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CONSTRUCTIVE APPROXIMATION

Locally Supported, Piecewise Polynomial Biorthogonal Wavelets on Nonuniform Meshes

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Abstract. In this paper, biorthogonal wavelets are constructed on nonuniform meshes. Both primal and dual wavelets are locally supported, continuous piecewise polynomials. The wavelets generate Riesz bases for the Sobolev spaces H^s for $|s| < \frac{3}{2}$. The wavelets at the primal side span standard Lagrange finite element spaces.

1. Introduction

This paper is concerned with the construction of locally supported biorthogonal wavelets on nonuniform meshes. We consider meshes that are generated by uniform refinements starting from an arbitrary initial triangulation of some domain $\Omega \subset \mathbb{R}^n$. In the wavelet literature this is also referred to as a semiregular setting [12].

The wavelets at the primal side will span standard Lagrange (C^0) finite element spaces, with or without essential boundary conditions of, in principal, any order and with any number of vanishing moments. For any $|s| < \frac{3}{2}$, after a proper scaling, the infinite union of the wavelets is a Riesz basis for the Sobolev space $H^s(\Omega)$ (or for the corresponding space from a modified scale incorporating essential boundary conditions). The wavelet construction directly extends to Lipschitz' manifolds consisting of patches, where each patch can be described by a parametrization with a constant Jacobian determinant.

The wavelets satisfy all conditions to use them as ingredients in various wavelet-based algorithms for solving operator equations. For an overview of such algorithms, see [8] and [3]. Key aspects include optimal preconditioning, matrix compression, and adaptive schemes.

An alternative approach to construct wavelet bases on domains or manifolds that cannot be fitted with a uniform grid structure, is to write them as a disjoint union of parametric images of a unit cube, map wavelets living on the cube to the subdomains using the parametrizations, and finally stitch them together. Such constructions yielding wavelet bases suitable for solving operator equations can be found in [9], [1], [10].

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This work can be viewed as a continuation of [11]. A novel aspect is that in the present paper the dual wavelets are also locally supported. As a consequence, the field of applications is extended to all "classical" wavelet applications as signal analysis and image compression.

Another remarkable aspect is that the dual wavelets will be continuous piecewise polynomials. This allows the application of simple standard quadrature formulas for computing wavelet coefficients. Wavelet constructions, also of higher regularity, where the dual functions are piecewise polynomials were discussed earlier in [13], [14], [15]. These constructions concern shift-invariant settings in one or, in [14], two dimensions. In [13], [14] extensions are discussed to uniform meshes on bounded domains Ω. Yet, there the property of polynomial reproduction is lost, which means that the wavelets can only be shown to generate a Riesz basis for $L_2(\Omega)$ and, furthermore, that wavelets near the boundary do not have cancellation properties.

Our construction distinguishes from most other wavelet constructions on nonuniform meshes ("second generation wavelets") in the sense that, as in the shift-invariant case ("first generation wavelets"), the wavelets are proven to generate Riesz bases for a scale of Sobolev spaces. In this respect, note that any compression algorithm based on deleting small wavelet coefficients can only be meaningful when there is some notion of stability.

Locally supported biorthogonal wavelets where the primal spaces are finite element spaces with respect to semiregular meshes were also constructed in [5]. Yet, that construction applies to one case of P_1 elements in two dimensions, and it yields wavelets with one vanishing moment. These wavelets generate a Riesz basis for $H^s(\Omega)$ or $H_0^s(\Omega)$ when $s \in (-\tilde{s}, \frac{3}{2})$ with $\tilde{s} \approx 0.114$. The dual wavelets are given as solutions of refinement equations and are not piecewise polynomials. Compared to P_1 wavelets that will be constructed in this paper, an advantage of the wavelets from [5] is that their implementation is easier and that they have a smaller support. On the other hand, the P_1 wavelets from this paper have four vanishing moments.

This paper is organized as follows: In Section 2, we recall the theory concerning stability of biorthogonal space decompositions, which originates from [7]. To construct bases for the subspaces that make up these space decompositions, that is, the wavelets, we follow the construction known as that of the "stable completions" [2], which is related to the "lifting scheme" [17]. We give a new and short proof of stability of these bases, which is not based on matrix arguments and, therefore, which is fully separated from issues related to implementation.

In Section 3.1 we reduce the whole construction of biorthogonal bases on nonuniform meshes to a construction on a reference element. We give general criteria for local biorthogonal bases to give rise to global biorthogonal continuous scaling functions and wavelets, all with supports that are restricted to a uniform bounded number of mesh-cells. Necessarily, these global functions depend on the (local) topology of the mesh. Yet, this dependence will be explicitly given.

In Sections 3.2–3.5 we give four concrete realizations of biorthogonal bases on nonuniform meshes. With n denoting the space dimension and $d-1$, $\tilde{d}-1$ being the degrees of polynomial exactness at primal and dual sides, these examples are characterized by $(n, d, \tilde{d}) = (1, 2, 4)$, $(1, 5, 4)$, $(2, 2, 4)$, and $(2, 5, 4)$. Although in two dimensions, the constructions are rather complex, we show how the wavelet and inverse wavelets transform can be implemented at relatively low cost.

2. General Mechanism to Construct Stable Wavelet Bases

Let H be a separable Hilbert space with scalar product $\langle\ ,\ \rangle$ and norm $\|\ \|$. Let Σ be some countable collection of functions in H.

We start by recalling some convenient compact notations that, for example, can be found in [8]. Let us formally view Σ as a column vector. Then, for a column vector $\mathbf{c} = (c_\sigma)_{\sigma\in\Sigma}$ of scalars, $\mathbf{c}^T\Sigma := \sum_{\sigma\in\Sigma} c_\sigma \sigma$ is a natural notation. We always consider the spaces of scalar vectors as being equipped with the ℓ_2-norm and, consequently, the spaces of possibly infinite matrices as being equipped with the corresponding operator norm. For $x \in H$, with $\langle\Sigma, x\rangle$ and $\langle x, \Sigma\rangle$, we will mean the column- and row-vectors with coefficients $\langle\sigma, x\rangle$ and $\langle x, \sigma\rangle$, $\sigma \in \Sigma$. More generally, when $\hat{\Sigma}$ is another countable collection in H, with $\langle\Sigma, \hat{\Sigma}\rangle$ is meant the matrix $(\langle\sigma, \hat{\sigma}\rangle)_{\sigma\in\Sigma,\hat{\sigma}\in\hat{\Sigma}}$.

With these notations, a collection Σ is called a *Riesz system* when

$$\|\mathbf{c}^T\Sigma\| \eqsim \|\mathbf{c}\|, \tag{2.1}$$

and Σ is called a *Riesz basis* when it is, in addition, a basis for H. Two collections Σ and $\tilde{\Sigma}$ are called *biorthogonal* or $\tilde{\Sigma}$ is *dual* to Σ or vice versa, when

$$\langle\Sigma, \tilde{\Sigma}\rangle = \mathbf{id}. \tag{2.2}$$

Part (a) of the following lemma will be used in the forthcoming Theorem 2.3 concerning stability of biorthogonal space decompositions, whereas part (b) will be applied to construct Riesz bases for the subspaces that make up these space decompositions.

Lemma 2.1. *Let V and $\tilde{V}$ be closed subspaces of H.*

(a) *The following statements are equivalent*:
  (i) *There exist Riesz bases Σ and $\tilde{\Sigma}$ for V and $\tilde{V}$ such that $\langle\Sigma, \tilde{\Sigma}\rangle$ is bounded invertible.*
  (ii)

$$\inf_{0\neq\tilde{v}\in\tilde{V}} \sup_{0\neq v\in V} \frac{|\langle\tilde{v}, v\rangle|}{\|\tilde{v}\|\|v\|} > 0, \tag{2.3}$$

  and for any $v \in V$ there holds $\sup_{0\neq\tilde{v}\in\tilde{V}} \frac{|\langle\tilde{v}, v\rangle|}{\|\tilde{v}\|\|v\|} > 0$.
  (iii) *There exists a* (*unique*) *bounded projector $Q : H \to H$ with* $\operatorname{Im} Q = V$ *and* $\operatorname{Im}(\mathrm{id} - Q) = \tilde{V}^\perp$.
  (iv) *To any Riesz basis for $\tilde{V}$ there corresponds a unique dual collection in V. Moreover, this collection is a Riesz basis for V.*

(b) *Let any of the equivalent conditions* (i)–(iv) *from* (a) *be satisfied. Let X, W be subspaces of H such that $X = W + V$ and*

$$\cos\angle(W, V) := \sup_{0\neq w\in W, 0\neq v\in V} \frac{|\langle w, v\rangle|}{\|w\|\|v\|} < 1. \tag{2.4}$$

  Then $(\mathrm{id} - Q)|_W : W \to X \cap \tilde{V}^\perp$ *is bounded invertible*, *see Figure* 1.

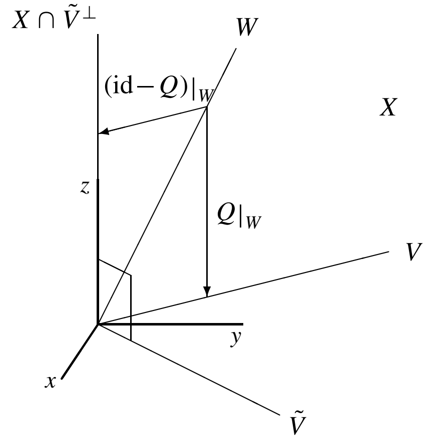

Fig. 1. Illustration for Lemma 2.1(b). H and X are represented by $\mathbb{R}^3$ and the plane $x = 0$, respectively. $\tilde{V}$ is contained in the plane $z = 0$.

Proof. (a) (i) $\to$ (ii): This follows easily by expressing v and $\tilde{v}$ in terms of the Riesz bases from (i).

(ii) $\to$ (iii): For this part we refer to [11, Theorem 2.1(a)].

(iii) $\to$ (iv): Let $\tilde{\Sigma}$ be a Riesz basis for $\tilde{V}$. Let $\tilde{V}'$ be the dual space of $\tilde{V}$ equipped with the operator norm. In [6] it was proved that there exists a Riesz basis $\tilde{\Sigma}'$ for $\tilde{V}'$ which is dual to $\tilde{\Sigma}$, here in the sense that $\tilde{\Sigma}'(\tilde{\Sigma}) := (\tilde{\sigma}'(\tilde{\sigma}))_{\tilde{\sigma}' \in \tilde{\Sigma}', \tilde{\sigma} \in \tilde{\Sigma}} = \mathbf{id}$.

Let $\tilde{R} : \tilde{V}' \to \tilde{V}$ be the Riesz map, i.e., $\langle \tilde{v}, \tilde{R}\tilde{f}\rangle = \tilde{f}(\tilde{v})$ for all $\tilde{f} \in \tilde{V}'$, $\tilde{v} \in \tilde{V}$, and let Q be the projector onto V from (iii). From

$$\langle \tilde{\Sigma}, Q\tilde{R}\tilde{\Sigma}'\rangle = \langle \tilde{\Sigma}, \tilde{R}\tilde{\Sigma}'\rangle = \tilde{\Sigma}'(\tilde{\Sigma}),$$

we see that $\tilde{\Sigma}$ and $Q\tilde{R}\tilde{\Sigma}'$ are biorthogonal systems. Since $\tilde{R}$ is an isomorphism, we may conclude that $Q\tilde{R}\tilde{\Sigma}'$ is a Riesz basis for V when $Q|_{\tilde{V}} : \tilde{V} \to V$ is a homeomorphism.

For $\tilde{v} \in \tilde{V}$, there holds $\|Q\tilde{v}\| \geq |\langle Q\tilde{v}, \tilde{v}\rangle|/\|\tilde{v}\| = \|\tilde{v}\|$. Since $\tilde{V}$ is closed, this property of Q and its boundedness show that $\mathrm{Im}(Q|_{\tilde{V}})$ is closed. Now suppose that $\mathrm{Im}(Q|_{\tilde{V}}) \neq V$, then there would be a $0 \neq v \in V$, such that

$$0 = \langle Q\tilde{v}, v\rangle = \langle \tilde{v}, Q^* v\rangle \qquad (\tilde{v} \in \tilde{V}). \tag{2.5}$$

One easily verifies that $\mathrm{Im}\, Q^* = \tilde{V}$ and $\mathrm{Im}(\mathrm{id} - Q^*) = V^\perp$. The first property together with (2.5) shows that $Q^* v = 0$, whereas the second property gives $\|Q^* v\| \geq |\langle Q^* v, v\rangle|/\|v\| = \|v\|$, which contradicts $v \neq 0$. We conclude that, indeed, $Q|_{\tilde{V}} : \tilde{V} \to V$ is a homeomorphism.

There remains to show that there is only one collection in V that is dual to $\tilde{\Sigma}$. Suppose this is wrong. Then there would be a $0 \neq v \in V$ such that $\langle v, \tilde{\Sigma}\rangle = 0$ and thus $\langle v, \tilde{v}\rangle = 0$ for all $\tilde{v} \in \tilde{V}$. Since $Q|_{\tilde{V}} : \tilde{V} \to V$ is a homeomorphism there exists a $0 \neq \tilde{y} \in \tilde{V}$ with $Q\tilde{y} = v$. From $\mathrm{Im}(\mathrm{id} - Q) = \tilde{V}^\perp$, we get $\langle \tilde{y}, \tilde{v}\rangle = 0$ for all $\tilde{v} \in \tilde{V}$, contradicting $\tilde{y} \neq 0$.

(iv) $\to$ (i): Any separable Hilbert space has an orthonormal basis. Starting with such a basis for $\tilde{V}$ and applying (iv) shows (i), where $\langle \Sigma, \tilde{\Sigma}\rangle$ is even the identity matrix.

(b) Write $x \in X$ as $x = w + v$ where $w \in W$, $v \in V$. Formula (2.4) shows that this decomposition is unique and that $\|x\|^2 \eqsim \|w\|^2 + \|v\|^2$. Taking $x \in X \cap \tilde{V}^\perp$, we have $Qx = 0$ and so $v = Qv = -Qw$, i.e., $x = (\mathrm{id} - Q)w$ and $\|x\|^2 \eqsim \|w\|^2 + \|Qw\|^2 \eqsim \|w\|^2$. ■

Remarks 2.2.

(a) Since (i) is symmetric in V and $\tilde{V}$, so are (ii)–(iv), i.e., the roles of V and $\tilde{V}$ may everywhere be interchanged. In that case, as was already mentioned in the proof, the projector from (iii) is nothing other than Q^*. Pairs of spaces V, $\tilde{V}$ that satisfy any, and thus all, of (i)–(iv) will be said to satisfy the *maximum angle condition*.

(b) Estimate (2.4) is known as the *strengthened Cauchy–Schwarz inequality*. Pairs of spaces W, V that satisfy (2.4) will be said to satisfy the *minimum angle condition*.

(c) If Σ, $\tilde{\Sigma}$ are Riesz bases for V and $\tilde{V}$ such that $\langle \Sigma, \tilde{\Sigma} \rangle$ is bounded invertible, then the projector Q from (iii) can be computed by

$$Qx = \langle x, \tilde{\Sigma} \rangle \langle \Sigma, \tilde{\Sigma} \rangle^{-1} \Sigma,$$

and, similarly, $Q^* y = \langle y, \Sigma \rangle \langle \tilde{\Sigma}, \Sigma \rangle^{-1} \tilde{\Sigma}$.

(d) Below we will apply Lemma 2.1 to an infinite *sequence* of pairs of closed subspaces V, $\tilde{V}$ of some Hilbert space H, together with corresponding sequences of spaces X and W. We will be interested in results that hold *uniformly* over these sequences. The proof of the lemma shows that if we replace in (i), (iii), and (b) "bounded" by "*uniformly* bounded," and the conditions for being a Riesz system or satisfying (2.3) or (2.4) by corresponding conditions that hold *uniformly* over the sequences, then the resulting lemma remains valid. In this respect, we will speak about *uniform Riesz systems*, *uniform Riesz bases*, and *uniform maximum* or *minimum angle conditions*.

In the following, let $\mathcal{H}^s$ for $s \in \mathbb{R}$ or $|s| \le t$ denote a scale of Sobolev spaces, possibly incorporating essential boundary conditions, on an n-dimensional domain or sufficiently smooth manifold. We will denote $\mathcal{H}^0$ also as L_2, and when $s < 0$ the space $\mathcal{H}^s$ is understood to be the *dual* of $\mathcal{H}^{-s}$. From now on, the role of the general Hilbert space H will be played by L_2, and so $(\,)^*$ will mean an adjoint with respect to the L_2-scalar product, and $\perp$ denotes orthogonality with respect to this scalar product.

Theorem 2.3 (Biorthogonal Space Decompositions). *Let $V_0 \subset V_1 \subset V_2 \subset \cdots$ and $\tilde{V}_0 \subset \tilde{V}_1 \subset \tilde{V}_2 \subset \cdots$ be sequences of nested closed subspaces of L_2, and let $\rho > 1$ be some constant, that in applications will be the* refinement factor.

Assume that $(V_j, \tilde{V}_j)_j$ satisfies the uniform maximum L_2-angle condition. Let (Q_j) be the sequence of uniformly bounded projectors $Q_j : L_2 \to L_2$ with $\mathrm{Im}\, Q_j = V_j$ *and* $\mathrm{Im}(\mathrm{id} - Q_j) = \tilde{V}_j^\perp$ *from Lemma* 2.1(a)(iii).

Assume that there exist $0 < \gamma < d$ *such that*

$$(\mathcal{J}) \qquad \inf_{v_j \in V_j} \|v - v_j\|_{L_2} \lesssim \rho^{-sj} \|v\|_{\mathcal{H}^s} \qquad (v \in \mathcal{H}^s,\ 0 \le s \le d)$$

(direct *or* Jackson estimate), *and*

$$(\mathcal{B}) \quad \|v_j\|_{\mathcal{H}^s} \lesssim \rho^{sj} \|v_j\|_{L_2} \quad (v_j \in V_j,\ 0 \le s < \gamma) \quad \text{(inverse } or \text{ Bernstein estimate)},$$

and that analogous assumptions $(\tilde{\mathfrak{J}})$ *and* $(\tilde{\mathfrak{B}})$ *with constants* $0 < \tilde{\gamma} < \tilde{d}$ *hold for* $(\tilde{V}_j)$.

Then, with $Q_{-1} := 0$, *one has*

$$\left\| \sum_{j=0}^{\infty} w_j \right\|_{\mathcal{H}^s}^2 \lesssim \sum_{j=0}^{\infty} \rho^{2sj} \|w_j\|_{L_2}^2 \qquad (w_j \in \operatorname{Im}(Q_j - Q_{j-1}),\ s \in (-\tilde{d}, \gamma)) \tag{2.6}$$

and

$$\sum_{j=0}^{\infty} \rho^{2sj} \|(Q_j - Q_{j-1})v\|_{L_2}^2 \lesssim \|v\|_{\mathcal{H}^s}^2 \qquad (v \in \mathcal{H}^s,\ s \in (-\tilde{\gamma}, d)). \tag{2.7}$$

For $s \in (-\tilde{\gamma}, \gamma)$, *the mappings* $(w_j) \mapsto \sum_{j=0}^{\infty} w_j$ *and* $v \mapsto ((Q_j - Q_{j-1})v)$, *which are bounded in the sense of* (2.6) *and* (2.7), *are each others' inverse.*

Analogous results are valid with (Q_j) *replaced by* (Q_j^*) *and with interchanged roles of* (γ, d) *and* $(\tilde{\gamma}, \tilde{d})$.

Remark 2.4. An earlier theorem, demonstrating stability of biorthogonal space decompositions in an even more general context, can be found in [7]. See also [8], [3] and the references cited there, for example, for generalizations to Besov norms. A proof of the theorem in its present form can be found in [11, Theorem 2.1].

The essential point of the present formulation is that explicit knowledge of some biorthogonal bases for V_j and $\tilde{V}_j$ is *not* assumed. In [11] the conditions of Theorem 2.3 were verified for both (V_j) and $(\tilde{V}_j)$ being sequences of standard finite element spaces.

In the remainder of this section, we will assume that we are in the situation as indicated in Theorem 2.3. The nesting $\tilde{V}_j \subset \tilde{V}_{j+1}$ gives $Q_j^* = Q_{j+1}^* Q_j^*$ or $Q_j = Q_j Q_{j+1}$, from which we deduce that

$$\operatorname{Im}(Q_{j+1} - Q_j) = V_{j+1} \cap \tilde{V}_j^{\perp}.$$

A direct consequence of Theorem 2.3 is that *if* we have uniform L_2-Riesz bases Ψ_j for the spaces $V_{j+1} \cap \tilde{V}_j^{\perp}$, and an L_2-Riesz basis Φ_0 for V_0, then, for $s \in (-\tilde{\gamma}, \gamma)$,

$$\Phi_0 \cup \bigcup_{j=0}^{\infty} \rho^{-sj} \Psi_j$$

is a Riesz basis for $\mathcal{H}^s$. The elements of the Ψ_j are called *wavelets*.

Remark 2.5. Since, in particular, $\Psi := \Phi_0 \cup \bigcup_j \Psi_j$ is a Riesz basis for L_2, an application of Lemma 2.1(a) with "V" = "$\tilde{V}$" = "H" = L_2 shows that there exists a unique dual collection $\tilde{\Psi} := \tilde{\Phi}_0 \cup \bigcup_j \tilde{\Psi}_j$ in L_2 which, moreover, is a Riesz basis for L_2. Exploiting biorthogonality shows that the $\tilde{\Psi}_j$ are uniform L_2-Riesz bases for the spaces $\tilde{V}_j \cap V_{j-1}^{\perp}$ and that $\tilde{\Phi}_0$ is an L_2-Riesz basis for $\tilde{V}_0$. From Theorem 2.3 we conclude that for $s \in (-\gamma, \tilde{\gamma})$, $\tilde{\Phi}_0 \cup \bigcup_j \rho^{-sj} \tilde{\Psi}_j$ is a Riesz basis for $\mathcal{H}^s$. The elements of the $\tilde{\Psi}_j$ are called *dual wavelets*.

For $s \in (-\tilde{\gamma}, \gamma)$ and $v \in \mathcal{H}^s$, the unique expansion of v in terms of Ψ is given by

$$v = \langle v, \tilde{\Psi} \rangle \Psi. \tag{2.8}$$

Remark 2.6. The fact that the dual sequence $(\tilde{V}_j)$ satisfies a Jackson estimate is closely related to the fact that integration of a resulting biorthogonal wavelet against a smooth function produces something that is small. Indeed, for simplicity, restricting ourselves to the domain case (for the manifold case, see, e.g., [11, Prop. 4.7]), the Jackson estimate $(\tilde{\mathcal{J}})$ is usually enforced by demanding that $\tilde{V}_j$ contains all piecewise polynomials up to degree $\tilde{d}-1$ satisfying some global smoothness conditions with respect to a quasi-uniform mesh with mesh-size $\sim \rho^{-j}$. Now the fact that $\psi_j \in \Psi_j$ satisfies $\psi_j \perp_{L_2} \tilde{V}_j$ shows that, for smooth v, there holds $\langle v, \psi_j \rangle_{L_2} = \langle v - p, \psi_j \rangle_{L_2}$ where p is a Taylor polynomial of v of order $\tilde{d} - 1$ around some point in supp ψ_j. Assuming that diam(supp ψ_j) $\eqsim \rho^{-j}$, by estimating the remainder term we find that

$$|\langle v, \psi_j \rangle_{L_2}| \lesssim \rho^{-(\tilde{d}+n/2)j} \|v\|_{W^{\infty,\tilde{d}}(\text{supp } \psi_j)},$$

which property of the wavelets is referred to as the *cancellation property of order* $\tilde{d}$.

Obviously, when also diam(supp$\tilde{\psi}_j$) $\eqsim \rho^{-j}$, the dual wavelets have the cancellation property of order d. The wavelets and dual wavelets we are going to construct will satisfy an even slightly stronger condition on their supports (see Definition 2.7). The cancellation property of the wavelets (or dual wavelets) is essential for finding sparse approximate wavelet representations of operators (or functions).

Usually, it is not a problem to equip V_0 with some L_2-Riesz basis Φ_0. Below we discuss the *construction of the wavelets*. Suppose that we can identify:

- uniform L_2-Riesz bases $\Theta_j \cup \Xi_j$ for V_{j+1} and $\tilde{\Phi}_j$ for $\tilde{V}_j$ with $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2} = \mathbf{id}$.

Then with $W_j := \text{cl}_{L_2} \text{ span } \Xi_j$, $Z_j := \text{cl}_{L_2} \text{ span } \Theta_j$, we have:

- $V_{j+1} = W_j + Z_j$;
- $(Z_j, \tilde{V}_j)_j$ satisfies the uniform maximum L_2-angle condition; and
- $(W_j, Z_j)_j$ satisfies the uniform minimum L_2-angle condition.

Lemma 2.1 now shows that there exist unique uniformly L_2-bounded projectors P_j with $\text{Im } P_j = Z_j$ and $\text{Im}(\text{id} - P_j) = \tilde{V}_j^{\perp}$ where, moreover, $(\text{id} - P_j)|_{W_j} : W_j \to V_{j+1} \cap \tilde{V}_j^{\perp}$ is invertible, with a uniformly L_2-bounded inverse. We conclude that these $(\text{id} - P_j)|_{W_j}$ map uniform L_2-Riesz bases to uniform L_2-Riesz bases, and thus that

$$\Psi_j := (\text{id} - P_j)\Xi_j \tag{2.9}$$

are uniform L_2-Riesz bases for the spaces $V_{j+1} \cap \tilde{V}_j^{\perp}$. From Remark 2.2(c) we learn that

$$\Psi_j = \Xi_j - \langle \Xi_j, \tilde{\Phi}_j \rangle_{L_2} \Theta_j. \tag{2.10}$$

Note that these wavelets Ψ_j depend on V_{j+1}, $\tilde{V}_j$, Z_j, and Ξ_j but, as follows from (2.9), not on the choice of the bases Θ_j and $\tilde{\Phi}_j$ for Z_j and $\tilde{V}_j$.

In view of applications, we will be mainly interested in wavelets that are uniformly local in the following sense:

Definition 2.7. Let $(\Sigma_j)_{j\in\mathbb{N}}$ be a sequence of collections of functions. We will call the (functions from) Σ_j *uniformly local* when

$$\operatorname{diam}(\operatorname{supp}\sigma_j) \lesssim \rho^{-j} \qquad \text{and} \qquad \sup_{\alpha\in\mathbb{Z}^n} \#\{\sigma_j \in \Sigma_j : \operatorname{supp}\sigma_j \cap \rho^{-j}(\alpha + [0,1]^n)\} \lesssim 1.$$

Having two sequences Σ_j, $\hat{\Sigma}_j$ of collections of uniformly local functions, matrices $\mathbf{A}_j$ indexed with Σ_j, $\hat{\Sigma}_j$ will be called *uniformly local* when $(\mathbf{A}_j)_{\sigma_j,\hat{\sigma_j}} = 0$ when $\operatorname{dist}(\operatorname{supp}\sigma_j, \operatorname{supp}\hat{\sigma_j}) \gtrsim \rho^{-j}$.

Note that such matrices $\mathbf{A}_j$ are *uniformly sparse* and, furthermore, that functions from the collections $\mathbf{A}_j\hat{\Sigma}_j$ are uniformly local. A particular instance of uniformly local matrices is given by $\langle \Sigma_j, \hat{\Sigma}_j\rangle_{L_2}$.

From the above definition and its consequences we learn that if the Ξ_j, $\tilde{\Phi}_j$, and Θ_j are uniformly local, then the collections of wavelets resulting from (2.10) are uniformly local.

Remark 2.8. For $Z_j := \operatorname{span}\Theta_j = V_j$, the wavelet construction (2.9)/(2.10) is known as the construction via “stable completions” [2], which is related to the so-called “lifting scheme” [17]. Our derivation of the fact that the Ψ_j are uniform L_2-Riesz systems is new in the sense that it is not based on matrix arguments, which means that it is fully separated from issues related to the implementation.

With $Z_j = V_j$, Θ_j is a basis for V_j, and so the conditions for getting uniformly local wavelets we derived now read as assuming that we have uniformly local, biorthogonal L_2-Riesz bases for the spaces V_j and $\tilde{V}_j$ at our disposal. In practice, this condition is much more restrictive than assuming $\langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2} = \mathbf{id}$ for some uniform L_2-Riesz system $\Theta_j \subset V_{j+1}$, which lead us in [11] to consider the generalization $Z_j \neq V_j$, which suffices for all applications for which uniformly local dual wavelets are not needed. Examples of such applications are wavelet-based algorithms for solving operator equations (see [8]). On the other hand, for “classical” wavelet applications, like signal analysis and image compression, having uniformly local dual wavelets is essential.

In many applications, one needs to switch from a representation of a function $v \in V_J$ with respect to the “multi-scale basis” $\Phi_0 \cup \bigcup_{j=0}^{J-1}\Psi_j$, to a representation with respect to some “single-scale” basis Φ_J.

Since $V_{j+1} = V_j \oplus (V_{j+1} \cap \tilde{V}_j^{\perp})$, there exist matrices $\mathbf{M}_{j,0}$ and $\mathbf{M}_{j,1}$ such that $\Phi_j^T = \Phi_{j+1}^T\mathbf{M}_{j,0}$ and $\Psi_j^T = \Phi_{j+1}^T\mathbf{M}_{j,1}$, and

$$\mathbf{M}_j = [\mathbf{M}_{j,0} \quad \mathbf{M}_{j,1}]$$

is invertible. Writing $v \in V_J$ in both forms $\mathbf{c}_0^T\Phi_0 + \sum_{j=0}^{J-1}\mathbf{d}_j^T\Psi_j^T$ and $\mathbf{c}_J^T\Phi_J$, the basis transformation $\mathbf{T}_J$ mapping the “multi-scale coefficients” $(\mathbf{c}_0^T, \mathbf{d}_0^T, \ldots, \mathbf{d}_{J-1}^T)^T$ to the “single-scale coefficients” $\mathbf{c}_J$, satisfies

$$\mathbf{T}_J = [\mathbf{M}_{J-1,0}\mathbf{T}_{J-1} \quad \mathbf{M}_{J-1,1}] = \mathbf{M}_{J-1}\begin{bmatrix}\mathbf{T}_{J-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{id}\end{bmatrix}, \tag{2.11}$$

and $\mathbf{T}_0 = \mathbf{id}$. So, assuming a geometrical increase of dim V_J as a function of J, we see that $\mathbf{T}_J$ can be performed in $\mathcal{O}(\dim V_J)$ operations when the $\mathbf{M}_j$ are uniformly sparse.

Writing $\Theta_j^T = \Phi_{j+1}^T \mathbf{N}_{j,0}$, $\Xi_j^T = \Phi_{j+1}^T \mathbf{R}_{j,1}$, $\tilde{\Phi}_j^T = \tilde{\Phi}_{j+1}^T \tilde{\mathbf{M}}_{j,0}$ for some matrices $\mathbf{N}_{j,0}$, $\mathbf{R}_{j,1}$, and $\tilde{\mathbf{M}}_{j,0}$, we infer that (2.10) is equivalent to

$$\mathbf{M}_{j,1} = (\mathbf{id} - \mathbf{N}_{j,0}\tilde{\mathbf{M}}_{j,0}^* \langle \Phi_{j+1}, \tilde{\Phi}_{j+1} \rangle_{L_2}^T)\mathbf{R}_{j,1}.$$

We conclude that the $\mathbf{M}_j$ are uniformly sparse whenever this holds for $\mathbf{M}_{j,0}$, $\mathbf{N}_{j,0}$, $\tilde{\mathbf{M}}_{j,0}$, $\langle \Phi_{j+1}, \tilde{\Phi}_{j+1} \rangle_{L_2}$, and $\mathbf{R}_{j,1}$.

Formula (2.11) shows that if one also needs an implementation of optimal complexity of $\mathbf{T}_J^{-1}$, mapping the "single-scale coefficients" to the "multi-scale coefficients," then it is necessary that also the $\mathbf{M}_j^{-1}$ are uniformly sparse. Only under special circumstances, the inverse of a sparse matrix is again sparse, and with the construction (2.10), $\mathbf{M}_j^{-1}$ will generally be a densely populated matrix.

In the wavelet literature, $\mathbf{T}_J^{-1}$ and $\mathbf{T}_J$ are called the *wavelet transform* and *inverse wavelet transform*, respectively.

From now on we will focus on the special case $Z_j = V_j$. In this case, Θ_j is a basis for V_j, and we take $\Phi_j = \Theta_j$, so that $\mathbf{N}_{j,0} = \mathbf{M}_{j,0}$ and $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2} = \mathbf{id}$. It holds that $\mathbf{M}_{j,0} = \langle \tilde{\Phi}_{j+1}, \Phi_j \rangle_{L_2}$, $\tilde{\mathbf{M}}_{j,0} = \langle \Phi_{j+1}, \tilde{\Phi}_j \rangle_{L_2}$, and $\mathbf{R}_{j,1} = \langle \tilde{\Phi}_{j+1}, \Xi_j \rangle_{L_2}$, meaning that these matrices are uniformly local, and so in particular indeed are uniformly sparse, when the collections Φ_j, $\tilde{\Phi}_j$, and Ξ_j are uniformly local.

Furthermore, we now get

$$\mathbf{M}_j = [\mathbf{M}_{j,0} \quad \mathbf{R}_{j,1}] \begin{bmatrix} \mathbf{id} & -\tilde{\mathbf{M}}_{j,0}^* \mathbf{R}_{j,1} \\ \mathbf{0} & \mathbf{id} \end{bmatrix}, \tag{2.12}$$

and we conclude that the $\mathbf{M}_j^{-1}$ are uniformly local, and thus uniformly sparse, under the additional condition that the initial supplements Ξ_j are selected such that:

- the basis transformations $[\mathbf{M}_{j,0} \quad \mathbf{R}_{j,1}]^{-1}$ from Φ_{j+1} to $\Phi_j \cup \Xi_j$ are uniformly local.

A closely related additional advantage of having a uniformly local $\mathbf{M}_j^{-1}$ is that uniformly local dual wavelets become available: In Remark 2.5 the set of dual wavelets $\tilde{\Psi}_j$ was defined as the unique collection in $\tilde{V}_{j+1} \cap V_j^{\perp}$ that is dual to Ψ_j. From $[\Phi_j^T \quad \Psi_j^T] = \Phi_{j+1}^T \mathbf{M}_j$ and $\langle \mathbf{M}_j^T \Phi_{j+1}, (\bar{\mathbf{M}}_j)^{-1} \tilde{\Phi}_{j+1} \rangle_{L_2} = \mathbf{id}$, we infer that

$$[\tilde{\Phi}_j^T \quad \tilde{\Psi}_j^T] = \tilde{\Phi}_{j+1}^T (\mathbf{M}_j^*)^{-1}.$$

We conclude that the $\tilde{\Psi}_j$ are uniformly local when the bases $\tilde{\Phi}_{j+1}$ and the matrices $\mathbf{M}_j^{-1}$ are uniformly local.

3. Biorthogonal Scaling Functions on Nonuniform Meshes

In the remainder of this paper we will construct biorthogonal, uniformly local, uniform L_2-Riesz bases Φ_j, $\tilde{\Phi}_j$ for spaces V_j, $\tilde{V}_j$ on some domain $\Omega \subset \mathbb{R}^n$, that are nested *as function of* j, *and that*, *with* $\rho = 2$, *satisfy Bernstein estimates with* $\gamma = \tilde{\gamma} = \frac{3}{2}$ *and Jackson estimates for certain values* $d, \tilde{d} > \frac{3}{2}$. *By Lemma* 2.1(a), *the fact that such biorthogonal bases are available implies that* $(V_j, \tilde{V}_j)_j$ *also satisfies the uniform maximum* L_2*-angle condition and*, *thus*, *that all the conditions of Theorem* 2.3 *are satisfied*.

Furthermore, *we will construct uniformly local collections* Ξ_j (the initial completions), *such that* $\Phi_j \cup \Xi_j$ *are uniform* L_2*-Riesz bases for the spaces* V_{j+1} *and such that the basis transformations from* Φ_{j+1} *to* $\Phi_j \cup \Xi_j$ *are uniformly local*.

From the previous section we learn then that both the wavelet collections

$$\Psi_j := \Xi_j - \langle \Xi_j, \tilde{\Phi}_j \rangle_{L_2} \Phi_j,$$

and their dual collections $\tilde{\Psi}_j$ *are uniformly local*, *uniform* L_2*-Riesz bases for the spaces* $V_{j+1} \cap \tilde{V}_j^{\perp}$ *and* $\tilde{V}_{j+1} \cap V_j^{\perp}$, *respectively*, *and that for* $|s| < \frac{3}{2}$, *both* $\Phi_0 \cup \bigcup_{j=0}^{\infty} 2^{-sj} \Psi_j$ *and* $\tilde{\Phi}_0 \cup \bigcup_{j=0}^{\infty} 2^{-sj} \tilde{\Psi}_j$ *are a Riesz bases for* $H^s(\Omega)$ (*or*, *for the corresponding space from a modified scale incorporating essential boundary conditions*, *see Remark* 3.7).

That is, in contrast to our earlier joint paper with W. Dahmen ([11]), here we obtain wavelets for which their duals are also uniformly local, at the cost of getting wavelets with larger supports.

The primal spaces V_j will be *standard Lagrange finite element spaces* with respect to meshes that are generated by *uniform dyadic refinements* starting with an *arbitrary initial mesh*. Both Φ_j and $\tilde{\Phi}_j$, and so Ψ_j and $\tilde{\Psi}_j$, will be defined *explicitly*.

Remark 3.1. Usually, at least the $\tilde{\Phi}_j$ are only given as solution of some refinement equation (see [4]). Exceptions are given by [13], [14], [15] dealing with uniform mesh cases. An advantage of knowing $\tilde{\Psi}_j$ explicitly is that there is much more freedom in making efficient and accurate numerical approximations of expansions like (2.8).

3.1. *Reduction to a Reference Element*

In this subsection, we will explain the general mechanism to reduce the construction of Φ_j, $\tilde{\Phi}_j$, and Ξ_j to a construction on the reference (macro-)element. Examples of concrete realizations will be given in the subsequent subsections.

Consider the closed reference n-simplex,

$$\boldsymbol{T} = \left\{ \lambda \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \geq 0 \right\}.$$

We fix a refinement of $\boldsymbol{T}$ into 2^n congruent subsimplices $\boldsymbol{T}_1, \ldots, \boldsymbol{T}_{2^n}$, each of them determined by some ordered set of vertices.

For any closed n-simplex T, let $\lambda_T(x) \in \boldsymbol{T}$ denote the barycentric coordinates of $x \in T$ with respect to the ordered set of vertices of T. The above dyadic refinement of $\boldsymbol{T}$ induces such a refinement of T into 2^n congruent subsimplices $(\lambda_T^{-1} \circ \lambda_{\boldsymbol{T}_k}^{-1} \circ \lambda_T)(T)$ $(1 \leq k \leq 2^n)$.

Let τ_0 be a fixed collection of closed n-simplices, or elements, such that $\bigcup_{T \in \tau_0} T$ is a partition, also called triangulation, of the closure of some domain $\Omega \subset \mathbb{R}^n$. We assume that the triangulation is conforming, i.e., the intersection of any two elements is either empty or a common face. Here, with a face of T, we mean any k-simplex spanned by $k+1$ vertices of T, where $0 \leq k < n$. Starting from τ_0, we obtain an infinite sequence of collections of simplices $(\tau_j)_{j\geq 0}$ by defining τ_{j+1} as the collection of all simplices that arise by applying the above refinement to all simplices from τ_j. To avoid some technical complications, we will always assume that $n \leq 3$, meaning that automatically all these triangulations are conforming.

We will consider collections of functions $\boldsymbol{\Sigma} = \{\boldsymbol{\sigma}_\lambda : \lambda \in \boldsymbol{I}\}$ with some index set $\boldsymbol{I} \subset \boldsymbol{T}$ that satisfy:

($\mathcal{C}$) $\boldsymbol{\sigma}_\lambda \in C(\boldsymbol{T})$;
($\mathcal{V}$) $\boldsymbol{\sigma}_\lambda$ vanishes on any face that does not include λ;
($\mathcal{S}$) $\pi(\boldsymbol{I} \cap \partial\boldsymbol{T}) = \boldsymbol{I} \cap \partial\boldsymbol{T}$ and $\boldsymbol{\sigma}_\lambda|_{\partial\boldsymbol{T}} = (\boldsymbol{\sigma}_{\pi(\lambda)} \circ \pi)|_{\partial\boldsymbol{T}}$ for any permutation

$$\pi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1};$$

($\mathcal{I}$) for $\boldsymbol{e} = \boldsymbol{T}$, or for $\boldsymbol{e}$ being any face of $\boldsymbol{T}$, $\{\boldsymbol{\sigma}_\lambda|_{\boldsymbol{e}} : \lambda \in \boldsymbol{I} \cap \boldsymbol{e}\}$ is independent.

An easy example one may think of is $\boldsymbol{I}$ being the set of vertices of $\boldsymbol{T}$ and $\boldsymbol{\Sigma}$ the set of linear nodal basis functions, i.e., $\boldsymbol{\sigma}_\lambda(\mu) = \delta_{\lambda,\mu}$ $(\lambda, \mu \in \boldsymbol{I})$. Later this set will be denoted by $\boldsymbol{\Delta}^{(1,0)}$.

These "local" functions from such collections can be assembled into collections of "global" functions in a way known from finite element methods: For $j \geq 0$ and with

$$I_j = \{x \in \Omega : \lambda_T(x) \in \boldsymbol{I} \text{ for some } T \in \tau_j\},$$

we define the collection $\Sigma_j = \{\sigma_{j,x} : x \in I_j\}$ of functions on Ω by

$$\sigma_{j,x}(y) = \begin{cases} \mu(x;\tau_j)\boldsymbol{\sigma}_{\lambda_T(x)}(\lambda_T(y)) & \text{if } x, y \in T \in \tau_j, \\ 0 & \text{elsewhere}, \end{cases} \tag{3.1}$$

with scaling factor $\mu(x;\tau_j) := (\sum_{\{T\in\tau_j : T\ni x\}} \mathrm{vol}(T)/\mathrm{vol}(\boldsymbol{T}))^{-1/2}$. So these global functions result from connecting the local basis functions over the interfaces. The assumptions ($\mathcal{C}$), ($\mathcal{V}$), ($\mathcal{S}$), and ($\mathcal{I}$) show that the Σ_j are collections of well-defined, uniformly local, continuous, and independent functions on Ω.

In the following paragraphs we collect some general properties of this assembling procedure.

Suppose that we have two collections $\boldsymbol{\Sigma}^{(1)} = \{\boldsymbol{\sigma}_\lambda^{(1)}; \lambda \in \boldsymbol{I}^{(1)}\}$ and $\boldsymbol{\Sigma}^{(2)} = \{\boldsymbol{\sigma}_\lambda^{(2)}; \lambda \in \boldsymbol{I}^{(2)}\}$ satisfying ($\mathcal{C}$)–($\mathcal{I}$). Then, for the resulting $(\Sigma_j^{(1)})$ and $(\Sigma_j^{(2)})$, there holds span $\Sigma_j^{(1)} \subset$ span $\Sigma_j^{(2)}$ $(j \in \mathbb{N})$, if and only if

$$\operatorname{span} \boldsymbol{\Sigma}^{(1)} \subset \operatorname{span} \boldsymbol{\Sigma}^{(2)}.$$

To show the if-statement, let $\mathbf{Q} = (q_{\nu,\mu})_{\nu\in\boldsymbol{I}^{(2)},\mu\in\boldsymbol{I}^{(1)}}$ be such that $(\boldsymbol{\Sigma}^{(1)})^T = (\boldsymbol{\Sigma}^{(2)})^T\mathbf{Q}$ or

$$\boldsymbol{\sigma}_\mu^{(1)} = \sum_{\nu\in\boldsymbol{I}^{(2)}} q_{\nu,\mu}\boldsymbol{\sigma}_\nu^{(2)}. \tag{3.2}$$

Then, there holds that, for $x \in I_j^{(1)}$,

$$\frac{\sigma_{j,x}^{(1)}}{\mu(x;\tau_j)} = \sum_{\{y \in I_j^{(2)} : \exists T \in \tau_j,\, x,y \in T\}} q_{\lambda_T(y),\lambda_T(x)} \frac{\sigma_{j,y}^{(2)}}{\mu(y;\tau_j)}. \tag{3.3}$$

Indeed, it is not difficult to verify that both sides of (3.3) agree on supp $\sigma_{j,x}^{(1)}$. Note that by ($\mathcal{S}$), the coefficient $q_{\lambda_T(y),\lambda_T(x)}$ in front of $\sigma_{j,y}^{(2)}$ is uniquely defined, also when x and y are included on a face shared by elements in τ_j. Furthermore, the conditions ($\mathcal{V}$) on $\mathbf{\Sigma}^{(1)}$ and ($\mathcal{I}$) on $\mathbf{\Sigma}^{(2)}$ ensure that the right-hand side of (3.3) vanishes outside supp $\sigma_{j,x}^{(1)}$.

Formula (3.3) shows that the representations of the inclusions Incl : span $\Sigma_j^{(1)} \to$ span $\Sigma_j^{(2)}$ with respect to $\Sigma_j^{(1)}$ and $\Sigma_j^{(2)}$ are uniformly local and, furthermore, it shows how they can be constructed from the representation $\mathbf{Q}$ of Incl : span $\mathbf{\Sigma}^{(1)} \to$ span $\mathbf{\Sigma}^{(2)}$ with respect to $\mathbf{\Sigma}^{(1)}$ and $\mathbf{\Sigma}^{(2)}$.

As a special case, suppose that $(\Sigma_j^{(1)})$ and $(\Sigma_j^{(2)})$ satisfy ($\mathcal{C}$)–($\mathcal{I}$) and that

$$\text{span}\,\mathbf{\Sigma}^{(1)} = \text{span}\,\mathbf{\Sigma}^{(2)}.$$

With $(\Sigma_j^{(1)})$ and $(\Sigma_j^{(2)})$ being the sequences of the corresponding global bases, an application of the foregoing shows that *both the basis transformations from* $\Sigma_j^{(1)}$ *to* $\Sigma_j^{(2)}$ *and their inverses are uniformly local*.

The question whether, for given $\mathbf{\Sigma}$, there holds span $\Sigma_j \subset$ span Σ_{j+1} ($j \in \mathbb{N}$) can also be reduced to a special case of the foregoing analysis. Indeed, with the refined index set being defined by

$$\boldsymbol{I}^{(r)} := \bigcup_{k=1}^{2^n} \lambda_{\boldsymbol{T}_k}^{-1}(\boldsymbol{I}),$$

let us define $\mathbf{\Sigma}^{(r)} = \{\boldsymbol{\sigma}_\lambda^{(r)} : \lambda \in \boldsymbol{I}^{(r)}\}$, satisfying ($\mathcal{C}$), ($\mathcal{V}$), ($\mathcal{S}$), and ($\mathcal{I}$), by

$$\boldsymbol{\sigma}_\nu^{(r)}(\mu) = \begin{cases} \boldsymbol{\sigma}_{\lambda_{\boldsymbol{T}_k}(\nu)}(\lambda_{\boldsymbol{T}_k}(\mu)) & \text{if } \nu, \mu \in \boldsymbol{T}_k \text{ for some } 1 \le k \le 2^n, \\ 0 & \text{elsewhere on } \boldsymbol{T}. \end{cases} \tag{3.4}$$

Then for the resulting $(\Sigma_j^{(r)})$ it holds that $\Sigma_j^{(r)} = 2^{-n/2}\Sigma_{j+1}$ and so span $\Sigma_j \subset$ span Σ_{j+1} ($j \in \mathbb{N}$) if and only if

$$\text{span}\,\mathbf{\Sigma} \subset \text{span}\,\mathbf{\Sigma}^{(r)}. \tag{$\mathcal{R}$}$$

Such a collection $\mathbf{\Sigma}$ is called *refinable* and $\mathbf{\Sigma}^{(r)}$ is the *refinement* of $\mathbf{\Sigma}$. Formulas (3.2) and (3.3) show how, the uniformly local, representation of Incl : span $\Sigma_j \to$ span Σ_{j+1} can be constructed from the representation of the local inclusion.

We note the trivial equality

$$\langle u, v\rangle_{L_2(\Omega)} = \sum_{T \in \tau_j} \frac{\text{vol}(T)}{\text{vol}(\boldsymbol{T})} \langle u \circ \lambda_T^{-1}, v \circ \lambda_T^{-1}\rangle_{L_2(\boldsymbol{T})}. \tag{3.5}$$

Let Σ be a collection satisfying $(\mathcal{C})$–$(\mathcal{I})$. From (3.5), and the fact that $\boldsymbol{\Sigma}$ is an independent set and thus an $L_2(\boldsymbol{T})$-Riesz system, we obtain that

$$\begin{aligned}
\|\mathbf{c}_j^T \Sigma_j\|_{L_2(\Omega)}^2 &= \sum_{T\in\tau_j} \frac{\mathrm{vol}(T)}{\mathrm{vol}(\boldsymbol{T})} \left\| \sum_{x\in I_j\cap T} c_{j,x}\,\mu(x;\tau_j)\boldsymbol{\sigma}_{\lambda_T(x)} \right\|_{L_2(\boldsymbol{T})}^2 \\
&\eqsim \sum_{T\in\tau_j} \frac{\mathrm{vol}(T)}{\mathrm{vol}(\boldsymbol{T})} \sum_{x\in I_j\cap T} |c_{j,x}|^2 \mu(x;\tau_j)^2 \\
&= \sum_{x\in I_j} |c_{j,x}|^2 \mu(x;\tau_j)^2 \sum_{\{T\in\tau_j : T\ni x\}} \frac{\mathrm{vol}(T)}{\mathrm{vol}(\boldsymbol{T})} \\
&= \|\mathbf{c}_j\|^2,
\end{aligned}$$

i.e., the Σ_j are *uniform* $L_2(\Omega)$*-Riesz systems*.

Having two collections $\boldsymbol{\Sigma}^{(1)} = \{\boldsymbol{\sigma}_\lambda^{(1)}; \lambda \in \boldsymbol{I}^{(1)}\}$ and $\boldsymbol{\Sigma}^{(2)} = \{\boldsymbol{\sigma}_\lambda^{(2)}; \lambda \in \boldsymbol{I}^{(2)}\}$ satisfying $(\mathcal{C})$–$(\mathcal{I})$, for $x \in I_j^{(1)}$, $y \in I_j^{(2)}$, it holds that

$$\langle \sigma_{j,x}^{(1)}, \sigma_{j,y}^{(2)}\rangle_{L_2(\Omega)} = \mu(x;\tau_j)\mu(y;\tau_j) \sum_{\{T\in\tau_j : T\ni x,y\}} \frac{\mathrm{vol}(T)}{\mathrm{vol}(\boldsymbol{T})} \langle \boldsymbol{\sigma}_{\lambda_T(x)}^{(1)}, \boldsymbol{\sigma}_{\lambda_T(y)}^{(2)}\rangle_{L_2(\boldsymbol{T})}, \tag{3.6}$$

where, when $\{T \in \tau_j : T \ni x, y\} \neq \emptyset$, the factors $\langle \boldsymbol{\sigma}_{\lambda_T(x)}^{(1)}, \boldsymbol{\sigma}_{\lambda_T(y)}^{(2)}\rangle_{L_2(\boldsymbol{T})}$ in the sum on the right-hand side are independent of T. We see that the uniformly local matrices $\langle \Sigma_j^{(1)}, \Sigma_j^{(2)}\rangle_{L_2(\Omega)}$ can easily be assembled from $\langle \boldsymbol{\Sigma}^{(1)}, \boldsymbol{\Sigma}^{(2)}\rangle_{L_2(\boldsymbol{T})}$ using some information about the geometry of τ_j.

Now that we have collected general properties of the assembling of global functions from local ones, we come to the construction of Φ_j, Φ_j, and Ξ_j. With $\boldsymbol{\Phi}$, $\tilde{\boldsymbol{\Phi}}$, and $\boldsymbol{\Phi}$ being particular instances of a collection $\boldsymbol{\Sigma}$, the idea is to define Φ_j, $\tilde{\Phi}_j$, and Ξ_j as the corresponding collections of global functions according to (3.1). Yet, not in all concrete realizations will we be able to construct $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$ such that Φ_j and $\tilde{\Phi}_j$ are biorthogonal. In these cases, we redefine the dual scaling functions by $\langle \tilde{\Phi}_j, \Phi_j\rangle_{L_2(\Omega)}^{-1} \tilde{\Phi}_j$ which indeed is dual to Φ_j. The matrices $\langle \tilde{\Phi}_j, \Phi_j\rangle_{L_2(\Omega)}$ will be of a special structure such that their inverses, and so the redefined dual scaling functions, are uniformly local anyhow.

At the primal side, the collection $\boldsymbol{\Phi}$ will always be selected such that it satisfies $(\mathcal{C})$, $(\mathcal{V})$, and $(\mathcal{S})$ and such that, for *some fixed d and m*,

$$\mathrm{span}\,\boldsymbol{\Phi} = P_{d-1,m}(\boldsymbol{T}),$$

being defined as *the space of continuous piecewise polynomials on $\boldsymbol{T}$ of degree $d-1$ with respect to an m-times repeated dyadic partition of $\boldsymbol{T}$.*

We define

$$\boldsymbol{I}_q = \{\lambda \in \boldsymbol{T} : \lambda_i/q \in \mathbb{N}\},$$

which is sometimes called the principal lattice of order q. It is well known that

$$\mathrm{card}(\boldsymbol{I}_{(d-1)2^m}) = \dim(P_{d-1,m}(\boldsymbol{T})).$$

We will always assume that the index set of $\mathbf{\Phi}$ is given by

$$\boldsymbol{I} = \boldsymbol{I}_{(d-1)2^m}, \tag{3.7}$$

which guarantees that $\mathbf{\Phi}$ satisfies the conditions $(\mathcal{I})$ and $(\mathcal{R})$ as well. Indeed, for $\boldsymbol{e} = \boldsymbol{T}$, or for $\boldsymbol{e}$ being a face of $\boldsymbol{T}$, by $(\mathcal{V})$ there holds

$$\operatorname{span}\{\varphi_\lambda|_{\boldsymbol{e}} : \lambda \in \boldsymbol{I}_{(d-1)2^m} \cap e\} = \operatorname{span} \mathbf{\Phi}|_{\boldsymbol{e}} = P_{d-1,m}(\boldsymbol{T})|_{\boldsymbol{e}} = P_{d-1,m}(\boldsymbol{e}),$$

and so $\operatorname{card}(\boldsymbol{I}_{(d-1)2^m} \cap \boldsymbol{e}) = \dim(P_{d-1,m}(\boldsymbol{e}))$ shows $(\mathcal{I})$. Furthermore, it is clear that $\operatorname{span} \mathbf{\Phi}^{(r)} \subset P_{d-1,m+1}(\boldsymbol{T})$. Now from

$$\boldsymbol{I}^{(r)}_{(d-1)2^m} = \boldsymbol{I}_{(d-1)2^{m+1}},$$

we conclude that $\operatorname{span}\mathbf{\Phi}^{(r)} = P_{d-1,m+1}(\boldsymbol{T})$ and thus that $(\mathcal{R})$ is valid.

A particular collection $\mathbf{\Phi}$ satisfying the above conditions is the *nodal* one $\mathbf{\Phi} = \mathbf{\Delta}^{(d-1,m)} = \{\boldsymbol{\delta}^{(d-1,m)}_\lambda : \lambda \in \boldsymbol{I}_{(d-1)2^m}\} \subset P_{d-1,m}(\boldsymbol{T})$ defined by

$$\boldsymbol{\delta}^{(d-1,m)}_\lambda(\mu) = \begin{cases} 1, & \lambda = \mu, \\ 0, & \lambda \neq \mu \in \boldsymbol{I}_{(d-1)2^m}. \end{cases}$$

Note that $(\mathbf{\Delta}^{(d-1,m)})^{(r)} = \mathbf{\Delta}^{(d-1,m+1)}$.

Remark 3.2. We included the possibility of $m > 0$ to introduce some freedom in the choice of $\mathbf{\Phi}$. Indeed, note that for $d = 2$ and $m = 0$, the only possibility is $\mathbf{\Phi} = \mathbf{\Delta}^{(1,0)}$ (up to scalar multiples).

For the resulting sequence of collections (Φ_j), of functions on Ω defined by (3.1) corresponding to $\mathbf{\Phi}$, there holds $\operatorname{cl}_{L_2(\Omega)} \operatorname{span} \Phi_j = V_j$, being the space of *continuous piecewise polynomials of order* $d-1$ *with respect to* τ_{j+m} *having finite* $L_2(\Omega)$*-norm*. In view of this, the elements of τ_j will also be called *macro*elements in case $m > 0$. The sequence (V_j) satisfies the Bernstein estimate $(\mathcal{B})$ with $\gamma = \frac{3}{2}$ and the Jackson estimate $(\mathcal{J})$ for this value of d.

At the dual side, we will select $\tilde{\mathbf{\Phi}}$ satisfying $(\mathcal{C})$, $(\mathcal{S})$, $(\mathcal{V})$, $(\mathcal{I})$, and $(\mathcal{R})$. Aiming at biorthogonality, for the resulting $(\tilde{\Phi}_j)$ defined by (3.1) corresponding to $\tilde{\mathbf{\Phi}}$, there should hold $\operatorname{card}(\tilde{\Phi}_j) = \operatorname{card}(\Phi_j)$, independent of τ_0. This means that the index set $\tilde{\boldsymbol{I}}$ of $\tilde{\mathbf{\Phi}}$ should satisfy $\operatorname{card}(\tilde{\boldsymbol{I}}) = \operatorname{card}(\boldsymbol{I}_{(d-1)2^m})$ and $\operatorname{card}(\tilde{\boldsymbol{I}} \cap \boldsymbol{e}) = \operatorname{card}(\boldsymbol{I}_{(d-1)2^m} \cap \boldsymbol{e})$ for any face $\boldsymbol{e}$ of $\boldsymbol{T}$, which means that it is no restriction to take $\tilde{\boldsymbol{I}} = \boldsymbol{I}_{(d-1)2^m}$.

Because of $(\mathcal{R})$, the sequence $(\tilde{V}_j)$, defined by $\tilde{V}_j := \operatorname{cl}_{L_2(\Omega)} \operatorname{span} \tilde{\Phi}_j$, is nested. Since the $\tilde{\varphi}_{j,x}$ are continuous, standard arguments (see [16, §2.4]) show that $(\tilde{V}_j)$ satisfies the Bernstein estimate $(\tilde{\mathcal{B}})$ with $\tilde{\gamma} = \frac{3}{2}$. The set $\tilde{\mathbf{\Phi}}$ will selected such that, for some $\tilde{d}$, its span includes $P_{\tilde{d}-1,0}(\boldsymbol{T})$, so that $(\tilde{V}_j)$ satisfies the Jackson estimate $(\tilde{\mathcal{J}})$ for this value of $\tilde{d}$. In view of the cancellation property, we are aiming at making $\tilde{d}$ as large as possible. A dimension argument shows that $\tilde{d} - 1 \leq (d-1)2^m$ where, in practice, the upper bound cannot be attained because of the other requirements.

In some cases (Sections 3.2, 3.3), we will be able to construct biorthogonal $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$. From (3.6) we conclude that then Φ_j, $\tilde{\Phi}_j$ are biorthogonal, uniformly local, uniform $L_2(\Omega)$-Riesz systems.

In other cases (Sections 3.4, 3.5), with respect to some partitioning of the index set $\boldsymbol{I}$ into $\boldsymbol{I}^{(1)}, \ldots, \boldsymbol{I}^{(q)}$, where for each $1 \leq k \leq q$, $\pi(\boldsymbol{I}^{(k)}) = \boldsymbol{I}^{(k)}$ or, more generally, $\pi(\boldsymbol{I}^{(k)} \cap \partial \boldsymbol{T}) = \boldsymbol{I}^{(k)} \cap \partial \boldsymbol{T}$, $\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})}$ will be a block lower triangular matrix with diagonal blocks equal to identity matrices. As an example of such a partition, one may think of $\boldsymbol{I}$ being the set of vertices and midpoints of edges split into the set of vertices $\boldsymbol{I}^{(1)}$ and the set of midpoints $\boldsymbol{I}^{(2)}$.

Then, defining

$$I_j^{(k)} = \{x \in \Omega : \lambda_T(x) \in \boldsymbol{I}^{(k)} \text{ for some } T \in \tau_j\},$$

we have $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2} = [(\varphi_{j,x}, \tilde{\varphi}_{j,y})_{x \in I_j^{(k)},\, y \in I_j^{(\ell)}}]_{1 \leq k, \ell \leq q}$ and, again, (3.6) shows that this is a block lower triangular matrix, with diagonal blocks equal to identity matrices. We infer that both the $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}$ and their inverses are uniformly local and uniformly bounded matrices. So, we conclude that

$$\Phi_j, \ \langle \tilde{\Phi}_j, \Phi_j \rangle_{L_2(\Omega)}^{-1} \tilde{\Phi}_j$$

are biorthogonal, uniformly local, uniform $L_2(\Omega)$-Riesz systems. We will refer to this step as the *a posteriori biorthogonalization*.

Remark 3.3. The reason why we apply the a posteriori biorthogonalization, instead of biorthogonalizing $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$ before constructing the global scaling functions, is that, in the cases in question, such a "local" biorthogonalization would violate $(\mathcal{V})$.

Finally, we come to the specification of the uniformly local collections Ξ_j, such that $\Phi_j \cup \Xi_j$ are uniform L_2-Riesz bases for the spaces V_{j+1} and such that the basis transformations from Φ_{j+1} to $\Phi_j \cup \Xi_j$ are uniformly local.

With

$$\mathbf{\Xi}^{(d-1,m)} := \{\boldsymbol{\delta}_\lambda^{(d-1,m+1)} : \lambda \in \boldsymbol{I}_{(d-1)2^{m+1}} \backslash \boldsymbol{I}_{(d-1)2^m}\},$$

it is well known that

$$P_{d-1,m+1}(\boldsymbol{T}) = \operatorname{span} \mathbf{\Xi}^{(d-1,m)} \oplus P_{d-1,m}(\boldsymbol{T}).$$

Without returning to this point, in all examples, we take Ξ_j to be the "global" collection defined by (3.1) *corresponding to $\Xi^{(d-1,m)}$. Note that Ξ_j is nothing other than the* "hierarchical surplus," *that is, the collection of all "global" nodal basis functions corresponding to the "new nodes."* With the canonical application of $\boldsymbol{I}_{(d-1)2^{m+1}}$ as an index set for $\mathbf{\Phi} \cup \mathbf{\Xi}^{(d-1,m)}$, this collection satisfies $(\mathcal{C})$, $(\mathcal{V})$, $(\mathcal{S})$ and, since it spans $P_{d-1,m+1}(\boldsymbol{T})$, also $(\mathcal{I})$. We conclude that indeed the $\Phi_j \cup \Xi_j$ are uniformly local, uniform L_2-Riesz bases for the spaces V_{j+1}.

The set $\mathbf{\Phi}^{(r)}$ also satisfies $(\mathcal{C})$–$(\mathcal{I})$ and $\operatorname{span} \mathbf{\Phi}^{(r)} = P_{d-1,m+1}(\boldsymbol{T}) = \operatorname{span} \mathbf{\Phi} \cup \mathbf{\Xi}^{(d-1,m)}$. As we have shown on page OF12, *this means that the basis transformations in both directions between the corresponding global bases, which are $2^{-n/2}\Phi_{j+1}$ and $\Phi_j \cup \Xi_j$,* *are uniformly local*, *where formulas* (3.2) *and* (3.3) *show how they are defined in terms of the local basis transformations*.

In the following Remarks 3.4–3.6 we discuss some issues concerning getting efficient implementations.

Remark 3.4. To compute the wavelet and inverse wavelet transforms, formula (2.12) shows that, apart from $[\mathbf{M}_{j,0} \quad \mathbf{R}_{j,1}]^{-1}$ and $[\mathbf{M}_{j,0} \quad \mathbf{R}_{j,1}]$, one needs the application of the matrices $\tilde{\mathbf{M}}^*_{j,0}\mathbf{R}_{j,1}$. Taking into account the possibility that an a posteriori biorthogonalization is needed, meaning that the collections of the dual scaling functions are given by $\langle \tilde{\Phi}_j, \Phi_j \rangle^{-1}_{L_2(\Omega)} \tilde{\Phi}_j$, we have

$$\mathbf{M}^*_{j,0}\mathbf{R}_{j,1} = \langle \Xi_j, \langle \tilde{\Phi}_j, \Phi_j \rangle^{-1}_{L_2(\Omega)} \tilde{\Phi}_j \rangle^T_{L_2(\Omega)} = \langle \Phi_j, \tilde{\Phi}_j \rangle^{-T}_{L_2(\Omega)} \langle \Xi_j, \tilde{\Phi}_j \rangle^T_{L_2(\Omega)}.$$

In the case that $\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})} \neq \mathbf{id}$, the last equality in the above display indicates an efficient way to apply $\tilde{\mathbf{M}}^*_{j,0}\mathbf{R}_{j,1}$ in a factorized way. Formula (3.6) shows how $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}$ and $\langle \Xi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}$ can be computed from $\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})}$ and $\langle \mathbf{\Xi}^{(d-1,m)}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})}$. Since $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}$ is assumed to have a block lower triangular structure with diagonal blocks equal to identity matrices, $\langle \Phi_j, \tilde{\Phi}_j \rangle^{-1}_{L_2(\Omega)}$ can easily be constructed from $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}$, where its application takes as many operations as applying $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}$.

Remark 3.5. For the case that $\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})} \neq \mathbf{id}$, we applied a correction at the dual side, that is we considered the biorthogonal system Φ_j, $\langle \tilde{\Phi}_j, \Phi_j \rangle^{-1}_{L_2(\Omega)} \tilde{\Phi}_j$. The motivation not to consider the biorthogonal system $\langle \Phi_j, \tilde{\Phi}_j \rangle^{-1}_{L_2(\Omega)} \Phi_j$, $\tilde{\Phi}_j$ is that, in that case, $[\mathbf{M}_{j,0} \quad \mathbf{R}_{j,1}]$ should be replaced by

$$\langle \Phi_{j+1}, \tilde{\Phi}_{j+1} \rangle^T_{L_2(\Omega)} [\mathbf{M}_{j,0} \quad \mathbf{R}_{j,1}] \begin{bmatrix} \langle \Phi_j, \tilde{\Phi}_j \rangle^{-T}_{L_2(\Omega)} & \mathbf{0} \\ \mathbf{0} & \mathbf{id} \end{bmatrix}, \tag{3.8}$$

being the basis transformation from $\langle \Phi_j, \tilde{\Phi}_j \rangle^{-1}_{L_2(\Omega)} \Phi_j \cup \Xi_j$ to $\langle \Phi_{j+1}, \tilde{\Phi}_{j+1} \rangle^{-1}_{L_2(\Omega)} \Phi_{j+1}$. Comparison with Remark 3.4 tells us that for computing the inverse wavelet transform this correction at the primal side demands an additional application of $\langle \Phi_{j+1}, \tilde{\Phi}_{j+1} \rangle^T_{L_2(\Omega)}$. A similar observation holds for the wavelet transform. Note that since the supports of functions from $\langle \Phi_j, \tilde{\Phi}_j \rangle^{-1}_{L_2(\Omega)} \Phi_j$ extend to several macroelements, one cannot expect to obtain a cheaper implementation by a "direct" computation of the above basis transformation, that is, not using the factorization (3.8).

Remark 3.6. Reversing the last argument from Remark 3.5 leads to the insight that, regardless of whether $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$ are biorthogonal or not, for $m > 0$ particular efficient implementations of wavelet and inverse wavelet transforms can be expected, when as scaling functions at the primal side the collections of nodal basis functions $\Delta_j^{(d-1,m)}$ are applied, which are defined by (3.1) corresponding to $\mathbf{\Delta}^{(d-1,m)}$. Indeed, since the supports of functions from $\Delta_j^{(d-1,m)}$ are restricted to *elements* (i.e., $T \in \tau_{j+m}$) instead of *macroelements*, and Ξ_j is just a subset of $\Delta_{j+1}^{(d-1,m)}$, the basis transformations between $\Delta_j^{(d-1,m)} \cup \Xi_j$ and $\Delta_{j+1}^{(d-1,m)}$ can be implemented very efficiently. Let $\mathbf{G}_j$ now be the matrices such that $(\Delta_j^{(d-1,m)})^T = \Phi_j^T \mathbf{G}_j$. Both $\mathbf{G}_j$ and $\mathbf{G}_j^{-1}$ are uniformly bounded and uniformly local, and they can easily be constructed from the corresponding local transformations. The pairs $\Delta_j^{(d-1,m)}$, $\bar{\mathbf{G}}^{-1}\langle\tilde{\Phi}_j, \Phi_j\rangle_{L_2(\Omega)}^{-1}\tilde{\Phi}_j$ are biorthogonal, uniformly local, uniformly $L_2(\Omega)$-Riesz systems. With these systems applied, the matrix $\mathbf{M}_{j,0}^*\mathbf{R}_{j,1}$ reads as

$$\mathbf{G}_j^{-1}\langle\Phi_j, \tilde{\Phi}_j\rangle_{L_2(\Omega)}^{-T}\langle\Xi_j, \tilde{\Phi}_j\rangle_{L_2(\Omega)}^T.$$

The same arguments that were used in Remark 3.1 show that if the basis transformations between $\Phi_j \cup \Xi_j$ and Φ_{j+1} are most efficiently implemented as a composition of transformations from Φ_j to $\Delta_j^{(d-1,m)}$, $\Delta_j^{(d-1,m)} \cup \Xi_j$ to $\Delta_{j+1}^{(d-1,m)}$, and $\Delta_{j+1}^{(d-1,m)}$ to Φ_{j+1} or vice versa, then the approach of applying the nodal basis functions as scaling functions is more efficient.

Remark 3.7. So far we have considered the construction of bases for the "full" spaces. Homogeneous *Dirichlet conditions* on the boundary, or on a part of the boundary consisting of the union of $(n-1)$-dimensional faces of $T \in \tau_0$, can be incorporated into the construction by excluding those $\varphi_{j,x}$, $\tilde{\varphi}_{j,x}$ and $\xi_{j,x}$ from Φ_j, $\tilde{\Phi}_j$ and Ξ_j for which x is on (that part of) the boundary. The conditions $(\mathcal{V})$ and $(\mathcal{I})$ ensure that the resulting sequences (V_j), $(\tilde{V}_j)$, defined by $V_j = \mathrm{cl}_{L_2(\Omega)}\operatorname{span}\Phi_j$ and $\tilde{V}_j = \mathrm{cl}_{L_2(\Omega)}\operatorname{span}\tilde{\Phi}_j$ are still nested. The space V_j is the standard Lagrange finite element space in which the boundary conditions are incorporated. Basis transformations between the "reduced" sets $\Phi_j \cup \Psi_j$ and Φ_{j+1} and vice versa are obtained by simply deleting those rows and columns with indices corresponding to basis functions that have been removed. By replacing the scale of Sobolev spaces by the scale of subspaces that incorporate the essential boundary conditions, the Jackson and Bernstein estimates remain valid, and so the wavelets generate Riesz bases for the same range in the scale. On the other hand, wavelets from the resulting Ψ_j or $\tilde{\Psi}_j$ with supports that intersect the interiors of $T \in \tau_j$ that have a nonempty intersection with the boundary, will generally not have cancellation properties.

Finally, as demonstrated in [11], a construction like this carries directly over to finite element-type spaces on certain Lipschitz *manifolds*. More precisely, those manifolds are covered that consist of patches, each of them the parametric image of a domain with triangulations generated by uniform refinements, such that the images of the triangulations match at the interfaces, and on each domain the Jacobian determinant is piecewise constant with respect to the initial triangulation.

In the next subsections, for a number of examples of $(n, d, m, \tilde{d})$, we construct sets $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$. Using these two ingredients, the general theory presented in this subsection shows how the global scaling and dual scaling functions, and wavelets and dual wavelets, can be constructed and, furthermore, how the wavelet and inverse wavelet transforms can be computed.

3.2. *The Case* $(n, d, m, \tilde{d}) = (1, 2, 2, 4)$

In order to easily formulate conditions $(\mathcal{S})$ and $(\mathcal{V})$, in Section 3.1 we used, as an index set for $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$, the subset $\boldsymbol{I}_{(d-1)2^m}$ of the barycentric coordinates. Yet, to view $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ as vectors, the index set $\{1, 2, \ldots, \#\boldsymbol{I}_{(d-1)2^m}\}$ would be more appropriate. Therefore, in

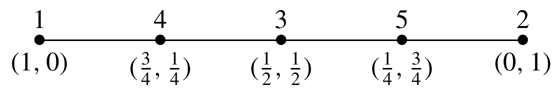

Fig. 2. Numbering of $\boldsymbol{I}_4$.

Figure 2 we fix a numbering of $\boldsymbol{I}_{(d-1)2^m} = \boldsymbol{I}_4$, so that we can switch between both index sets at our convenience.

We start with $\boldsymbol{\Phi}^{(0)} = \boldsymbol{\Delta}^{(1,2)}$, see Figure 3. It satisfies $(\mathcal{C})$, $(\mathcal{S})$, $(\mathcal{V})$, $(\mathcal{I})$, and $(\mathcal{R})$ and it spans $P_{1,2}(\boldsymbol{T})$.

Using a numbering of the elements of $\boldsymbol{\Delta}^{(3,0)}$ as indicated in Figure 4, at the dual side we start with $\tilde{\boldsymbol{\Phi}}^{(0)}$, where $\tilde{\boldsymbol{\varphi}}_i^{(0)} = \boldsymbol{\delta}_i^{(3,0)}$ for $i \in \{1, 2, 4, 5\}$. Later, the missing $\tilde{\boldsymbol{\varphi}}_3^{(0)}$ will be selected from $P_{3,1}(\boldsymbol{T}) \backslash P_{3,0}(\boldsymbol{T})$, such that it vanishes on $\partial \boldsymbol{T}$ and $\tilde{\boldsymbol{\varphi}}_3^{(0)}(\lambda_1, \lambda_2) = \tilde{\boldsymbol{\varphi}}_3^{(0)}(\lambda_2, \lambda_1)$. We infer that $\tilde{\boldsymbol{\Phi}}^{(0)}$ satisfies $(\mathcal{C})$, $(\mathcal{S})$, $(\mathcal{V})$, and $(\mathcal{I})$ and that

$$P_{3,0}(\boldsymbol{T}) \subset \operatorname{span} \tilde{\boldsymbol{\Phi}}^{(0)} \subset P_{3,1}(\boldsymbol{T})$$

showing $(\mathcal{R})$.

Remark 3.8. Note that refinements of the still unknown $\tilde{\boldsymbol{\varphi}}_3^{(0)}$ are not used to ensure $(\mathcal{R})$. As a consequence, we will be able to construct explicitly given dual scaling functions.

On the other hand, allowing for implicitly defined dual scaling functions would introduce additional freedom in the construction, which might mean that smaller macroelements can be used, resulting in wavelets with smaller support. However, in that case, $\tilde{d}$ will be also smaller, giving weaker cancellation properties. We are planning to discuss this approach in a forthcoming paper.

Together, the above conditions mean that

$$\tilde{\boldsymbol{\varphi}}_3^{(0)} \in \operatorname{span}\{\tilde{\boldsymbol{\varphi}}_4^{(0)} + \tilde{\boldsymbol{\varphi}}_5^{(0)}, \boldsymbol{\delta}^{(3,1)}_{(\frac{5}{6},\frac{1}{6})} + \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{6},\frac{5}{6})}, \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{2},\frac{1}{2})}\}, \tag{3.9}$$

see Figure 5.

Apart from fixing $\tilde{\boldsymbol{\varphi}}_3^{(0)}$, in the following we apply some (invertible) basis transformations to both collections $\boldsymbol{\Phi}^{(0)}$ and $\tilde{\boldsymbol{\Phi}}^{(0)}$, which preserve $(\mathcal{S})$ and $(\mathcal{V})$. Obviously, a basis

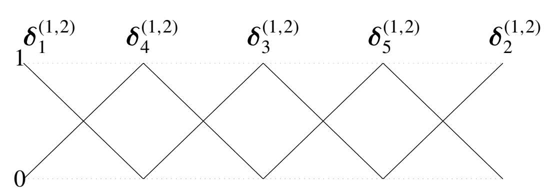

Fig. 3. $\boldsymbol{\Delta}^{(1,2)}$.

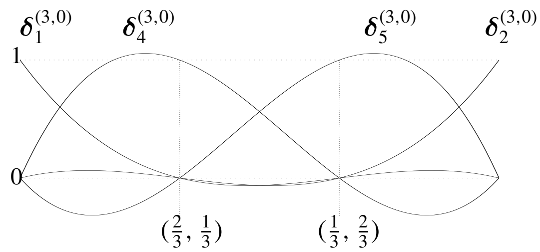

Fig. 4. $\boldsymbol{\Delta}^{(3,0)}$.

transformation always preserves $(\mathcal{C})$. Moreover, a basis transformation is represented by an invertible matrix. The fact that $(\mathcal{V})$ is preserved means that any principal submatrix of this matrix corresponding to all indices associated to some face is necessarily invertible, which means that $(\mathcal{I})$ is preserved as well. Since the basis transformations do not change the spans and preserve $(\mathcal{S})$, $(\mathcal{V})$, and $(\mathcal{I})$, we conclude that $(\mathcal{R})$ is also preserved. We will end up with biorthogonal sets $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$.

Now we come to the description of the basis transformations and the selection of $\tilde{\varphi}_3^{(0)}$:

(I) We search

$$\varphi_1 \in \varphi_1^{(0)} + \operatorname{span}\{\varphi_3^{(0)}, \varphi_4^{(0)}, \varphi_5^{(0)}\},$$

such that $\varphi_1 \perp \tilde{\varphi}_2^{(0)}, \tilde{\varphi}_4^{(0)}, \tilde{\varphi}_5^{(0)}$. Obviously, φ_2 defined by $\varphi_2(\lambda_1, \lambda_2) = \varphi_1(\lambda_2, \lambda_1)$ then satisfies $\varphi_2 \perp \tilde{\varphi}_1^{(0)}, \tilde{\varphi}_4^{(0)}, \tilde{\varphi}_5^{(0)}$. For $i \in \{3, 4, 5\}$, we take $\varphi_i = \varphi_i^{(0)}$.

(II) We select $\tilde{\varphi}_3^{(0)}$ by imposing $\tilde{\varphi}_3^{(0)} \perp \varphi_1$ (and thus $\tilde{\varphi}_3^{(0)} \perp \varphi_2$). Since $\tilde{\varphi}_4^{(0)} + \tilde{\varphi}_5^{(0)} \perp \varphi_1$, the span of the resulting $\tilde{\boldsymbol{\Phi}}$ does not change if, instead of (3.9), we search $\tilde{\varphi}_3^{(0)}$ in the smaller space $\operatorname{span}\{\boldsymbol{\delta}^{(3,1)}_{(\frac{5}{6},\frac{1}{6})} + \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{6},\frac{5}{6})}, \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{2},\frac{1}{2})}\}$.

(III) With $\tilde{\boldsymbol{\Phi}} := \langle \tilde{\boldsymbol{\Phi}}^{(0)}, \boldsymbol{\Phi}\rangle^{-1}_{L_2(\boldsymbol{T})} \tilde{\boldsymbol{\Phi}}^{(0)}$, we get $\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}\rangle_{L_2(\mathbf{T})} = \mathbf{id}$. Since by the previous steps, in the first two columns of $\langle \tilde{\boldsymbol{\Phi}}^{(0)}, \boldsymbol{\Phi}\rangle_{L_2(\boldsymbol{T})}$ only the diagonal element is nonzero, this transformation preserves $(\mathcal{V})$.

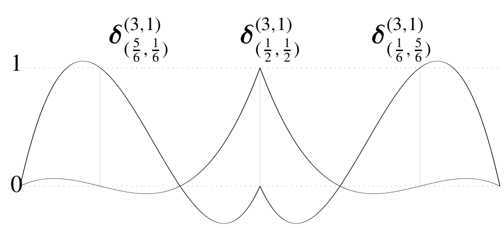

Fig. 5.

By substituting

$$\langle\{\boldsymbol{\delta}_1^{(1,2)},\boldsymbol{\delta}_3^{(1,2)},\boldsymbol{\delta}_4^{(1,2)}\},\{\boldsymbol{\delta}_1^{(3,0)},\boldsymbol{\delta}_2^{(3,0)},\boldsymbol{\delta}_4^{(3,0)},\boldsymbol{\delta}_5^{(3,0)},\boldsymbol{\delta}_{(\frac{5}{6},\frac{1}{6})}^{(3,1)}+\boldsymbol{\delta}_{(\frac{1}{6},\frac{5}{6})}^{(3,1)},\boldsymbol{\delta}_{(\frac{1}{2},\frac{1}{2})}^{(3,1)}\}\rangle_{L_2(\mathbf{T})}$$

$$= \mathrm{vol}(\boldsymbol{T})\begin{bmatrix} \frac{2413}{30720} & \frac{167}{30720} & \frac{687}{10240} & \frac{-267}{10240} & \frac{117}{1280} & \frac{17}{3840} \\ \frac{-5}{512} & \frac{-5}{512} & \frac{69}{512} & \frac{69}{512} & \frac{-27}{640} & \frac{193}{1920} \\ \frac{45}{1024} & \frac{7}{1024} & \frac{237}{1024} & \frac{-33}{1024} & \frac{15}{128} & \frac{1}{128} \end{bmatrix},$$

the above procedure results in $\varphi_3 = \boldsymbol{\delta}_3^{(1,2)}$, $\varphi_4 = \boldsymbol{\delta}_4^{(1,2)}$,

$$\varphi_1 = \boldsymbol{\delta}_1^{(1,2)} + \frac{23}{150}\boldsymbol{\delta}_3^{(1,2)} - \frac{23}{60}\boldsymbol{\delta}_4^{(1,2)} - \frac{3}{100}\boldsymbol{\delta}_5^{(1,2)},$$

$$\tilde{\varphi}_3^{(0)} = \boldsymbol{\delta}_{(\frac{5}{6},\frac{1}{6})}^{(3,1)} + \boldsymbol{\delta}_{(\frac{1}{6},\frac{5}{6})}^{(3,1)} - \frac{657}{299}\boldsymbol{\delta}_{(\frac{1}{2},\frac{1}{2})}^{(3,1)},$$

$$\begin{bmatrix}\tilde{\varphi}_1\\ \tilde{\varphi}_3\\ \tilde{\varphi}_5\end{bmatrix} = \frac{1}{\mathrm{vol}(\boldsymbol{T})}\begin{bmatrix} \frac{50}{3} & \frac{-299}{162} & \frac{-64}{27} & \frac{-2}{81} \\ 0 & \frac{-5083}{2025} & \frac{2552}{2025} & \frac{2552}{2025} \\ 0 & \frac{6877}{4050} & \frac{7196}{2025} & \frac{-484}{2025} \end{bmatrix}\begin{bmatrix}\boldsymbol{\delta}_1^{(3,0)}\\ \tilde{\varphi}_3^{(0)}\\ \boldsymbol{\delta}_4^{(3,0)}\\ \boldsymbol{\delta}_5^{(3,0)}\end{bmatrix},$$

see Figure 6.

The analysis from Section 3.1 shows that the resulting global sets Φ_j, $\tilde{\Phi}_j$ are biorthogonal, uniformly local, uniform $L_2(\Omega)$-Riesz systems. The collection Φ_j is a basis for the space of continuous piecewise linears with respect to τ_{j+2}. Furthermore, the spaces $\tilde{V}_j := \mathrm{cl}_{L_2(\Omega)}\,\mathrm{span}\,\tilde{\Phi}_j$ are nested and satisfy $(\tilde{\mathcal{B}})$ and $(\tilde{\mathcal{J}})$ with $\tilde{\gamma} = \frac{3}{2}$ and $\tilde{d} = 4$.

3.3. *The Case* $(n, d, m, \tilde{d}) = (1, 5, 0, 4)$

As in Section 3.2, $(d-1)2^m = 4$, and we use the same numbering from Figure 2 of the index set $\boldsymbol{I}_4$ for $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$. We now take $\boldsymbol{\Phi}^{(0)} = \boldsymbol{\Delta}^{(4,0)}$.

As in Section 3.2, at the dual side we take $\tilde{\varphi}_i^{(0)} = \boldsymbol{\delta}_i^{(3,0)}$ for $i \in \{1, 2, 4, 5\}$ and search $\tilde{\varphi}_3^{(0)} \in \mathrm{span}\{\boldsymbol{\delta}_{(\frac{5}{6},\frac{1}{6})}^{(3,1)} + \boldsymbol{\delta}_{(\frac{1}{6},\frac{5}{6})}^{(3,1)}, \boldsymbol{\delta}_{(\frac{1}{2},\frac{1}{2})}^{(3,1)}\}$. To fix $\tilde{\varphi}_3^{(0)}$, and to biorthogonalize $\boldsymbol{\Phi}^{(0)}$, $\tilde{\boldsymbol{\Phi}}^{(0)}$, we follow the same procedure as described in Section 3.2.

By substituting

$$\langle\{\boldsymbol{\delta}_1^{(4,0)},\boldsymbol{\delta}_3^{(4,0)},\boldsymbol{\delta}_4^{(4,0)}\},\{\boldsymbol{\delta}_1^{(3,0)},\boldsymbol{\delta}_2^{(3,0)},\boldsymbol{\delta}_4^{(3,0)},\boldsymbol{\delta}_5^{(3,0)},\boldsymbol{\delta}_{(\frac{5}{6},\frac{1}{6})}^{(3,1)}+\boldsymbol{\delta}_{(\frac{1}{6},\frac{5}{6})}^{(3,1)},\boldsymbol{\delta}_{(\frac{1}{2},\frac{1}{2})}^{(3,1)}\}\rangle_{L_2(\boldsymbol{T})}$$

$$= \mathrm{vol}(\boldsymbol{T})\begin{bmatrix} \frac{151}{2520} & 0 & \frac{1}{28} & \frac{-1}{56} & \frac{29}{560} & \frac{1}{336} \\ \frac{-13}{210} & \frac{-13}{210} & \frac{9}{70} & \frac{9}{70} & \frac{-3}{14} & \frac{23}{210} \\ \frac{2}{21} & \frac{2}{63} & \frac{2}{7} & \frac{-2}{35} & \frac{17}{70} & \frac{1}{210} \end{bmatrix},$$

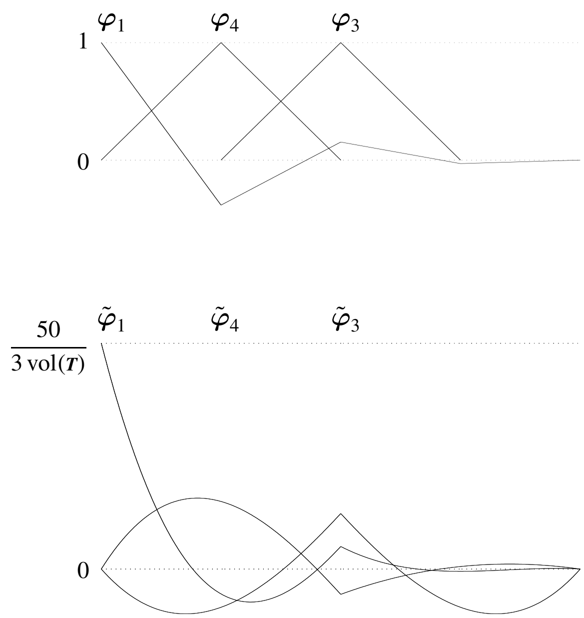

Fig. 6. Biorthogonal $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$ ($\varphi_2, \varphi_5, \tilde{\varphi}_2, \tilde{\varphi}_5$ by permuting barycentric coordinates) .

this procedure now results in $\varphi_3 = \boldsymbol{\delta}_3^{(4,0)}$, $\varphi_4 = \boldsymbol{\delta}_4^{(4,0)}$,

$$\varphi_1 = \boldsymbol{\delta}_1^{(4,0)} - \frac{15}{128}\boldsymbol{\delta}_4^{(4,0)} + \frac{5}{128}\boldsymbol{\delta}_5^{(4,0)},$$

$$\tilde{\varphi}_3^{(0)} = \boldsymbol{\delta}^{(3,1)}_{(\frac{5}{6},\frac{1}{6})} + \boldsymbol{\delta}^{(3,1)}_{(\frac{1}{6},\frac{5}{6})} - \frac{63}{5}\boldsymbol{\delta}^{(3,1)}_{(\frac{1}{2},\frac{1}{2})},$$

$$\begin{bmatrix}\tilde{\varphi}_1\\ \tilde{\varphi}_3\\ \tilde{\varphi}_5\end{bmatrix} = \frac{1}{\mathrm{vol}(\boldsymbol{T})}\begin{bmatrix} 20 & \frac{-40}{27} & \frac{-56}{9} & \frac{-68}{27} \\ 0 & \frac{-5}{9} & \frac{4}{9} & \frac{4}{9} \\ 0 & \frac{5}{16} & \frac{163}{48} & \frac{23}{48}\end{bmatrix}\begin{bmatrix}\boldsymbol{\delta}_1^{(3,0)}\\ \tilde{\varphi}_3^{(0)}\\ \boldsymbol{\delta}_4^{(3,0)}\\ \boldsymbol{\delta}_5^{(3,0)}\end{bmatrix},$$

and φ_2, φ_5 and $\tilde{\varphi}_2$, $\tilde{\varphi}_5$ by permuting barycentric coordinates.

The resulting global sets Φ_j, $\tilde{\Phi}_j$ are biorthogonal, uniformly local, uniform $L_2(\Omega)$-Riesz systems. The collection Φ_j is a basis for the space of continuous piecewise quartics with respect to τ_j. Note that, in contrast to Section 3.2, for each $x \in I_j$, the basis function $\varphi_{j,x}$ has the same support as the nodal basis function corresponding to that point.

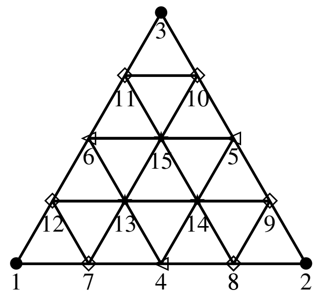

Fig. 7. Numbering of $\boldsymbol{I}_4$, and its partitioning into $\{\bullet\} \cup \{\triangleleft\} \cup \{\diamond\} \cup \{\star\}$.

3.4. *The Case* $(n, d, m, \tilde{d}) = (2, 2, 2, 4)$

We number the index set $\boldsymbol{I}_{(d-1)2^m} = \boldsymbol{I}_4$ of $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$ as in Figure 7, and switch between these numbers and the corresponding barycentric coordinates at our convenience. We take $\boldsymbol{\Phi}^{(0)} = \boldsymbol{\Delta}^{(1,2)}$. It satisfies $(\mathcal{C})$, $(\mathcal{S})$, $(\mathcal{V})$, $(\mathcal{I})$, and $(\mathcal{R})$, and it spans $P_{1,2}(\boldsymbol{T})$.

We define $\tilde{\varphi}^{(0)}_{1..3,7..12} = \boldsymbol{\delta}^{(3,0)}_{1..3,7..12}$ using a numbering of $\boldsymbol{I}_3$, and with that of the elements of $\boldsymbol{\Delta}^{(3,0)}$ as given in Figure 8. Later, we will define the missing $\tilde{\varphi}^{(0)}_{4..6,13..15}$ such that $\tilde{\boldsymbol{\Phi}}^{(0)} := \{\tilde{\varphi}^{(0)}_{1..15}\}$ satisfies $(\mathcal{C})$, $(\mathcal{S})$, $(\mathcal{V})$, and $(\mathcal{I})$, as well as

$$\boldsymbol{\delta}^{(3,0)}_{13} \in \operatorname{span}\{\tilde{\varphi}^{(0)}_{13..15}\}, \tag{3.10}$$

$$\tilde{\varphi}^{(0)}_{4..6} \in P_{3,1}(\boldsymbol{T}), \tag{3.11}$$

and

$$\tilde{\varphi}^{(0)}_{13..15} \in P_{3,1}(\boldsymbol{T}) \cup \operatorname{span}\{\tilde{\varphi}^{(0)}_{4..6}\}^{(r)}, \tag{3.12}$$

where $\{\tilde{\varphi}^{(0)}_{4..6}\}^{(r)}$ is defined in (3.4) as the refinement of $\{\tilde{\varphi}^{(0)}_{4..6}\}$. From (3.10) we have $P_{3,0}(\boldsymbol{T}) \subset \operatorname{span}\{\tilde{\varphi}^{(0)}_{1..3,7..15}\}$, and so (3.11) and (3.12) show that $(\mathcal{R})$ is valid and, moreover, that

$$P_{3,0}(\boldsymbol{T}) \subset \operatorname{span} \tilde{\boldsymbol{\Phi}}^{(0)} \subset P_{3,2}(\boldsymbol{T}). \tag{3.13}$$

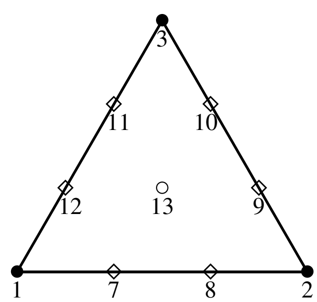

Fig. 8. Numbering of $\boldsymbol{I}_3$.

Apart from specifying the missing $\tilde{\varphi}^{(0)}_{4..6,13..15}$, in the following we describe invertible basis transformations on both $\mathbf{\Phi}^{(0)}$ and $\tilde{\mathbf{\Phi}}^{(0)}$ that preserve $(\mathcal{S})$ and $(\mathcal{V})$. The same reasoning as in Section 3.2 shows that then $(\mathcal{C})$, $(\mathcal{I})$, and $(\mathcal{R})$ are preserved as well. As a consequence of $(\mathcal{S})$, we only have to specify $\varphi^{(0)}_i$ and $\tilde{\varphi}^{(0)}_i$ for i running over any element of the sets 1..3, 4..6, 7..12, 13..15 (corresponding to $\{\bullet\}$, $\{\triangleleft\}$, $\{\diamond\}$, $\{\star\}$ from Figure 7), since the other functions then follow by permuting the barycentric coordinates.

We will not be able to end up with biorthogonal $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$. Instead, we derive $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$, such that with respect to a partitioning of 1..15 into $\{\bullet\}$, $\{\triangleleft\}$, $\{\diamond\}$, $\{\star\}$, the matrix $\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}}\rangle_{L_2(\boldsymbol{T})}$ is of the form

$$\begin{bmatrix} \mathbf{id} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \mathbf{id} & \mathbf{0} & \mathbf{0} \\ * & * & \mathbf{id} & \mathbf{0} \\ * & * & * & \mathbf{id} \end{bmatrix}. \tag{3.14}$$

With respect to a corresponding partitioning of I_j, the matrix $\langle \Phi_j, \tilde{\Phi}_j\rangle_{L_2(\Omega)}$ of the global basis functions Φ_j, $\tilde{\Phi}_j$ defined by (3.1) then inherits the same block form. The pairs Φ_j, $\langle \tilde{\Phi}_j, \Phi_j\rangle^{-1}_{L_2(\Omega)}\tilde{\Phi}_j$ will be biorthogonal, uniformly local, uniform $L_2(\Omega)$-Riesz systems.

The sets $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$ are obtained by performing the steps **(I)–(VI)**:

(I) In view of $(\mathcal{S})$ and $(\mathcal{V})$, φ_1 is searched in

$$\varphi^{(0)}_1 + \operatorname{span}\{\varphi^{(0)}_7 + \varphi^{(0)}_{12}, \varphi^{(0)}_4 + \varphi^{(0)}_6, \varphi^{(0)}_8 + \varphi^{(0)}_{11}, \varphi^{(0)}_{13}, \varphi^{(0)}_{14} + \varphi^{(0)}_{15}\}$$

such that

$$\varphi_1 \perp \tilde{\varphi}^{(0)}_{2,7,8,9}, \boldsymbol{\delta}^{(3,0)}_{13}, \tag{3.15}$$

which determines φ_1 uniquely. Clearly, (3.15) is equivalent to $\varphi_1 \perp \tilde{\varphi}^{(0)}_{2,3,7..12}, \boldsymbol{\delta}^{(3,0)}_{13}$. Since $\boldsymbol{\delta}^{(3,0)}_{13} \in \operatorname{span}\{\tilde{\varphi}^{(0)}_{13..15}\}$ by (3.10), and forthcoming transformations at the dual side have to preserve (3.1), condition (3.15) is necessary for obtaining the first row in (3.14). We define $\tilde{\varphi}^{(1)}_1 = \tilde{\varphi}^{(0)}_1 / \langle \tilde{\varphi}^{(0)}_1, \varphi_1\rangle_{L_2(\boldsymbol{T})}$.

(II) In view of $(\mathcal{V})$, φ_4, φ_7 (and φ_8) are searched in $\operatorname{span}\{\varphi^{(0)}_{4,7,8,13..15}\}$, and, in view of $(\mathcal{S})$, in particular $\varphi_4 \in \varphi^{(0)}_4 + \operatorname{span}\{\varphi^{(0)}_7 + \varphi^{(0)}_8, \varphi^{(0)}_{13} + \varphi^{(0)}_{14}, \varphi^{(0)}_{15}\}$.

To get the zeros in the second row in (3.14), φ_4 must satisfy

$$\varphi_4 \perp \tilde{\varphi}^{(0)}_{9,10},\ \boldsymbol{\delta}^{(3,0)}_{13}, \tag{3.16}$$

which determines φ_4 uniquely and which is equivalent to $\varphi_4 \perp \tilde{\varphi}^{(0)}_{9..12},\ \boldsymbol{\delta}^{(3,0)}_{13}$.

To get the zero in the third row in (3.14), it is necessary that $\varphi_7 \perp \boldsymbol{\delta}^{(3,0)}_{13}$. Furthermore, for obtaining the identity matrix in this row, φ_7 should be orthogonal to $\tilde{\varphi}_{9..12}$. If $\operatorname{span}\{\tilde{\varphi}_{9..12}\}$ would be equal to $\operatorname{span}\{\tilde{\varphi}^{(0)}_{9..12}\}$, then these conditions on φ_7 could only mean that φ_7 is a multiple of φ_4. Yet, since $\tilde{\varphi}^{(0)}_{9,10}$ ($\tilde{\varphi}^{(0)}_{7,8}$, $\tilde{\varphi}^{(0)}_{11,12}$) can be updated by the same multiple of $\tilde{\varphi}^{(0)}_5$ ($\tilde{\varphi}^{(0)}_4$, $\tilde{\varphi}^{(0)}_6$) that still has to be defined, it might be sufficient when only

$$\varphi_{7,8} \perp \tilde{\varphi}^{(0)}_9 - \tilde{\varphi}^{(0)}_{10},\ \tilde{\varphi}^{(0)}_{11} - \tilde{\varphi}^{(0)}_{12},\ \boldsymbol{\delta}^{(3,0)}_{13}. \tag{3.17}$$

Indeed, in the case that $\tilde{\varphi}_5^{(0)}$ is selected such that

$$\frac{\langle \varphi_7, \tilde{\varphi}_9^{(0)} \rangle_{L_2(\boldsymbol{T})}}{\langle \varphi_7, \tilde{\varphi}_5^{(0)} \rangle_{L_2(\boldsymbol{T})}} = \frac{\langle \varphi_8, \tilde{\varphi}_9^{(0)} \rangle_{L_2(\boldsymbol{T})}}{\langle \varphi_8, \tilde{\varphi}_5^{(0)} \rangle_{L_2(\boldsymbol{T})}} =: \alpha, \tag{3.18}$$

then with

$$\tilde{\varphi}_9^{(1)} := \tilde{\varphi}_9^{(0)} - \alpha \tilde{\varphi}_5^{(0)}$$

(similarly, $\tilde{\varphi}_{10}^{(1)}, \tilde{\varphi}_{7,8}^{(1)}, \tilde{\varphi}_{11,12}^{(1)}$), (3.17) gives

$$\varphi_{7,8} \perp \tilde{\varphi}_{9..12}^{(1)}, \boldsymbol{\delta}_{13}^{(3,0)}.$$

Together (3.16) and (3.17), and the fact that $\{\varphi_{4,7,8}\}$ should be an independent set, determine $\operatorname{span}\{\varphi_{4,7,8}\}$ uniquely. We fix φ_7 by selecting it from $\varphi_7^{(0)} + \operatorname{span}\{\varphi_{4,13..15}^{(0)}\}$.

Defining $\tilde{\varphi}_{7,8}^{(2)} \in \operatorname{span}\{\tilde{\varphi}_{7,8}^{(1)}\}$ (and with that $\tilde{\varphi}_{7..12}^{(2)}$) by imposing $\langle \varphi_{7,8}, \tilde{\varphi}_{7,8}^{(2)} \rangle_{L_2(\mathbf{T})} = \mathbf{id}$ now yields $\langle \varphi_{7..12}, \tilde{\varphi}_{7..12}^{(2)} \rangle_{L_2(\mathbf{T})} = \mathbf{id}$.

Remark 3.9. A consequence of the above procedure is that $\tilde{\varphi}_5^{(0)} \not\perp \varphi_{7,8}$. Since orthogonality cannot be restored by any transformation at the dual side that preserves $(\mathcal{V})$, we conclude that we cannot end up with biorthogonal $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$.

To ensure that (3.15) and (3.16), in which the $\tilde{\varphi}_{7..12}^{(0)}$ are replaced by $\tilde{\varphi}_{7..12}^{(2)}$, remain valid it is, furthermore, necessary that

$$\tilde{\varphi}_5^{(0)} \perp \varphi_{1,4}, \tag{3.19}$$

which is desirable on its own. Finally, since we also want $\varphi_4 \perp \tilde{\varphi}_7^{(2)}(, \tilde{\varphi}_8^{(2)})$ or, equivalently, $\varphi_5 \perp \tilde{\varphi}_9^{(2)}$, the function $\tilde{\varphi}_5^{(0)}$ should satisfy

$$\frac{\langle \varphi_5, \tilde{\varphi}_9^{(0)} \rangle_{L_2(\boldsymbol{T})}}{\langle \varphi_5, \tilde{\varphi}_5^{(0)} \rangle_{L_2(\boldsymbol{T})}} = \alpha. \tag{3.20}$$

(III) We take $\varphi_{13} = \varphi_{13}^{(0)}$.

At this point, we have fixed $\boldsymbol{\Phi}$. Further definitions and transformations take place at the dual side. First we specify $\tilde{\varphi}_{4..6}^{(0)}$ and $\tilde{\varphi}_{13..15}^{(0)}$.

(IV) We search $\tilde{\varphi}_{4..6}^{(0)} \in P_{3,1}(\boldsymbol{T})$. A basis for this space is given by

$$\{\tilde{\varphi}_{1..3,7..12}^{(0)}\} \cup \{\boldsymbol{\delta}_{13}^{(3,0)}\} \cup \{\boldsymbol{\delta}_{\lambda}^{(3,1)} : \lambda \in \boldsymbol{I}_6 \backslash \boldsymbol{I}_3\}.$$

To save some space in the expressions, we introduce a numbering of $\boldsymbol{I}_6 \backslash \boldsymbol{I}_3$ given in Figure 9.

Because of $(\mathcal{S})$ and $(\mathcal{V})$, we may search

$$\tilde{\varphi}_5^{(0)} \in \operatorname{span}\{\boldsymbol{\delta}_2^{(3,1)}, \boldsymbol{\delta}_6^{(3,1)} + \boldsymbol{\delta}_7^{(3,1)}, \boldsymbol{\delta}_{11}^{(3,1)} + \boldsymbol{\delta}_{12}^{(3,1)}, \boldsymbol{\delta}_{15}^{(3,1)} + \boldsymbol{\delta}_{16}^{(3,1)}, \boldsymbol{\delta}_{14}^{(3,1)} + \boldsymbol{\delta}_{17}^{(3,1)}, \boldsymbol{\delta}_{13}^{(3,1)} + \boldsymbol{\delta}_{18}^{(3,1)}, \boldsymbol{\delta}_{10}^{(3,1)}\}.$$

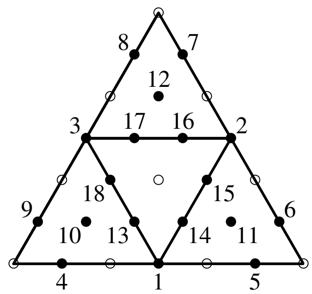

Fig. 9. $\boldsymbol{\delta}^{(3,1)}_{1..18}$.

In fact, we may also add $\boldsymbol{\delta}^{(3,0)}_{13}$ and $\tilde{\varphi}^{(0)}_9 + \tilde{\varphi}^{(0)}_{10}$ to this set of generators. However, one may verify that both these functions satisfy all homogeneous linear conditions on $\tilde{\varphi}^{(0)}_5$ given below, and thus that adding these functions will not change the span of the resulting $\tilde{\boldsymbol{\Phi}}$. In **(II)**, we already imposed on $\tilde{\varphi}^{(0)}_5$ the conditions (3.18), (3.19) (two conditions) and (3.20). Here we add the conditions

$$\tilde{\varphi}^{(0)}_5 \perp \varphi_2, \tag{3.21}$$

and (3.24) to be discussed below. Together, these six conditions determine span$\{\tilde{\varphi}^{(0)}_5\}$ uniquely.

We define $\tilde{\varphi}^{(1)}_5 = \tilde{\varphi}^{(0)}_5 / \langle \tilde{\varphi}^{(0)}_5, \varphi_5 \rangle_{L_2(\boldsymbol{T})}$. Note that (3.19) and (3.21) are equivalent to $\tilde{\varphi}^{(1)}_5 \perp \varphi_{1..4,6}$ resulting in the zero and the identity matrix in the second column of (3.14).

(V) We search $\tilde{\varphi}^{(0)}_{13..15}$ satisfying

$$\boldsymbol{\delta}^{(3,0)}_{13} \in \operatorname{span}\{\tilde{\varphi}^{(0)}_{13} + \tilde{\varphi}^{(0)}_{14} + \tilde{\varphi}^{(0)}_{15}\}, \tag{3.22}$$

which is equivalent to (3.10), and

$$\varphi_{2,4,7,8} \perp \tilde{\varphi}^{(0)}_{13}. \tag{3.23}$$

By ($\mathbb{S}$), $\varphi_2 \perp \tilde{\varphi}^{(0)}_{13}$ implies $\varphi_3 \perp \tilde{\varphi}^{(0)}_{13}$ and so $\varphi_1 \perp \tilde{\varphi}^{(0)}_{14,15}$. Since, furthermore, $\varphi_1 \perp \boldsymbol{\delta}^{(3,0)}_{13}$, we get

$$\langle \varphi_1, \tilde{\varphi}^{(0)}_{13} \rangle_{L_2(\boldsymbol{T})} = \langle \varphi_1, \tilde{\varphi}^{(0)}_{13} + \tilde{\varphi}^{(0)}_{14} + \tilde{\varphi}^{(0)}_{15} \rangle_{L_2(\boldsymbol{T})} - \langle \varphi_1, \tilde{\varphi}^{(0)}_{14} + \tilde{\varphi}^{(0)}_{15} \rangle_{L_2(\boldsymbol{T})} = 0.$$

By applying the same argument to $\varphi_4 \perp \tilde{\varphi}^{(0)}_{13}, \boldsymbol{\delta}^{(3,0)}_{13}$ and $\varphi_{7,8} \perp \tilde{\varphi}^{(0)}_{13}, \boldsymbol{\delta}^{(3,0)}_{13}$ we see that (3.22) and (3.23) imply that

$$\varphi_{1..12} \perp \tilde{\varphi}^{(0)}_{13,14,15},$$

giving the zeros in the last column of (3.14).

It turns out not to be possible to find $\tilde{\varphi}^{(0)}_{13} \in P_{3,1}(\boldsymbol{T})$ satisfying (3.22) and (3.23). Therefore, we enlarge this space with the span of the refinement of $\{\tilde{\varphi}^{(0)}_{4..6}\}$, which is a

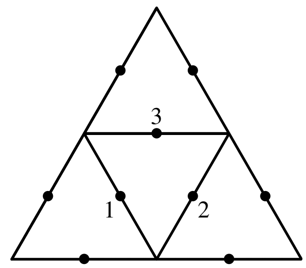

Fig. 10. $\boldsymbol{I}_4\backslash\boldsymbol{I}_2$ and $\boldsymbol{\eta}_{1..3}$.

collection of functions defined in (3.4), with index set $\boldsymbol{I}_4\backslash\boldsymbol{I}_2$. Since $\tilde{\varphi}_{13}^{(0)}$ should vanish on $\partial\boldsymbol{T}$, it is sufficient to consider only those functions from this collection corresponding to "interior points" of $\boldsymbol{I}_4\backslash\boldsymbol{I}_2$. We will denote these functions by $\boldsymbol{\eta}_{1..3}$, according to the numbering given in Figure 10.

In view of $(\mathcal{S})$ and $(\mathcal{V})$, we may search

$$\begin{aligned}\tilde{\varphi}_{13}^{(0)} \;\in\; \operatorname{span}\{&\boldsymbol{\delta}_{10}^{(3,1)},\ \boldsymbol{\delta}_{11}^{(3,1)}+\boldsymbol{\delta}_{12}^{(3,1)},\ \boldsymbol{\delta}_{13}^{(3,1)}+\boldsymbol{\delta}_{18}^{(3,1)},\ \boldsymbol{\delta}_{14}^{(3,1)}\\ &+\boldsymbol{\delta}_{17}^{(3,1)},\ \boldsymbol{\delta}_{15}^{(3,1)}+\boldsymbol{\delta}_{16}^{(3,1)}, \boldsymbol{\eta}_1,\ \boldsymbol{\eta}_2+\boldsymbol{\eta}_3,\ \boldsymbol{\delta}_{13}^{(3,0)}\}.\end{aligned}$$

Any choice of $\tilde{\varphi}_{13}^{(0)}$ fixes $\tilde{\varphi}_{14,15}^{(0)}$ by permuting the barycentric coordinates. Since $\boldsymbol{\delta}_{13}^{(3,0)} \notin \operatorname{span}(\{\boldsymbol{\delta}_{1..18}^{(3,1)}\} \cup \{\boldsymbol{\eta}_{1..3}\})$, condition (3.22) can be rewritten as

$$\tilde{\varphi}_{13}^{(0)} \in \lambda\boldsymbol{\delta}_{13}^{(3,0)} + \operatorname{span}\boldsymbol{\Theta},$$

with a scalar $\lambda \neq 0$ and with $\boldsymbol{\Theta} = \{\boldsymbol{\theta}_{1..4}\}$ being defined by

$$\begin{aligned}\boldsymbol{\theta}_1 &= \boldsymbol{\delta}_{11}^{(3,1)}+\boldsymbol{\delta}_{12}^{(3,1)}-2\boldsymbol{\delta}_{10}^{(3,1)},\\ \boldsymbol{\theta}_2 &= \boldsymbol{\delta}_{13}^{(3,1)}+\boldsymbol{\delta}_{18}^{(3,1)}-\boldsymbol{\delta}_{14}^{(3,1)}-\boldsymbol{\delta}_{17}^{(3,1)},\\ \boldsymbol{\theta}_3 &= \boldsymbol{\delta}_{15}^{(3,1)}+\boldsymbol{\delta}_{16}^{(3,1)}-\boldsymbol{\delta}_{14}^{(3,1)}-\boldsymbol{\delta}_{17}^{(3,1)},\\ \boldsymbol{\theta}_4 &= \boldsymbol{\eta}_2+\boldsymbol{\eta}_3-2\boldsymbol{\eta}_1.\end{aligned}$$

Moreover, since $\boldsymbol{\delta}_{13}^{(3,0)}$ may not be a multiple of $\tilde{\varphi}_{13}^{(0)}$, since that would mean $\tilde{\varphi}_{13}^{(0)} = \tilde{\varphi}_{14}^{(0)} = \tilde{\varphi}_{15}^{(0)}$ and, furthermore, $\varphi_{2,4,7,8} \perp \boldsymbol{\delta}_{13}^{(3,0)}$, condition (3.23) now means that

$$\tilde{\varphi}_{13}^{(0)} = \lambda\boldsymbol{\delta}_{13}^{(3,0)} + \mathbf{c}^T\boldsymbol{\Theta},$$

where $0 \neq \mathbf{c} \in \operatorname{Ker}\langle\varphi_{2,4,7,8}, \boldsymbol{\Theta}\rangle_{L_2(\boldsymbol{T})}$. A computation shows that the first three columns of $\langle\varphi_{2,4,7,8}, \boldsymbol{\theta}_{1..3}\rangle_{L_2(\boldsymbol{T})}$ are independent, and so $\boldsymbol{\theta}_4$, and thus $\tilde{\varphi}_4^{(0)}$, should be selected such that

$$\operatorname{Ker}\langle\varphi_{2,4,7,8}, \boldsymbol{\Theta}\rangle_{L_2(\boldsymbol{T})} \neq \{0\}, \tag{3.24}$$

which condition on $\tilde{\varphi}_4^{(0)}$ was already announced in step **(II)**.

One may verify that $\mathrm{span}\{\tilde{\varphi}^{(0)}_{13..15}\}$ does not depend on the choice of $\lambda \neq 0$ and $\mathbf{c} \neq 0$ in the one-dimensional space $\mathrm{Ker}\langle \varphi_{2,4,7,8}, \Theta\rangle_{L_2(\boldsymbol{T})}$. We define $\tilde{\varphi}^{(1)}_{13..15} \in \mathrm{span}\{\tilde{\varphi}^{(0)}_{13..15}\}$ by imposing $\langle \varphi_{13..15}, \tilde{\varphi}^{(1)}_{13..15}\rangle_{L_2(\boldsymbol{T})} = \mathbf{id}$.

By steps **(I)**–**(V)**, with $\tilde{\varphi}^{(2)}_{1..6,13..15} := \tilde{\varphi}^{(1)}_{1..6,13..15}$, the matrix $\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}^{(2)}\rangle_{L_2(\boldsymbol{T})}$ has the desired block-lower triangular form (3.14), which we more specifically denote by

$$\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}^{(2)}\rangle_{L_2(\boldsymbol{T})} = \begin{bmatrix} \mathbf{id} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A} & \mathbf{id} & \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} & \mathbf{id} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} & \mathbf{F} & \mathbf{id} \end{bmatrix}.$$

As was already pointed out in Remark 3.9, it is not possible to obtain a biorthogonal system. Indeed $\langle \tilde{\boldsymbol{\Phi}}^{(2)}, \boldsymbol{\Phi}\rangle^{-1}_{L_2(\mathbf{T})}\tilde{\boldsymbol{\Phi}}^{(2)}$ will violate $(\mathcal{V})$, since by this transformation some $\tilde{\varphi}^{(2)}_i$ will be updated by $\tilde{\varphi}^{(2)}_j$ with j corresponding to points on edges that do not include point i. Yet, as will be shown in step **(VI)**, by performing some "partial" transformations at the dual side, which do preserve $(\mathcal{C})$, $(\mathcal{S})$, $(\mathcal{V})$, $(\mathcal{I})$, and $(\mathcal{R})$, it is possible to introduce a number of zeros in the lower block triangular part.

(VI) With

$$\tilde{\boldsymbol{\Phi}}^{(3)} := \begin{bmatrix} \mathbf{id} & \mathbf{0} & \mathbf{0} & -\mathbf{D}^* \\ \mathbf{0} & \mathbf{id} & \mathbf{0} & -\mathbf{E}^* \\ \mathbf{0} & \mathbf{0} & \mathbf{id} & -\mathbf{F}^* \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{id} \end{bmatrix} \tilde{\boldsymbol{\Phi}}^{(2)},$$

we have

$$\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}^{(3)}\rangle_{L_2(\boldsymbol{T})} = \begin{bmatrix} \mathbf{id} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A} & \mathbf{id} & \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} & \mathbf{id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{id} \end{bmatrix}.$$

In view of $(\mathcal{V})$, note that each $\tilde{\varphi}^{(3)}_i$ is obtained by adding to $\tilde{\varphi}^{(2)}_i$ a linear combination of $\tilde{\varphi}^{(2)}_{13..15}$, whose functions vanish on $\partial\boldsymbol{T}$.

Let $\hat{\mathbf{A}}$ be the matrix obtained from $\mathbf{A} = (\langle \varphi_i, \tilde{\varphi}^{(3)}_j\rangle_{L_2(\boldsymbol{T})})_{i\in\{4..6\},\, j\in\{1..3\}}$ by replacing those entries by zeros which correspond to pairs of points on different edges. With

$$\tilde{\boldsymbol{\Phi}}^{(4)} := \begin{bmatrix} \mathbf{id} & -\hat{\mathbf{A}}^* & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{id} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{id} \end{bmatrix} \tilde{\boldsymbol{\Phi}}^{(3)},$$

we get

$$\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}^{(4)}\rangle_{L_2(\boldsymbol{T})} = \begin{bmatrix} \mathbf{id} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}-\hat{\mathbf{A}} & \mathbf{id} & \mathbf{0} & \mathbf{0} \\ \mathbf{G} & \mathbf{C} & \mathbf{id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{id} \end{bmatrix},$$

where $\mathbf{G} := \mathbf{B} - \mathbf{C}\hat{\mathbf{A}}$.

Finally, with $\hat{\mathbf{G}}$, $\hat{\mathbf{C}}$ being the matrices obtained from

$$\mathbf{G} = (\langle \varphi_i, \tilde{\varphi}_j^{(4)} \rangle_{L_2(\boldsymbol{T})})_{i\in\{7..12\}, j\in\{1..3\}}, \qquad \mathbf{C} = (\langle \varphi_i, \tilde{\varphi}_j^{(4)} \rangle_{L_2(\boldsymbol{T})})_{i\in\{7..12\}, j\in\{4..6\}},$$

respectively, by replacing those entries by zeros which correspond to pairs of points on different edges, and

$$\tilde{\boldsymbol{\Phi}} := \begin{bmatrix} \mathbf{id} & \mathbf{0} & -\hat{\mathbf{G}}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{id} & -\hat{\mathbf{C}}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{id} \end{bmatrix} \tilde{\boldsymbol{\Phi}}^{(4)},$$

we get

$$\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}} \rangle_{L_2(\boldsymbol{T})} = \begin{bmatrix} \mathbf{id} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A} - \hat{\mathbf{A}} & \mathbf{id} & \mathbf{0} & \mathbf{0} \\ \mathbf{G} - \hat{\mathbf{G}} & \mathbf{C} - \hat{\mathbf{C}} & \mathbf{id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{id} \end{bmatrix}.$$

From the definitions of $\tilde{\boldsymbol{\Phi}}^{(3)}$, $\tilde{\boldsymbol{\Phi}}^{(4)}$, and $\tilde{\boldsymbol{\Phi}}$ it follows that the matrix $\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}} \rangle_{L_2(\boldsymbol{T})}$ only contains possibly nonzero off-diagonal entries $\langle \varphi_i, \tilde{\varphi}_j \rangle_{L_2(\boldsymbol{T})}$ on the positions $(i, j) = (5, 1), (9, 1), (9, 4)$, and $(10, 4)$, as well as those that correspond to permuting barycentric coordinates. All these entries correspond to pairs of points that are included on different edges.

Remark 3.10. The fact that $\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}} \rangle_{L_2(\boldsymbol{T})} \neq \mathbf{id}$ and thus $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)} \neq \mathbf{id}$ has clearly an adverse effect on the sizes of the supports of the dual scaling functions from $\langle \tilde{\Phi}_j, \Phi_j \rangle_{L_2(\Omega)}^{-1} \tilde{\Phi}_j$ and thus on that of the wavelets and dual wavelets. Yet, by computing the wavelet and inverse wavelet transforms in the way as exposed in Remark 3.4, the fact that $\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}} \rangle_{L_2(\boldsymbol{T})} \neq \mathbf{id}$ only affects the computation of these transforms in the sense that on each level $j + 1$, in addition an application of the matrix $\langle \Phi_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)}^{-T}$ has to be performed. Assuming a uniform square grid, a simple calculation using the fact that $\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}} \rangle_{L_2(\boldsymbol{T})}$ has only a few nonzero off-diagonal entries shows that the total number of operations needed for these computations is less than half the number of degrees of freedom on the highest level.

Together, steps **(I)–(VI)** fully describe the procedure to find $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$. A sufficient ingredient for the actual calculations is the matrix $\langle \boldsymbol{\Delta}^{(1,2)}, \boldsymbol{\Delta}^{(3,2)} \rangle_{L_2(\boldsymbol{T})}$. These calculations result in a collection $\boldsymbol{\Phi}$ defined by

$$\begin{aligned}
\varphi_1 &= \boldsymbol{\delta}_1^{(1,2)} + \frac{101}{2490}(\boldsymbol{\delta}_4^{(1,2)} + \boldsymbol{\delta}_6^{(1,2)} + \boldsymbol{\delta}_{13}^{(1,2)}) - \frac{173}{996}(\boldsymbol{\delta}_7^{(1,2)} + \boldsymbol{\delta}_{12}^{(1,2)}) \\
&\quad - \frac{9}{1660}(\boldsymbol{\delta}_8^{(1,2)} + \boldsymbol{\delta}_{11}^{(1,2)} + \boldsymbol{\delta}_{14}^{(1,2)} + \boldsymbol{\delta}_{15}^{(1,2)}), \\
\varphi_4 &= \boldsymbol{\delta}_4^{(1,2)} + \frac{361}{658}(\boldsymbol{\delta}_7^{(1,2)} + \boldsymbol{\delta}_8^{(1,2)}) - \frac{1219}{3290}(\boldsymbol{\delta}_{13}^{(1,2)} + \boldsymbol{\delta}_{14}^{(1,2)}) + \frac{8}{35}\boldsymbol{\delta}_{15}^{(1,2)}, \\
\varphi_7 &= \boldsymbol{\delta}_7^{(1,2)} - \frac{353029}{564499}\boldsymbol{\delta}_4^{(1,2)} - \frac{1033547}{2822495}\boldsymbol{\delta}_{13}^{(1,2)} + \frac{131990}{564499}\boldsymbol{\delta}_{14}^{(1,2)} + \frac{342166}{2822495}\boldsymbol{\delta}_{15}^{(1,2)}, \\
\varphi_{13} &= \boldsymbol{\delta}_{13}^{(1,2)}.
\end{aligned}$$

At the dual side, $\tilde{\mathbf{\Phi}}^{(2)}$ is defined by

$$\begin{aligned}
\tilde{\varphi}_1^{(2)} &= \frac{1}{\mathrm{vol}(\boldsymbol{T})}\frac{415}{3}\boldsymbol{\delta}_1^{(3,0)},\\
\tilde{\varphi}_4^{(2)} &= \frac{1}{\mathrm{vol}(\boldsymbol{T})}\Bigg[\frac{-9301424162156}{1912996185027}\boldsymbol{\delta}_1^{(3,1)} + \frac{111448863524740}{17216965665243}(\boldsymbol{\delta}_4^{(3,1)} + \boldsymbol{\delta}_5^{(3,1)})\\
&\quad + \frac{120098054733160}{5738988555081}(\boldsymbol{\delta}_{10}^{(3,1)} + \boldsymbol{\delta}_{11}^{(3,1)}) + \frac{791219875405708}{17216965665243}(\boldsymbol{\delta}_{13}^{(3,1)} + \boldsymbol{\delta}_{14}^{(3,1)})\\
&\quad + \frac{349505115151472}{17216965665243}(\boldsymbol{\delta}_{18}^{(3,1)} + \boldsymbol{\delta}_{15}^{(3,1)}) + \frac{545882055813164}{17216965665243}(\boldsymbol{\delta}_{17}^{(3,1)}\\
&\quad + \boldsymbol{\delta}_{16}^{(3,1)}) + \frac{29746337340748}{17216965665243}\boldsymbol{\delta}_{12}^{(3,1)}\Bigg],\\
\tilde{\varphi}_7^{(2)} &= \frac{1}{\mathrm{vol}(\boldsymbol{T})}\Bigg[\frac{16214441833474060}{183117091220847}\boldsymbol{\delta}_7^{(3,0)} + \frac{9269556596061196}{183117091220847}\boldsymbol{\delta}_8^{(3,0)}\Bigg]\\
&\quad - \frac{359961477817185491}{89252626683938760}\tilde{\varphi}_4^{(2)},\\
\tilde{\varphi}_{13}^{(2)} &= \frac{1}{\mathrm{vol}(\boldsymbol{T})}\Bigg[\frac{512}{135}\boldsymbol{\delta}_{13}^{(3,0)} - \frac{429691798688}{26453357865}(\boldsymbol{\delta}_{11}^{(3,1)} + \boldsymbol{\delta}_{12}^{(3,1)} - 2\boldsymbol{\delta}_{10}^{(3,1)})\\
&\quad + \frac{146540371984}{5290671573}(\boldsymbol{\delta}_{13}^{(3,1)} + \boldsymbol{\delta}_{18}^{(3,1)} - \boldsymbol{\delta}_{14}^{(3,1)} - \boldsymbol{\delta}_{17}^{(3,1)})\\
&\quad - \frac{403973483368}{26453357865}(\boldsymbol{\delta}_{15}^{(3,1)} + \boldsymbol{\delta}_{16}^{(3,1)} - \boldsymbol{\delta}_{14}^{(3,1)} - \boldsymbol{\delta}_{17}^{(3,1)})\Bigg]\\
&\quad + \frac{637665395009}{1289356257420}(\boldsymbol{\eta}_2 + \boldsymbol{\eta}_3 - 2\boldsymbol{\eta}_1),
\end{aligned}$$

where $\boldsymbol{\eta}_{1..3}$ are the functions that correspond to "interior points" (see Figure 10) from the refinement of the above $\{\tilde{\varphi}_{4..6}^{(2)}\}$ defined by (3.4). The transformations described in step **(VI)** yield the collection $\tilde{\mathbf{\Phi}}$ given by

$$\begin{aligned}
\tilde{\varphi}_1 &= \tilde{\varphi}_1^{(2)} - \frac{10209}{21056}(\tilde{\varphi}_4^{(2)} + \tilde{\varphi}_6^{(2)}) - \frac{110772169122200294 4137}{737201106569595885568}(\tilde{\varphi}_7^{(2)} + \tilde{\varphi}_{12}^{(2)})\\
&\quad - \frac{193438650565173948439}{737201106569595885568}(\tilde{\varphi}_8^{(2)} + \tilde{\varphi}_{11}^{(2)}) - \frac{269103595837}{10869175296}\tilde{\varphi}_{13}^{(2)}\\
&\quad - \frac{140609892845}{5434587648}(\tilde{\varphi}_{14}^{(2)} + \tilde{\varphi}_{15}^{(2)}),\\
\tilde{\varphi}_4 &= \tilde{\varphi}_4^{(2)} + \frac{2496527831240624965}{17278150935224903568}(\tilde{\varphi}_7^{(2)} + \tilde{\varphi}_8^{(2)})\\
&\quad - \frac{2034877615278695}{36035450441065728}(\tilde{\varphi}_{13}^{(2)} + \tilde{\varphi}_{14}^{(2)}) + \frac{16741222248937735}{36035450441065728}\tilde{\varphi}_{15}^{(2)},
\end{aligned}$$

$$\begin{aligned}\tilde{\varphi}_7 &= \tilde{\varphi}_7^{(2)} + \frac{4978122426946063}{651082991007456}\tilde{\varphi}_{13}^{(2)} + \frac{6063260745291823}{651082991007456}\tilde{\varphi}_{14}^{(2)} \\ &\quad + \frac{4163663044298017}{651082991007456}\tilde{\varphi}_{15}^{(2)}, \\ \tilde{\varphi}_{13} &= \tilde{\varphi}_{13}^{(2)}.\end{aligned}$$

The nonzero off-diagonal entries of $\langle \mathbf{\Phi}, \tilde{\mathbf{\Phi}} \rangle_{L_2(\boldsymbol{T})}$ are given by

$$\begin{aligned}\langle \varphi_5, \tilde{\varphi}_1 \rangle_{L_2(\boldsymbol{T})} &= \frac{747}{2632}, & \langle \varphi_9, \tilde{\varphi}_1 \rangle_{L_2(\boldsymbol{T})} &= \frac{-769556495}{8164913536}, \\ \langle \varphi_9, \tilde{\varphi}_4 \rangle_{L_2(\boldsymbol{T})} &= \frac{115709629}{9185527728}, & \langle \varphi_{10}, \tilde{\varphi}_4 \rangle_{L_2(\boldsymbol{T})} &= \frac{1601470997}{9185527728},\end{aligned} \tag{3.25}$$

with, as always, equal values for those entries that correspond to permuting barycentric coordinates.

The resulting collections Φ_j, $\langle \tilde{\Phi}_j, \Phi_j \rangle_{L_2(\Omega)}^{-1} \tilde{\Phi}$ are biorthogonal, uniformly local, uniform $L_2(\Omega)$-Riesz systems. The primal collection is a basis for the space of continuous piecewise linears with respect to τ_{j+2}. The spans of the dual collections are nested as functions of j, and satisfy $(\tilde{\mathfrak{B}})$ and $(\tilde{\mathfrak{J}})$ with $\tilde{\gamma} = \frac{3}{2}$ and $\tilde{d} = 4$.

3.5. *The Case* $(n, d, m, \tilde{d}) = (2, 5, 0, 4)$

As in Section 3.4, $(d-1)2^m = 4$, and to construct $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$, we follow exactly the same procedure from that section described in steps **(I)–(VI)**, except that we now start with $\mathbf{\Phi}^{(0)} = \mathbf{\Delta}^{(4,0)}$ instead of $\mathbf{\Delta}^{(1,2)}$. The actual computations using $\langle \Delta^{(4,0)}, \Delta^{(3,2)} \rangle_{L_2(\boldsymbol{T})}$ now result in a collected $\mathbf{\Phi}$ defined by

$$\begin{aligned}\varphi_1 &= \boldsymbol{\delta}_1^{(4,0)} - \frac{1}{40}(\boldsymbol{\delta}_4^{(4,0)} + \boldsymbol{\delta}_6^{(4,0)} + \boldsymbol{\delta}_{13}^{(4,0)}) - \frac{3}{640}(\boldsymbol{\delta}_7^{(4,0)} + \boldsymbol{\delta}_{12}^{(4,0)}) \\ &\quad + \frac{13}{640}(\boldsymbol{\delta}_8^{(4,0)} + \boldsymbol{\delta}_{11}^{(4,0)} + \boldsymbol{\delta}_{14}^{(4,0)} + \boldsymbol{\delta}_{15}^{(4,0)}), \\ \varphi_4 &= \boldsymbol{\delta}_4^{(4,0)} + \frac{3}{4}(\boldsymbol{\delta}_7^{(4,0)} + \boldsymbol{\delta}_8^{(4,0)}) - \frac{1}{8}(\boldsymbol{\delta}_{13}^{(4,0)} + \boldsymbol{\delta}_{14}^{(4,0)}), \\ \varphi_7 &= \boldsymbol{\delta}_7^{(4,0)} - \frac{224}{285}\boldsymbol{\delta}_4^{(4,0)} - \frac{259}{1140}\boldsymbol{\delta}_{13}^{(4,0)} - \frac{23}{380}\boldsymbol{\delta}_{14}^{(4,0)} + \frac{23}{190}\boldsymbol{\delta}_{15}^{(4,0)}, \\ \varphi_{13} &= \boldsymbol{\delta}_{13}^{(4,0)}.\end{aligned}$$

At the dual side, $\tilde{\mathbf{\Phi}}^{(2)}$ is defined by

$$\begin{aligned}\tilde{\varphi}_1^{(2)} &= \frac{150}{\mathrm{vol}(\boldsymbol{T})}\boldsymbol{\delta}_1^{(3,0)}, \\ \tilde{\varphi}_4^{(2)} &= \frac{1}{\mathrm{vol}(\boldsymbol{T})}\left[\frac{10534545}{112976}\boldsymbol{\delta}_1^{(3,1)} - \frac{837515}{112976}(\boldsymbol{\delta}_4^{(3,1)} + \boldsymbol{\delta}_5^{(3,1)})\right.\end{aligned}$$

$$- \frac{319865}{56488}(\boldsymbol{\delta}_{10}^{(3,1)} + \boldsymbol{\delta}_{11}^{(3,1)}) - \frac{1398915}{112976}(\boldsymbol{\delta}_{13}^{(3,1)} + \boldsymbol{\delta}_{14}^{(3,1)})$$

$$- \frac{1385055}{56488}(\boldsymbol{\delta}_{18}^{(3,1)} + \boldsymbol{\delta}_{15}^{(3,1)}) - \frac{2232895}{112976}(\boldsymbol{\delta}_{17}^{(3,1)} + \boldsymbol{\delta}_{16}^{(3,1)}) - \frac{93205}{112976}\boldsymbol{\delta}_{12}^{(3,1)}\Bigg],$$

$$\tilde{\varphi}_7^{(2)} = \frac{1}{\mathrm{vol}(\boldsymbol{T})}\left[\frac{90905}{3528}\boldsymbol{\delta}_7^{(3,0)} - \frac{32575}{3528}\boldsymbol{\delta}_8^{(3,0)}\right] - \frac{5833}{12348}\tilde{\varphi}_4^{(2)},$$

$$\tilde{\varphi}_{13}^{(2)} = \frac{1}{\mathrm{vol}(\boldsymbol{T})}\Bigg[\frac{35}{12}\boldsymbol{\delta}_{13}^{(3,0)} - \frac{67744}{12339}(\boldsymbol{\delta}_{11}^{(3,1)} + \boldsymbol{\delta}_{12}^{(3,1)} - 2\boldsymbol{\delta}_{10}^{(3,1)})$$

$$- \frac{51068}{12339}(\boldsymbol{\delta}_{13}^{(3,1)} + \boldsymbol{\delta}_{18}^{(3,1)} - \boldsymbol{\delta}_{14}^{(3,1)} - \boldsymbol{\delta}_{17}^{(3,1)})$$

$$+ \frac{11380}{12339}(\boldsymbol{\delta}_{15}^{(3,1)} + \boldsymbol{\delta}_{16}^{(3,1)} - \boldsymbol{\delta}_{14}^{(3,1)} - \boldsymbol{\delta}_{17}^{(3,1)})\Bigg] - \frac{112976}{431865}(\eta_2 + \eta_3 - 2\eta_1),$$

where $\boldsymbol{\eta}_{1..3}$ are the functions that correspond to "interior points" (see Figure 10) from the refinement of the above $\{\tilde{\varphi}_{4..6}^{(2)}\}$ defined by (3.4). Finally, the collection $\tilde{\boldsymbol{\Phi}}$ is given by

$$\tilde{\varphi}_1 = \tilde{\varphi}_1^{(2)} - \frac{10}{21}(\tilde{\varphi}_4^{(2)} + \tilde{\varphi}_6^{(2)}) - \frac{162721}{40831}(\tilde{\varphi}_7^{(2)} + \tilde{\varphi}_{12}^{(2)}) - \frac{14913}{5833}(\tilde{\varphi}_8^{(2)} + \tilde{\varphi}_{11}^{(2)})$$

$$+ \frac{128480}{7203}\tilde{\varphi}_{13}^{(2)} + \frac{21800}{7203}(\tilde{\varphi}_{14}^{(2)} + \tilde{\varphi}_{15}^{(2)}),$$

$$\tilde{\varphi}_4 = \tilde{\varphi}_4^{(2)} + \frac{119012}{87495}(\tilde{\varphi}_7^{(2)} + \tilde{\varphi}_8^{(2)}) - \frac{4807}{3087}(\tilde{\varphi}_{13}^{(2)} + \tilde{\varphi}_{14}^{(2)}) + \frac{57628}{71001}\tilde{\varphi}_{15}^{(2)},$$

$$\tilde{\varphi}_7 = \tilde{\varphi}_7^{(2)} - \frac{3008}{1029}\tilde{\varphi}_{13}^{(2)} + \frac{1108}{1029}\tilde{\varphi}_{14}^{(2)} - \frac{29545}{94668}\tilde{\varphi}_{15}^{(2)},$$

$$\tilde{\varphi}_{13} = \tilde{\varphi}_{13}^{(2)}.$$

As in Section 3.4, $\boldsymbol{\Phi}$, $\tilde{\boldsymbol{\Phi}}$ are not biorthogonal. The nonzero off-diagonal entries of $\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}} \rangle_{L_2(\boldsymbol{T})}$ are given by

$$\langle \varphi_5, \tilde{\varphi}_1 \rangle_{L_2(\boldsymbol{T})} = \frac{10}{21}, \qquad \langle \varphi_9, \tilde{\varphi}_1 \rangle_{L_2(\boldsymbol{T})} = -\frac{64}{171},$$

$$\langle \varphi_9, \tilde{\varphi}_4 \rangle_{L_2(\boldsymbol{T})} = \frac{181}{570}, \qquad \langle \varphi_{10}, \tilde{\varphi}_4 \rangle_{L_2(\boldsymbol{T})} = \frac{371}{570},$$

with, as always, equal values for those entries that correspond to permuting barycentric coordinates.

The resulting collections Φ_j, $\langle \tilde{\Phi}_j, \Phi_j \rangle_{L_2(\Omega)}^{-1} \tilde{\Phi}_j$ are biorthogonal, uniformly local, uniform $L_2(\Omega)$-Riesz systems. The primal collection is a basis for the space of continuous piecewise quartics with respect to τ_j. The spans of the dual collections are nested as functions of j, and satisfy $(\tilde{\mathcal{B}})$ and $(\tilde{\mathcal{J}})$ with $\tilde{\gamma} = \frac{3}{2}$ and $\tilde{d} = 4$.

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