range of significance of this function may be empty. If we assume that $\Psi_{\ell,s}$ is well defined on the interval $[t_\ell, t_{\ell+s+1}]$, i.e., if we only assume the set $\mathcal{B}$ to contain all n-tuples $({t_{\ell+1}}^{[m_{\ell+1}]}, \dots, {t_{\ell+s}}^{[m_{\ell+s}]}, x^{[p]})$, $x \in [t_\ell, t_{\ell+s+1}]$, then we cannot say much about its properties on that interval. On the other hand, if Φ were supposed to be a piecewise smooth Chebyshev function on the interval $[t_\ell, t_{\ell+s+1}]$, i.e., under the assumption $\mathcal{B} \supset [t_\ell, t_{\ell+s+1}]^n$, not only the function $\Psi_{\ell,s}$ would be well defined on $[t_\ell, t_{\ell+s+1}]$, but the subblossoming principle on $[t_\ell, t_{\ell+s+1}]$ would also allow us to assert that $\Psi_{\ell,s}$ would be a piecewise smooth Chebyshev function of order p on that interval.

In Proposition 4.1 below, we consider the intermediate assumption that $\mathcal{B}$ contains all admissible n-tuples belonging to $[t_\ell, t_{\ell+s+1}]^n$, that is, $\mathcal{B} \supset \mathcal{A} \cap [t_\ell, t_{\ell+s+1}]^n$. This assumption does not ensure that Φ is a piecewise smooth Chebyshev function on $[t_\ell, t_{\ell+s+1}]$, since it is much weaker than $\mathcal{B} \supset [t_\ell, t_{\ell+s+1}]^n$. Therefore a priori it is not sufficient to allow a subblossoming principle directly on $[t_\ell, t_{\ell+s+1}]$. Nonetheless, we shall see that it leads to the same conclusion on the function $\Psi_{\ell,s}$ as in the case $\mathcal{B} \supset [t_\ell, t_{\ell+s+1}]^n$.

Proposition 4.1. *Given two integers $\ell \in \mathbf{Z}$ and $s \in \mathbf{N}$ such that $p := n - \sum_{i=\ell+1}^{\ell+s} m_i \geq 1$, we assume that $\mathcal{B} \supset \mathcal{A} \cap [t_\ell, t_{\ell+s+1}]^n$. Then the function $\Psi_{\ell,s}$ introduced in* (4.1) *is a piecewise smooth Chebyshev function of order p on $[t_\ell, t_{\ell+s+1}]$. Its blossom $\psi_{\ell,s}$ is given by*

$$\psi_{\ell,s}(x_1, \dots, x_p) := \varphi({t_{\ell+1}}^{[m_{\ell+1}]}, \dots, {t_{\ell+s}}^{[m_{\ell+s}]}, x_1, \dots, x_p), \quad x_1, \dots, x_p \in [t_\ell, t_{\ell+s+1}]. \tag{4.2}$$

For any $x_1, \dots, x_p \in [t_\ell, t_{\ell+s+1}]$, the n-tuple $({t_{\ell+1}}^{[m_{\ell+1}]}, \dots, {t_{\ell+s}}^{[m_{\ell+s}]}, x_1, \dots, x_p)$ is admissible. Hence, under the assumption $\mathcal{B} \supset \mathcal{A} \cap [t_\ell, t_{\ell+s+1}]^n$, the function $\psi_{\ell,s}$ is well defined on $[t_\ell, t_{\ell+s+1}]^p$. The proof of Proposition 4.1 will use the following lemma:

Lemma 4.2. *In order to prove Proposition* 4.1, *it is sufficient to show that the following properties are satisfied*:

(1) *$\Psi_{\ell,s}$ is a piecewise smooth geometrically regular function of order p on $[t_\ell, t_{\ell+s+1}]$;*
(2) *for any $x \in [t_\ell, t_{\ell+s+1}]$, and any integer $i \leq p$, the ith-order osculating flat of $\Psi_{\ell,s}$ at x (meant as $\mathrm{Osc}_i \Psi_{\ell,s}(t_\ell^+)$ if $x = t_\ell$ and $i > n - m_\ell$, and as $\mathrm{Osc}_i \Psi_{\ell,s}(t_{\ell+s+1}^-)$ if $x = t_{\ell+s+1}$ and $i > n - m_{\ell+s+1}$), is given by*

$$\mathrm{Osc}_i \Psi_{\ell,s}(x) = \mathrm{Osc}_{i+\sum_{j=\ell+1}^{\ell+s} m_j} \Phi(x) \cap \bigcap_{j=\ell+1}^{\ell+s} \mathrm{Osc}_{n-m_j} \Phi(t_j), \quad \text{if } x \notin \{t_{\ell+1}, \dots, t_{\ell+s}\}, \tag{4.3}$$

$$\mathrm{Osc}_i \Psi_{\ell,s}(t_{\ell+k}) = \mathrm{Osc}_{i+\sum_{j=\ell+1, j\neq \ell+k}^{\ell+s} m_j} \Phi(t_{\ell+k}) \cap \bigcap_{\substack{j=\ell+1\\ j\neq\ell+k}}^{\ell+s} \mathrm{Osc}_{n-m_j} \Phi(t_j), \quad 1 \leq k \leq s. \tag{4.4}$$

Proof. Suppose that the two conditions (1) and (2) are satisfied. In order to prove Proposition 4.1 it is sufficient to check that, for any pairwise distinct $\tau_1, \dots, \tau_r \in [t_\ell, t_{\ell+s+1}]$ and any positive integers $\mu_1, \dots, \mu_r$ such that $\sum_{i=1}^{r} \mu_i = p$,

$$\bigcap_{i=1}^{r} \mathrm{Osc}_{p-\mu_i} \Psi_{\ell,s}(\tau_i) = \{\psi_{\ell,s}({\tau_1}^{[\mu_1]}, \dots, {\tau_r}^{[\mu_r]})\}. \tag{4.5}$$

Let us first suppose that none of the τ_i's belongs to $\{t_{\ell+1}, \dots, t_{\ell+s}\}$. According to (4.3), for all $i = 1, \dots, r$,

$$\text{Osc}_{p-\mu_i}\,\Psi_{\ell,s}(\tau_i) = \text{Osc}_{n-\mu_i}\,\Phi(\tau_i) \cap \bigcap_{j=\ell+1}^{\ell+s} \text{Osc}_{n-m_j}\,\Phi(t_j). \tag{4.6}$$

Hence

$$\bigcap_{i=1}^{r} \text{Osc}_{p-\mu_i}\,\Psi_{\ell,s}(\tau_i) = \bigcap_{i=1}^{r} \text{Osc}_{n-\mu_i}\,\Phi(\tau_i) \cap \bigcap_{j=\ell+1}^{\ell+s} \text{Osc}_{n-m_j}\,\Phi(t_j). \tag{4.7}$$

The n-tuple $({t_{\ell+1}}^{[m_{\ell+1}]}, \dots, {t_{\ell+s}}^{[m_{\ell+s}]}, {\tau_1}^{[\mu_1]}, \dots, {\tau_r}^{[\mu_r]})$ belongs to $[t_\ell, t_{\ell+s+1}]^n$ and it is admissible. Therefore, due to the assumption $\mathcal{B} \supset \mathcal{A} \cap [t_\ell, t_{\ell+s+1}]^n$, the right-hand side of (4.7) consists of the single point $\varphi({t_{\ell+1}}^{[m_{\ell+1}]}, \dots, {t_{\ell+s}}^{[m_{\ell+s}]}, {\tau_1}^{[\mu_1]}, \dots, {\tau_r}^{[\mu_r]})$. On account of (4.2), equality (4.7) is nothing but the expected equality (4.5).

Assume now that $\tau_1 = t_{\ell+1}$, and that the other τ_i's do not belong to $\{t_{\ell+2}, \dots, t_{\ell+s}\}$. Then, according to (4.4),

$$\text{Osc}_{p-\mu_1}\Psi_{\ell,s}(\tau_1) = \text{Osc}_{n-\mu_1-m_{\ell+1}}\Phi(t_{\ell+1}) \cap \bigcap_{j=\ell+2}^{\ell+s} \text{Osc}_{n-m_j}\Phi(t_j),$$

while (4.6) is still valid for $i = 2, \dots, r$. Taking the inclusion $\mathcal{B} \supset \mathcal{A} \cap [t_\ell, t_{\ell+s+1}]^n$ into consideration, it follows that

$$\bigcap_{i=1}^{r} \text{Osc}_{p-\mu_i}\,\Psi_{\ell,s}(\tau_i) = \{\varphi({t_{\ell+1}}^{[m_{\ell+1}+\mu_1]}, {t_{\ell+2}}^{[m_{\ell+2}]}, \dots, {t_{\ell+s}}^{[m_{\ell+s}]}, {\tau_2}^{[\mu_2]}, \dots, {\tau_r}^{[\mu_r]})\},$$

which is again the expected equality (4.5). The general case follows in a similar way. ■

Proof of Proposition 4.1. We shall prove by induction on s that the two conditions (1) and (2) of Lemma 4.2 are fulfilled. Since $\mathcal{A} \cap [t_\ell, t_{\ell+1}]^n = [t_\ell, t_{\ell+1}]^n$, for $s = 0$, our assumption is that $\mathcal{B} \supset [t_\ell, t_{\ell+1}]^n$ (i.e., Φ is a Chebyshev function on I_ℓ). There is nothing to prove since $\Psi_{\ell,0}$ is the restriction of Φ to the interval I_ℓ.

We now assume that $s \geq 1$, that $p := n - \sum_{i=\ell+1}^{\ell+s} m_i \geq 1$, and that $\mathcal{B} \supset \mathcal{A} \cap [t_\ell, t_{\ell+s+1}]^n$. We also assume the two properties (1) and (2) of Lemma 4.2 to be proved for $s - 1$, and we shall prove that they are still valid for s. Accordingly, setting $q := p + m_{\ell+s} = n - \sum_{i=\ell+1}^{\ell+s-1} m_i$ and $q_1 := p + m_{\ell+1} = n - \sum_{i=\ell+2}^{\ell+s} m_i$, the two functions

$$\begin{aligned} \Psi_{\ell,s-1}(x) &= \varphi({t_{\ell+1}}^{[m_{\ell+1}]}, \dots, {t_{\ell+s-1}}^{[m_{\ell+s-1}]}, x^{[q]}), \qquad & x \in [t_\ell, t_{\ell+s}], \\ \Psi_{\ell+1,s-1}(x) &= \varphi({t_{\ell+2}}^{[m_{\ell+2}]}, \dots, {t_{\ell+s}}^{[m_{\ell+s}]}, x^{[q_1]}), \qquad & x \in [t_{\ell+1}, t_{\ell+s+1}], \end{aligned}$$

satisfy (1) and (2) on $[t_\ell, t_{\ell+s}]$ and $[t_{\ell+1}, t_{\ell+s+1}]$, respectively. As shown in Lemma 4.2, this implies the function $\Psi_{\ell,s-1}$ to be a piecewise smooth Chebyshev function of order q on $[t_\ell, t_{\ell+s}]$, its blossom being given by

$$\psi_{\ell,s-1}(x_1, \dots, x_q) = \varphi({t_{\ell+1}}^{[m_{\ell+1}]}, \dots, {t_{\ell+s-1}}^{[m_{\ell+s-1}]}, x_1, \dots, x_q), \qquad x_1, \dots, x_q \in [t_\ell, t_{\ell+s}].$$

We are thus allowed to apply the subblossoming principle to $\Psi_{\ell,s-1}$ on the interval $[t_\ell, t_{\ell+s}]$. Therefore, for any integer k, $0 \le k \le m_{\ell+s}$, the function

$$\Psi_{\ell,s-1,k}(x) := \psi_{\ell,s-1}({t_{\ell+s}}^{[k]}, x^{[q-k]}), \qquad x \in [t_\ell, t_{\ell+s}], \tag{4.8}$$

is a piecewise smooth Chebyshev function of order $q-k$ on the interval $[t_\ell, t_{\ell+s}]$, with blossom

$$\psi_{\ell,s-1,k}(x_1, \dots, x_{q-k}) := \psi_{\ell,s-1}({t_{\ell+s}}^{[k]}, x_1, \dots, x_{q-k}), \quad x_1, \dots, x_{q-k} \in [t_\ell, t_{\ell+s}]. \tag{4.9}$$

Using again Lemma 4.2, we can also state that $\Psi_{\ell+1,s-1}$ is a piecewise smooth Chebyshev function of order q_1 on $[t_{\ell+1}, t_{\ell+s+1}]$, its blossom being given by

$$\psi_{\ell+1,s-1}(x_1, \dots, x_{q_1}) = \varphi({t_{\ell+2}}^{[m_{\ell+2}]}, \dots, {t_{\ell+s}}^{[m_{\ell+s}]}, x_1, \dots, x_{q_1}), \quad x_1, \dots, x_{q_1} \in [t_{\ell+1}, t_{\ell+s+1}].$$

The subblossoming principle allows us to similarly state that, for any integer k, $0 \le k \le m_{\ell+1}$, the function

$$\overline{\Psi}_{\ell+1,s-1,k}(x) := \psi_{\ell+1,s-1}({t_{\ell+1}}^{[k]}, x^{[q_1-k]}), \qquad x \in [t_{\ell+1}, t_{\ell+s+1}], \tag{4.10}$$

is a piecewise smooth Chebyshev function of order $q_1 - k$ on the interval $[t_{\ell+1}, t_{\ell+s+1}]$.

According to (4.8) and (4.10), we have

$$\Psi_{\ell,s}(x) = \begin{cases} \Psi_{\ell,s-1,m_{\ell+s}}(x) & \text{if } x \in [t_\ell, t_{\ell+s}], \\ \overline{\Psi}_{\ell+1,s-1,m_{\ell+1}}(x) & \text{if } x \in [t_{\ell+1}, t_{\ell+s+1}]. \end{cases} \tag{4.11}$$

So, $\Psi_{\ell,s}$ is a piecewise smooth Chebyshev function (hence, in particular, a piecewise smooth geometrically regular function) of order p on both intervals $[t_\ell, t_{\ell+s}]$ and $[t_{\ell+1}, t_{\ell+s+1}]$ separately. We want to draw the reader's attention on the fact that this does not allow us to directly conclude that $\Psi_{\ell,s}$ is a piecewise smooth Chebyshev function on $[t_\ell, t_{\ell+s+1}]$. However, if the two intervals overlap, that is, if $s \ge 2$, it is sufficient to conclude that $\Psi_{\ell,s}$ is a piecewise smooth geometrically regular function of order p on the whole interval $[t_\ell, t_{\ell+s+1}]$. On the contrary, it is not sufficient in the case $s = 1$, because in this case we do not know yet what kind of a connection there exists between the left and right derivatives of $\Psi_{\ell,s}$ at the point $t_{\ell+1}$. Our proof of (1) by induction is therefore not complete yet.

In order to solve the particular case $s = 1$, and also in order to prove (2), we do need to give a sketch of how to prove the subblossoming principle. Given $k \le m_{\ell+s} - 1$, suppose that $\Psi_{\ell,s-1,k}$ is a piecewise smooth Chebyshev function of order $q-k$ on $[t_\ell, t_{\ell+s}]$, with blossom $\psi_{\ell,s-1,k}$ (which is satisfied for $k = 0$ since $\Psi_{\ell,s-1,0} = \Psi_{\ell,s-1}$) and let us show how to prove that $\Psi_{\ell,s-1,k+1}$ is in turn a piecewise smooth Chebyshev function of order $q-k-1$ on $[t_\ell, t_{\ell+s}]$. Since $\Psi_{\ell,s-1,k+1}(x) = \psi_{\ell,s-1,k}(t_{\ell+s}, x^{[q-k-1]})$ for all $x \in [t_\ell, t_{\ell+s}]$, we have

$$\begin{aligned} \Psi_{\ell,s-1,k+1}(t_{\ell+s}) &= \Psi_{\ell,s-1,k}(t_{\ell+s}), \\ \{\Psi_{\ell,s-1,k+1}(x)\} &= \mathrm{Osc}_1 \Psi_{\ell,s-1,k}(x) \cap \mathrm{Osc}_{q-k-1} \Psi_{\ell,s-1,k}(t_{\ell+s}^-), \\ & \qquad x \in [t_\ell, t_{\ell+s}[. \end{aligned} \tag{4.12}$$

For all $x \in [t_\ell, t_{\ell+s}]$, we can derive from relations (4.12) the existence of a number $\lambda_{\ell,s-1,k}(x)$ such that

$$\Psi_{\ell,s-1,k+1}(x) = \Psi_{\ell,s-1,k}(x) + \lambda_{\ell,s-1,k}(x)\Psi_{\ell,s-1,k}{}'(x). \tag{4.13}$$

We want to draw the reader's attention to the fact that when $x = t_j$, $\ell < j < \ell + s$, the equality (4.13) represents in fact two equalities, one corresponding to $x = t_j^-$ and the other to t_j^+. Indeed, due to the regularity of $\Psi_{\ell,s-1,k}$, there exists a positive number a such that $\Psi_{\ell,s-1,k}{}'(t_j^+) = a\Psi_{\ell,s-1,k}{}'(t_j^-)$, which implies that $\lambda_{\ell,s-1,k}(t_j^+) = \lambda_{\ell,s-1,k}(t_j^-)/a$. On the other hand, we have

$$\lambda_{\ell,s-1,k}(t_{\ell+s}) = 0, \qquad \lambda_{\ell,s-1,k}(x) \neq 0 \quad \text{for} \quad x \in [t_\ell, t_{\ell+s}[. \tag{4.14}$$

The latter relation results from $\psi_{\ell,s-1,k}(x^{[q-k-1]}, \cdot)$ being one-to-one since it is a piecewise smooth Chebyshev function of order 1 on $[t_\ell, t_{\ell+s}]$. What is less obvious a priori is that, on each $[t_j, t_{j+1}]$, $\ell \le j \le \ell + s - 1$, the function $\lambda_{\ell,s-1,k}$ is C^∞, with

$$\lambda_{\ell,s-1,k}{}'(t_{\ell+s}^-) = -\frac{1}{q-k}. \tag{4.15}$$

For the proof we refer to [8]. This actually constitutes one of the difficult parts of the subblossoming principle. It implies in particular the piecewise smoothness of $\Psi_{\ell,s-1,k+1}$, so that, after (possibly left or right) differentiation of (4.13) up to order $i \ge 1$, we obtain

$$\begin{aligned} \Psi_{\ell,s-1,k+1}^{(i)}(x) = &\sum_{j=1}^{i-1} \binom{i}{j} \Psi_{\ell,s-1,k}^{(j)}(x) \\ &+ (1 + i\lambda_{\ell,s-1,k}{}'(x))\Psi_{\ell,s-1,k}^{(i)}(x) + \lambda_{\ell,s-1,k}(x)\Psi_{\ell,s-1,k}^{(i+1)}(x), \\ &\qquad x \in [t_\ell, t_{\ell+s}]. \end{aligned} \tag{4.16}$$

We will not insist on why the left and right derivatives at each t_j, $\ell + 1 \le j \le \ell + s - 1$, are linked by lower triangular matrices with positive diagonal elements. For this we refer the reader to [8]. We are rather interested in focusing on what occurs at the point $t_{\ell+s}$. On account of (4.15) and of the left part of (4.14), the equalities (4.16) lead to

$$\begin{aligned} &(\Psi_{\ell,s-1,k+1}{}'(t_{\ell+s}^-), \ldots, \Psi_{\ell,s-1,k+1}^{(q-k-1)}(t_{\ell+s}^-))^T \\ &\qquad = \mathcal{R}_{\ell,s-1,k}^- \cdot (\Psi_{\ell,s-1,k}{}'(t_{\ell+s}^-), \ldots, \Psi_{\ell,s-1,k}^{(q-k-1)}(t_{\ell+s}^-))^T, \end{aligned} \tag{4.17}$$

where $\mathcal{R}_{\ell,s-1,k}^-$ is a lower triangular matrix of order $q - k - 1$ with diagonal elements $1 - i/(q - k)$, $1 \le i \le q - k - 1$, hence with positive diagonal elements. In particular, this yields the equality

$$\mathrm{Osc}_i \Psi_{\ell,s-1,k+1}(t_{\ell+s}^-) = \mathrm{Osc}_i \Psi_{\ell,s-1,k+1}(t_{\ell+s}^-), \qquad i \le q - k - 1. \tag{4.18}$$

For $x \in [t_\ell, t_{\ell+s}[$, the linear independence of the vectors $\Psi_{\ell,s-1,k+1}{}'(x), \ldots, \Psi_{\ell,s-1,k+1}^{(q-k-1)}(x)$ (with, as usual, the meaning of either left or right derivatives if x is one of the points $t_{\ell+1}, \ldots, t_{\ell+s-1}$) readily follows from (4.16) and the right part of (4.14).

Hence $\Psi_{\ell,s-1,k+1}$ is a piecewise smooth geometrically regular function of order $q-k-1$ on $[t_\ell, t_{\ell+s}]$. Due to (4.12), $\Psi_{\ell,s-1,k+1}$ takes its values in $\mathrm{Osc}_{q-k-1}\Psi_{\ell,s-1,k}(t_{\ell+s}^-)$. From (4.16) we can thus derive the following inclusion, valid for all $i \le q-k-1$ and all $x \in [t_\ell, t_{\ell+s}[$,

$$\text{(4.19)} \qquad \mathrm{Osc}_i\Psi_{\ell,s-1,k+1}(x) \subset \mathrm{Osc}_{i+1}\Psi_{\ell,s-1,k}(x) \cap \mathrm{Osc}_{q-k-1}\Psi_{\ell,s-1,k}(t_{\ell+s}^-).$$

Because $\Psi_{\ell,s-1,k}$ is a piecewise smooth Chebyshev function on $[t_\ell, t_{\ell+s}]$, the right-hand side of (4.19) is of dimension i (see [8, Corollary 3.5]). Accordingly, the latter inclusion is in fact an equality. Along with (4.18) this enables us to achieve our expected conclusion that $\Psi_{\ell,s-1,k+1}$ in turn is a piecewise smooth Chebyshev function of order $q-k-1$ on $[t_\ell, t_{\ell+s}]$, the proof being similar to that of Lemma 4.2. Moreover, on account of (4.11), by iteration of (4.18) we obtain

$$\text{(4.20)} \qquad \mathrm{Osc}_i\Psi_{\ell,s}(t_{\ell+s}^-) = \mathrm{Osc}_i\Psi_{\ell,s-1}(t_{\ell+s}^-), \qquad i \le p.$$

Due to (4.8), (4.11), and to $\Psi_{\ell,s-1}$ being a piecewise smooth Chebyshev function on $[t_\ell, t_{\ell+s}]$, with blossom $\psi_{\ell,s-1}$, the function $\Psi_{\ell,s}$ takes its values in $\mathrm{Osc}_p\Psi_{\ell,s-1}(t_{\ell+s})$. Iterating (4.19) and using again [8, Corollary 3.5], we also obtain

$$\text{(4.21)} \qquad \mathrm{Osc}_i\Psi_{\ell,s}(x) \;=\; \mathrm{Osc}_{i+m_{\ell+s}}\Psi_{\ell,s-1}(x) \cap \mathrm{Osc}_p\Psi_{\ell,s-1}(t_{\ell+s}^-), \qquad \text{if} \quad x \in [t_\ell, \dots, t_{\ell+s}[.$$

Finally, iteration of (4.17) proves the existence of a lower triangular matrix $\mathcal{R}_{\ell,s-1}^-$ with positive diagonal elements such that

$$\text{(4.22)} \qquad (\Psi_{\ell,s}{}'(t_{\ell+s}^-), \dots, \Psi_{\ell,s}{}^{(p)}(t_{\ell+s}^-))^T = \mathcal{R}_{\ell,s-1}^- \cdot (\Psi_{\ell,s-1}{}'(t_{\ell+s}^-), \dots, \Psi_{\ell,s-1}^{(p)}(t_{\ell+s}^-))^T,$$

Applying the same kind of arguments to $\Psi_{\ell+1,s-1}$ on the interval $[t_{\ell+1}, t_{\ell+s+1}]$ similarly yields

$$\text{(4.23)} \qquad \mathrm{Osc}_i\Psi_{\ell,s}(t_{\ell+1}^+) = \mathrm{Osc}_i\Psi_{\ell+1,s-1}(t_{\ell+1}^+), \qquad i \le p.$$

and, for $x \in\,]t_{\ell+1}, \dots, t_{\ell+s+1}]$ and $i \le p$,

$$\text{(4.24)} \qquad \mathrm{Osc}_i\Psi_{\ell,s}(x) = \mathrm{Osc}_{i+m_{\ell+1}}\Psi_{\ell+1,s-1}(x) \cap \mathrm{Osc}_p\Psi_{\ell+1,s-1}(t_{\ell+1}^+).$$

It also proves the existence of a lower triangular matrix $\mathcal{R}_{\ell,s-1}^+$ with positive diagonal elements such that

$$\text{(4.25)} \qquad (\Psi_{\ell,s}{}'(t_{\ell+1}^+), \dots, \Psi_{\ell,s}^{(p)}(t_{\ell+1}^+))^T = \mathcal{R}_{\ell+1,s-1}^+ \cdot (\Psi_{\ell+1,s-1}{}'(t_{\ell+1}^+), \dots, \Psi_{\ell+1,s-1}^{(p)}(t_{\ell+1}^+))^T.$$

In the particular case $s = 1$, equalities (4.22) and (4.25) can be written as follows:

$$(\Psi_{\ell,1}{}'(t_{\ell+1}^-), \dots, \Psi_{\ell,1}^{(n-m_{\ell+1})}(t_{\ell+1}^-))^T \;=\; \mathcal{R}_{\ell,0}^- \cdot (\Phi'(t_{\ell+1}^-), \dots, \Phi^{(n-m_{\ell+1})}(t_{\ell+1}^-))^T,$$
$$(\Psi_{\ell,1}{}'(t_{\ell+1}^+), \dots, \Psi_{\ell,1}^{(n-m_{\ell+1})}(t_{\ell+1}^+))^T \;=\; \mathcal{R}_{\ell+1,0}^+ \cdot (\Phi'(t_{\ell+1}^+), \dots, \Phi^{(n-m_{\ell+1})}(t_{\ell+1}^+))^T.$$

Taking the piecewise smooth geometric regularity of Φ into account, the latter two equalities show that the left and right derivatives of $\Psi_{\ell,1}$ at $t_{\ell+1}$ are linked by a lower triangular matrix with positive diagonal elements, which completes the proof of $\Psi_{\ell,1}$ being piecewise smooth geometrically regular on $[t_\ell, t_{\ell+2}]$. Finally, both functions $\Psi_{\ell,s-1}$ and $\Psi_{\ell+1,s-1}$ satisfying property (2) of Lemma 4.2 on $[t_\ell, t_{\ell+s}]$ and $[t_{\ell+1}, t_{\ell+s+1}]$, respectively, the equalities (4.20), (4.21), (4.23), and (4.24) prove that the function $\Psi_{\ell,s}$ satisfies it in turn on $[t_\ell, t_{\ell+s+1}]$. ■

Proposition 4.3. *Assume that $\mathcal{B} \supset \mathcal{A}$. Then, for any admissible q-tuple $(\zeta_1, \ldots, \zeta_q)$, $q \leq n-1$, the function $x \mapsto \widetilde{\Phi}(x) := \varphi(\zeta_1, \ldots, \zeta_q, x^{[n-q]})$ is a piecewise smooth Chebyshev function of order $n-q$ on $\mathcal{D}(\zeta_1, \ldots, \zeta_q)$. Its blossom is defined by*

$$\widetilde{\varphi}(x_1, \ldots, x_{n-q}) = \varphi(\zeta_1, \ldots, \zeta_q, x_1, \ldots, x_{n-q}), \quad x_1, \ldots, x_{n-q} \in \mathcal{D}(\zeta_1, \ldots, \zeta_q). \tag{4.26}$$

Proof. Suppose that $\mathcal{D}(\zeta_1, \ldots, \zeta_q) = [t_\ell, t_{\ell+s+1}]$, $\ell \in \mathbf{Z}$, $s \in \mathbf{N}$. Then, $m_\ell \neq 0$, $m_{\ell+s+1} \neq 0$ and, according to (2.7), up to a permutation,

$$(\zeta_1, \ldots, \zeta_q) = ({t_{\ell+1}}^{[m_{\ell+1}]}, \ldots, {t_{\ell+s}}^{[m_{\ell+s}]}, y_1, \ldots, y_r),$$

with $r := q - \sum_{i=\ell+1}^{\ell+s} m_i$, and where the points $y_1, \ldots, y_r$ all belong to $[t_\ell, t_{\ell+s+1}]$. Since $\mathcal{B} \supset \mathcal{A}$, we can apply Proposition 4.1 to the corresponding function $\Psi_{\ell,s}$. This function is thus a piecewise smooth Chebyshev function of order $p := n - \sum_{i=\ell+1}^{\ell+s} m_i$ on $[t_\ell, t_{\ell+s+1}]$ and its blossom is the function $\psi_{\ell,s}$ defined in (4.2). We are thus allowed to apply the subblossoming principle to $\Psi_{\ell,s}$ on $[t_\ell, t_{\ell+s+1}]$, which proves that $x \mapsto \psi_{\ell,s}(y_1, \ldots, y_r, x^{[p-r]}) = \widetilde{\Phi}(x)$ is a Chebyshev function of order $p-r = n-q$ on $[t_\ell, t_{\ell+s+1}]$ and that its blossom is the function $(x_1, \ldots, x_{p-r}) \mapsto \psi_{\ell,s}(y_1, \ldots, y_r, x_1, \ldots, x_{p-r}) = \widetilde{\varphi}(x_1, \ldots, x_{n-q})$. ■

As a particular case of the latter proposition, we can state the following result:

Corollary 4.4. *Suppose that $\mathcal{B} \supset \mathcal{A}$. Given any admissible $(n-1)$-tuple $(\zeta_1, \ldots, \zeta_{n-1})$, and any $a, b \in \mathcal{D}(\zeta_1, \ldots, \zeta_{n-1})$, $a \neq b$, there exists a strictly monotone piecewise smooth function $\beta(\zeta_1, \ldots, \zeta_{n-1}; a, b; \cdot) : \mathcal{D}(\zeta_1, \ldots, \zeta_{n-1}) \to \mathbf{R}$ such that*

$$\begin{aligned} \varphi(\zeta_1, \ldots, \zeta_{n-1}, x) = {} & [1 - \beta(\zeta_1, \ldots, \zeta_{n-1}; a, b; x)]\varphi(\zeta_1, \ldots, \zeta_{n-1}, a) \\ & + \beta(\zeta_1, \ldots, \zeta_{n-1}; a, b; x)\varphi(\zeta_1, \ldots, \zeta_{n-1}, b), \\ & \qquad x \in \mathcal{D}(\zeta_1, \ldots, \zeta_{n-1}). \end{aligned} \tag{4.27}$$

Proof. This readily results from the function $\varphi(\zeta_1, \ldots, \zeta_{n-1}, \cdot)$ being a piecewise smooth Chebyshev function of order 1 on $\mathcal{D}(\zeta_1, \ldots, \zeta_{n-1})$, that is, from its being piecewise smooth and strictly monotone on $\mathcal{D}(\zeta_1, \ldots, \zeta_{n-1})$, with values in an affine line. ■

Proof of (i) Implies (ii). Suppose that $\mathcal{B} \supset \mathcal{A}$. It is actually sufficient to prove the existence of a B-spline basis in the space $\mathcal{S}$ itself. With this aim in mind, given an $\mathcal{S}$-spline Σ and $x \in \mathbf{R}$, as classical, we shall develop a de Boor-like algorithm to compute the point $\Sigma(x) = \sigma(x^{[n]})$. All n-tuples $(\xi_{\ell+1}, \ldots, \xi_{\ell+n})$ being admissible, we can introduce the following points:

$$T_\ell := \sigma(\xi_{\ell+1}, \ldots, \xi_{\ell+n}), \qquad \ell \in \mathbf{Z}. \tag{4.28}$$

The knot vector $\mathcal{K}$ being bi-infinite, for all $k \in \mathbf{Z}$, we can consider the greatest integer j such that $\xi_j \le t_k$. We shall denote it by $j(k)$. It is thus the only integer such that $\xi_{j(k)} \le t_k$, $\xi_{j(k)+1} \ge t_{k+1}$.

According to Remark 2.5, from (4.27) and (2.8) it follows that the pseudoaffinity property $(\mathrm{SB})_3$ is satisfied. The function $\beta(\zeta_1, \ldots, \zeta_{n-1}; a, b; \cdot)$ involved in (2.12) comes from (4.27). Therefore it does not depend on the $\mathcal{S}$-spline Σ and it is strictly monotone on $\mathcal{D}(\zeta_1, \ldots, \zeta_{n-1})$.

Suppose that $x \in [t_k, t_{k'}]$, where $m_k \neq 0, m_{k+1} = \cdots = m_{k'-1} = 0, m_{k'} \neq 0$. With the notations introduced above, we have $\xi_{j(k)} = t_k$, $\xi_{j(k)+1} = t_{k'}$. Moreover, for $0 \le r \le n$ and $j(k)-n+r \le \ell \le j(k)$, the n-tuple $(\xi_{\ell+1}, \ldots, \xi_{\ell+n-r}, x^{[r]})$ is admissible. Therefore we can set

$$T_\ell^k := \sigma(\xi_{\ell+1}, \ldots, \xi_{\ell+n-r}, x^{[r]}), \qquad j(k) - n + r \le \ell \le j(k). \tag{4.29}$$

On account of (2.12), for any $r \le n-1$ and any $\ell = j(k) - n + r + 1, \ldots, j(k)$, we have

$$\begin{aligned} T_\ell^{r+1} \;=\; & [1 - \beta(\xi_{\ell+1}, \ldots, \xi_{\ell+n-r-1}; \xi_\ell, \xi_{\ell+n-r}; x)]T_{\ell-1}^r \\ & + \beta(\xi_{\ell+1}, \ldots, \xi_{\ell+n-r-1}; \xi_\ell, \xi_{\ell+n-r}; x)T_\ell^r. \end{aligned} \tag{4.30}$$

At the nth step, we obtain $\Sigma(x) = T_{j(k)}^n$ as an affine combination of the $n+1$ starting points $T_\ell^0 = T_\ell$, $j(k) - n \le \ell \le j(k)$, which we can write

$$\Sigma(x) = \sum_{\ell=j(k)-n}^{j(k)} \mathcal{N}_\ell(x) T_\ell\,, \qquad \sum_{\ell=j(k)-n}^{j(k)} \mathcal{N}_\ell(x) = 1. \tag{4.31}$$

For $j(k) - n + r + 1 \le \ell \le j(k)$, the interval $[t_k, t_{k'}]$ is contained in $[\xi_\ell, \xi_{\ell+n-r}]$. Accordingly, all affine combinations involved in (4.30) have positive coefficients as soon as $x \in]t_k, t_{k'}[$. Hence $\mathcal{N}_\ell(x) > 0$ for $\ell = j(k) - n + r + 1, \ldots, j(k)$. Setting $\mathcal{N}_\ell(x) := 0$ for all other integers ℓ, a standard argument leads to

$$\Sigma(x) = \sum_{\ell \in \mathbf{Z}} \mathcal{N}_\ell(x) T_\ell, \qquad \sum_{\ell \in \mathbf{Z}} \mathcal{N}_\ell(x) = 1, \qquad x \in \mathbf{R}, \tag{4.32}$$

where the $\mathcal{N}_\ell$'s do not depend on the chosen $\mathcal{S}$-spline Σ, and satisfy $(\mathrm{BSB})_1$ and $(\mathrm{BSB})_3$. Due to (4.32) they satisfy $(\mathrm{BSB})_2$ and $(\mathrm{BSB})_4$ too.

Due to (2.10), the equality (4.28) means that the poles T_ℓ, with respect to the $\mathcal{N}_\ell$'s, satisfy (3.4). Accordingly, Proposition 3.2 automatically garantees that the functions $\mathcal{N}_\ell$, $\ell \in \mathbf{Z}$, form a B-spline basis. ■

5. Final Remarks

We have established the equivalence between existence of B-spline bases and existence of blossoms, but we have given no explicit conditions ensuring the latter existence. We shall conclude the paper by a few remarks on how to guarantee the existence of blossoms, along with some words about the possibility of relaxing some of our assumptions.

5.1. *How to Obtain B-Spline Bases*?

For any $\ell \in \mathbf{Z}$, the set ${I_\ell}^n$ is contained in $\mathcal{A}$. Hence, the condition $\mathcal{B} \supset \mathcal{A}$ implies that any $(x_1, \ldots, x_n) \in {I_\ell}^n$ satisfies (2.3). Due to $\mathcal{E}_\ell$ being a W-space on I_ℓ, this actually means that the space $D\mathcal{E}_\ell$ is an n-dimensional extended Chebyshev space on I_ℓ (see [8]). Therefore, in our context of W-splines, the existence of a B-spline basis both in the space $\mathcal{S}$ and in all other spline spaces obtained by insertion of knots automatically implies that we are dealing with Chebyshev splines, that is, splines with sections in extended Chebyshev spaces.

Suppose that, for any $\ell \in \mathbf{Z}$, the space $D\mathcal{E}_\ell$ is an extended Chebyshev space on I_ℓ. With $\mathcal{E}_\ell$ it is classical to associate differential operators $L_1^\ell, \ldots, L_n^\ell$, of order $1, \ldots, n$, respectively, so that $\mathcal{E}_\ell$ is the kernel of $D \circ L_n^\ell$. Instead of expressing the connections through the ordinary derivatives, we can do it through these differential operators. This gives, on the one hand in the space $\mathcal{E}$ [8],

$$(L_1^\ell U, \ldots, L_n^\ell U)^T(t_\ell^+) = \mathcal{Q}_\ell \cdot (L_1^\ell U, \ldots, L_n^\ell U)^T(t_\ell^-), \qquad \ell \in \mathbf{Z},$$

where $\mathcal{Q}_\ell$ is a lower triangular matrix of order n with positive diagonal, and in the space $\mathcal{S}$,

$$(L_1^\ell S, \ldots, L_{n-m_\ell}^\ell S)^T(t_\ell^+) = \widehat{\mathcal{Q}}_\ell \cdot (L_1^\ell S, \ldots, L_{n-m_\ell}^\ell S)^T(t_\ell^-), \qquad \ell \in \mathbf{Z},$$

where $\widehat{\mathcal{Q}}_\ell$ is obtained from $\mathcal{Q}_\ell$ by deleting its last m_ℓ rows and columns. If each matrix $\mathcal{Q}_\ell$ is totally positive (i.e., if all its minors are nonnegative), then Φ is a piecewise smooth Chebyshev function on $\mathbf{R}$, that is, $\mathcal{B} = \mathbf{R}^n$ (see [8]).

Let us start with the space $\mathcal{S}$ defined by means of the sequence $\mathcal{E}_\ell$ and with connection matrices $\widehat{\mathcal{M}}_\ell$ of order $n - m_\ell$ with positive diagonal (or $\widehat{\mathcal{Q}}_\ell$ as well). Whenever we shall be able to complete all these matrices into nth-order matrices $\mathcal{M}_\ell$ (resp., $\mathcal{Q}_\ell$) with positive diagonal elements so as to ensure the blossoms to be defined at least on $\mathcal{A}$ in the corresponding space $\mathcal{E}$, then the de Boor-like algorithm will provide a B-spline basis satisfying all properties $(\mathrm{BSB})_i$, $1 \le i \le 5$, that is in particular the end point property. If $\widehat{\mathcal{Q}}_\ell$ is supposed to be totally positive, it is always possible to complete it into a regular totally positive matrix $\mathcal{Q}_\ell$ of order n. Hence, according to the results recalled above, total positivity of all matrices $\widehat{\mathcal{Q}}_\ell$ ensures the existence of a B-spline basis not only in $\mathcal{S}$ but also in any spline space derived from $\mathcal{S}$ by insertion of knots. Note that the problem of constructing B-spline bases in such a context was first considered by P. J. Barry [1].

Total positivity is thus a sufficient condition, but it may too restrictive. In [9] we gave a necessary and sufficient condition on each connection matrix $\mathcal{M}_\ell$ for the inclusion $\mathcal{B} \supset \mathcal{A}$ to be satisfied in the case $n = 3$, with simple knots. This can be easily adapted to any nonzero multiplicities: we just have to require the condition to be satisfied at all knots which are simple. It would be interesting to establish a similar result for any dimensions, but so far the problem is open. Note that it is not even solved in the four-dimensional case when allowing some multiplicities to be zero.

5.2. *Relaxing Some of the Assumptions*

5.2.1. Due to our assumption $m_\ell \le n$ for all $\ell \in \mathbf{Z}$, the splines we dealt with were continuous on the whole real line. However, it is possible to relax this assumption by allowing each multiplicity to be less than or equal to $n+1$ (without changing the definition of admissibility). This does not alter the equivalence stated in Theorem 3.3. If $m_k = n+1$, exactly two n-tuples $(\xi_{\ell+1}, \dots, \xi_{\ell+n})$ are equal to $(t_k{}^{[n]})$, namely those corresponding to $\ell = j(k) - n - 1$ and $\ell = j(k) - n$. The first one is involved in the restriction of $\mathcal{S}$ to $]-\infty, t_k]$ and the second one in its restriction to $[t_k, +\infty[$. All B-splines of a possible B-spline basis in the space $\mathcal{S}$ will then vanish at the point t_k except those of indices $j(k) - n - 1$ and $j(k) - n$, which thus satisfy $\mathcal{N}_{j(k)-n-1}(t_k) = \mathcal{N}_{j(k)-n}(t_k) = 1$.

Suppose now that $m_k = m_{k'} = n + 1$ $(k < k')$ and $0 \le m_j \le n$ for $k < j < k'$. Any admissible n-tuple $(\zeta_1, \dots, \zeta_n)$ either belongs to $]-\infty, t_k]^n$, or to $[t_k, t_{k'}]^n$, or to $[t_{k'}, +\infty[^n$. Denoting by $\mathcal{S}_{k,k'}$ the restriction of $\mathcal{S}$ to the interval $[t_k, t_{k'}]$, we obtain the classical situation of continuous spline functions over the closed bounded interval $[t_k, t_{k'}]$, associated with the knot vector $\mathcal{K}_{k,k'} := (t_\ell{}^{[m_\ell]})_{k \le \ell \le k'}$.

Conversely, any such space $\mathcal{S}_{k,k'}$ can be extended into a spline space $\mathcal{S}$ on $\mathbf{R}$, corresponding to some bi-infinite knot vector $\mathcal{K} = (t_\ell{}^{[m_\ell]})_{\ell \in \mathbf{Z}}$ and the associated set of admissible n-tuples $\mathcal{A}$, in such a way as to ensure that blossoms exist on $\mathcal{A}$ iff they exist on the set $\mathcal{A}_{k,k'} := \mathcal{A} \cap [t_k, t_{k'}]^n$ of admissible n-tuples belonging to $[t_k, t_{k'}]^n$. For instance, we can require that the splines be polynomial outside $[t_k, t_{k'}]$ with C^{n-m_k} connections at all knots other than $t_k, \dots, t_{k'}$. Setting $m := \sum_{k<i<k'} m_i$, we have $j(k') = j(k)+m+n+1$. If $\mathcal{N}_\ell$, $\ell \in \mathbf{Z}$, is a B-spline basis of $\mathcal{S}$, the functions $\mathcal{N}_{j(k)-n}, \dots, \mathcal{N}_{j(k)+m}$ (restricted to $[t_k, t_{k'}]$) do not depend on the extension $\mathcal{S}$. They form the unique B-spline basis of the space $\mathcal{S}_{k,k'}$, that is, the unique sequence of $n + m + 1$ functions satisfying the axioms $(\mathrm{BSB})_i$, $1 \le i \le 5$, on $[t_k, t_{k'}]$. Note that here, the decomposition axiom can be replaced by the fact that all functions $\mathcal{N}_{j(k)-n}, \dots, \mathcal{N}_{j(k)+m}$ belong to $\mathcal{S}_{k,k'}$.

Hence, in $\mathcal{S}_{k,k'}$, we can state the equivalence between existence of blossoms (defined on $\mathcal{A}_{k,k'}$) and existence of a B-spline basis in any spline space derived from $\mathcal{S}_{k,k'}$ by knot insertion. Let us recall that existence of blossoms automatically guarantees that each B-spline basis is the optimal normalized totally positive basis of $\mathcal{S}_{k,k'}$ (see [11]).

5.2.2. Throughout the paper we also assumed the knot vector $\mathcal{K}$ to be bi-infinite. Let us now see what occurs when it is finite, that is, when only a finite number of multiplicities are not equal to 0. Given a positive integer q, suppose that m_1 and m_q are positive and that $m_\ell = 0$ whenever either $\ell < 1$ or $\ell > q$. Denote the knot vector $\mathcal{K} = (t_\ell{}^{[m_\ell]})_{1 \le \ell \le q}$ by $(\xi_1, \dots, \xi_N)$, with $\xi_\ell \le \xi_{\ell+1}$ and $N := \sum_{i=1}^q m_i$. The domain of an admissible tuple may now be composed of an infinite number of intervals. Indeed, it cannot contain I_0 (resp. I_q) without containing $\bigcup_{\ell \le 0} I_\ell$ (resp., $\bigcup_{\ell \ge q} I_\ell$) too. The spline space $\mathcal{S}$ is of finite dimension $N + n + 1$. In order to obtain as many n-tuples $(\xi_{\ell+1}, \dots, \xi_{\ell+n})$ as the dimension of $\mathcal{S}$, we can complete the knot vector $\mathcal{K}$ into $\mathcal{K}' := (\xi_{-n+1}, \dots, \xi_{N+n})$ by fixing $2n$ arbitrary additional points $\xi_{-n+1} \le \cdots \le \xi_0 < \xi_1 = t_1$ and $t_q = \xi_N < \xi_{N+1} \le \cdots \le \xi_{N+n}$. The equivalence stated in Theorem 3.3 is still valid, however, after modifying slightly the axioms of a B-spline basis as follows:

- $(\mathrm{BSB})_1$ *support condition*: for all $\ell \in \{-n, \dots, N\}$, the support of $\mathcal{N}_\ell$ is equal to $\mathcal{D}(\xi_{\ell+1}, \xi_{\ell+n})$;
- $(\mathrm{BSB})_2$ *decomposition condition*: the sequence $\mathcal{N}_\ell$, $-n \le \ell \le N$, spans (hence, is a basis of) the finite-dimensional space $\mathcal{S}$;

(BSB)$_3$ *positivity condition*: for all $\ell \in \{-n, \ldots, N\}$, $\mathcal{N}_\ell$ is positive on $\mathcal{D}(\xi_{\ell+1}, \xi_{\ell+n}) \cap]\xi_0, \xi_{N+1}[$;

(BSB)$_4$ *normalization condition*: $\sum_{\ell=-n}^{N} \mathcal{N}_\ell(x) = 1$ for all $x \in \mathbf{R}$; and

(BSB)$_5$ *end point condition*: for all $1 \leq k \leq q$ and all $-n \leq \ell \leq N$ such that $\mathcal{D}(\xi_{\ell+1}, \ldots, \xi_{\ell+n})$ has t_k as its right-hand (resp., left) point, $\mathcal{N}_\ell$ is exactly C^{n-m_k+p} (resp., $C^{n-m_k+p'}$) at t_k, where p (resp., p') is defined as in (3.3).

Again, we can replace the decomposition condition by the fact that all functions $\mathcal{N}_\ell$, $-n \leq \ell \leq N$, belong to $\mathcal{S}$. If $\mathcal{B} \supset \mathcal{A}$, the de Boor-like algorithm still provides a B-spline basis. However, this algorithm depending on the chosen additional points $\xi_{-n+1}, \ldots, \xi_0$, $\xi_{N+1}, \ldots, \xi_{N+n}$, unlike the previous case where the knot vector was supposed to be bi-infinite, there now exist several such B-spline bases. For instance the de Boor algorithm makes it obvious that modifying the only point ξ_{-n+1} does modify the two B-splines $\mathcal{N}_{-n}, \mathcal{N}_{-n+1}$.

All these considerations can easily be adapted to the case where the knot vector is infinite on one side only. Theorem 3.3 is still valid (unicity omitted) when the knot vector is not supposed to be bi-infinite. In particular, considering the case where all multiplicities are equal to 0, we obtain the following result:

Proposition 5.1. *The following two properties are equivalent*:

(i) *any spline space based on $\mathcal{E}$ possesses a B-spline basis*;
(ii) *$\mathcal{E}$-blossoms exist on the whole of $\mathbf{R}^n$, i.e., $\mathcal{B} = \mathbf{R}^n$.*

5.2.3. In [5] we proved that it was possible to extend the theory of blossoms beyond the framework of extended Chebyshev spaces. Here too we can go beyond it. For simplicity, we have assumed that each $\mathcal{E}_\ell$ was a W-space on I_ℓ. Still, it is possible to do without this hypothesis. Let us start with a piecewise smooth function $\Phi = (\Phi^1, \ldots, \Phi^n)$ assumed to satisfy only the condition (II) introduced in Subsection 2.1, but not necessarily the condition (I). According to the study developed in [5], as soon as the set $\mathcal{B}$ contains ${I_\ell}^n$, for all $x \in I_\ell$, the $n-1$ vectors $\Phi'(x), \ldots, \Phi^{(n-1)}(x)$ are linearly independent. For this reason we can consider replacing condition (I) by the following weaker one:

(I)$'$ *for all $x \in \mathbf{R}$, there exists a positive integer $r(x)$ such that the n vectors $\Phi'(x), \ldots$, $\Phi^{(n-1)}(x)$, $\Phi^{(r(x))}(x)$ are linearly independent*.

If so, each $\mathcal{E}_\ell$ is a quasi-Chebyshev space on I_ℓ. One can prove Theorem 3.3 to be still valid in this new context of quasi-Chebyshevian splines, as it is, provided that all multiplicities are positive. The case $n = 3$ with simple knots was treated in [10], where the necessary and sufficient explicit condition was illustrated by the study obtained of the so-called variable degree polynomial splines.

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